CALTECH LECTURE
NOTES
Ph 129
FOX QUESTIONS

Evaluation of Integrals and Infinite Series I

10D. Let $a > 0$, $b > 0$. By integrating

$$\frac{\log(z+a)}{z^2 + b^2}$$

round an appropriate contour, or otherwise, evaluate the integral

$$F(a, b) = \int_{-\infty}^{\infty} \frac{\log(x^2 + a^2)}{x^2 + b^2} dx.$$

[As a check, verify that $F(1, 1) = \pi \log 2$.]

Outline one method that could be used to discover whether the result is still valid when $a = 0$.

2. (i) Obtain the coefficients A_n in the expansion

$$\cosh ax = \sum_{n=0}^{\infty} A_n \cos nx$$

for x in the range $(-\pi, \pi)$.

(ii) Evaluate

$$\int_0^{\pi/2} \log(1 + p \tan^2 x) dx,$$

where p is any positive real number.

8. Determine for what values of the real parameter x (i) the sequence $nx/(x^2 + n^2)$, (ii) the series $\Sigma nx/(x^2 + n^2)$, is uniformly convergent in the range $x \geqslant 1$.

6. Define the principal value of the logarithm of a complex number, and obtain, with proof, the power series for the principal value of $\log(1+x)$ where $|x| \leqslant 1$ and $x \neq -1$. (You may assume known any general theorem on power series, but any such theorem which you use should be accurately quoted.)

Sum the series

$$\sin\theta + \tfrac{1}{2}\sin 2\theta + \tfrac{1}{3}\sin 3\theta + \tfrac{1}{4}\sin 5\theta + \dots$$

for $0 < \theta < \pi$. By substituting suitable values of θ, or otherwise, prove that

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \dots - \frac{1}{8n-1} + \frac{1}{8n+1} - \dots = \tfrac{1}{8}\pi(1 + \sqrt{2}).$$

7. By a suitable contour integration evaluate

$$\int_0^{\infty} \frac{\log x \, dx}{1 + x^2}.$$

6. 1 H. Discuss the convergence and absolute convergence of the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2 - (-1)^n}$$

for all real values of x and all real non-zero values of α, stating fully any general theorems on convergence that are used.

5 H. Prove that a uniformly convergent series of integrable functions may be integrated term by term over a bounded closed interval.

Prove that

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n+2)!} = -2 \int_0^{\frac{1}{2}\pi} \log\left(1 - \frac{1}{4} \sin^2 \theta\right) \frac{d\theta}{\sin^2 \theta}.$$

8. 9 G. Let $I_\alpha = \int_0^\infty \frac{\sqrt{x} \log x \, dx}{x^2 + 1}$. Prove that $I_0 = \infty$, $I_2 = 0$, and evaluate I_4 by contour integration or otherwise.

9. 2 H. Show that, if α_n is real and $1 + \alpha_n > 0$ for every positive integer n, and if $\Sigma \alpha_n^2$ converges, then $\Pi(1 + \alpha_n)$ and $\Sigma \alpha_n$ both converge or both diverge. Show also that, if $\Sigma \alpha_n$ converges, $\Pi(1 + \alpha_n)$ converges or diverges to zero according as $\Sigma \alpha_n^2$ converges or diverges.

Deduce that $\Pi[1 - (-1)^n n^{-\frac{1}{2}}]$ diverges to zero.

Prove also that $\Pi(1 + \alpha_n)$ converges, while $\Sigma \alpha_n$, $\Sigma \alpha_n^2$ both diverge, when

$$a_{2n-1} = r^{-1} + r^{-\frac{1}{2}} + r^{-\frac{1}{4}}, \quad a_{2n} = -r^{-\frac{1}{2}}.$$

10. 1. The series $\sum_{n=0}^{\infty} u_n(s)$ of real or complex terms converges uniformly for $s = 1, 2, 3, \ldots$. Suppose that

$$\lim_{n \to \infty} u_n(s) = a_s$$

exists for each r. Show that $\sum_r a_r$ is convergent and that

$$\lim_{n \to \infty} \sum_{r=0}^{\infty} u_r(s) = \sum_{r=0}^{\infty} a_r.$$

Hence, or otherwise, show that

$$\lim_{s \to \infty} s[(1 + s)^{1/s} - 1] = \sum_{r=1}^{\infty} (-1)^{r+1} r^2 / r!$$

for all s with $|s| < 1$.

[Any standard test for uniformity of convergence may be used without proof but must be clearly stated. The binomial theorem for positive rational exponent may be assumed without proof but no properties of the exponential or logarithmic functions may be assumed.]

11. 2. Define the residue of a function at an isolated singularity.

Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{(1 + x^2)^2} dx$$

by the theory of residues, or otherwise.

12. 9 B. Prove that, if $-\pi < \alpha < \pi$,

$$\int_0^\infty \frac{\sinh \alpha x}{\sinh \pi x} dx = \tan \frac{1}{2} \alpha.$$

D. Define a *regular singularity* of the equation

$$P(z)\frac{d^2w}{dz^2} + Q(z)\frac{dw}{dz} + R(z)w = 0,$$

where $P(z)$, $Q(z)$ and $R(z)$ are regular functions of the complex variable z.

Verify that $z = 0$ is a regular singularity of the equation

$$4z\frac{d^2w}{dz^2} + 2\frac{dw}{dz} + w = 0,$$

and obtain two linearly independent solutions involving power series in z. Describe domains in which these solutions are valid.

D. Given a differential equation of the form

$$(Az + B)\frac{d^2w}{dz^2} + (Cz + D)\frac{dw}{dz} + (Ez + F)w = 0,$$

show how to determine a function $f(t)$ and a contour Γ so that

$$w = \int_\Gamma e^{zt} f(t)\, dt$$

is a solution of the equation.

Apply the method to find two linearly independent solutions of the equation

$$z\frac{d^2w}{dz^2} + \frac{dw}{dz} - w = 0,$$

giving reasons for supposing that the solutions obtained are linearly independent.

E. The function $\psi(x, \lambda)$ is defined for $a \leqslant x \leqslant b$ and satisfies the equation

$$\frac{d^2\psi(x, \lambda)}{dx^2} + \{\lambda - r(x)\}\psi(x, \lambda) = 0$$

with the boundary condition $\psi(a, \lambda) = 0$; the function $\psi(x, \mu)$ is defined similarly. If $\mu > \lambda$, show that $\psi(x, \mu)$ vanishes at least once between $x_n(\lambda)$ and $x_{n+1}(\lambda)$, where $x_n(\lambda)$, $x_{n+1}(\lambda)$ denote successive zeros of $\psi(x, \lambda)$. Deduce that $\psi(x, \mu)$ has at least as many zeros in (a, b) as $\psi(x, \lambda)$.

By comparing the above equation with an equation with constant coefficients, or otherwise, show that the number of zeros of $\psi(x, \lambda)$ in (a, b) tends to infinity with λ.

J. Define the Green's function of a second-order ordinary linear non-homogeneous differential equation with given boundary conditions.

Give a method of constructing this function, given two independent solutions of the associated homogeneous equation which do not satisfy the boundary conditions. Explain how the function may be used to solve the non-homogeneous equation. Illustrate your exposition by considering the equation $y'' = x$, with the boundary conditions $y(0) = y'(1) = 0$.

5. Obtain formal expansions in powers of x of two independent solutions of the differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + \alpha y = 0 \qquad (1)$$

where α is a constant. Show that one solution is a polynomial in x if

$$\alpha = s^2 - 1 \quad (s = 1, 2, 3, \ldots).$$

If α is not of this form prove that there are no non-trivial power-series solutions of (1) that are convergent at both $x = 1$ and $x = -1$.

6. Examine each of the following statements about the differential equation

$$w'' + f(z)w' + g(z)w = 0,$$

say whether or not it is true, and justify your answer:

(i) if $f(z)$ and $g(z)$ are periodic, with period 2π, then every solution of the equation also has period 2π;

(ii) at least one of $f(z), g(z)$ has a singularity at z_0, then every non-trivial solution of the equation has a singularity at z_0;

(iii) in the case $f(z) = \sec z$, $g(z) = \csc z$, there is a non-trivial solution that is a polynomial in $\sin z$ and $\cos z$;

(iv) in the case $f(z) = \sec z$, $g(z) = \csc z$, there is a non-trivial solution that is regular for all z.

7. Explain what is meant by the eigenvalues and eigenfunctions of the boundary-value problem

$$u''(x) + [\lambda - q(x)] u(x) = 0 \quad (a < x < b),$$
$$u(a)\cos\alpha + u'(a)\sin\alpha = 0, \quad u(b)\cos\beta + u'(b)\sin\beta = 0,$$

where $q(x)$ is continuous, a, b, α, and β are real constants, and λ is a parameter. Prove that the eigenvalues are real, and that eigenfunctions associated with two distinct eigenvalues are orthogonal.

If $q(x) \equiv 0$, $\alpha = 0$, and $\pi = 0$, obtain an equation for the eigenvalues and discuss its solution when $\tan\beta \geqslant 0$ and when $\tan\beta < -b$. How are these results modified if either

$$-b < \tan\beta < 0 \quad \text{or} \quad \tan\beta = -b?$$

Find the eigenfunctions in these four cases.

8. Obtain the general solution of

$$x^2y'' + xy' + (x^2 - 2)y = 0$$

in the form of series in ascending powers of x, when $2n$ is not an even integer. Indicate briefly how to obtain the general solution when $2n$ is an even integer.

9

3C. The equation

$$y'' + \rho(x)y' + q(x)y = 0$$

has two linearly independent solutions $y = u(x)$, $y = v(x)$ valid in $a \leqslant x \leqslant b$ chosen so that $u(a) = v(b) = 0$, and it may be assumed that $u(x)$, $v(x)$ are non-zero. Obtain the solution of the equation

$$y'' + p(x)y' + q(x)y = f(x),$$

subject to $u(a) = y(b) = 0$, in the form

$$y(x) = \int_a^b G(x, \xi) f(\xi) d\xi,$$

giving explicit formulae for the Green's function $G(x, \xi)$ in the ranges $a \leqslant \xi \leqslant x$ and $x \leqslant \xi \leqslant b$.

Show also that, in the case where the condition $u(a) = 0$ leads to $u(b) = 0$, the problem as posed has in general no solution, but has infinitely many solutions if $f(x)$ satisfies a certain integral condition.

10

D. Prove that the differential equation

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + \lambda^2 y = 0 \tag{1}$$

has a non-zero solution regular for all finite x if and only if $\lambda = \lambda_n$ where for some non-negative integer n, the value λ_n has the value $2n(2n+3)$ if n is even and $(2n-1)(2n+2)$ if n is odd.

If, for each n, $y_n(x)$ is such a solution of (1) for $\lambda = \lambda_n$, prove that

$$\int_{-1}^{1} x^2 y_m(x) y_n(x) dx = 0 \quad \text{if} \quad m \neq n.$$

11

Find the eigenvalues of the differential equation

$$xy'' + (1-x)y' + \alpha y = 0 \quad (\alpha = \text{constant}),$$

for the range $0 \leqslant x < \infty$ and for the boundary conditions that y is finite at $x = 0$ and as x tends to infinity y does not become infinite of an order higher than a positive power of x.

Find the self-adjoint form of the differential equation and obtain the orthogonality relation for the eigenfunctions.

12

2D. If $f(t)$ and $g(t)$ are continuous, and a and n are constants, obtain by the method of variation of parameters, or otherwise, the general solution of

$$\frac{dx}{dt} + nx - ny = f(t),$$

$$\frac{dy}{dt} + nx + ny = g(t).$$

If n is an integer, show that there is no solution for which x vanishes at $t = 0$ and at $t = \pi$, unless

$$\int_0^\pi e^{nt}\{f\cos nt - g\sin nt\} dt = 0.$$

If such a solution exists, is it unique?

13. The function $f(x, y)$ is continuous in the rectangle R defined by

$$|x - x_0| < a, \quad |y - y_0| < b,$$

and satisfies the Lipschitz condition

(*) $\qquad |f(x, y_1) - f(x, y_2)| < K|y_1 - y_2| \quad \{(x, y_1), (x, y_2) \in R\}.$

Prove that there is an interval $|x - x_0| < h \ (\leqslant a)$ in which the differential equation

$$y' = f(x, y) \qquad (y' = dy/dx)$$

has a unique solution $y(x)$ with $y(x_0) = y_0$.

By considering the equation $y' = y^2$ and taking $x_0 = y_0 = 0$, or otherwise, show that the full conclusion does not necessarily hold if (*) is not satisfied. State which part of the conclusion breaks down in this example.

14. State a theorem on existence and uniqueness of solutions, corresponding to given initial conditions, of the differential equation

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} \qquad (\mathbf{x}' = d\mathbf{x}/dt)$$

for the $n \times 1$ matrix (or column vector) $\mathbf{x} = \mathbf{x}(t)$, where $\mathbf{A}(t)$ is a given (complex) $n \times n$ matrix continuous on the real t interval I. Assuming such a theorem, prove that the solutions $\mathbf{x}$ form an n-dimensional vector space over the field of complex numbers.

Define a fundamental matrix of solutions. Show that, if $\mathbf{U} = \mathbf{U}(t)$ is a given fundamental matrix, any solution $\mathbf{x} = \mathbf{x}(t)$ can be expressed uniquely in the form $\mathbf{x} = \mathbf{Uc}$, where $\mathbf{c}$ is a constant column vector. Show also that a set of n solutions $\mathbf{x}_i = \mathbf{Uc}_i$ $(i = 1, \ldots, m)$ is linearly dependent if and only if the set of constant vectors $\mathbf{c}_i$ is linearly dependent.

15. Obtain, using series in ascending powers of z, the general solution

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - 1)w = 0.$$

For what values of z do your series converge?

Find in the same form the solution of

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - 1)w = z^2.$$

16. (i) Solve the differential equation (of Riccati's form):

$$z(x^2 - 1)y' + x^2 - (x^2 - 1)y - y^2 = 0,$$

where $y' = dy/dx$.

(ii) Prove the orthogonality property of solutions of the real Sturm-Liouville equation

$$\{p(x)y'\}' + \{q(x) + \lambda r(x)\}y = 0,$$

in the case $p(x) > 0$, and $y(a) = ky'(a)$, $y(b) = ky'(b)$, where k is a constant and $[a, b]$ is the range of integration.

Hence show that all the eigenvalues of the equation are real.

8 D Define the Wronskian of n solutions of a linear differential equation of order n with leading coefficient 1. Show that it vanishes (identically) if and only if they are linearly dependent solutions.

Show how an inhomogeneous equation can be solved by the method of variation of parameters when the solutions of the associated homogeneous equation are known.

Find by inspection one solution of

$$L(x,y) \equiv y''(\tfrac{1}{4}x^2 - x) + y'(-\tfrac{1}{2}x^2 + 1) + y(x - 1) = 0,$$

in the range $[\tfrac{1}{3}, \tfrac{1}{2}]$, and hence find the general solution. Hence solve $L(x,y) = 1$.

(18) *No question.*

19 D Show how the equation

$$(a_n x + b_n) w^{(n)} + \ldots + (a_1 x + b_1) w' + (a_0 x + b_0) w = 0,$$

where the a_n, b_n are constants, and dashes denote differentiations with respect to x, may be solved using contour integration. Indicate what types of contour are suitable when different contours may be expected to produce independent solutions. Illustrate by considering (i) the trivial equation $w'' = 0$, (ii) $w'' = xw$.

20 Define the *Wronskian* $W(f_1, f_2, ..., f_n)$ of a set $(f_1, f_2, ..., f_n)$ of real functions of a real variable, each of which is differentiable at least $(n-1)$ times. Show that, if $n \geqslant 2$,

$$W(f_1, f_2, ..., f_n) = 0 \quad \text{for all} \quad x \in [a, b]$$

is a necessary, but not sufficient, condition for the functions to be linearly dependent.

Suppose now that each f_i has a continuous nth derivative in $[a, b]$. Show that a necessary and sufficient condition that there should exist a homogeneous linear differential equation of the form

$$y^{(n)} + a_1(x) y^{(n-1)} + ... + a_{n-1}(x) y' + a_n(x) y = 0$$

(where each $a_i(x)$ is continuous in $[a, b]$), of which $(f_1, f_2, ..., f_n)$ is a fundamental system of solutions in $[a, b]$, is that $W(f_1, f_2, ..., f_n)$ should be non-zero everywhere in $[a, b]$.

[You may quote without proof any results you may need about the existence and uniqueness of solutions of equations such as (1) subject to suitable initial conditions.]

21 Find a function $f(z)$, and finite contours Γ_1 and Γ_2 (of which Γ_1 but not Γ_2 is closed), such that, for $i = 1, 2$,

$$w_i(z) = \int_{\Gamma_i} e^{z\zeta} f(\zeta) \, d\zeta$$

is a non-trivial solution of the differential equation

$$zw'' + w' - zw = 0.$$

Express $w_1(z)$ as a polynomial in z, and $w_2(z)$ as a power-series in z, convergent for all z.

[You may assume the validity of term-by-term integration in obtaining the power-series for $w_2(z)$.]

Find also an infinite contour Γ_3 such that

$$w = w_3(z) = \int_{\Gamma_3} e^{z\zeta} f(\zeta) \, d\zeta$$

is a solution valid for all z such that $\Re(z) > 0$.

22 Let

$$L(y) = \frac{d}{dx}\left[p(x) \frac{dy}{dx} \right] + q(x) y,$$

where $p(x)$ is differentiable, $q(x)$ is continuous and $p(x) > 0$ if $a < x < b$, and let $y = y_i(x)$ $(i = 1, 2)$ be non-trivial solutions of $L(y) = 0$ valid in $[a, b]$. Show that

(i) $y_1(x)$ has only finitely-many zeros in $[a, b]$;

(ii) if $y_1(x)$, $y_2(x)$ have a common zero, they are linearly dependent;

(iii) if $y_1(x)$, $y_2(x)$ are linearly independent, one zero of $y_1(x)$ lies between each pair of zeros of $y_2(x)$.

Find a solution, valid in $|x| < 1$, of the equation

$$\frac{d}{dx}\left[(1 - x^2) \frac{dy}{dx} \right] + 12y = 0$$

in the form of a power-series in x^2, and show that it vanishes for at least 2 and at most 4 values of x satisfying $-1 < x < 1$.

[General theorems on existence and uniqueness of solutions of linear equations, and on the convergence of power-series solutions, may be quoted without proof.]

23C If $\phi(x)$ is the unique real solution of

$$\phi^3 + \phi + x = 0,$$

prove that $y = \phi(x)$ is a solution of the differential equation

$$(27x^2 + 4)y'' + 27xy' - 2y = 0.$$

Hence obtain the first three terms, and a recurrence relation for the general term, of a power series expansion for $\phi(x)$ valid in some neighbourhood of $x = 0$.

24A (i) Show that every solution of the equation

$$xy'' - (1 + x)y' + 2(1 - x)y = 0$$

is regular at every point of the x-plane.

(ii) Find in terms of power series the general solution of

$$xy'' + y' + xy = 0.$$

25F The confluent hypergeometric equation is

$$xu'' + (c - x)u' - au = 0.$$

Discuss briefly the singular points of this equation and obtain the solution $\Phi(a, c; x)$ regular about the origin as a power series in x. (Φ is normalized to 1 at $x = 0$.)

Verify that with a suitably chosen contour L,

$$\int_L dt\, e^t t^{a-1} (t - x)^{-a}$$

is a solution of the differential equation. If L runs between 0 and 1 along the real axis show that the result is proportional to $\Phi(a, c; x)$ and find the constant of proportionality. [You may use the result

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} = \int_0^1 dt\, t^{x-1}(1 - t)^{y-1}.]$$

Deduce that

$$\Phi(a, c; x) = e^x \Phi(c - a, c; -x).$$

26 6 B. The linear second order differential equation

$$Lx(t) \equiv x''(t) - q(t)x(t) = 0 \qquad (a \leqslant t \leqslant b)$$

(where q is a continuous real-valued function on $[a, b]$) has no non-trivial solutions satisfying the boundary conditions

$$f(x) = \alpha_1 x(a) + \alpha_2 x'(a) = 0,$$
$$g(x) = \beta_1 x(b) + \beta_2 x'(b) = 0.$$

Show that there exist non-trivial solutions u and v of the equation $L(x) = 0$ such that $f(u) = 0$, $g(v) = 0$ and such that $u - v'v = -1$.

Show how u and v can be used to define a Green's function $k(s, t)$, and establish the basic properties of the Green's function.

$y(t)$ is a given complex-valued function on $[a, b]$. Show that x is a complex-valued C^2 function on $[a, b]$ satisfying the equations

$$L(x) = y, \qquad f(x) = g(x) = 0$$

if and only if

$$x(s) = -\int_a^b k(s, t)y(t)\, dt \qquad (a \leqslant s \leqslant b).$$

27 9. (P55309) Find a particular integral of the equation

$$\frac{d^2 y}{dx^2} - \frac{5y}{x^2} = x \log x.$$

28 12. (T56309) Obtain two independent solutions in series of the equation

$$x^2 (1 - x) \frac{d^2 y}{dx^2} - x(1 + 3x) \frac{dy}{dx} + (1 - x) y = 0.$$

29 14. If $u(x)$ and $v(x)$ are linearly independent solutions of the equation

$$(r(x)y')' + t(x)y = 0,$$

and if x_1, x_2 are two consecutive zeros of $u(x)$ such that $r(x) \neq 0$ in $x_1 \leqslant x \leqslant x_2$, prove that $v(x)$ has one and only one zero in $x_1 < x < x_2$.

Formal Asymptotic Expansions

(i) Define what the asymptotic expansion $f(Z) \sim \sum_{n=0}^{\infty} A_n/Z^n$ as $Z \to \infty$ means. Prove from your definition that if the asymptotic expansion is valid, the coefficients A_n are unique.

(ii) Find the asymptotic expansion of

$$I = \int_0^\infty \exp(-xt) \, dt / \left[1 + t^2\right]$$

for large real positive x.

Asymptotic Expansions: Saddle Point Method

(i) Let $\phi(x)$, $h(x)$ be two nice functions and suppose $h'(a) = 0$, $h''(a) < 0$ and $h(a)$ is the maximum value of h in the range $a \le x \le b$. (Assume a does not coincide with a or b.) Then prove:

$$\int_a^b \phi(x) \exp\left[\gamma h(x)\right] \, dx \sim \phi(a) \left[\frac{-2\pi}{\gamma h''(a)}\right]^{\frac{1}{2}} \exp\left[\gamma h(a)\right] \text{ as } \gamma \to \infty \, .$$

(ii) Define the Legendre polynomial $P_n(\mu)$ by:

$$P_n(\mu) = \frac{1}{\pi} \int_0^\pi \left[\mu + (\mu^2 - 1)^{\frac{1}{2}} \cos\theta\right]^n \, d\theta \text{ for } \mu > 1.$$

Find the asymptotic behavior of $P_n(\mu)$ as $n \to \infty$.

3. Mathews and Walker: problem 3-38

4. Lagrange's Formula

Let $u = f(Z)$ be analytic at $Z = Z_0$ with $u_0 = f(Z_0)$. This defines an inverse function $Z = g(u)$.

Prove Lagrange's Formula:

$$\frac{d^n}{du^n} g(u) \bigg|_{u=u_0} = \frac{d^{n-1}}{dZ^{n-1}} \left[\frac{Z-Z_0}{f(Z)-u_0} \right]^n \bigg|_{Z=Z_0}$$

Hint: consider

$$\frac{1}{2\pi i} \oint_C \frac{dZ}{\left[f(Z) - u_0 \right]^n}$$

where C is any sensible contour surrounding Z_0.

5. Airy's Integral

Show that the full asymptotic expansion of

$$I(x) = \int_{-\infty}^{+\infty} \exp ix \left[t + t^3/3 \right] dt$$

is $I(x) \sim \frac{1}{\sqrt{x}} \exp (-2x/3) \sum_{n}^{\infty} \frac{\Gamma(3n+\frac{1}{2})}{(2n)!} \cdot (-9x)^{-n}$

for x real $> 0 \to +\infty$. Hint: write I in the form

$$a \int_{-\infty}^{+\infty} du \ dt/du \ \exp (-xu^2)$$

where t is expressed in a power series in u using above problem. You may also need the duplication law for the Γ function.

6. The Method of Stationary Phase -1

$$I(\gamma) = \int_a^b \exp\left[i\gamma f(x)\right] \phi(x) \, dx \qquad -(*)$$

where γ is real > 0 and the integral runs over the real axis from a to b.

$f(x)$, $\phi(x)$ are wonderful regular functions and at $x = \alpha$, $f'(x) = 0$
and $f''(\alpha) > 0$. Distort the integration range into a suitable complex
contour near $x = \alpha$ and use the method of steepest descent to show that
for large γ the contribution to I from x near α is:

$$I(\gamma) \sim \left[\frac{2\pi}{\gamma f''(\alpha)}\right]^{\frac{1}{2}} \phi(\alpha) \, e\left[i\gamma f(\alpha) + i\pi/4\right] \qquad -(\text{IF})$$

7. The Method of Stationary Phase -2

$I(\gamma)$ is as defined in the previous problem but now $f'(x)$ does not
vanish in the range $a \le x \le b$. By integration by parts - or otherwise -
show

$$I(\gamma) = \frac{\phi(\beta)}{i\gamma f'(\beta)} e^{i\gamma f(\beta)} - \frac{\phi(\alpha)}{i\gamma f'(\alpha)} e^{i\gamma f(\alpha)} + 0(1/\gamma^2).$$

Combine this with the result of the previous problem to discuss the
error in (IF).

11 F If t sufficiently small

$$f(t) = \sum_{n=0}^{\infty} a_n t^{n+\rho},$$

prove that for large positive p

$$\int_0^A dt\, e^{-pt} f(t) \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(n+\rho+1)}{p^{n+\rho+1}},$$

where $A > 0$. Deduce the corresponding result for an integral of the form

$$\int_0^A dt\, e^{-pt^2} f(t).$$

Explain the method of steepest descent for obtaining the asymptotic behaviour as $p \to \infty$ of $h(p)$ where
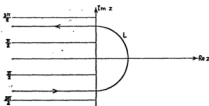
$$h(p) = \int_C dz\, e^{-p\phi(z)} \psi(z)$$

and where $\phi(z)$ and $\psi(z)$ are analytic functions and C is some contour in the complex z-plane.

The Bessel function of order ν, $J_\nu(x)$, is given by

$$J_\nu(x) = \frac{1}{2\pi i} \int_L dz \exp(-x \sinh z + \nu z),$$

where L is the contour shown in the diagram below. Obtain the leading term of the asymptotic expansion as $\nu \to \infty$ of

$$J_\nu\left(\frac{\nu}{\cosh \beta}\right), \quad \text{where } \beta > 0.$$

1. P54402. Give a brief discussion of Lagrange's method of determining the stationary values of a function of several variables which are restricted by certain relations.

The positive variables $x_1, x_2, \ldots, x_r$ are restricted by the relations

$$\sum_{i=1}^{r} x_i = N, \qquad \sum_{i=1}^{r} a_i x_i = E,$$

where a_i, N, E are given positive constants. Show that the values of the x_i for which the function

$$\sum_{i=1}^{r} (x_i \log x_i - x_i)$$

is stationary are given by

$$x_i = \frac{N e^{\mu a_i}}{\sum\limits_{j=1}^{r} e^{\mu a_j}},$$

where μ is a solution of the equation

$$\frac{\sum\limits_{i=1}^{r} a_i e^{\mu a_i}}{\sum\limits_{i=1}^{r} e^{\mu a_i}} = \frac{E}{N}.$$

In the particular case in which $r = 3$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, show that if $1 < E/N < 3$ there is a unique solution for x_1, x_2 and x_3.

2. P 55402 A real function $F(x)$ and n linearly independent real functions $u_r(x)$ $(r = 1,2,...,n)$ are given in the interval $-1 \leqslant x \leqslant n$. It is required to find a set of coefficients a_r which give stationary values to the integral

$$\int_{-1}^{1} F(x) \left\{ \sum_{r=1}^{n} a_r u_r(x) \right\}^2 dx$$

subject to the restriction

$$\int_{-1}^{1} \left\{ \sum_{r=1}^{n} a_r u_r(x) \right\}^2 dx = 1.$$

Show that the constants a_s must satisfy

$$\sum_{s=1}^{n} (F_{rs} - \lambda_r g_{rs}) a_s = 0 \quad (r = 1,2,...,n), \quad \sum_{r=1}^{n} \sum_{s=1}^{n} a_r a_s g_{rs} = 1,$$

where $F_{rs} = \int_{-1}^{1} F(x) u_r(x) u_s(x) dx$, $g_{rs} = \int_{-1}^{1} u_r(x) u_s(x) dx$,

and λ is a root of the equation $\det \left\{ F_{rs} - \lambda g_{rs} \right\} = 0$.

Show further that the roots of this equation are the required stationary values of the integral.

3. P 57403. The n variables $x_1,...,x_n$ are restricted by the conditions

$$g_j(x_1,...,x_n) = 0 \quad (j = 1,...,m; \ m < n).$$

Prove that in general the stationary values of a function $f(x_1,...,x_n)$ can be determined by solving the equations

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0 \quad (i = 1,2,...,n),$$

where $\lambda_1,...,\lambda_m$ are undetermined multipliers, together with $g_j = 0$ $(j = 1,...,m)$.

The real variables x,y,z satisfy $x^2 + y^2 + z^2 = 3$, $xyz = k$, where $0 < k < 1$. Show that $x + y + z$ has the stationary values

$$a \pm 2 \left(\frac{k}{a}\right)^{\frac{1}{2}}, \quad b \pm 2 \left(\frac{k}{b}\right)^{\frac{1}{2}},$$

where a and b are the positive roots of the cubic equation

$$\gamma^3 - 3\gamma + 2k = 0.$$

4. P 61401. If $f(x_1, \ldots, x_n) = \sum_{ij} a_{ij} x_i x_j$, $g(x_1, \ldots, x_n) = \sum_i x_i^2$, where

the matrix $\underline{a}$ is symmetric, show that the stationary points and values of f
when the variables are subject to the restriction $g = 1$ are given by the
eigenvectors and eigenvalues of $\underline{a}$. Show also that these same eigenvectors
and eigenvalues give the stationary points and values of the function f/g
when the variables are unrestricted.

If $\underline{a}$ is non-singular, explain how, in a similar way, these eigenvectors
and eigenvalues can be used to give the stationary points and values of g
when the variables are subject to the restriction $f = 1$.

5. P 44210. (i) Establish Euler's condition for a function $y = y(x)$ to give
a stationary value of

$$\int_a^b P(x,y,p)dx, \qquad (*)$$

where $p = dy/dx$ and $y(a)$ and $y(b)$ are given.

(ii) Suppose that there is a continuously differentiable function
$P(x,y)$ defined for $x > 0$ with the property that any solution $y = Y(x)$ to
the differential equation $p = P(x,y)$ makes the integral $(*)$ stationary
for any positive a, b and the end conditions

$$y(a) = Y(a), \quad y(b) = Y(b).$$

Show that

$$P_y(x,y,P)dx + P_p(x,y,P)(dy - Pdx) \qquad (\dagger\dagger)$$

where $P = P(x,y,p)$, is an exact differential.

(iii) When $P(x,y,p) = 9p^2 + 2y^2x^{-2}$

show that $P = 2y/3x$ has the required property. By considering $P(x,y,P)dx - dW$
where dW is the exact differential $(\dagger\dagger)$, or otherwise, deduce that

$$9 \int_0^1 p^2 dx \ge 6 + 2 \int_0^1 (y/x)^2 dx$$

for every continuously differentiable function $y(x)$ $(0 \le x \le 1)$ with
$y(0) = 0$, $y(1) = 1$.

6. P 58402. The function $r = r(\theta)$ is a geodesic from θ_1 to θ_2 on the surface of revolution $x = r\cos\theta$, $y = r\sin\theta$, $z = f(r)$. Prove that the function $r(\theta)$ minimises the integral

$$\int_{\theta_1}^{\theta_2} F(r', r) \, d\theta ,$$

where $r' = \dfrac{dr}{d\theta}$, and $F(r', r) \equiv \left[r'^2 \cdot \left\{ 1 + \left(\dfrac{df}{dr} \right)^2 \right\} + r^2 \right]^{\frac{1}{2}}$.

Prove that along the geodesic

$$r' \frac{\partial F}{\partial r'} - F = \text{constant}.$$

Hence, or otherwise, show that $r(\theta)$ satisfies an equation of the form

$$r \sqrt{(r^2 - k^2)} = \frac{dr}{d\theta} \sqrt{\left\{ 1 + \left(\frac{df}{dr} \right)^2 \right\}} .$$

7. P 45305. A disturbance travels by a ray-path in a diametral plane of a sphere in which the velocity v is a function of radius r only. The path is such as to make stationary the travel-time t which is given by

$$\int_A^B \frac{ds}{v(r)} ,$$

where A and B are fixed end points and ds is an element of path. Show that on the path $r\sin\phi = Cv$, where ϕ is the angle between tangent and radius vector, and C is constant for the path.

If A, B lie on $r = R$ with angular separation α, find expressions for α and t as integrals with respect to r. Hence show that $dt/d\alpha = C$ if α is varied while R remains fixed.

8. P 61402. In geometrical optics the rays satisfy Fermat's condition

$$\delta \int_A^B \mu \, ds = 0,$$

where the positive function of position μ is the refractive index, and the line integral is taken between fixed end-points. Prove that:

(a) If, in a horizontally stratified medium, $\mu = \sqrt{(a - bz)}$, where a,b are positive constants and z is the height, the rays are inverted parabolas with their directrices in the plane $z = a/b$. It may be assumed that the rays lie in vertical planes.

(b) If f is a single-valued function of position such that $|\nabla f| = \mu$, show that

$$\int_A^B |L \, ds \ge |f(B) - f(A)|,$$

with equality only if the path of integration is an orthogonal trajectory of the family of surfaces $f = $ constant. Deduce that these orthogonal trajectories satisfy Fermat's condition.

9. P 62303. The function $y(x)$ is an extremal of $\int_0^1 x \left(\dfrac{dy}{dx} \right)^2 dx$ subject to the conditions (i) $\int_0^1 xy^2 dx$ is constant, (ii) $y(0) = 1$, $y(1) = 0$. Obtain the function $y(x)$ in the form

$$y = \sum_{j=0}^{\infty} a_j (\mu x)^j,$$

where μ is a zero of the Bessel function of order zero. Determine a_j in terms of a_0. [Bessel's equation of order zero is $xy'' + y' + xy = 0$.]

4C. Explain without proof how to minimise the integral

$$\int_P^Q f(x, y, y') \, dx \quad \text{subject to a condition} \quad \int_P^Q g(x, y, y') \, dx = \text{constant.}$$

The ends of a uniform heavy inextensible chain of length L are fixed at points P, Q ($PQ < L$). Starting from the hypothesis that the potential energy of the chain is a minimum in equilibrium, establish the Cartesian equation for a catenary.

1A. If $f(x_1, \ldots, x_n) = \sum_{r,s} x_r \, a_{rs} \, x_s \, g(x_1, \ldots, x_n) = \sum_r x_r^2$ where the matrix a is symmetric, show that the stationary points and values of f when the variables are subject to the restriction $g = 1$ are given by the eigenvectors and eigenvalues of a. Show also that these same eigenvectors and eigenvalues give the stationary points and values of the function f/g when the variables are unrestricted.

If a is non-singular, explain how in a similar way, these eigenvectors and eigenvalues can be used to give the stationary points and values of g when the variables are subject to the restriction $f = 1$.

10. (i) F is a given function of two variables. It is required to find a function $y = y(x)$ which at given boundary values $y(a) = \alpha$ and $y(b) = \beta$ which will give a stationary value to the integral

$$\int_a^b F(y, p) \, dx, \quad \text{where} \quad p = \frac{dy}{dx}.$$

Show that, under suitable conditions, y will satisfy the differential equation

$$p(\partial F/\partial p) - F = \text{constant.}$$

If the initial value of y is given but the final value is allowed to vary, show that y must satisfy the same differential equation as above with the boundary conditions $y(a) = \alpha$ and $\partial F/\partial p = 0$ at $x = b$.

(ii) Find the curve $y = f(x)$ for $0 \leqslant x \leqslant 1$, having length l_0 and with $f(0) = 0$, which maximizes the area of the region $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant f(x)$. You may assume that the maximum is given by a function with a continuous derivative for $x > 0$.

13 &E. The value integral

$$\int_V F(\phi, \operatorname{grad} \phi) \, dV$$

is stationary with respect to variations of ϕ that are zero on the boundary S of V.

Show that

$$\frac{\partial}{\partial x_r} \left(\frac{\partial F}{\partial \phi_r} \right) = \frac{\partial F}{\partial \phi},$$

where $\phi_r = \partial \phi / \partial x_r$.

Deduce that

$$\int_V F \, dS_r = \int_S \phi_r \frac{\partial F}{\partial \phi_r} \, dS_r.$$

14 C Find the values of c_0, c_1, c_2, c_3 such that the function

$$y = c_0 + c_1 \cos \pi x + c_2 \cos 2\pi x + c_3 \cos 3\pi x$$

is the best approximation to the function $f(x) = 2x - 1$ over the interval $(0, 1)$ in the sense that

$$I = \frac{\int_0^1 [y - f(x)]^2 \, dx}{\int_0^1 [f(x)]^2 \, dx}$$

has its minimum value. Show that the minimum value of I in the above sense is approximately 0·0023. Sketch the two curves $y(x)$, $f(x)$ in $(-1, 2)$.

$$\left[\frac{96}{\pi^4} = 0.9855. \right]$$

15 D State and prove Euler's equation for extremals of the integral

$$\int_{x_1}^{x_2} f(x, y, y') \, dx.$$

Show that for integrals of the type

$$\int_{s=a}^{s=b} g(x, y) \, ds,$$

where ds is the element of arc length, the equation may be written

$$g_y - g_x y' - g y'' / (1 + y'^2) = 0.$$

Hence or otherwise show that, if there is a smooth extremal joining 2 given points that minimizes the surface area generated by rotating the curve about the x-axis, it is given by

$$y = c \cosh \{(x + a)/c\},$$

where a, c are constants.

① C. An n-dimensional manifold carries a positive-definite Riemannian metric $g_{ij} dx^i dx^j$, and

$$\Gamma^l_{ij} = \tfrac{1}{2} g^{lk} \left[\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right].$$

If u_i is a covariant vector and u_{ij} a covariant tensor, prove that

$$u_{i,k} = \frac{\partial u_i}{\partial x^k} - \Gamma^r_{ik} u_r$$

and

$$u_{ij,k} = \frac{\partial u_{ij}}{\partial x^k} - \Gamma^r_{ik} u_{rj} - \Gamma^r_{jk} u_{ir}$$

are tensors.

Show that $u_{i,s,t} = u_{i,t,s}$ for all covariant vectors u_i if and only if the curvature tensor

$$R^t_{ism} = \frac{\partial}{\partial x^s} \Gamma^t_{im} - \frac{\partial}{\partial x^m} \Gamma^t_{is} + \Gamma^r_{im} \Gamma^t_{rs} - \Gamma^r_{is} \Gamma^t_{rm}$$

is zero.

② C. Obtain the equations

$$\frac{d^2 x^l}{ds^2} + \Gamma^l_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = 0$$

for the geodesics on a Riemannian manifold.

If the metric is given by

$$ds^2 = \phi^2 (dx^1)^2 + [1 + \phi_1^2] (dx^1)^2 + \phi_2 \phi_3 dx^2 dx^3 + [1 + \phi_3^2] (dx^3)^2,$$

where ϕ is a function (twice differentiable) of x^2 and x^3, and ϕ_2, ϕ_3 are its partial derivatives, show that

$$\phi \frac{dx^1}{ds}$$

is constant along any geodesic.

③ E. A threefold carries a Riemannian metric

$$ds^2 = \phi(dx^1)^2 + (dx^2)^2 + (dx^3)^2,$$

where ϕ is a positive infinitely differentiable function of x^1, x^2 and x^3. Show that a field of parallel contravariant vectors can be constructed for an arbitrary choice of vector at a fixed point if, and only if,

$$\phi = (A + Bx^2 + Cx^3)^2,$$

where A, B, C are functions of x^1 only.

If $\phi = 1 + (A + Bx^2 + kBx^3)^2$, where A, B are functions of x^1 only and k is a constant, show that there is a field of parallel vectors on the threefold whose first component is zero everywhere.

2 E Define a differentiable manifold, and explain in terms of the differential structure the difference between orientable and non-orientable manifolds.

Prove that a 2-sphere is an orientable differentiable manifold, and that the real projective plane is a non-orientable differentiable manifold.

2 A Define parallel displacement of a tangent vector to a differentiable manifold, and explain how infinitesimal parallelism is given by the differential equation

$$d\varepsilon_i = v_i^j \varepsilon_j$$

for a moving frame $(\varepsilon_1, \ldots, \varepsilon_n)$. Prove that if

$$\Omega_i^j = dv_i^j - v_i^k \wedge v_k^j,$$

then Ω_i^j are components of a tensorial form.

Show that a necessary and sufficient condition that in a coordinate neighbourhood U there should exist a moving frame $(f_1, \ldots, f_n)$ whose parallel displacement is given by

$$df_i = 0$$

is that

$$\Omega_i^j = 0.$$

2 A A surface S in Euclidean 3-space (referred to orthogonal coordinates (x^1, x^2, x^3)) is given by

$$x^i = x^i(u^1, u^2) \quad (i = 1, 2, 3).$$

If $x_\alpha^i = \partial x^i / \partial u^\alpha$, show that, for each i, x_α^i is a covariant tensor on S. Covariant differentiation on S is defined by means of the Christoffel symbols of the metric induced on S, and is denoted by a comma. Prove that

$$x_{\alpha,\beta}^i = L_{\alpha\beta} X^i,$$

where (X^1, X^2, X^3) is the normal to S, and $L_{\alpha\beta}$ is a symmetric covariant tensor on S.

By calculating

$$\sum_i x_i^i (x_{\alpha,\beta}^i - x_{\alpha,\gamma}^i),$$

or otherwise, prove that the Riemannian curvature tensor on the surface is given by

$$R_{\alpha\beta\gamma} = L_{\alpha\beta} L_{\beta\gamma} - L_{\alpha\gamma} L_{\beta\beta}.$$

2 A Define a vector field on a differentiable manifold, and prove that if ξ^i, η^i are two vector fields

$$\xi^\alpha \frac{\partial \eta^\lambda}{\partial x^\alpha} - \eta^\alpha \frac{\partial \xi^\lambda}{\partial x^\alpha}$$

is also a vector field. If h_i^j is a tensor field of weight zero, prove that

$$H_{ik}^j = h_i^\lambda \frac{\partial h_k^j}{\partial x^\lambda} - h_k^\lambda \frac{\partial h_i^j}{\partial x^\lambda} + h_\lambda^j \frac{\partial h_i^\lambda}{\partial x^k} + h_\lambda^j \frac{\partial h_k^\lambda}{\partial x^i}$$

is also a tensor field.

2A If A is a $(n-1)$-differential form defined in a Euclidean space $(x^1, ..., x^n)$ and D is a simple domain in this space, prove that

$$\int_{\partial D} A = \int_D dA,$$

where ∂ is the boundary operator and dA the exterior derivative of A.

From this result, derive Stokes's Theorem for p-forms on an n-dimensional manifold.

2A Defining parallel displacement in a differentiable manifold by the formulae

$$d\varepsilon_\alpha = \omega_\alpha^\beta \varepsilon_\beta$$

for the parallel displacement of a moving frame, show that

$$\Omega_\alpha^\beta = d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta$$

is invariant to a change of frame.

Show that if

$$A_{\alpha\beta} = \Omega_\alpha^{\mu_1} \wedge \Omega_{\mu_1}^{\mu_2} \wedge ... \wedge \Omega_{\mu_n}^\beta$$

then

$$dA_{\alpha\beta} = 0.$$

2A In a neighbourhood of a point O on a surface the geodesic distance of a point P from O is denoted by r, and θ is the angle at O between the geodesic OP and a fixed geodesic through O. If (r, θ) are taken as local coordinates in a neighbourhood of P, show that the distance element is of the form

$$ds^2 = dr^2 + G\, d\theta^2.$$

Show that the Gaussian curvature K is given by

$$K = -\frac{1}{\sqrt{G}}\frac{\partial^2 \sqrt{G}}{\partial r^2}.$$

2A Let U be an open subset of $\mathbb{R}^n$, with standard coordinates (x^i), let $\mathbb{R}^m$ have standard coordinates (y^i) and let $\alpha : U \to \mathbb{R}^m$ be differentiable. Define the concept of a $(0, 2)$-tensor field ω on $\mathbb{R}^m$ and of the induced field $\alpha^*(\omega)$ on U. Prove that if $\omega = g_{ij} dy^i \otimes dy^j$ then

$$\alpha^*(\omega) = (g_{ij} \circ \alpha)\frac{\partial \alpha^i}{\partial x^r}\frac{\partial \alpha^j}{\partial x^s} dx^r \otimes dx^s.$$

Suppose that $U = (0, \infty) \times (0, \tfrac{1}{2}\pi) \times (0, 2\pi) \subset \mathbb{R}^3$, and $\alpha : U \to \mathbb{R}^3$ is defined by

$$x^1 = x^1 \sin x^2 \cos x^3,$$
$$x^2 = x^1 \sin x^2 \sin x^3,$$
$$x^4 = x^1 \cos x^2.$$

Find the Riemannian metric on U induced by α from the standard Riemannian metric on $\mathbb{R}^3$, and for each $x \in U$ find an orthonormal base of the tangent space at x.

12. 2A Define a differentiable (4-manifold) structure on the space $M(2)$ of all 2×2 real matrices, such that the maps from $M(2) \times M(2)$ (with the product differentiable structure) to $M(2)$ defined by addition and multiplication of matrices are both differentiable.

Show that the groups of orthogonal matrices with determinant $+1$ forms a sub-manifold of $M(2)$ diffeomorphic to the unit circle S^1.

Show that the matrices with non-zero determinant also form a submanifold. Is it orientable!

13. 2A Define the *curvature* κ and *torsion* τ of a unit speed curve τ in R^3. State and prove the Serret-Frenet formulae.

If β lies on a sphere centre the origin, show that

$$\tau \kappa' \beta + \tau \kappa' = \kappa' \beta,$$

where N and B are the principal normal and binormal of β. Hence or otherwise find in terms of κ and τ the radius of the sphere on which β lies.

14. 2A Define a connection on a differentiable manifold, and the *Riemannian connection* on a Riemannian manifold. Give an explicit description of the standard Riemannian connection on R^3 in terms of co-ordinates; state, without proof, the Gauss equation for the relation between this and the Riemannian connection induced on a surface S in R^3, explaining the terms involved.

By showing that the curvature of R^3 vanishes and then examining its tangential and normal components at the surface S, prove that

(i) $R(X, Y)Z = \langle L.Y \cdot Z \rangle LX - \langle LX \cdot Z \rangle LY$,

(ii) $D_X(LY) - D_Y(LX) = L(\langle X, Y \rangle)$,

(iii) $K(p) = \langle R(X, Y) Y \cdot X \rangle_p$.

In the above formulae, D is the induced connection on S, R is its curvature; X, Y and Z are vector fields on S, L is the Weingarten map, and $K(p)$ is the Gauss curvature of S at p.

Assuming that K is constant and strictly positive, show that k, the greater of the principal curvatures of S, cannot have a local maximum at a non-umbilic point.

[You may assume the existence of fields of principal vectors in a neighbourhood of such a point.]

Partial Differential Equations I

9F. Write down the differential equation satisfied by the spherically symmetric solutions of the equation

$$(\nabla^2 - c^2 + 8(c^2-1)^{\frac{1}{2}}) \psi = 0.$$

Expressing ϕ in the form $\phi = (1/r)e^{-cr}f(r)$, where $c = 1 - \epsilon$, find a power series for $f(r)$. Hence determine the eigenvalues of c that yield a solution for $r\phi$ regular at $r = 0$ and such that $r\phi \to 0$ as $r \to \infty$.

2 Obtain this expression for the Laplacian operator in terms of general orthogonal curvilinear co-ordinates.

For a particular choice (x_1, x_2, x_3) of non-co-ordinates, Laplace's equation takes the form

$$(\xi_2 - \xi_3)\frac{\partial^2 L_1}{\partial \xi_1^2} + (\xi_3 - \xi_1)\frac{\partial^2 L_2}{\partial \xi_2^2} + (\xi_1 - \xi_2)\frac{\partial^2 L_3}{\partial \xi_3^2} = 0,$$

where $\xi_i = L_i(x_i)$ ($i = 1, 2, 3$).

Show that, if this equation has a solution of the form

$$\phi = L_1(x_1)\, L_2(x_2)\, L_3(x_3),$$

then

$$\frac{d^2 L_i}{d x_i^2} = (A + B\xi_i)\, L_i \qquad (i = 1, 2, 3).$$

where A, B are constants.

3 **5G.** Interpret geometrically the three equations:

(1) $P\,dx + Q\,dy + R\,dz = 0;$ (2) $P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R;$ (3) $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$

State a necessary and sufficient condition for the integrability of (1), and prove its necessity.

In each of the following two cases determine whether or not the equation is integrable, and, if so, solve it:

(i) $z\,dx + z\,dy + y\,dz = 0;$ (ii) $(y^2 + z^2 - x^2)\,dx - 2xy\,dy - 2xz\,dz = 0.$

7C. Define the characteristics of the partial differential equation

$$a u_{xx} + 2b u_{xy} + c u_{yy} + d u_x + e u_y + f u = g$$

where $a, b, \ldots, g$ are functions of x and y. Prove that, if there are two distinct families of real characteristics, then there exist transformations $(x, y) \to (\xi, \eta)$ of the independent variables for which the second-order terms of the equation assume either the standard forms $u_{\xi\xi}$ or $u_{\xi\xi} - u_{\eta\eta}$.

Show that the equation

$$(x^2 - 1)u_{xx} + 2xyu_{xy} + (y^2 - 1)u_{yy} + 2xu_x + 2yu_y = 0$$

is hyperbolic in $x^2 + y^2 > 1$. By making an appropriate change of variable, show that its solution in this domain is of the form $F(\xi + G(\xi - \theta))$, where the functions F and G are arbitrary (but sufficiently regular) and $\xi = \cos^{-1}(1/r)$, r and θ being polar coordinates. Show also that the characteristics of the equation are the straight lines tangent to the unit circle.

5. (i) Show that $\nabla^2(1/r) = 0$ (provided $r \neq 0$), and that, if V is any volume containing the point $r = 0$, then

$$\int_V \nabla^2\left(\frac{1}{r}\right) dV = -4\pi.$$

(ii) Show that there are $2n + 1$ linearly independent solutions of Laplace's equation $\nabla^2\psi = 0$ of the form

$$\psi_n = \frac{x^u}{\delta x^s \delta y^t \delta z^u}\left(\frac{1}{r}\right), \quad (r \neq 0),$$

where $s, n, t,$ and u are positive integers such that $s + t + u = n$.

6C. Explain the meaning and importance of the *characteristics* of a hyperbolic partial differential equation of the second order in two independent variables. Illustrate your discussion by considering the equation for the oscillations of a non-uniform string

$$\rho(x)\frac{\delta^2 y}{\delta t^2} = c^2 \frac{\delta^2 y}{\delta x^2},$$

where c is a constant and $\rho(x)$ is a given function. Give formal expressions for the characteristic curves, and transform the equation to the standard form

$$\frac{\delta^2 y}{\delta \xi \delta \eta} + \text{terms of lower order} = 0.$$

7. **G.** Give formal descriptions of the solutions of

(i) the differential equation

$$P(x, y, z)\frac{\delta z}{\delta x} + Q(x, y, z)\frac{\delta z}{\delta y} = R(x, y, z),$$

(ii) the system of equations

$$\frac{dx}{P(x, y, z)} = \frac{dy}{Q(x, y, z)} = \frac{dz}{R(x, y, z)}.$$

Establish a relationship between them.

Find the general solution of the equation

$$z\left(x\frac{\delta z}{\delta x} - y\frac{\delta z}{\delta y}\right) + x^2 + y^2 = 0,$$

and obtain the particular solution which reduces to $x^2 + y^2 = 1$ when $z = 0$.

8E. Variables $q_1, q_2,$ and ϕ are defined by the relations

$$kx = (q_1 q_2)^{\frac{1}{2}}\cos\phi, \quad ky = (q_1 q_2)^{\frac{1}{2}}\sin\phi, \quad 2kz = q_1 - q_2,$$

where k is a constant. A solution to the equation

$$\left(\nabla^2 + \frac{2}{r} - k^2\right)\psi(r) = 0 \quad [\text{where } r = (x, y, z)]$$

is $\psi = f_1(q_1)f_2(q_2)$. Show that

$$\frac{d}{dq_i}[q_i^2 f_i'(q_i)] + \left(\frac{\beta_i}{4} - \mu\right)f_i(q_i) = 0 \quad (i = 1, 2),$$

where $\beta_1 + \beta_2 = 1$. Find the condition on β_i such that $e^{\frac{1}{2}q_i}f_i(q_i)$ can be polynomials in form, and deduce that if $1/k$ is a positive integer there is a solution $\psi(r)$ that is finite at both $r = 0$ and at $r = \infty$.

9 Solve the partial differential equation

$$\frac{\partial^2 f}{\partial x^2} - 2y\frac{\partial^2 f}{\partial x\,\partial y} - 3y^2\frac{\partial^2 f}{\partial y^2} - 2y\frac{\partial f}{\partial y} = \sin x,$$

subject to the conditions that $f = 0$ and $\partial f/\partial x = y$ when $x = 0$.

10 15C Show that the equation $\partial v/\partial t = k\nabla^2 v$ for conduction of heat in a uniform solid occupying space with rectangular coordinates x, y, z has a solution of the form

$$t^{-\frac12}\exp\{-x^2/4kt\}.$$

In a uniform solid of infinite extent the temperature at $t = 0$ is $f(x)$. Show that the temperature at a general point at time t is given by

$$\frac{1}{2\sqrt{\pi k t}}\int_{-\infty}^{\infty} f(\xi)\exp\{-(x-\xi)^2/4kt\}\,d\xi.$$

Obtain the corresponding result when the initial temperature is $F(x, y, z)$.

11 13E Define the Green's function $G(x; \xi)$ for the Dirichlet problem

$$\nabla^2 u = 0 \quad \text{in } \mathcal{D}, \qquad u = f \text{ on } S,$$

where S is a closed surface in R_3 bounding a domain $\mathcal{D}$. Show that

$$u(x) = -\int_S f(\xi)\frac{\partial}{\partial n}G(x; \xi)\,dS,$$

where $\partial/\partial n$ denotes differentiation along the outward normal on S.

Obtain the Green's function $G(x; \xi)$ explicitly when $\mathcal{D}$ is the half-space $z > 0$. Hence show that the bounded harmonic function satisfying $u(x, y, 0) = f(x, y)$, where f is bounded and continuous, is

$$u(x, y, z) = \frac{z}{2\pi}\iint\frac{f(\xi, \eta)\,d\xi\,d\eta}{[(x-\xi)^2 + (y-\eta)^2 + z^2]^{3/2}}.$$

12 13E Define the terms *hyperbolic*, *elliptic* and *parabolic* as applied to the differential equation

$$a u_{xx} + 2b u_{xy} + c u_{yy} = f(x, y, u, u_x, u_y),$$

where a, b and c are continuous functions of x and y. Show that, when the equation is hyperbolic, it may be reduced by real change of variable $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ to the canonical form

$$u_{\xi\eta} = f(\xi, \eta, u, u_\xi, u_\eta).$$

Assuming that u, u_x and u_y are everywhere continuous, and that F is a continuous function of all its arguments, show that if u_{xx}, u_{xy} and u_{yy} suffer discontinuities as the point (x, y) crosses a curve C, then C must be a member of one of the families

$$\xi(x, y) = \text{const.}, \qquad \eta(x, y) = \text{const.}$$

Defining

$$s(x) = [u_{xx}] = \lim_{\varepsilon\to 0}\left[u_x(x+\varepsilon, y) - u_x(x-\varepsilon, y)\right],$$

show that $s(x)$ satisfies a differential equation of the form

$$\frac{ds}{dq} = g(\eta)\,s(x).$$

In particular, show that if $f = \nabla u_x + \xi_\nu u_y$, then

$$s(x) = s(0)\,e^{\lambda x}.$$

13 11D The temperature $\theta(x,t)$ in a semi-infinite rod $x > 0$ satisfies the equation

$$\frac{\partial\theta}{\partial t} = \kappa\frac{\partial\theta}{\partial x^2}.$$

At the end $x = 0$, the temperature is prescribed as the step-function

$$\theta(0,t) = \begin{cases} 0 & (t < \tau), \\ 1 & (t > \tau), \end{cases}$$

and the initial condition is $\theta(x,\tau) = 0$ ($x > 0$). Find $\theta(x,t)$ for $t > \tau$.

Infer the solution (in the form of an integral) satisfying

$$\theta(0,t) = 0, \qquad \theta(0,t) = f(t) \quad (t > 0),$$

where $f(t)$ is a suitably well-behaved function.

14 10C The continuous function $u(x,y)$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} = 0 \qquad (*)$$

in the quadrant $x > 0$, $y > 0$, of which the straight sides $x = 0$ and $y = 0$ are part of the boundary. With a grid of side $h = 1$ obtain in explicit numerical form the approximate linear equations relating the values of u at the grid points, which are the difference replacements of $(*)$,

(i) when the boundary condition on the curved side of the boundary is $u = xy$;

(ii) when the boundary condition on the curved side of the boundary is that the inward normal derivative $\partial u/\partial n = xy$.

[You are not expected to remove needs, nor to solve the equations.]

15 11D A simple closed surface S bounds a volume V. Show that eigenfunctions $u_m(\mathbf{x})$, $u_n(\mathbf{x})$ of the problem

$$Lu \equiv (\nabla^2 + k^2)u = 0 \quad \text{in } V, \qquad u = 0 \quad \text{on } S,$$

corresponding to distinct eigenvalues k_m, k_n, satisfy the orthogonality condition

$$\int_V u_m u_n \, dV = 0.$$

Assuming that an arbitrary (suitably well-behaved) function $f(\mathbf{x})$ can be expanded as a linear sum of normalised eigenfunctions, and that k is not an eigenvalue of the above problem, obtain the solution of the problem

$$Lu = f \quad \text{in } V, \qquad u = 0 \quad \text{on } S,$$

and deduce Green's function for the problem

$$G(\mathbf{x},\boldsymbol{\xi}) = \sum_n \frac{u_n(\mathbf{x})\,u_n(\boldsymbol{\xi})}{k^2 - k_n^2}.$$

Hence obtain an expression in the form of a surface integral for the function $v(\mathbf{x})$ satisfying

$$Lv = 0 \quad \text{in } V, \qquad v = g(\mathbf{x}) \quad \text{on } S.$$

16 11D Explain in general terms what is meant by a 'correctly posed' problem in the theory of second-order partial differential equations. Give examples of correctly posed problems and of incorrectly posed problems of hyperbolic and elliptic type.

17 **11D** Solve the equation

$$\frac{\partial^2 \phi}{\partial x^2} + 3x \frac{\partial^2 \phi}{\partial x \partial y} + 2x^2 \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial \phi}{\partial y} = 0,$$

subject to the conditions

$$\phi = x, \quad \frac{\partial \phi}{\partial y} = 0 \quad \text{on} \quad y = 0 \quad (0 < x < 1).$$

Sketch the region of the x, y plane in which your solution is unique.

18 **11D** a, b, c, f are given continuous functions of the real variables x, y. State, but do not prove, conditions under which the solution $u(x, y)$ of the equation

$$\frac{\partial^2 u}{\partial x \partial y} + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + cu = f$$

is uniquely determined by information on the line segment $y = x \ (0 \leq x \leq 1)$. Define the Riemann function for this problem, and show how the general solution may be expressed in terms of it.

19 **12E** Discuss the role of eigenfunctions in constructing solutions of second order linear partial differential equations.

20 13. A function $V(r, \theta, z)$ satisfies Laplace's equation in cylindrical polars (r, θ, z),

$$\frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r}\right) + r \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{for} \quad a < r < b, \ z > 0,$$

and the boundary conditions

$$V(a, z) = V(b, z) = 0, \qquad (z > 0),$$
$$V(r, 0) = F(r), \quad V(r, z) \to 0 \text{ as } z \to \infty \quad (a < r < b).$$

Show that

$$V(r, z) = \sum_{n=1}^{\infty} A_n \exp(-\alpha_n z) \overline{\Psi}(\alpha_n r)$$

where $\overline{\Psi}(\alpha_n r) = J_o(\alpha_n r) Y_o(\alpha_n a) - J_o(\alpha_n a) Y_o(\alpha_n r)$, the numbers α_n are the roots of $\overline{\Psi}(\alpha_n b) = 0$, and

$$A_n = \frac{\int_a^b r F(r) \overline{\Psi}(\alpha_n r) dr}{\int_a^b r \left[\overline{\Psi}(\alpha_n r)\right]^2 dr}$$

5. Show that the function

$$P_n^m(\mu) = (1-\mu^2)^{m/2} \frac{d^m}{d\mu^m} P_n(\mu) \qquad (m = 0,1,2,\dots n)$$

satisfies the equation

$$\left[(1-\mu^2) P'(\mu)\right]' - \frac{m^2 P(\mu)}{1-\mu^2} + n(n+1) P(\mu) = 0.$$

Hence show that there are $2n + 1$ linearly independent solutions of

$\nabla^2 \varphi = 0$ of the form $\varphi = r^n e^{\pm im\chi} P_n^m(\cos\theta)$, in

spherical polars (r, θ, χ).

6. Show that if $\varphi(r, \theta, \chi)$ is a solution of $\nabla^2\varphi = 0$, then

$$\bar\varphi(r, \theta, \chi) = \frac{1}{r} \varphi(\frac{1}{r}, \theta, \chi)$$ is also a solution (a generalised 'image' theorem).

7. If $\underline{s}_1, \underline{s}_2, \dots \underline{s}_n$ are unit vectors in the x-y plane making equal angles $2\pi/n$ with each other, and $\underline{k}$ is a unit vector in the z-direction, show that $(\underline{s}_1 \cdot \nabla)(\underline{s}_2 \cdot \nabla) \dots (\underline{s}_n \cdot \nabla)(\underline{k} \cdot \nabla)^{n-m} \frac{1}{r} = \frac{A}{r^{n+1}} \cos m(\chi-\chi_0) P_n^m(\cos\theta)$ ~~from~~

$$+ \frac{B}{r^{n+1}} P_n(\cos\theta)$$

where $x = r\sin\theta \cos\chi$, $y = r\sin\theta \sin\chi$, $z = r\cos\theta$, and $P_n^m(\mu)$ is as defined in question 5, and A, B and χ_0 are constants.

(24)

Given the equation:

$$a \frac{\partial^2 u}{\partial x^2} + 2 h \frac{\partial^2 u}{\partial x \partial y} + b \frac{\partial^2 u}{\partial y^2} = 0$$

where a, h and b are constants such that $ab \neq h^2$; show that it is possible to transform the equation by a linear map $(x, y) \to (\xi, \eta)$ to the form:

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

Hence solve the original equation. Explain the difference between the two cases $ab < h^2$ and $ab > h^2$ and obtain the general solution to the exceptional case $ab = h^2$.

Find the complete solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 6 \frac{\partial^2 u}{\partial y^2} = \gamma \quad .$$

1. Murphy's Formula

Define the Legendre Polynomial by Rodriquez's formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(x^2-1)^n\right]$$

By expanding this in powers of $1-x$ or otherwise, prove Murphy's formula:

$$P_n(x) = {}_2F_1\left[-n, \; 1+n, \; 1, \; 1/2 - 1/2 \; x\right]$$

2. Laplace Transform

Use the Laplace Transform to show that a solution of the equation

$$\left[xf(D) + F(D)\right]\phi = 0$$

where $D\phi = d\phi/dx$, $D^2\phi = d^2\phi/dx^2$, etc., and where f and F are finite polynomials in powers of D is

$$\phi = \int_a^b \exp\left\{xt + \int^t F(u)du/f(u)\right\} \frac{dt}{f(t)}$$

where a and b satisfy

$$\left[\exp\left\{xt + \int^t F(u)du/f(u)\right\}\right]_a^b = 0$$

Use this result to find an integral representation for the confluent hypergeometric function ${}_1F_1$ (a; c:z) valid for Re c > Re a > 0.

3. Elliptic Integrals

Define $y = sn(x,k)$ by:

$$x = \int_0^y \frac{du}{\sqrt{(1-u^2)(1-k^2 u^2)}}$$

Express $I = \int_0^q \frac{dp}{(1+p^4)^{\frac{1}{2}}}$

in terms of $sn^{-1}y$ for suitable values of y and k.

4. Let $y = \frac{1}{2} Sin^{-1} z$

and put $q = z^2$

By differentiating $q^{\frac{1}{2}} = Sin(q^{\frac{1}{2}}y)$ twice – or otherwise – show that

$$q(1-q) \, d^2y/dq^2 + (3/2 - 2q) \, dy/dq - y/4 = 0$$

Derive an expression for y in terms of a hypergeometric function.

⑤

5.   Let x = F(a, b + 2; c + 1, z)

y = F(a + 1, b + 1, c + 1, z)

w = F(a + 2, b; c + 1, z)

It follows from a general theorem that x, y and w satisfy a linear
relation. Show that this is:

$$c_2(b+1) x - c_1(a+1) w + \left[c_1(a+1) - c_2(b+1)\right] y$$

$$= \left[c_2 a - c_1 b - c_2(b+1) + c_1(a+1)\right] zy \qquad -(*)$$

where

$$c_1 = a(2b+1-c) - (b+1)(a+b-c)$$

$$c_2 = b(2a+1-c) - (a+1)(a+b-c)$$

(Hint: compare coefficients of z^n on each side of (*).)

Define the Chelyshev polynomials $T_n(x)$ by

$$T_n(x) = \cos n (\cos^{-1} x)^{\cdot}$$

Derive a recurrence relation for T_n, T_{n+1} and T_{n+2} and by comparison
with (*) or otherwise, prove

$$T_n(x) = F(-n, n; 1/2:1/2 - 1/2 x)$$

6. **An Exponential Integral**

One of the exponential integrals is defined by

$$\text{Ein}(z) = \int_0^z \frac{(1-e^{-t})}{t} \, dt$$

Show that Ein(z) can be related to a confluent hypergeometric function by

$$\text{Ein}(z) = -3/3a \; {}_1F_1(a; \, 1; \, -z)\Big|_{a=0}$$

7. **Continued Fractions**

The eigenvalues α of Mathieu's equation are determined by the continued fraction:

$$\frac{-\alpha}{\beta} = \frac{-\beta}{2(a-4) - \dfrac{\beta^2}{2(a-16) - \dfrac{\beta^2}{2(a-36) - \dfrac{\beta^2}{2} \ldots \ldots \ldots}}}$$

The eigenvalue α corresponding to $\text{ce}_2(\phi)$ may be expanded as

$$\alpha = 4 + c \, \beta^2 + O(\beta^4)$$

Find c from the continued fraction.

8.

$y = \operatorname{sn} x$ obeys the differential equation

$$dy/dx = \sqrt{(1-y^2)(1-k^2 y^2)} \qquad -(*)$$

and satisfies $\operatorname{sn}(o) = 0$

Mathews and Walker (p. 209) define cn and dn by:

$$\operatorname{cn} x = \sqrt{1 - \operatorname{sn}^2 x}$$

$$\operatorname{dn} x = \sqrt{1 - k^2 \operatorname{sn}^2 x} \qquad -(\text{IP})$$

By constructing differential equations for cn and dn - or otherwise - prove:

$$\operatorname{cn}^{-1} y = \int_y^1 \frac{du}{\sqrt{(1-u^2)(1-k^2+k^2 u^2)}}$$

$$\operatorname{dn}^{-1} y = \int_y^1 \frac{du}{\sqrt{(1-u^2)(u^2+k^2-1)}}$$

9.

The function $y = \operatorname{sn}(x)$ defined by (*) in the previous problem is known to be an elliptic function with periods $4K$ and $2iK'$ and 2 poles at respectively iK' and $2K + iK'$ in the unit cell

$\operatorname{cn}(x)$ and $\operatorname{dn}(x)$ are defined by (**) in the previous problem.

(i) Prove $\operatorname{cn}(x)$ and $\operatorname{dn}(x)$ are meromorphic.

(ii) Prove $\operatorname{cn}(x)$ and $\operatorname{dn}(x)$ are elliptic functions.

(iii) Prove $\operatorname{cn}(x)$ and $\operatorname{dn}(x)$ have two and only two zeros in the unit cell.

10. E. Give a definition of the Legendre polynomial $P_n(x)$ of order n, and from your definition prove that

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{2\delta_{mn}}{2n+1} \quad (m = 0, 1, 2, \ldots).$$

Prove that, for any function $f(x)$ continuous in $(-1, 1)$, the polynomial $q(x)$ of degree n that minimises the integral

$$\int_{-1}^{1} [f(x) - q(x)]^2\, dx$$

is given by

$$q(x) = \sum_{r=0}^{n} a_r P_r(x),$$

where

$$a_r = (r + \tfrac{1}{2}) \int_{-1}^{1} f(x) P_r(x)\, dx.$$

5. The Legendre polynomial $P_l(x)$ is defined by the relation

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l.$$

Show that

$$\int_{-1}^{1} P_l(x) P_m(x) \, dx = \frac{2}{2l+1} \delta_{lm}$$

and that

$$P_l(1) = (\pm 1)^l,$$

where δ_{lm} is the Kronecker delta. Hence show that

$$\frac{dP_l^l}{dx} = \sum_{r=1}^{l} (2l - 4r + 3) P_{l-r-1}(x),$$

where

$$K = \frac{2l+1-(-1)^l}{6}.$$

6. Prove that

$$Ae^{z} \int_C e^{iz} (1 - t^2)^{\nu - \frac{1}{2}} dt$$

is the solution $J_\nu(x)$ of Bessel's equation

$$z^2 y'' + zy' + (z^2 - \nu^2) y = 0,$$

provided that the constant C and the contour C are suitably chosen.

Prove that if α_m, α_n are distinct zeros of J_ν, where $\nu > -1$, then

$$\int_0^1 x J_\nu(\alpha_m x) J_\nu(\alpha_n x) \, dx = 0.$$

7. An arbitrary function is approximated in the mean throughout an arbitrary range of its argument and with respect to an arbitrary positive weight function by a series of polynomials in such a way that the coefficients of the polynomials are independent of the order of the approximation. Show that the polynomials form an orthogonal set, and determine the coefficients in terms of the polynomials. Calculate the polynomials of orders zero, one and two for a weight function of unity and the range $(-1, 1)$.

8. $P_n(t)$ is defined to be the coefficients of t^n in the expansion of $(1 - 2\mu t + t^2)^{-\frac{1}{2}}$ in ascending powers of t. Prove the identities

$$\mu \frac{dP_n}{d\mu} - \frac{dP_{n-1}}{d\mu} = nP_n,$$

$$\frac{dP_{n+1}}{d\mu} - \mu \frac{dP_n}{d\mu} = (n+1)P_n,$$

and deduce the

$$\frac{d}{d\mu} \left(1 - \mu^2 \right) \frac{dP_n}{d\mu} + n(n+1)P_n = 0.$$

Evaluate $\int_{-1}^{1} \mu^m P_n(\mu) \, d\mu$ for all positive integers m.

(15) 1. If $J_n(x)$ is defined in the coefficient of t^n in the expansion of $e^{\frac{1}{2}x(t-t^{-1})}$ in powers of t, prove that

$$J_n(x) = \sum_{m=0}^{\infty} (-1)^r (\tfrac{1}{2}x)^{2n+m}/m \; (n+m) \; !$$

(16) 2. The Bessel function of the second kind $Y_\nu(x)$ is defined, for ν ≠ integer, by

$$Y_\nu(x) = (\sin \nu \pi)^{-1} \left[J_\nu(x) \cos \nu \pi - J_{-\nu}(x) \right],$$

and $Y_n(x)$ (n = integer) is defined as $\lim_{\nu \to n} Y_\nu(x)$. Prove that

$$Y_n(x) = \frac{1}{\pi} \left\{ \frac{\partial J_\nu}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}}{\partial \nu} \right\}_{\nu = n}$$

(17) 3. Show that

$$J_{\frac{1}{2}}(x) = \left(\frac{2}{\pi x} \right)^{\frac{1}{2}} \sin x,$$

$$J_{m+\frac{1}{2}}(x) = \frac{(-1)^m (2x)^{m+\frac{1}{2}}}{\pi^{\frac{1}{2}}} \cdot \frac{d^m}{d(x^2)^m} \left(\frac{\sin x}{x} \right). \; (m=1,2,\ldots)$$

(18) 4. Show that the general solution of the equation

$$4 y'' + 9 x y = 0$$

may be written in the form

$$y = Ax^{\frac{1}{2}} J_{1/3} (x^{3/2}) + B x^{\frac{1}{2}} J_{-1/3} (x^{3/2})$$

(the Airy integral).

(19) 5. From Rodrigues' formula, or otherwise, prove that

$$P'_n(x) - P'_{n-2}(x) = (2n-1) P_{n-1}(x),$$

and hence

$$P'_n(x) = \sum_{r=1}^{N} (2n - 4r + 3) P_{n-2r+1}$$

where $N = \frac{1}{2}n$ or $\frac{1}{2}(n+1)$ according as n is even or odd.

. Assuming that $P_n(x)$ satisfies the Legendre equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad (1)$$

verify that the function $Q_n(x)$ defined by

$$Q_n(x) = \tfrac{1}{2} P_n(x) \log \frac{x+1}{x-1} - \sum_{r=1}^{N} \frac{2n-4r+3}{(2r-1)(n-r+1)} P_{n-2r+1}$$

is a second solution. (The result of question 5 will be found useful). (Note the behaviour of $Q_n(x)$ near the points $x = \pm 1$ which are regular singularities of (1). The general solution is, of course,

$AP_n(x) + BQ_n(x)$, where A and B are constants).

7. The hanging chain (D. Bernoulli 1732).

A uniform chain AB of density ρ per unit length and length $\mathcal{L}$ is suspended from the point A, the end B being free. If x is measured vertically upwards from B, show that small oscillations of the chain in a horizontal plane about the equilibrium position are described by the equation

$$\frac{\partial^2 y}{\partial t^2} = g \frac{\partial}{\partial x} \left(x \frac{\partial y}{\partial x} \right)$$

where $y(x,t)$ is the horizontal displacement. Hence show that the normal modes of oscillation are of the form

$$y = C J_0 \left(n \, \mathcal{T} \right) \cos \left(nt + \epsilon \right),$$

where $\mathcal{T} = 2 \left(\frac{x}{g} \right)^{\frac{1}{2}}$ and n is any solution of $J_0(2n \left(\frac{\mathcal{L}}{g} \right)^{\frac{1}{2}}) = 0$.
Sketch the first few modes.

8. (P58405) The real function $u(x)$ satisfies the equation

$$\frac{d^2 u}{dx^2} + (E - x^2)u = 0,$$

(where E is a constant) and is related to the function $H(x)$ by

$$u(x) = H(x) \exp(-\tfrac{1}{2}x^2).$$

Prove that

$$\frac{d^2 H}{dx^2} - 2x \frac{dH}{dx} + (E - 1)H = 0.$$

By considering the series solution for $H(x)$, prove that if $E = 2m+1$, where n is an integer, there is a bounded solution for $u(x)$. Obtain the bounded form of $u(x)$ in the cases $n = 0$ and 1.

10. (P55310) Show that

$$\int_0^1 x\, J_o(k_m x) J_o(k_n x)\, dx = 0$$

where k_m and k_n are distinct positive zeros of the Bessel function $J_o(x)$.

11. (P55606) If the functions $P_n(\cos\theta)$ are defined by the equations

$$\sum_{n=0}^{\infty} x^n P_n(\cos\theta) = (1 - x\, e^{i\theta})^{-\frac{1}{2}} (1 - x\, e^{-i\theta})^{-\frac{1}{2}} \qquad (\,|x|\, < 1)$$

show that

$$P_{2n}(\cos\theta) = c_o + c_2 \cos 2\theta + \dots + c_{2n} \cos 2n\theta$$

where $c_o,\ c_2,\ \dots c_{2n}$ are independent of θ, and

$$c_o = \left[\frac{1.3.5\ \dots(2n-1)}{2.4.6\ \dots\ 2n}\right]^2 \quad,\quad \frac{c_2}{c_o} = \frac{2n(2n+1)}{(n+1)(2n-1)}$$

Hence, or otherwise, express $\sin\theta$ as a series in the functions $P_n(\cos\theta)$ for values of θ between 0 and π.

2. A sequence of polynomials $f_0(x)$, $f_1(x)$, $f_2(x)$, ... , of degrees $0, 1, 2, ...$ respectively, is constructed by the following orthogonalization process: $f_0(x) = 2^{-\frac{1}{2}}$, and the $r + 1$ coefficients in $f_r(x)$ $(r = 1, 2, ...)$ are defined by the conditions

$$\int_{-1}^{1} f_r(x) f_s(x) \, dx = 0 \quad (s = 0, 1, ..., r - 1)$$

$$= 1 \quad (s = r)$$

Show that $f_r(x) = (n + \tfrac{1}{2})^{\frac{1}{2}} P_n(x)$, where $P_n(x)$ is the Legendre polynomial of degree n.

3. Prove that if

$$f_n(x) = x^n + a_{n-1} x^{n-1} + ... + a_0,$$

then $\int_{-1}^{1} \left[f_n(x) \right]^2 \, dx$ is least when the a_r are such that $f_n(x) \propto P_n(x)$.

4. Show that the functions $T_0(x) = 1$, $T_n(x) = \cos(n \cos^{-1} x)$ are polynomials, orthogonal with weight function $\rho(x) = (1 - x^2)^{-\frac{1}{2}}$ over the interval $-1 \leq x \leq 1$. [Tchebycheff 1859].

1. Evaluate the following integrals in terms of $\Gamma(z)$ under suitable restrictions on z

$$\int_0^\infty dt\, t^{z-1} e^{-it}, \qquad \int_0^\infty dt\, t^{z-1}\begin{matrix}\cos t\\ \sin t\end{matrix}, \qquad \int_0^\infty e^{-st}\, t^{z-1}\, dt.$$

2. Show by a suitable change of variables that

$$\int_A dx\, dy\, x^{\alpha-1}\, y^{\beta-1}\, f(x+y) = B(\alpha,\beta) \int_0^1 dx\, f(x)\, x^{\alpha+\beta-1}$$

$$A = \{x \geq 0\} \cap \{y \geq 0\} \cap \{x+y \leq 1\}$$

3. Show that

$$B(\alpha,\beta) \sim \Gamma(\beta)\,(\alpha)^{-\beta} \qquad \alpha \to \infty$$

4. Compare the convergence of the two series

$$\sum \frac{a_n\, n!}{(z)_{n+1}} \qquad \sum a_n n^{-z}$$

where

$$(z)_n = z(z+1)\cdots(z+n-1)$$

5. Prove that $\Gamma(z^k) = [\Gamma(z)]^k$ Use this in conjunction with

$$\Gamma(z)\,\Gamma(1-z) = \pi / \sin \pi z$$

to show that for y real

$$|\Gamma(iy)| = [\pi / y \sinh \pi y]^{1/2}$$

6. Use the asymptotic form of $\Gamma(z)$ to show that for $y \to \infty$

$$|\Gamma(x+iy)| \sim |y|^x \left[2\pi\, e^{-\pi y/|y|}\right]^{1/2}$$

7. Examine the singular points of Legendre's and Bessel's equations.

8. Show that in Riemann's notation the Legendre equation is

$$w = P\left\{\begin{matrix} -1 & 0 & 1 \\ 0 & n+1 & 0 & z \\ 0 & -n & 0 \end{matrix}\right\}$$

and obtain the transformation which reduces the Legendre equation to a hypergeometric equation. Hence show that

$$P_n(z) = F(-n,\, n+1;\, 1;\, (1-z)/2)$$

9. **Verify that**

(36) $$J_\nu(z) = [\Gamma(\nu+1)]^{-1} \left(\frac{z}{2}\right)^\nu e^{-iz} \Phi(\nu+\tfrac{1}{2}, 2\nu+1; 2iz)$$

(37) Using the fact that $J_\nu(z) \cong \left(\frac{z}{2}\right)^\nu / \Gamma(\nu+1)$ $z \approx 0$

show that for suitable restrictions on ν, $J_\nu(z) = \int_C e^{z(t-\frac{1}{t})} \frac{dt}{t^{\nu+1}}$

where is the contour shown below:

(38) Using the transformation $t = \frac{2u}{z}$ obtain another representation

for $J_\nu(z)$ which holds for all z.

(39) Using the transformation $t = e^u$ show that

$$J_\nu(z) = \frac{1}{2\pi i} \int_{\infty - i\pi}^{\infty + i\pi} e^{z \sinh u - \nu u} \, du$$

and sketch the contour indicated the limits in the above equation.

Functional Analysis I

1. Axiomatic Approach

Prove from the axioms of a vector space that

(i) there is only one vector $|a'\rangle$ satisfying $|a\rangle + |a'\rangle = |0\rangle$
where $|a\rangle$ is any vector and $|0\rangle$ is the null vector

(ii) $\alpha \, |0\rangle = |0\rangle$ for any scalar α.

2. $1 \times 1 = 1$?

(i) Write down the axioms of a vector space.

(ii) Write down the additional axioms necessary to get a Normed vector space.

(iii) Write down the additional axioms necessary to get a Banach space.

(iv) Write down the additional axioms necessary to get a Hilbert space.

(v) The space S satisfies **all** the axioms in your list (i) **except** $1.|a\rangle = |a\rangle$ for vectors $|a\rangle \in S$. Show that if S satisfies the additional axioms in (ii), S was the whole time a Normed vector space.

3. The Positive Approach

Let $||x||$ be the norm of a Normed vector space N. Prove from the axioms that $||x|| \geq 0$.

4. Metric Mystery

Let X be a metric space with metric ρ . The sequence $\{x_n\}$ converges to x in X. Prove that $\lim_{n \to \infty} \rho(x_n, y) = \rho(x, y)$ for any $y \in X$.

5. Mr. Banach Meet Mr. Hilbert

(i) Let H be a Hilbert space with scalar product $\langle f|g \rangle$.

Define $||f||^2 = \langle f|f \rangle$.

Prove the parallelogram law:

$$||f + g||^2 + ||f - g||^2 = 2 ||f||^2 + 2 ||g||^2$$

(ii) Let C be the Banach space, considered in the lectures, of continuous functions $f(x)$ defined over range $[a, b]$ in x, with norm

$$||f|| = \max_{a \leq x \leq b} |f(x)|$$

Prove that C is <u>not</u> a Hilbert space with this norm.

<u>6.</u> Prove the polarization identity relating the inner product to the norm, i.e.,

$$<x|y> = \tfrac{1}{4} \{ ||x + y||^2 - ||x - y||^2 - i ||x + iy||^2 + i ||x - iy||^2 \}$$

This can be used to show that a normed linear space can be converted to an inner product space if its norm satisfies the ||gram law. (But you needn't go this far!)

<u>7.</u> <u>ℓ_2</u>

The inner product space ℓ_2 is defined to be the set of infinite sequences of complex numbers $(x_1, \ldots x_n, \ldots)$ satisfying

$$\sum_{i=1}^{\infty} |x_i|^2 < \infty$$

The inner product is defined by:

$$<y|x> = \sum_{i=1}^{\infty} y_i^* x_i$$

Prove that ℓ_2 is a Hilbert space.

8. ∞-tuples

In the lectures, we showed that the space ℓ_∞ of infinite sequences of real numbers: $f = (a_1, \ldots, a_n \ldots)$ for which $\sup_i |a_i| < \infty$, [$\sup$ = least upper bound] formed a Banach space with norm:

$$||f|| = \sup_i |a_i|$$

show

(i) Space c of all such sequences with (a_n) a __convergent__ sequences, is a Banach space.

(ii) Space c_0 of all such sequences with $a_n \to 0$ as $n \to \infty$, is a Banach space.

9. Let V be an inner product space and $|a_i>$ $i = 1 \ldots N$ an orthonormal set [not necessarily complete]. Show that

$$||x - \sum_{n=1}^{N} c_n e_n||$$

is minimized by $c_n = <e_n|x>$ for any $|x> \epsilon V$. $(||\ldots||$ is norm in space V).

10. Unsung Heroes

Explain what is meant by saying 1, x, x^2... is a complete basis for $L^2_w[0, 1]$. Given a function $f(x)$, $x \in [0, 1]$, I define a new function $F(u)$ by

$$F(u) = f(\sqrt{u})$$

is 1, u, u^2... a complete set for $F(u)$? If so, does it mean that 1, $x^2 = u$, $x^4 = u^2$... was complete in $L^2_w[0, 1]$?

11. Christoffel Numbers

(i) Let $p_n(x)$ be an orthogonal set of polynomials over the range [a, b] with respect to weight function $w(x)$. Show how they can be used to find formulae

$$\int_a^b \rho(x)w(x)\,dx = \sum_{i=1}^n \lambda_i \rho(x_i)$$

exact for any polynomial $\rho(x)$ of degree $2n - 1$ or less. Identify the numbers x_i, prove $\lambda_i > 0$ and find $\sum_{i=1}^n \lambda_i$.

(ii) Let [a,ß] be any subinterval of [a, b]. Show that for some n, $p_n(x)$ vanishes at least once in [a,ß].

[Hint: use Weierstrass's Approximation Theorem.]

I2. The Compleat Hermite

State the orthogonality properties of the Laguerre $L_n^a(x)$ and
Hermite polynomials $H_n(x)$. Assume that the Laguerre functions
$e^{-x/2} x^{a/2} L_n^a(x)$ are closed in $L^2(0, +\infty)$.

(i) What does this mean?

(ii) Prove that the Hermite functions $\exp(-x^2/2) H_n(x)$ are
closed in $L^2(-\infty, +\infty)$.

(Hint: break arbitrary f(x) into even and odd parts.)

13. Consider the linear vector space of real continuous functions with continuous first derivatives in the closed interval [0, 1]. Which (if any) of the following expressions define a scalar product satisfying our axioms?

   (i) $\langle f|g \rangle = \int_0^1 f'(t)g'(t) \, dt + f(0)g(0)$

   (ii) $\langle f|g \rangle = \int_0^1 f'(t)g'(t) \, dt$

14. Orthogonalize – with the Schmidt method – the set of vectors 1, x, and x^2 in the space of functions $f(x)$ in range $-1 \le x \le 1$ and scalar product

$$\langle f|g \rangle = \int_{-1}^{1} f(x)g(x) \, dx/\sqrt{1-x^2}$$

15. (i) Does the set of entire functions constitute a linear vector space? (An entire function is an analytic function of z that has no singularities for finite z).

(ii) Describe qualitatively (i.e., you need not write out all the axioms) the properties and differences between linear vector spaces, metric spaces, Hilbert spaces and Banach spaces.

(iii) S is the set of all complex numbers z with $|z| = 1$. Addition is defined by the normal addition of complex numbers and a distance $\rho(z_1, z_2)$ is defined as $\sqrt{(z_1-z_2)^*(z_1-z_2)}$. What sort of space is S? Is it complete?

(iv) S is the same set but "addition" is defined by $z = e^{i\theta} = z_1 z_2$ where $\theta = \theta_1 + \theta_2$ and $z_i = e^{i\theta_i}$ ($i=1,2$). The norm of a vector z is defined as $||z|| = |\theta|$ (choosing $-\pi \le \theta \le \pi$). Does this satisfy required axioms for a norm? What sort of space is S? Is it complete?

16. (i) Find the Fourier series of

$$f(x) = \begin{cases} 1 & -1 \le x < 0 \\ x & 0 < x \le 1 \end{cases}$$

What (from general principles) happens at $x = 0$?

Deduce the value of the infinite series

$$1 + 1/_{3^2} + 1/_{5^2} + \ldots$$

(ii) $f(x)$ and $g(x)$ have Fourier series

$$\left. \begin{matrix} f(x) \\ g(x) \end{matrix} \right\} = \sum_{n=-\infty}^{+\infty} \left\{ \begin{matrix} f_n \\ g_n \end{matrix} \right\} e^{inx}/\sqrt{2\pi}$$

What is the Fourier series of $f(x)g(x)$?

17. Let $\alpha(x)$ and $\beta(x,\varepsilon)$ be generalized functions of x in some neighborhood of $\varepsilon = 0$. We define

$$\lim_{\varepsilon \to 0} \beta(x,\varepsilon) = \alpha(x)$$

if for any good function $g(x)$

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} \beta(x,\varepsilon) g(x) dx = \int_{-\infty}^{+\infty} \alpha(x) g(x) dx$$

Show that the Fourier transform of $\alpha(x)$ is equal to $\lim_{\varepsilon \to 0}$ (Fourier transform of $\beta(x,\varepsilon)$). [You may assume the Parseval's relation

$$\int_{-\infty}^{+\infty} g_1(t) \, g_2^{\ *}(t) \, dt = \int_{-\infty}^{+\infty} f_1(x) \, f_2^{\ *}(x) \, dx$$

where g_i are the F. T.'s of the $\underline{\text{good functions}}$ f_i].

18. Find the Fourier transform of the following functions (or indicate that they do not exist in the sense of generalized functions).

   (i) $\delta(2x)$
   (ii) $\delta(x^2)$
   (iii) $\epsilon/$
          $\pi(\epsilon^2 + x^2)$
   (iv)   Use the result of the previous question (even if you can't prove it!) to show

$$\lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + x^2} = \delta(x)$$

   (v) By taking Fourier transforms prove

$$x^n \delta^{(m)}(x) = (-1)^n \frac{m!}{(m-n)!} \delta^{(m-n)}(x) \qquad m \geq n.$$

$$= 0 \qquad m < n$$

Note: $\delta^{(k)}(x)$ is k'th derivative of $\delta(x)$.

1. Evaluate the integral

$$I = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp[i \sum_{k=1}^{n} t_k x_k - \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j] dx_1 \ldots dx_n$$

arising in the theory of Fourier transforms of a function of n variables. Assume t_k and a_{ij} are real.

You can assume

$$\int_{-\infty}^{+\infty} e^{-t^2} dt = \sqrt{\pi}$$

2. Let l_2 be, as usual, set of all sequences $(a_1, a_2, \ldots)$ with finite norm

$$||a|| = \sqrt{a_1^2 + a_2^2 + \ldots}$$

Let C, D and G be the operators

$$C(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots)$$

$$D(a_1, a_2, a_3, \ldots) = (a_2, a_3, \ldots)$$

$$G(a_1, a_2, a_3, \ldots) = (a_1, a_2/2, a_3/3, \ldots)$$

Discuss linearity, boundedness, adjoint, complete continuity, eigenvectors, and solutions of the equation $0x = a$ for the three operators $0 = C, D, G$.

3. Let T_1 T_2, H, S and T be bounded operators in some normed vector space N. Prove

(1) $||T_1 T_2|| \leq ||T_1|| \ ||T_2||$

(ii) $||H^n|| = ||H||^n$ if H hermitian and n any integer (hint: use Schwarz inequality)

(iii) The operator $1 + S + S^2 + \ldots + S^n + \ldots$ exists if $||S|| < 1$ and N is complete.

(iv) T^{-1} exists if $||T - I|| < 1$, where I is identity operator.

4. T is an hermitian operator in a Hilbert space H: T is positive, i.e.,

$$\langle Tf|f\rangle \geq 0 \quad \text{for all } |f\rangle \ \epsilon H.$$

(i) Prove the generalized Schwarz inequality

$$|\langle Tf|g\rangle|^2 \leq \langle Tf|f\rangle \ \langle Tg|g\rangle$$

for all $|f\rangle$ and $|g\rangle$ in H.

(ii) Prove that if for some $|f\rangle$, $\langle Tf|f\rangle = 0$, then $|f\rangle = 0$.

5. Let H be the Hilbert space $L^2_{\omega=1}[0,1]$ and define the operator T on H by

$$T f(x) = x \ f(x) \quad \text{where } f(x) \text{ is any function (i.e., vector)}$$

ϵ H. Show

    (a) T is hermitian

    (b) $||T|| = 1$

    (c) T possesses no eigenvalues

    (d) The spectrum of T consists exactly of the interval [0, 1].

    (e) is T completely continuous?

6. Prove that the identity operator I (I|x> = |x>) is <u>not</u> completely continuous.

7. Let f(x) be any member of $L^2_{w=1}[-\infty,\infty]$ and define the Fourier transform of f(x)

$$g(t) = 1/\sqrt{2\pi} \int_{-\infty}^{+\infty} f(x) \, e^{-ixt} dx$$

Show that this can be regarded as an operator T from the space $L^2_1[-\infty,\infty]$ onto itself. Further show T is unitary ~~and T⁻¹ = T where T⁻²~~ ~~is unitary~~ and $T^4 = I$ where I is the identity operator.

General theorems on Fourier analysis should be stated but need not be proved.

8. Write Volterra's integral equation

$$f(x) = u(x) + \int_a^x k(x,t) \, f(t); \quad a \leq x \leq b \quad (\ast)$$

in operator form as

$$|f> = |u> + K|f>$$

Take the norm appropriate for the Banach space B of continuous functions C[a,b] : a ≤ x ≤ b.

Suppose k(x,t) is bounded: $|k(x,t)| \leq K$ all x,t in [a,b]. Bound $|K^n f|$ and hence show that $1 + K + K^2 + \ldots$ exists. Deduce that $(\ast)$ is always soluble.

9. Solve the system of equations

$$\frac{du}{dt} + au(t) = f(t)$$

$$u(0) = 0$$

where a is a complex constant and f(t) is a known function by first finding the Green's function for the problem.

10. Solve

$$du(t)/dt + \int_0^1 \sin k(s-t) \, u(s) \, ds = a(t)$$

subject to the boundary condition

$$u(0) = 0.$$

11. Find a solution to the transport equation

$$1/v \; \partial/\partial t \; u(\underline{x},t) + (\underline{n} \cdot \underline{\nabla}) \; u(\underline{x},t) + \alpha \; u(\underline{x},t) = \delta(\underline{x},t)$$

$$u = 0, \; t = 0$$

where:

(a) $\underline{n} \cdot \underline{\nabla} = n_x \; \partial/\partial x + n_y \; \partial/\partial_y + n_z \; \partial/\partial_z$ with $|\underline{n}| = 1$

(b) v and α are constants

(c) $\delta(\underline{x},t) = \delta(x) \; \delta(y) \; \delta(z) \; \delta(t)$

12. Find a function $u(x,y)$ defined over the rectangle $0 \leq x \leq 1$, $0 \leq y \leq 1$ such that

(a) $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = \delta(x-x') \; \delta(y-y')$

(b) $u(0,y) = u(1,y) = u(x,0) = u(x,1) = 0$.

Problem 1: Iteration of Property A

A matrix B is said to have the "property A" if it can be written in the form $\begin{bmatrix} I & -R \\ -Q & I \end{bmatrix}$ where I is the unit $n \times n$ matrix and R and Q are arbitrary $n \times n$ matrices. If $\tilde{I} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$, $\tilde{R} = \begin{bmatrix} 0 & R \\ 0 & 0 \end{bmatrix}$ and $\tilde{Q} = \begin{bmatrix} 0 & 0 \\ Q & 0 \end{bmatrix}$, then the equation $B \underline{x} = \underline{d}$ is solved iteratively by (a) Jacobi: $\underline{x}^{(n+1)} = (\tilde{Q} + \tilde{R}) \underline{x}^{(n)} + \underline{d}$

(b) Gauss-Seidel: $\underline{x}^{(n+1)} = \tilde{Q} \underline{x}^{(n+1)} + \tilde{R} \underline{x}^{(n)} + \underline{d}$.

By considering the eigenvalues of the iteration matrix in the two cases, show that if the Gauss-Seidel method converges it does so twice as fast as the Jacobi method.

Problem 2: Relax

One way (slightly different from that in the lectures) of accelerating the convergence of the Gauss-Seidel method is to consider the iteration

$$x^{(n+1)} = (I-w) \, x^{(n)} + w \left[\underline{d} + \tilde{Q} \, x^{(n+1)} + \tilde{R} \, x^{(n)} \right] \qquad (*)$$

where the matrices have the same meaning as in the previous problem. Suppose that the simple Jacobi iteration matrix $(\tilde{Q} + \tilde{R})$ has largest eigenvalue $\theta < 1$. Show that the best value of w that ensures the fastest convergence of (*) is

$$w_b = 2 / \left[1 + \sqrt{1 - \theta^2} \right]$$

and that the largest eigenvalue of the iteration matrix has modulus $w_b - 1$. ~~The~~

Problem 2 (continued)

In the lectures we considered the alternative over-relaxation scheme

$$(1+\alpha) \ (I - \tilde{Q}) \ x^{(n+1)} = \alpha \ (I - \tilde{Q}) \ x^{(n)} \\ + \tilde{R} \ x^{(n)} + d \qquad (**)$$

Show that for (**), the best value of α is $\alpha_b = -g^2/2$ and its iteration matrix has largest eigenvalue $\dfrac{g^2}{2 - g^2}$.

Is it better to use (*) or (**)?

Problem 3/ Calculation of the Legendre Functions

A sequence of functions $P_n(x)$ for $-1 \leq x \leq \infty$ are defined by the recurrence relation

$$(n+1) \ P_{n+1}(x) = (2n+1) \ x \cdot P_n(x) - n \ P_{n-1}(x) \qquad (*)$$

plus prescribed values of $P_0(x)$ and $P_1(x)$. By considering trial solution of the form $n^{-\alpha} e^{\pm in\theta}$ $\left(x = \dfrac{\bullet}{\cos\theta}\right)$ for $-1 \leq x \leq 1$ and of the form $n^{-\alpha} e^{\pm n\psi} \left(x = \text{cosh}\psi\right)$ for $x > 1$, discuss the solution of (*) for large values of n.

Use your results to discuss the numerical computation of the Legendre functions $P_n(x)$, $Q_n(x)$ for the two ranges of x and, say, $0 \leq n \leq 20$. (See Abramowitz and Stegun pages 332 → 341)

Iterative Eigenvalues

Let A be any n x n matrix and I the unit matrix and p any (real) number:
The Iteration schemes I_1, I_2 are defined by:

I_1: Z_{k+1} = (A -pI)Y_k : Y_0 arbitrary

           Y_{k+1} = Z_{k+1} ÷ numerically largest

                      largest element of Z_{k+1}

I_2: $A^{(k)}$ = $A^{(k-1)}$ $A^{(k-1)}$: $A_{\bullet}^{(0)}$ = A-pI

Show how I_1 and I_2 may be used to find an eigenvalue and eigenvector of
A. (Which one is found?)

Discuss the rate of convergence of the two methods and compare both this
and number of arithmetic operations in I_1 and I_2. Show I_2 is better for large $n \geq n_0$
(say) and find n_0 in terms of the desired accuracy and a certain ratio of eigenvalues
of A-pI.

What is the best choice for p?

1. Poisson's Equation:

Sinbad the Sailor wanted to solve Poisson's equation $\nabla^2 \phi = \rho$ for the number of fish in one of flatland's smaller oceans (this is near the coast of Pasadena's population) by the finite difference method.

$$\phi_{i,j}^{(n+1)} = \phi_{i,j}^{(n)} + \tfrac{1}{4}\alpha(\phi_{i+1,j}^{(n)} + \phi_{i-1,j}^{(n)} + \phi_{i,j+1}^{(n)} + \phi_{i,j-1}^{(n)} - 4\phi_{i,j}^{(n)} - 4h^2\rho_{i,j})$$

Here we have set up a grid, of Size h, labeled by integers i,j with $-N \le i,j \le N$ and N a large integer. (You can obtain the latter by feeding a small integer peanut butter sandwiches.) Again $\rho_{i,j} = \rho_{-i,-j}$ are the values of ρ (the density of sharks) at the grid points. $\phi_{i,j}^{(n)}$ ($n \ge 0$) is the n'th approximation to the true solution $\phi_{i,j}$ at the grid points. The boundary conditions are $\phi_{i,j} = 0$ ($i,j = \pm N$) — this being the density of fish as seen by an observer with a fishing rod on the shore. Further Sinbad starts with the pessimistic approximation $\phi_{i,j}^{(0)} = 0$. Let $\varepsilon_{i,j}^{(n)} = \phi_{i,j}^{(n)} - \phi_{i,j}$ be the error after n iterations. Disregard the possibility that Sinbad is eaten while in the middle of the ocean and consider the eigenvalues/vectors of the recursion equation for $\varepsilon^{(n)}$. Hence show that $\varepsilon^{(n)}$ may be written as a finite sum of terms of the form

$$a(p,q)\left[\delta(p,q)\right]^n \cos\frac{p\pi i}{2N} \cos\frac{q\pi j}{2N}$$

where p, q are integers in the range 0 to N, a(p,q) depends on the value of ϕ, and

$$\delta(p,q) = 1 - \alpha\left(\sin^2\frac{p\pi}{4N} + \sin^2\frac{q\pi}{4N}\right)$$

Find both the range of α for which Sinbad's process converges and also the optimum value of α (i.e., the value for which the process converges fastest and for which choice Sinbad will be given tenure as regius professor of seaweed).

Note that Sinbad's mathematical brother, Albert, has interpreted a negative fish density $\rho < 0$ as corresponding to minnows, with fins on their faces, swimming backwards.

2. Mathews & Walker, 16-1:

Which of the following are groups?

(a) All real numbers (group multiplication = ordinary multiplication)
(b) All real numbers (group multiplication = addition)
(c) All complex numbers except zero (group multiplication = ordinary multiplication)
(d) All positive rational numbers ("product" of a and b is a/b)

3. Mathews & Walker, 16-2:

Consider the following two elements of the symmetric group S_5:

$$g_1 = [54123] = (135)(24)$$

$$g_2 = [21534] = (12)(345)$$

Find a third element g of this group such that

$$g^{-1}g_1 g = g_2$$

4. Mathews & Walker, 16-4:

Consider the symmetry group of a regular tetrahedron.

(a) What is the order of this group?
(b) Decompose it into classes.
(c) Construct its character table.

5. Mathews & Walker, 16-7:

Show that a representation D(g) is irreducible, if, and only if,

$$\overline{X(g)^* X(g)} = 1$$

where X(g) is the character of D(g). Suppose $\overline{X(g)^* X(g)} = 2$; what does this tell us about D(g)?

6. Define a **unitary matrix**. Show that the set of all unitary (n x n)-matrices forms a group under matrix multiplication.

If H is a positive definite hermitian form on a complex n-dimensional vector space V, show that the set of linear mappings f:V → V such that H(fx, fy) = H(x, y) for all x, y in V, forms a group isomorphic to the group of unitary (n x n)-matrices.

If S is a non-singular symmetric bilinear form on a real 2-dimensional vector space W, show that the set of linear mappings g:W → W such that S(gx, gy) = S(x, y) for all x, y in W is a group, and that it contains a subgroup isomorphic to either the circle group (complex numbers of unit modulus) or to the (additive) group of real numbers.

Problem1: Interpolation

The function $Y = f(x)$ is tabulated at equal intervals and $Y_n = f(x_o + nh)$ $(x_o, h$ fixed n an integer). Let Δ be the forward difference operator ($\Delta Y_n = Y_{n+1} - Y_n$) and E the displacement operator ($E Y_n = Y_{n+1}$).

(i) The effect of a copying error in our table for Y_n may be studied by the model function $Y_m = 1$ for $m = 0$ and 0 for $m \neq 0$. Show that the differences for this function are given by:

$$\Delta^n Y_m = 0 \; : \; m > 0$$
$$= 0 \; : \; m < -n$$
$$= {}_n C_{-m} (-1)^{n+m} \qquad 0 \geq m \geq -n.$$

(Hint: use the relation between Δ and E)

(ii) The following table was prepared by our favorite secretary while watching Popeye.

X	Y
0	1.0
.1	.9048
.2	.8187
.3	.7408
.4	.6703
.5	.6056
.6	.5488
.7	.4966
.8	.4493
.9	.4066

Unfortunately she transposed two digits when Popeye was eaten by a herd of carnivorous seaweeds. Construct a difference table for Y and by comparing this with (i) correct her error.

<u>Problem 2: Euler's Transformation</u> (Mathews and Walker, page 54)

(i) Let S be the sum

$$S = \sum_{s=0}^{\infty} (-x)^s U_s \qquad (*)$$

where x is any number and U_s is any series. If Δ is the forward
difference operator -- by using the relation between it and the displacement
operator E, show that (*) can be formally transformed to:

$$S = \sum_{s=0}^{\infty} \frac{(-x)^s}{(1+x)^{s+1}} \Delta^s U_0 \qquad (**)$$

(ii) Use this method to evaluate π to four significant figures from the
series:

$$\pi = 4 \left\{ 1 - 1/3 + 1/5 - 1/7 \ldots \right\} \qquad (\overline{HP})$$

Hint: Sum the first six terms of $(\overline{HP})$ directly and apply
the transformation (**) to the remainder.

Problem * : Continuous Moment Sum Rules

Some years ago, "Continuous Moment Sum Rules" were popular in high
energy physics. These involve the integral

$$I(\alpha) \quad = \quad \int_{Y_1}^{Y_2} \oint(Y) \cdot (Y - Y_0)^{\alpha} \, dY \qquad (*)$$

where you can assume $Y_1 \geq Y_0$, and α is a variable parameter. $\oint(Y)$ is a
function which is known at the "grid-points" $Y_1 + m \, (Y_0 - Y_1)/N$
(m, N integers). Explain how (*) can be evaluated by repeated application
of Simpson's rule in the case $\alpha = 0$.

Generalize your discussion to the case $\alpha \neq 0$ expressing the formulae
in terms of the weights $W_{\pm 1, \, 0}$ of the model problem: ($x_0 \leq -a$)

$$\int_{-a}^{+a} (x - x_0)^{\alpha} \oint(x) \, dx \qquad (**)$$

$$\simeq W_{-1} \oint(-a) + W_0 \oint(0) + W_1 \oint(a)$$

where the weights w_i should be determined so that (**) is exact for
polynomials of degree ≤ 2.

① Positive Definite Iteration

The real matrix A is symmetric and positive definite. We write
it

$$A = D + L + L^T$$

where D diagonal and L lower triangular with zero diagonal elements.

Let Y be an eigenvector of (complex) eigenvalue λ for $(D+L)^{-1} L^T$.

Put

$$Y^* {}^T DY = \alpha$$

$$Y^* {}^T LY = \beta + i\gamma$$

Deduce

$$|\lambda|^2 = \frac{\beta^2 + \gamma^2}{\beta^2 + \gamma^2 + \alpha(\alpha+2\beta)}$$

and hence prove that Gauss-Seidel iteration will always converge for A.

What can you say about the Jacobi iteration scheme for such matrices A?

2. Quadratic Convergence

The equation

$$f(x) = 0 \qquad\qquad (*)$$

is solved iteratively by:

(a) Newton:

$$x_1 = x_0 - f(x_0)/f'(x_0)$$

(b) Interpolation:

$$x_2 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)}$$

If the true solution is x_{true} and $x_i = x_{true} + \delta x_i$ $i = 0\ldots 4$

Show

$$\delta x_1 \underset{\sim}{\gtrsim} 1/2\, f''(x_{true})/f'(x_{true}) \delta x_0^2$$

$$\delta x_2 \underset{\sim}{\gtrsim} 1/2\, f''(x_{true})/f'(x_{true}) \delta x_3\, \delta x_4$$

Use these formulae to compare the speed of convergence of methods (a) and (b).

Your computing budget is running low and for a particular $f(x)$ in (*), your computer programmer says that $f'(x)$ takes the same time to calculate as $f(x)$. Should you advise him to use method (a) or method (b) to solve (*)?

It is desired to represent the function $f(x) = x^3$ in the range $-1 \leq x \leq 1$ by the approximation $f(x) \approx a_0 + a_1 x + a_2 x^2$. (This is necessary when using x^3 in deep dark African jungles where the natives only count up to 2.)

Solomon suggests Taylor expansion about $x = 0$. Plato, who has just done partial wave analysis in his quantum mechanics class, suggests truncating the Legendre polynomial expansion. ($P_0 = 1$, $P_1 = x$, $P_2 = 1/2 \ (3x^2-1)$). Samson, while scratching a flea in his hair, suggests truncating the Chebyshev expansion ($T_0 = 1$, $T_1 = x$, $T_2 = 2x^2-1$). Find a_0, a_1 and a_2 for the three methods. Which gives the smallest maximum deviation $d = \max \ |f(x) - a_0 - a_1 x - a_2 x^2|$ $-1 \leq x \leq 1$

in

4. <u>π in the Sky</u>

It is desired to represent I = $\int_a^b f(x) \, dx = \sum_{r=1}^{n} W_r \, f(x_r)$ (*)

Describe how W_r and x_r are chosen for

(a) The Monte Carlo method

(b) Romberg's method of iterating Simpson's Rule

(c) Application of the m - order Gauss formula R times --

where n = (m-1) R + 1.

Pythagoras evaluates π numerically by

$$\pi/4 = \int_0^1 dx \int_0^{\sqrt{1-x^2}} f(x,y) dy$$ (1)

with f(x,y) = 1

while Cleopatra prefers

$$\pi/4 = \int_0^1 \int_0^1 dx \, dy \, G(x,y)$$ (2)

G(x,y) = 0 for $1 < x^2 + y^2$

= 1 for $1 \geq x^2 + y^2$.

Write the error in the three methods of tackling (*) as C $n^{-\alpha}$ where you should specify α and the form of C for each method. (The exact numerical coefficient in C need not be given.)

Use this to discuss the accuracy of π as calculated from (1) or (2) using n = q^2 points and our three methods.

5 7. Prove that the Gauss–Seidel method for the iterative solution of the set of linear equations

$$Ax = b$$

converges if A is symmetric and positive definite.

For what sort of system should this be used in preference to other methods, and why?

6 7 8. Let C be a closed curve with interior R and let f be a given function defined on C. If u is defined as the true solution of the problem

$$\nabla^2 u = 0 \quad \text{in} \quad R, \quad u = f \quad \text{on} \quad C,$$

and U is the solution of the corresponding set of difference equations on a grid, establish a bound for $|u - U|$ in terms of bounds for the higher derivatives of u.

7 13 9. Describe the method of Gaussian elimination for the solution of a set of simultaneous linear algebraic equations.

In what circumstances is elimination without interchange (a) possible, and (b) numerically satisfactory, and why?

8 13 10. The function $u(x, y)$ satisfies the equation

$$\nabla^2 u - \lambda u = 0$$

in a region inside a closed curve C in the (x, y) plane, and the boundary condition $u = f(x, y)$ on C. Explain and justify the use of the Monte Carlo method to obtain a numerical approximation to $u(x, y)$ when $\lambda \geq 0$. Why does the method fail if $\lambda < 0$?

9 14 11. Explain the meaning of the terms *explicit* and *implicit* as applied to finite difference approximations to the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

If the mesh points are at $x = mh$ $(m = 0, \pm 1, \pm 2, \ldots)$, $t = nk$ $(n = 0, 1, 2, \ldots)$, show that the simplest explicit scheme is stable if $k/h^2 \leq \frac{1}{2}$, but that the simplest implicit scheme is stable for any value of k/h^2.

10 10 12. Describe the Gauss–Seidel process for the iterative solution of the set of simultaneous linear equations $Ax = b$, explaining clearly what is meant by over- and under-relaxation. Prove that for the process to converge it is necessary that the constant β of over- or under-relaxation should satisfy the condition $0 < \beta < 2$.

11 18 13. With $r = \Delta t/(\Delta x)^2$, determine the truncation error in replacing the differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

by the finite difference scheme

$$u_{i,j+1} - u_{i,j} = r[\alpha(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}) + (1-\alpha)(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})],$$

where α is a constant satisfying $0 \leq \alpha \leq 1$. Under what conditions on r and α does an iterative scheme based on this equation converge?

Ph 129

Vectors and Matrices

1. V and W are vector spaces, and $\alpha \colon V \to W$ is a linear map. $\rho(F)$ is the set of vectors in W which are of the form αv; that is $v \in V$); $\alpha^{-1}(0)$ is the vectors in V such that $\alpha(v) = 0$. State and prove a relation between the dimensions r, s and u of $\rho(F)$, $\alpha^{-1}(0)$ and V.

It is now given that $W = V$ and that $\alpha^n = 0$ for a certain positive integer n. By considering the subspaces $\alpha^p(F)$ for $1 \le p \le r$, or otherwise, show that

$$r \le n(1 - r^{-1}).$$

2. Define the subspace $s(p)$ of a permutation p of the numbers $1, 2, \ldots, n$. If σ is also such a permutation, prove that $s(p\sigma) = s(p)s(\sigma)$.

Define the determinant $|A|$ of an $n \times n$ matrix, and prove that $|AB| = |A| \, |B|$.

A, B are $n \times n$ matrices; show that

$$\begin{vmatrix} A & -B \\ B & A \end{vmatrix} = |A + iB| \, |A - iB|.$$

3. V is the space of column vectors with n complex elements; $v_1, v_2, \ldots, v_n$ are vectors in V. Show that there is a base $b_1, b_2, \ldots, b_n$ of V such that

$$\tilde{b}_i b_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$$

and v_i is linearly dependent on $b_1, b_2, \ldots, b_i$.

Show that the matrix whose columns are $v_1, v_2, \ldots, v_n$ is the product of a unitary matrix and a triangular matrix.

4. $Q(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$ is a polynomial such that the equation $Q(x) = 0$ has distinct roots $\lambda_1, \lambda_2, \ldots, \lambda_n$. A square matrix A satisfies the equation $Q(A) = 0$; prove that A is similar to a diagonal matrix.

[Two matrices A, B are said to be similar if there is a non-singular matrix P such that $B = P^{-1}AP$. The following method is suggested. Define

$$Q_r(x) = \prod_{j \ne r} (x - \lambda_j);$$

show that

$$\frac{Q_1(x)}{Q_1(\lambda_1)} + \frac{Q_2(x)}{Q_2(\lambda_2)} + \ldots + \frac{Q_n(x)}{Q_n(\lambda_n)} = 1$$

and

$$\frac{Q_1(x)}{Q_1(\lambda_1)} + \frac{Q_2(x)}{Q_2(\lambda_2)} + \ldots + \frac{Q_n(x)}{Q_n(\lambda_n)} = 1.$$

Assuming that $Q(A) = 0$, show that any vector v such that Av is defined can be written in the form

$$v = v_1 + v_2 + \ldots + v_n$$

where v_r is either zero or an eigenvector with eigenvalue λ_r.

5 4B. Prove that, if a vector x is a common eigenvector of two square matrices A, B, then it is also an eigenvector of AB.

$x_1, ..., x_n$ are linearly independent vectors, each of which is an eigenvector of the $n \times n$ matrices A, B. Prove that AB = BA.

6 $R_{\rho\sigma\nu}$ is a tensor in n dimensions which satisfies the relations

$$R_{\rho\sigma\nu} = -R_{\rho\sigma\nu} = R_{\sigma\rho\nu} = R_{\rho\sigma\nu}.$$

How many independent components does $R_{\rho\sigma\nu}$ have?

Show that in three dimensions

$$R_{\rho\sigma\nu} + R_{\sigma\rho\nu} + R_{\rho\sigma\nu} = 0.$$

7 1F. Define the rank of a matrix, and prove that rank is unaltered by multiplication by a non-singular matrix.

Calculate the rank of the matrix

$$\begin{pmatrix} 2 & -3 & 6 & 11 \\ 1 & 4 & -5 & 0 \\ -3 & -2 & 1 & 4 \\ -3 & -3 & 3 & 19 \end{pmatrix}.$$

Prove that if A is a square matrix of order n, the rank of adj A is n, 1 or 0 according as the rank of A is n, $n-1$, or less than $n-1$.

8 2F. Show that if M is a square matrix such that $M^*M = 0$ then $M = 0$.

Show that if P and Q are Hermitian and $PQ = 0$ then $QP = 0$; deduce that if P and Q are normal and $PQ = 0$ then $QP = 0$.

[A matrix M is normal if $MM^* = M^*M$; an asterisk denotes the transposed complex conjugate matrix.]

9 3F. Prove that for any square matrix A there is a unique monic polynomial $m(x)$ of minimum degree such that $m(A) = 0$. [A monic polynomial is one in which the coefficient of the highest power of x that occurs is 1.]

Prove that a necessary and sufficient condition that there should exist a non-singular matrix P such that $P^{-1}AP$ is diagonal is that $m(x)$ should have no repeated factors.

5 D. Two right-handed triads 0123, 0'1'2'3' have common origin. If the direction cosines of the axis 0j' ($j = 1, 2, 3$) with respect to the triad 0123 are a_{ij} ($i = 1, 2, 3$), show that

$$a_{ij}a_{ik} = \delta_{jk}.$$

Prove that also $a_{ji} = 1$.

Define a tensor of rank n and deduce that ϵ_{ijk} is a tensor where

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (ijk) \text{ is an even permutation of } (123), \\ -1 & \text{if } (ijk) \text{ is an odd permutation of } (123), \\ 0 & \text{otherwise.} \end{cases}$$

Show also that if A_i, B_j are tensors of rank 1 then $\epsilon_{ijk}A_i B_j$ is a tensor of rank 1.

3 F. Let α be a linear transformation of an n-dimensional vector space V into itself. Explain what is meant by an eigen-vector of α and its associated eigen-value. Prove that the eigen-vectors of α having a given eigen-value λ, together with the zero vector, form a subspace V_λ of V; and show that if $\lambda_1, \lambda_2, \ldots, \lambda_r$ are distinct eigen-values of α, then the subspace $V_{\lambda_1} + V_{\lambda_2} + \ldots + V_{\lambda_r}$ is the direct sum of the subspaces $V_{\lambda_1}, V_{\lambda_2}, \ldots, V_{\lambda_r}$.

Prove that λ is an eigen-value of α if and only if it is a root of

$$\det (A - \lambda I) = 0, \tag{1}$$

where A is the matrix of α representing α with respect to an arbitrarily-chosen basis of V, and verify that the equation (1) is independent of the choice of basis. If λ is a root of (1) of multiplicity $m(\lambda)$, show that

$$\dim V_\lambda \leq m(\lambda),$$

and show by an example that the inequality may be strict.

4 G. If H is a Hermitian matrix of order n, show that there exist unitary matrices U with the property that U^*HU is a real diagonal matrix, where U^* is the transpose of the complex conjugate of U.

Prove that, if H is real, then the number of real unitary matrices U with the stated property, if finite, is $2^n(n!)$. In what circumstances are there infinitely-many such real matrices U!

3. Define a *linear mapping* from one vector space to another.

If U, V and W are finite dimensional vector spaces over a field F, and $f: U \to V$ and $g: V \to W$ are linear mappings, let $N(f)$ denote the set of vectors x in U such that $fx = 0$. Show that $N(f)$ is a linear subspace of U and that $\dim [N(f)] + \dim [f(U)] = \dim U$. Show also that

$$\dim [f(U) + \dim (gV) - \dim V \leqslant \dim [gf(U) \leqslant \min (f(U), \dim (gV)].$$

4. If F is an n-dimensional vector space over the field F and $f: V \to V$ is a linear mapping, define the *eigenvalues* and *eigenvectors* of f.

Let the polynomial $p(t)$ be $(t - a_1)(t - a_2) \ldots (t - a_n)$, where $a_1, a_2, \ldots, a_n$ are distinct elements of F. If

$$p_i(t) = \prod_{j \neq i} \frac{(t - a_j)}{(a_i - a_j)}$$

show that $\sum_{i=1}^{n} p_i(t) = 1$.

Hence, or otherwise, show that, if the linear mapping $f: V \to V$ satisfies the equation $p(f) = 0$, then it may be represented by a diagonal matrix.

5. Prove that, if A is a real $n \times n$ orthogonal matrix which does not have -1 as a characteristic root, then A can be expressed in the form $(I + S)^{-1}(I - S)$, where I denotes the $n \times n$ unit matrix and S is a suitable real skew-symmetric matrix.

Deduce that A leaves invariant a real quadratic form $x'Bx$ if and only if $SB = BS$.

Show that every improper orthogonal matrix has -1 as a characteristic root and that a proper orthogonal transformation of an odd-dimensional Euclidean space onto itself leaves fixed the points of at least one line through the origin.

① 7 D. Explain what is meant by saying that the mapping defined by $w = f(z)$ of a domain D of the z-plane into the w-plane is *conformal* at a point z_0 of D. Prove that, when $f(z)$ is a regular function of z in D, the mapping is conformal at points where $f'(z) \neq 0$.

Describe the domains into which the half-planes $\Re z > 0$, $\Im z > -\frac{1}{2}$ are taken by the mapping defined by $w = z^2 + z$. Find the largest value of ρ such that this mapping is $(1,1)$ for $|z| < \rho$.

② 8 D. Prove that, if $f(z)$ is regular for $|z| < R$, then $f(z)$ has an expansion as a power series

$$\sum_{n=0}^{\infty} a_n z^n$$

convergent for $|z| < R$, and obtain contour integral expressions for the coefficients a_n.

Given that $|f(re^{i\theta})| \leq M$, where $r < R$, prove that

$$|a_n| \leq M r^{-n} \quad (n = 0, 1, 2, \ldots).$$

Prove that, if $f(z)$ is regular in the whole plane and $|f(re^{i\theta})| \leq A r^k$ for all r, where $k > 0$, then

$$|a_n| \leq \frac{A r^{k/n}}{(n/k)^{n/k}} \quad (n = 1, 2, \ldots).$$

③ ✓ 9 D. Prove Rouché's theorem that, if $f(z)$ and $g(z)$ are regular on and within the simple closed contour γ, and $|f(z) - g(z)| < |f(z)|$ on γ, then $f(z)$ and $g(z)$ have the same number of zeros inside γ.

Prove that all the roots of the equation

$$z^5 - 5z^3 + 3 = 0$$

satisfy the inequalities $\frac{5}{4} < |z| < 2$. How many roots of the equation lie inside the circle $|z| = 1$?

④ 8. Discuss briefly the transformation $w = z^2$ of the complex plane. Specify the curves in the w-plane corresponding to the two lines (i) $z = a + iy$, a fixed and positive, y arbitrary; (ii) $z = x + ib$, b fixed and positive, x arbitrary. Explain why these curves cut two points while the given lines have only one point in common, and show how, by suitable limitations of the regions in which w and z are allowed to lie, the transformation may be considered as one-one and conformal. Verify the conformality for the intersection of the given lines and that of their transforms.

By the use of this transformation, or otherwise, find a function $f(w)$, regular in the closed region $4\rho > 4 - w^2$ (here w as usual $w = u + iv$, u and v real), and such that $f(w) = w$ when $4u = 4 - v^2$. Does there exist a function $f(w)$, regular everywhere, and equal to $f(w)$ in the region given? [Gives reasons for your answer, but a formal demonstration is not expected.]

10. Prove the theorem which gives the number of zeros of a regular function $f(z)$, lying within a simple closed contour C, in terms of the argument (or amplitude) of $f(z)$ on C. State without proof the extension of the theorem to the case when $f(z)$ has poles, but no other singularities.

By applying the theorem to its extension to the square with corners $\pm N(1 \pm i)$ (where N is an integer $\geqslant 2$), or otherwise, determine whether the equation $\tan \pi z = z$ has any roots other than its real roots, and, if so, how many.

8. State and prove Liouville's Theorem.

Deduce (i) that every polynomial has a root; and (ii) that every meromorphic function (that is, regular except for poles) on the complex sphere is rational.

Let

$$g(z) = a_0 + a_1 z + \ldots + a_N z^N$$

be a polynomial with real coefficients. By Cauchy's theorem, or otherwise, show that

$$2 \int_{-1}^{+1} \{g(x)\}^2 \, dx \le \int_{-\pi}^{\pi} |g(e^{i\theta})|^2 \, d\theta.$$

Deduce that

$$\sum_{m,n=0}^{N} \frac{a_m a_n}{m+n+1} \le \sum_{n=0}^{N} a_n^2,$$

the sign of equality being required only when all the a_n are zero.

(i) The function $f(z)$ is regular on and inside the simple closed curve Γ. It has no zeros on Γ and the zeros inside Γ are $z_1, \ldots, z_r$ with multiplicities $m_1, \ldots, m_r$ respectively. State and prove the 'principle of the argument', which gives $m_1 + \ldots + m_r$ in terms of the behaviour of the argument of $f(z)$ on Γ. Show also that

$$2\pi i \sum_{j=1}^{r} m_j z_j = \int_{\Gamma} z \frac{f'(z)}{f(z)} \, dz,$$

the integral being taken in the positive sense.

(ii) Let

$$g(z) = a_0 + a_1 z + \ldots + a_n z^n,$$

where the a's are real and

$$a_n > a_{n-1} > \ldots > a_1 > a_0.$$

By considering $(z-1)g(z)$, or otherwise, show that all the real or complex zeros of $g(z)$ are in $|z| < 1$. Deduce from the principle of the argument that

$$g(\theta) = a_0 + a_1 \cos\theta + \ldots + a_n \cos n\theta$$

has $2n$ distinct real zeros in $0 \le \theta < 2\pi$.

9. (i) Show that the transformation

$$w = \frac{\alpha z + \beta}{\gamma z + \delta},$$

where $\alpha, \beta, \gamma, \delta$ are any complex numbers such that $\alpha\delta \ne \beta\gamma$, takes straight lines or circles into straight lines or circles.

(ii) Find a function which maps the region

$$|z| < 2, \quad |z-1| > 1$$

conformally onto the interior of the unit circle.

8 E. Prove that if $F(z)$ is regular on the closed contour C then
$$\int_C F'(z)\,dz = 0.$$

Define $\log z$ as a one-valued function in the region R of all complex z, $z \neq 0$ and z not on the negative real axis, and show that this function is regular in R.

Hence or otherwise prove that if C is a simple closed contour and $f(z)$ and $g(z)$ are regular on and inside C and satisfy
$$|f(z) - g(z)| < |f(z)| + |g(z)|$$
on C, then f and g have the same number of zeros inside C.

[All contours may be assumed to have continuously varying tangents.]

9 E. State and prove a theorem which connects the integral of a function $f(z)$ round a closed contour C in the complex plane with the residues of f at singularities of f inside C, assuming the number of such singularities to be finite. By applying this result to $\{\cot \pi z\}/(a+z)^2$, or otherwise, prove that for complex non-integral a
$$\sum_{r=-\infty}^{\infty} \frac{1}{(a+r)^2} = \frac{\pi^2}{\sin^2 a\pi}.$$

[You may assume Cauchy's, and also that $\cot \pi z$ is uniformly bounded on the circles $|z| = n + \frac{1}{2}$ $(n = 1, 2, \ldots)$.]

8 E. State carefully, without proof, Riemann's fundamental theorem on the existence and uniqueness of a conformal mapping of a region D in the complex plane on the unit circle. Prove the theorem in the special case when D is a circular disc.

[You may assume general properties of mappings $z \to \dfrac{az+b}{cz+d}$.]

F. 2. The function $S(z)$ is defined for all complex z by
$$S(z) = \sum_{n=0}^{\infty} \frac{(-)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots.$$

Show that the (real or complex) zeros of $S(z)$ are precisely the numbers
$$nw \quad (n = 0, \pm 1, \pm 2, \ldots),$$
where w is a certain real number, $0 < w < 4$.

[You may assume, without seriously carrying out the estimation, that $S(4) < 0$. All properties of the exponential and trigonometric functions are used should be proved.]

8 C. The function $f(z)$ is regular for $|z| < R$. Prove that, for $|z| < R$,

$$f(z) = \sum_{0}^{\infty} a_n z^n, \quad \text{where} \quad a_n = \frac{1}{2\pi i} \int \frac{f(z)\, dz}{z^{n+1}},$$

the integral being taken round any circle $|z| = r$ with $0 < r < R$.

Prove also that, for $n > 0$,

$$a_n = \frac{1}{\pi r^n} \int_0^{2\pi} f(re^{i\theta}) \cos n\theta\, d\theta.$$

9 C. Prove that, if z is real and $0 < |z| < 1$,

$$\int_0^\pi \frac{z^2 \log x\, dx}{1 + z^2} = \frac{\pi^2 \sin \frac{1}{2}z\pi}{2(1 + \cos z\pi)}.$$

Is this result still valid if $z = 0$? Justify your answer.

10 C. Describe the singularities of the function

$$f(z) = \frac{z}{1 - e^{-z}}$$

in the closed complex plane.

Prove that, with $f(0)$ suitably defined, $f(z)$ is a regular function in the neighbourhood of the origin, and that, for $k = 0, 1, 2, \ldots$, the coefficient of z^k in the Taylor expansion of $\{f(z)\}^{k+1}$ about the origin is 1.

8 B. Suppose that a function $f(z)$, regular in the annulus $r < |z| < R$, is represented there by a series

$$f(z) = \sum_{-\infty}^{\infty} a_n z^n.$$

Show that a_n is uniquely determined by f, and obtain an expression for a_n in terms of f.

Show, assuming Laurent's theorem, that

$$\cosh\left(z + \frac{1}{z}\right) = a_0 + \sum_{n=1}^{\infty} a_n \left(z^n + \frac{1}{z^n}\right),$$

where

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \cosh\left(2\cos\theta\right) \cos n\theta\, d\theta,$$

the series being uniformly convergent in any closed annulus with centre at the origin.

10 B. Show that, if

$$f(z) = \sum_{0}^{\infty} a_n z^n$$

is convergent for all complex z, then

$$|a_n| \leqslant M(r)/r^n,$$

where $M(r) = \sup_{|z|=r} |f(z)|$.

Show further that, if $\log |f(z)| < |z|^k$ for all z such that $|z| > A$, where k and A are positive (finite) constants, then

$$\lim \frac{\log\left(1/|a_n|\right)}{n \log n} \geqslant 1/k.$$

A

4. Let $w = \phi(z)$ be a conformal map of the open set S on the z-plane onto the open set T on the w-plane. Show that a function $f(w)$ is regular in T precisely when $f(\phi(z))$ is regular in S.

Find a function $\phi(z)$ such that every function $g(z)$ regular in the quadrant

$$S: \Re z > 0, \quad \Im z > 0$$

can be developed as a power-series

$$g(z) = \sum_{0}^{\infty} a_n (\phi(z))^n,$$

the a_n being constants and the series converging throughout S.

20

5. State the 'principle of the argument'.

The function $f(z)$ is regular in a domain containing $|z| \leqslant 1$ and has a simple zero at $z = 0$ but no further zeros. Show that for sufficiently small γ there is precisely one $\zeta = \zeta(\gamma)$ in $|z| \leqslant 1$ with $f(\zeta) = \gamma$. By integrating

$$\frac{zf'(z)}{f(z) - \gamma}$$

round a suitable contour, or otherwise, show that $\zeta(\gamma)$ is a regular function of γ in some neighbourhood of the origin.

1. **Cos x**

   Use Hadamard's theorem to show

   $$\cos x = \prod_{n=1}^{\infty} \left\{ 1 - \frac{4x^2}{(2n-1)^2 \, \pi^2} \right\}$$

   Define

   $$\lambda(n) = \sum_{k=0}^{\infty} (2k+1)^{-n}$$

   Show

   $$\lambda(2) = \pi^2/8$$

2.

   Prove

   $$\frac{\cosh k - \cos x}{1 - \cos x} = \left\{ 1 + \left(\frac{k}{x}\right)^2 \right\} \left\{ 1 + \left(\frac{k}{2\pi - x}\right)^2 \right\}$$

   $$\left\{ 1 + \left(\frac{k}{2\pi + x}\right)^2 \right\} \left\{ 1 + \left(\frac{k}{4\pi - x}\right)^2 \right\} \quad \ldots\ldots$$

   Define

   $$J(n) = \sum_{m=1}^{\infty} m^{-n}$$

   Prove

   $$J(2) = \pi^2/6$$

3.

Prove from Hadamard's theorem that

(i) Any function of non-integral order has an infinity of zeros.

(ii) If $\lambda \neq 0$ and $p(Z)$ is any polynomial, the equation

$$\exp (\lambda Z) = p(Z)$$

has an infinity of solutions.

4. CoSec Z

Use Cauchy's representation to show:

$$\text{CoSec } Z = 1/Z + \sum_{n=1}^{\infty} (-1)^n \frac{2Z}{Z^2 - n^2 \pi^2}$$

$$\text{Define } \eta(n) = \sum_{R=1}^{\infty} (-1)^{R-1} R^{-n}$$

$$\text{Show} \quad \eta(2) = \pi^2/12$$

5. Sec Z

Use Cauchy's representation to find Sec Z as a sum over its poles.

$$\text{Define } \beta(n) = \sum_{R=0}^{\infty} (-1)^R (2R+1)^{-n}$$

$$\text{Show} \quad \beta(3) = \pi^3/32$$

6. When Men were Bold and Knights Breathed Fire*. (Trinity, 1898)

Define: $F(x) = \exp \left\{ \int_0^x yx \cot(yx) dy \right\}$

(i)   Show:
$$F(x) = \exp(x) \prod_{n=1}^{\infty} \left\{ (1 - x/n)^n \exp(x + x^2/2n) \right\}$$
$$\prod_{n=1}^{\infty} \left\{ (1 + x/n)^n \exp(-x + x^2/2n) \right\}$$

(ii) $F(-x) F(x) = 1$

(iii) $F(1-x)F(x) = 2 \operatorname{Sin}\pi x$

(iv) If $f(z) = z + z^2/2^2 + z^3/3^2 + \ldots = -\int_0^z \dfrac{\log(1-t)}{t} dt$

(which is related to Spence's dilogarithm)

Show that:
$$F(x) = \exp \left[1/2 \; \pi i x^2 - \frac{1}{2\pi i} \; f(1 - e^{-2\pi i x}) \right]$$

for $\left| 1 - e^{-2\pi i x} \right| < 1$

*And Ph d's meek clergymen, graduate students bright-eyed choirboys, and undergraduates
rowed for their land on the Cam.

NOTE: (i) is straightforward but quite long; (ii), (iv) very easy but I can't find
a simple proof of (iii).

3 A). The functions $f(x)$, $g(x)$, and $f(x)g(x)$, defined in $(0, 2\pi)$, have the Fourier series

$$\sum_{-\infty}^{\infty} A_n e^{inx}, \quad \sum_{-\infty}^{\infty} B_n e^{inx}, \quad \sum_{-\infty}^{\infty} C_n e^{inx},$$

respectively. By integrating term-by-term the first of these series multiplied by $g(x)e^{-inx}$ obtain an expression for C_n in terms of the coefficients A_n and B_n. The validity of this integration need not be discussed.

Verify this expression for

$$f(x) = \sin x, \quad g(x) = H(x - \pi),$$

where H is the Heaviside unit function. $[H(x) = 1$ (if $x > 0$), $= 0$ (if $x < 0$).]

2). Prove that if a function defined on the unit circle is differentiable k times, with a kth derivative that is piecewise differentiable, then its Fourier coefficients are of order $O(n^{-k-1})$.

Obtain the Fourier sine series and the Fourier cosine series of the function $f(\theta)$ defined by

$$f(\theta) = \frac{1}{2} + \theta \quad \text{for} \quad 0 \leqslant \theta \leqslant \pi,$$

and comment on the rate of convergence of the Fourier coefficients.

9 F. Describe in general terms a method of obtaining solutions of the differential equation

$$\sum_{r=0}^{n} (a_r + b_r x) u^{(r)} = 0 \quad (u^{(r)} \equiv d^r u/dx^r)$$

in the form of contour integrals

$$u(x) = \int_C e^{x\xi} g(\xi) \, d\xi.$$

[A discussion of non-triviality or linear independence of solutions is not expected.]

For the equation

$$x u'' - u = 0$$

obtain, in the form of integrals,

(1) a solution $u_1(x)$ that is valid for all x, and is such that

$$u_1(0) = 0, \quad u_1'(0) = 1;$$

(2) a solution $u_2(x)$ that is valid for all $x \equiv x + \frac{1}{2} y$ with $x > 0$, and is such that

$$u_2(x) \to 1 \quad \text{as} \quad x \to 0.$$

Deduce from the stated properties (or prove otherwise) that $u_1(x)$ and $u_2(x)$ are linearly independent.

9 C Prove that, for a suitable choice of contour,

$$y = \int \exp\left(iz\cos t\right) dt$$

satisfies the differential equation

$$zy'' + y' + zy = 0.$$

Show that one suitable contour is the interval $(0, \pi)$ and deduce that the corresponding solution of the differential equation is defined and bounded in $-\infty < x < \infty$.

Prove also that this solution is not identically zero.

10 F An infinite string has unit mass per unit length and is subject to unit tension. Explain the Fourier transform method for constructing the retarded Green's function $G(x, t)$ for the propagation of waves along the string. Show that

$$G(x, t) = \tfrac{1}{2}\theta(t)\left[\theta(x+t) - \theta(x-t)\right],$$

where

$$\theta(t) = \begin{cases} 1 & (t > 0) \\ 0 & (t < 0). \end{cases}$$

If the string is at rest for back in time and is subject to a force $F(t)$ applied at the origin, obtain the subsequent motion for $x > 0$.

10 F Define the Laplace transform $\mathscr{L}[f] = \hat f(p)$ of a function $f(t)$. Show that

$$\mathscr{L}[f'(t)] = -\hat f(0) + p\hat f(p).$$

Obtain also, $\mathscr{L}[f']$ and $\mathscr{L}[f'']$ in terms of $\hat f(p)$.

[Prime denotes differentiation with respect to t.]

Laguerre's differential equation is

$$tL_n'' + (1-t)L_n' + nL_n = 0$$

where n is an integer. Deduce an expression for the Laplace transform of L_n and verify that

$$\hat L_n(p) = C \,(p-1)^n/p^{n+1}$$

for some constant C. By showing that

$$\mathscr{L}\left\{\frac{e^{-xt/(1-x)}}{1-x}\right\} = \frac{1}{x + (1-x)p},$$

deduce a suitable normalization for $L_n(t)$.

$$\frac{e^{-xt/(1-x)}}{1-x} = \sum_0^\infty L_n(t)\frac{x^n}{n!}.$$

3

1. Find the Fourier coefficients $C_n(x)$ of the function

$$\varphi(x,y) = \sum_{n=-\infty}^{\infty} e^{-\pi(n+y)^2 x} \quad (x>0)$$

in the expansion $\varphi(x,y) = \sum_{n=-\infty}^{\infty} C_n(x) e^{2\pi i n y}$. Hence show

that the theta function $\theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} \quad (x>0)$,

satisfies the functional equation

$$\theta(x) = x^{-\frac{1}{2}} \theta(x^{-1}).$$

8. (Prelim 1936). Show that if $0 \le x \le \pi$,

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi} \left(\frac{\cos 2x}{1.3} + \frac{\cos 4x}{3.5} + \frac{\cos 6x}{5.7} + \dots \right).$$ What

function does the series represent in the range $-\pi \le x \le 0$? Is

term by term differentiation of the series legitimate; if so, what

function does the derived series represent? (This was not part of the

1936 question.)

9. **The Gibbs Phenomenon**

Let $f(x) = -1 \quad (-\pi < x < 0)$
$\qquad\qquad 1 \quad (0 < x < \pi)$

and let $f_N(x) = \sum_{1}^{2N+1} b_n \sin nx$, where the b_n are the Fourier

sine coefficients of $f(x)$. Obtain an expression for $\max_{0<x<\pi} f_N(x) = M$,

say, and show that $|M - 1|$ does not tend to zero as $N \to \infty$.

[This result exhibits the non-uniformity of the convergence of a Fourier

series near a point of discontinuity.]

13. Calculate the Fourier transforms and verify the inversion theorem
for the following functions:

$$\theta(t)\theta(1-t), \quad e^{-t^2/2\sigma^2}, \quad e^{-\mu|t|}$$

$$\theta(t)\, t\, e^{-\mu t}$$

14. A set of random variables $\{x_1, x_2 \cdots x_N\}$ (N large) are
distributed independently according to the probability densities
$p_1(x_1), p_2(x_2) \cdots$ and

$$\langle x_i \rangle = 0 \qquad \langle x_i^2 \rangle = \sigma_i^2$$

Using F. T. theory and making suitable assumptions about $\tilde{f}_i(\omega)$
show that $z = \sum_{i=1}^{N} x_i$ is distributed on a gaussian fashion and

$$\langle z \rangle = 0 \qquad \langle z^2 \rangle = \sum_{i=1}^{N} \sigma_i^2$$

15. Show that if

$$g(t) = f(t+\tau)$$

then

$$\tilde{g}(\omega) = e^{i\omega\tau} \tilde{f}(\omega)$$

Show that if $f(t)$ is periodic with period τ then

$$\sin \tfrac{1}{2}\omega\tau \; \tilde{f}(\omega) = 0$$

Hence deduce that

$$\tilde{f}(\omega) = \sum_{n=-\infty}^{\infty} c_n \, \delta\left(\omega - \frac{2n\pi}{\tau}\right)$$

where

$$c_n = \frac{1}{\tau} \int_0^\tau dt \; e^{-2\pi i n t/\tau} f(t)$$

⑬ ☆. Suppose $\tilde{K}(\omega)$ is the F.T. of a causal kernel K(t) and that $\tilde{K}(\omega) \simeq 0\left(\frac{1}{\omega}\right)$ as $\omega \to \infty$. Show by considering $\int\left[\tilde{K}(\omega)\right]^2 d\omega$ for a suitable contour C that

$$\int_{-\infty}^{\infty} d\omega \left[K_1(\omega)\right]^2 = \int_{-\infty}^{\infty} d\omega \left[K_2(\omega)\right]^2, \quad \tilde{K} = K_1 + i K_2, \quad \omega \text{ real.}$$

⑭ ☆. Suppose a linear amplifier is represented by a kernel function

$$\tilde{K}(\omega) = R(\omega) e^{-i\omega T}$$

where $\tilde{R}(\omega)$ is a rational function of ω. Show that if the system is subject to an input f(t) with a sharp front (e.g. f(t) = 0 $t < 0$) then the out put signal front is delayed by a time T. What frequency range then is important in determining the out put signal front delay?

⑮ ☆. Show using F.T. theory that the Green's function which yields solutions vanishing at infinity for the P.D.E.

$$(\nabla^2 - \beta^2) u(r) = \rho(r)$$

is

$$G(\beta, r) = -\frac{1}{4\pi} e^{\beta r}/r$$

16. Suppose the response function $K(\omega)$ of an amplifier is written

$$K(\omega) = e^{a(\omega) + i\theta(\omega)}$$

then $a(\omega)$ is called the logarithmic gain and $\theta(\omega)$ is the phase. Show that $a(\omega)$ is even and $\theta(\omega)$ is odd when $\omega \to -\omega$.

Suppose now that $K(\omega) \sim \frac{1}{\omega}, \omega \to \infty$ and that has no zeros in the lower half plane. Show by applying Cauchy's theorem to the function

$$[\log K(\omega)] / (\omega^2 - p^2)$$

that

$$\theta(p) = \frac{p}{\pi} P \int_{-\infty}^{\infty} \frac{d\omega \, a(\omega)}{\omega^2 - p^2}$$

Similarly by using the function

$$[\log K(\omega)] / [\omega (\omega^2 - p^2)]$$

show that

$$a(p) = a(\infty) + \frac{p^2}{\pi} P \int_{-\infty}^{\infty} \frac{d\omega \, \theta(\omega)}{\omega (p^2 - \omega^2)}$$

Can you see a simple way of altering the phase without changing the gain, at the expense of introducing zeroes in the lower half plane?

1) POISSON DISTRIBUTIONS

(i) The second edition of your favorite book contains 501 pages
and a total of 501 misprints. Estimate the chance that the second page
of Chapter 14 contains at least 3 misprints. (Assume the page exists.)

(ii) There are 500 warm human beings at Caltech. What is the
probability that 2 and only 2 of them will have a birthday tomorrow?
(Ignore leap years and complete problem set before going to their party.)

(iii) There are N parking lots for a particular Lakers-Bucks game.
Let $P_n(t)$ be the probability that n lots are occupied at time t. At
$t = 0$, all lots are empty and at a later time t, there is probability
$\lambda dt + \phi(dt)$ that a would-be parker arrives, in time interval t to t + dt.
Set up the differential equations for $P_n(t)$ and solve them... Assuming
that nobody leaves their lot after once parking.

Write down the differential equations for a bad game - parameterizing
this as a probability μ dt of any given parker leaving in cited time
interval. Prove from the differential equations that

$$\sum_{n=0}^{N} P_n(\tau) = 1 .$$

(iv) Let the normal distribution be:

$$\phi(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{x} \exp(-\tfrac{1}{2} y^2) dy$$

and the Poisson distribution be:

$$p(k, \lambda) = \exp(-\lambda) \ \lambda^k / k \ .$$

Prove:

$$\lim_{\lambda \to \infty} \sum_{\lambda + \alpha \lambda^{\frac{1}{2}} < k < \lambda + \beta \lambda^{\frac{1}{2}}} p(k, \lambda) = \phi(\beta) - \phi(\alpha) .$$

2) DESTRUCTION OF MYTHS

Myths are extracted from NAL and slowed down by a series of independent "collisions" in, say, a bubble chamber.

The probability $P(k, \lambda, x)$ of k collisions inside distance x is thus Poisson:

$$P(k, \lambda, x) = \exp (-\lambda x) \frac{(\lambda x)^k}{k!} \quad - (*)$$

POLLY measures a series of mythical tracks.

(i) Show (analytically) that the number of collisions inside fixed distance x is Gaussianly distributed if λx is large.

(ii) A myth is said (by POLLY) to have stopped when $k \geq K$ (K is a fixed large number) collisions have occurred. The value of x at the K'th collision is called the length of the track. If $(*)$ holds – show (analytically) that x is Gaussianly distributed.

(iii) Discuss what happens in part (ii) if we modify $(*)$ to correspond to independent collisions with the absolute cut off $k \leq K$ implied by (ii).

(iv) Comment on the Gaussian limits discovered in (i) and (ii) and the central limit theorem.

(v) Do problem 14-13 Mathews and Walker.

3) ALCATRAZ IN ATLANTIS

The American system of justice as applied to blacks and other malcontents is modeled on the following scheme devised by the regius professor of Seaweed in Atlantis.

A political prison consists of a row of N squirrel cages, numbered 1, 2,...N from left to right. A prisoner's sentence is to be placed initially in a specific cage. At noon each day, thereafter, he is moved either one cage to the left with probability 1/2 or one cage to the right with probability 1/2; moves on different days being independent random variables. If he is moved to the left from cage 1, he is released; if he is moved to the right from cage N he is executed.

One morning the people's revolution is successful and a member of the John Birch Society is placed in cage 6 during that afternoon. Discover (i) the probability that he will eventually be released and (ii) if he is released, the mean number of nights he has spent in prison.

(handwritten:) U_n, l a_s
You may find it helpful to the probability that our hero is released after tonights and put
$$U_n(2) = \sum_z u_{n,l} z^l$$

4)

Fairyland is a torus made up on N^2 square regions; a model for it may be obtained by sticking together opposite edges of an $N \times N$ chessboard. One square is occupied by the Palace of the Sleeping Beauty. A Prince travels randomly through Fairyland, moving each day from the square he is in to one of the four neighboring squares, each with probability 1/4; except that once he enters the Palace he never leaves it again.

For any square S, the Prince's mean time of travel from S to the Palace is denoted by T_s.

(i) Show that, wherever the Prince starts, there is a probability greater than 4^{-N} of his reaching the Palace within N days. Hence, deduce that the Prince has probability 1 of eventually reaching the Palace, and that each T_s is finite.

(ii) If he is initially placed at random in Fairyland, what is his probability distribution one day later?

(iii) Obtain in terms of the T_s a formula for the Prince's mean time of travel if he is initially placed at random in Fairyland. Using this and the result of (ii), obtain a relation between the T_s; and hence deduce that if the Prince is initially placed on a square next to that occupied by the Palace, his mean time of travel is $N^2 - 1$ days.

5) SNAP

Sinbad who had a loud mouth many little brains once made the mistake of playing snap with his mathematical brother Albert. After the latter had put his winnings to the bank, he made the following cryptic and typically staid entries in his dairy. Please fill in the proofs.

n objects arranged in a certain way are subject to a permutation chosen at random from the group S of all permutations on n objects. Let $\pi_n(z)$ denote the probability that exactly j of the objects remain in their original position and let $Q_{j,n}$ be the probability that a specified set of j objects remain in their positions, and no others do. Relate

$$\pi_n(Z) = \sum_{k=0}^{n} (Z-1)^k / k!$$

where

$$\pi_n(Z) = \sum_{j=0}^{n} P_{j,n} Z^j$$

Two well-shuffled packs of 52 cards are taken. The first cards in each are compared, then the second in each, and so on until the packs are exhausted. Show that the probability that no exactly matching pair of cards will be found this way is approximately e^{-1}.

Ph 129: Random Ravings

1) Do Problem 14-7 in Mathews and Walker.

2) Do Problem 14-8 in Mathews and Walker.

3) **Student's τ-shirts**

   It is desired to test the hypothesis

   $$H_o \; : \; \mu = 0$$

   against

   $$H_1 \; : \; \mu > 0$$

   where μ is the mean amount of shrinkage (in unspecified units) of τ-shirts of a new biodegradable detergent.

   A sample of 10 τ-shirts gave the following data:

Sample #	1	2	3	4	5	6	7	8	9	10
Shrinkage	1	-1	0	-1	0	-3	-4	-1	-2	-2

   Assuming shrinkage is normally distributed with mean μ and standard deviation σ, estimate μ and σ from this data. (Call these estimates $\bar{\mu}$ and $\bar{\sigma}$.)

   (a) Define $y = (\bar{\mu} - \mu)/\sigma$.
       What does the central limit theorem say for the distribution of y?
       Use this — plus the assumption that σ can be replaced by $\bar{\sigma}$ — to decide if H_o is tenable.

   (b) Define $\tau = (\bar{\mu} - \mu)/\bar{\sigma}$.
       What is the distribution of τ?
       Use this to decide if H_o is tenable.

   (c) Show that for large n, the distributions of y and τ become equal.

4) **Immorality**

   n measurements of the random variable x yield the results $x_1, x_2 \ldots x_n$. x has the Gaussian distribution:

   $$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-(x-a)^2/2\sigma^2\right]$$

   – just as in problem 14-8 of Mathews and Walker.

Use the central limit theorem to find the standard deviations of

$$\tilde{\sigma}^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

and $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ for large n.

Comment on the mathematical difference between this standard deviation and the "bias" discussed in 14-8. Comment on the physical significance of the numerical difference of these two concepts.

Moral: You can use the maximum likelihood estimates blindly for Gaussian distributed random variables.

Hint: Use the result of 14-7 that $\tilde{\sigma}^2$ can be written as $\frac{1}{(n-1)} \sum_{i=1}^{n-1} y_i^2$

where y_i are $n-1$ independent normally distributed r.v.'s with mean o and standard deviation σ'.

5) **All Roads Lead to Rome**

Consider, yet once more, the maximum likelihood method for finding estimates a^* and σ^{*2} respectively by measurements $x_1 \ldots x_n$ of a normally distributed random variable x - symbols are just as in previous immoral problem 4. Construct the likelihood function $L(x_i, a, \sigma)$ and consider $< \ln L(x_i, a, \sigma)>$. Hence show that the expected shape of $\ln L(x_i, a, \sigma)$ is Gaussian in a and σ for large n and also show that one "expects" these Gaussian distributions in a and σ to have standard deviations precisely equal to the standard deviations of the estimates, i.e. $\sqrt{<(a^*-a)^2>}$ and $\sqrt{<(\sigma^{*2}-\sigma^2)^2>}$ respectively.

Is this generally true?

Ph 109: Moving Finger is going down hill.

(Kirk McDonald - Guest Questions)

1). In a scattering experiment, we measure a cross-section X by observing N scattered particles in some apparatus with "detection efficiency" η . $(0 \le \eta \le 1)$. Further lump all the necessary beam intensity, target density, time of observation factors into a constant c so that the cross-section X is "expected" to be:

$$X = cN/\eta \qquad (*)$$

Now if the experiment were repeated many times (this requires a lot of graduate students), N would be distributed according to a Poisson distribution and so have error $\sqrt{N}$. Show that the maximum likelihood method predicts this result. Use this method under the two hypothesis's:

(a) The likelihood is given by the single observations of N events distributed according to a Poisson law.

(b) The likelihood is that of $m(\gg N)$ observations gotten by dividing the total running time into $1/m$ intervals. In each of these, there is negligible chance of 2 events and, in N of them we observe 1 and in $m-N$, zero events.

2). The detection efficiency η in (*) is often determined by a Monte Carlo computer program. Thus one generates n events with a random number generator and finds that m of these can be detected. What is the distribution of m ? Show that for large n, η becomes Gaussianly distributed with mean m/n and standard deviation $\sqrt{m(1-\eta)}/n$.

3). Redo the problem answered in question 1 of determining the error in the estimate of X using (*) , given that η is not known exactly but has some Gaussian distribution with mean η_0 and standard deviation σ . (Determined, as perhaps in question 2, independently of the scattering experiment.) Also comment on:

(a) Could I use (*) to solve this problem, assuming N and η are independent random variables?

(b) Can I use the formalism propounded on page 8 of McDonald's note. (CTSL Internal Report No. 57 - in my collection).

4). Often when performing an experiment, there is a background due to scattering off the walls of the target. To correct for this, data is taken with the target empty and so one measures two cross-sections σ (target full) and σ (target empty). The desired (by theoreticians in ivory towers) cross-section is the difference between these two measurements.

If a total time T is available to do an experiment, how should it be divided between target empty and full running so as to minimize the error on the true cross-section.

6). Use the bias formula in either McDonald's or Yellin's notes to find the bias in the maximum likelihood estimation of the standard deviation σ in a probability distribution

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-(x-\mu)^2/2\sigma^2\right)$$

(cf. Problem 14-8 in Mathews and Walker.)

7). A spin ½ particle elastically scatters on a spin 0 particle. The beam has polarization P in the y direction and momentum in the z direction. Under these circumstances the φ dependence of the scattered particle takes the form

$$(1 + P \cos \phi)d\phi/2\pi$$

normalized to 1 in $-180° \leq \phi \leq 180°$.

(a) In an experiment of N events we observe N values ϕ_i of φ . What is the equation satisfied by the maximum likelihood solution for P ?

(b) Now suppose our experiment can only detect scatters for φ between ±90°. Does the answer in (a) for P make sense? If not, what is the correct estimate?

Mainly Dull Questions from Cambridge

Moments v. Maximum Likelihood

$p(x, \alpha)$ is a one-dimensional probability distribution for a random variable x and some theoretical parameter α. It is correctly normalized

$$\left(\int_{-\infty}^{+\infty} p(x,\alpha)dx = 1 \right) \quad \text{and} \quad \text{we define}$$

$$L(x_i, \alpha) = \prod_{i=1}^{n} p(x_i, \alpha)$$

for n observations $x_1 \ldots x_n$.

(i) Consider ℓnL for α near α_0 (α_0 is the true value of α) and show ℓnL takes the form

$$\ell nL \approx -\frac{(\alpha - \alpha^*)^2}{2\sigma^2} + \text{const, where}$$

$$<\alpha^*> = \alpha_0$$

$$<\sigma^2> = 1 \bigg/ \left\{ \int_{-\infty}^{\infty} dx \; \frac{\{\partial p/\partial \alpha\}^2}{p} \right\}$$

as $n \to \infty$.

(ii) Comment on the difference between σ^2 defined above and $<(\alpha^* - \alpha_0)^2>$, where α^* is considered as the random variable that maximizes $^* L$.

(iii) Put $p(x, \alpha) = 1/\alpha \exp(-x/\alpha)$: $x \geq 0$

$\qquad\qquad\qquad = \alpha$ $\qquad\qquad x < 0$.

Show the maximum likelihood estimate α^* is identical to that given by the method of moments and hence verify the expression in (1) for $<\sigma^2>$.

<u>1-dimensional Particle Physics</u>

Events observed in an experiment are specified by an observable $x_i - \infty \leq x \leq \infty$. Theoretically the probability of observing an event in position x, $x+dx$ and time t, $t+dt$ is $p(x,\sigma)dxdt$ - where σ is some theoretical parameter. Derive the likelihood function for an experiment which in time t

(i) observes n events with positions $x_1 \ldots x_n$.

(ii) cannot tell the exact x_i of a particle but only if an event lies in a specific k-th (range) of x and the entire interval $-\infty \leq x < \infty$ is divided into K such non-overlapping bins and we observe n_k events in the k'th bin $(1 \leq k \leq K)$. Does the validity of the maximum likelihood method require K, all n_k or $\sum\limits_{k=1}^{K} n_k$ to be large?

(iii) Modify for the case where an event of true position y is assigned position x with probability distribution

$$\frac{1}{\sqrt{2\pi}\,\sigma} \, \exp\left[-(y-x)^2/2\sigma^2\right]$$

where σ is an a-priori-known function of y.

6Q. Define the characteristic function of a random real variable, and show that the characteristic function of the sum of two independent random real variables is the product of their characteristic functions.

Given that z_1, z_2, z_3 and z_4 are independent and are all distributed normally with zero mean and unit standard deviation, find the characteristic function of $y = z_1 z_2 + z_3 z_4$ and deduce its distribution function.

$$\left[\text{It may be assumed that } (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{i\omega - \frac{1}{2}\omega^2} d\omega = e^{-\frac{1}{2}k^2}.\right]$$

7Q. A random real variable has probability density function $f(x)$. Find in terms of $f(x)$ the probability density function of the largest member of a sample of n independent values of the variable.

Find the mean value of the largest member of a sample of five when $f(x) = \frac{1}{2}e^{-|x|}$.

7 G. An *experiment* consists of a succession of independent *attempts*, with probability p_i that the ith attempt is successful. Find the probability generating function of r_n, the number of successful attempts in the first n.

The experiment consists of 24 successive throws of an unbiased penny, and m is the number of times during the experiment at which equal numbers of heads and tails have been thrown. Show that the mean value of m is a large number of experiments is

$$\frac{(24+1)!}{2^{24}(12!)^2} - 1.$$

6 H. Explain what is meant by the terms *uncorrelated* and *independent* as applied to both finite and infinite sets of random variables.

The random variables z_1, z_2, z_3 have means p_1, p_2, p_3 and covariance matrix

$$\begin{pmatrix} p_1(1-p_1) & -p_1 p_2 & -p_1 p_3 \\ -p_1 p_2 & p_2(1-p_2) & -p_2 p_3 \\ -p_1 p_3 & -p_2 p_3 & p_3(1-p_3) \end{pmatrix}$$

where $p_1 + p_2 + p_3 = 1$. Find necessary and sufficient conditions for the linear functions $a_1 z_1 + a_2 z_2 + a_3 z_3$ and $b_1 z_1 + b_2 z_2 + b_3 z_3$ to be uncorrelated.

Find the values of p_1, p_2, p_3 for which $z_1 - z_2 - z_3$ and $z_2 - z_3$ are uncorrelated.

7 C. The function $u(x, y, z) = u(r)$ and its derivatives of the first and second orders are continuous in a domain D_a; D is a bounded domain contained in D_a with piece-wise smooth boundary S, and r is a point of D. Show that

$$u(\mathbf{r}) = -\int_S u(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} dS' - \int_D G(\mathbf{r}, \mathbf{r}') \nabla^2 u(\mathbf{r}') dV',$$

where $\partial/\partial n'$ denotes differentiation at $\mathbf{r}'$ in the direction of the normal to S at $\mathbf{r}'$ drawn away from D, and G is the Green's function of D.

Hence show that, if D is the sphere $r < a$, and $\nabla^2 u = 0$ in D, then

$$u(\mathbf{r}) = \frac{a(a^2 - r^2)}{4\pi} \int_S \frac{U(\theta', \phi')}{(r^2 + a^2 - 2ar \cos\gamma)^{3/2}} dU',$$

where r, θ, ϕ are spherical polar coordinates, and

$$\cos\gamma = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi'),$$
$$dU' = \sin\theta' d\theta' d\phi', \quad U(\theta', \phi') = u(a, \theta', \phi').$$

Prove that, if $U(\theta, \phi)$ is any continuous function on the unit sphere, and $u(r, \theta, \phi)$ is the function defined by the above integral, then u is harmonic in $r < a$, and

$$\lim_{r \to a} u(r, \theta, \phi) = U(\theta, \phi).$$

6 H. Events of a certain kind occur at random times in such a way that the probability of an event in a short interval of duration δt is $\epsilon \delta t + o(\delta t)$ and is independent of events at other times. Show that the probability that n such events will take place in any interval of duration t is $e^{-\epsilon t}(\epsilon t)^n/n!$.

Explain how these statements can be interpreted in terms of measures of sets in an appropriate probability space.

Determine the distribution of the time interval between successive events, and show that the expected number of events in an interval of duration equal to the mean interval between successive events is 1.

7 G. A child is given a birthday book and proceeds to enter in it first his own and then all his friends' birthdays, and finds that it is with the entry that (for the first time) every possible birthday occurs at least once in his book. Show (neglecting February 29th, twins, and the seasonal variation of births) that

$$\mathbf{E}s^n = \frac{n-1}{\prod_0^{N-1}} \frac{(N-s)s}{N-n}$$

where $N = 365$, and calculate the expectation and variance of n in terms of N.

8 G. The random variables $x_1, x_2, \ldots, x_n$ are independent and each has the negative-exponential distribution

Let

$$e^{-x} dx \quad (0 < x < \infty).$$

$$y = \max\,(x_1, x_2, \ldots, x_n)$$

and

$$z = \frac{x_1}{1} + \frac{x_2}{2} + \ldots + \frac{x_n}{n}.$$

Determine the distribution of y, and prove that y and z have exactly the same distribution. Hence (or otherwise) find the expectation and variance of y.

If $u = ne^{-y}$, show that, for large values of n, u has nearly the same distribution as x_1.

9 G. A random variable x has the distribution

$$\frac{1}{\sqrt{(2\pi)}} \frac{1}{x^{\frac{3}{2}}} e^{-\frac{1}{2x}} dx \quad (0 < x < \infty). \qquad (*)$$

Verify that the total probability-mass in this distribution is equal to unity.

Prove that

$$\mathbf{E}(e^{-sx}) = e^{-\sqrt{(2s)}} \quad (s > 0).$$

Hence (or otherwise) show that if $x_1, x_2, \ldots, x_n$ are independent random variables and each has the stated distribution $(*)$, and if

$$y = \frac{x_1 + x_2 + \ldots + x_n}{n^2},$$

then y also has the distribution $(*)$, for each $n \geqslant 1$.

Why does this result not contradict the central limit theorem?

[You may quote without proof any theorems you need concerning the Laplace-Stieltjes transform $\mathbf{E}(e^{-sx})$ $(s > 0)$ of the distribution of a positive random variable x.]

12 `4C` The limit inferior and limit superior of a sequence $A_1, A_2, \ldots$ of subsets of a space Ω are defined as follows:

$$\liminf_{n\to\infty} A_n = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} A_n, \quad \limsup_{n\to\infty} A_n = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_n;$$

when they are equal we say that $\lim A_n$ exists, and equate it to their common value. If the A_n's are measurable events associated with a probability space $(\Omega, \mathscr{A}, P)$, prove that

$$P(\liminf_{n\to\infty} A_n) \leq \liminf_{n\to\infty} P(A_n) \leq \limsup_{n\to\infty} P(A_n) \leq P(\limsup_{n\to\infty} A_n).$$

Deduce that $\lim P(A_n) = P(\lim A_n)$ when $\lim A_n$ exists. Does this statement remain true when P (on the $\mathscr{A}$-measurable subsets of Ω) is replaced by Lebesgue measure on the Borel-measurable subsets of the line)?

A random mechanism generates a random number N of random variables $x_1, x_2, \ldots, x_N$. It is known that (i) the x_i's take their values in the closed interval $[0, 1]$; (ii) with probability one, N is finite; (iii) with probability one, no two of the x_i's are equal. The event B_n occurs if and only if at least one of the sub-intervals

$$[0, 1/n], (1/n, 2/n], \ldots, (1-1/n, 1]$$

contains at least two of the x_i's. Prove that

$$\lim_{n\to\infty} P(B_n) = 1.$$

13 `4C` If x and y are independent Poisson random variables having expectations a and $\mathscr{M}$ respectively, where a, β and t are positive, write down the characteristic function $\phi(s) = E(e^{isx-isy})$ for the distribution of the random variable $z = x - y$.

When $a = \beta$ show that the distribution of $z/\sqrt{t}$ converges to the normal law as $t \to \infty$, and find the mean and variance of the limit distribution. [State carefully any general theorems about characteristic functions that you use.]

Whether or not a and β are equal, prove that

$$\phi(s) = \begin{cases} 1, & \text{if } s \text{ is an integer multiple of } 2\pi, \\ 0, & \text{if } s \text{ is not an integer multiple of } 2\pi. \end{cases}$$

If $a < \beta$, show that

$$P(z \geq m) \leq \left(\frac{a}{\beta}\right)^m P(z = -m) \qquad (m = 1, 2, \ldots).$$

Deduce that the distribution of z converges to a mass concentration at $+\infty$ when $a > \beta$, and to a mass concentration of 1 at $-\infty$ when $a < \beta$.

[You may assume that $\sum_n a^{|m|} m! / (m+n)! = O(a^n)$ as $n \to \infty$, for $a \geq 0$.]

4 C.) A particle describes a random walk on a line, and moves in unit time from position x to $x + 1$ or $x - 1$, each with probability $1/2$. Initially it is at $x = m$, where m is an integer in the range $(-N, N)$; there is an absorbing barrier at $x = N$ and another absorbing barrier at $x = -N$. Show that the particle is almost certain to be absorbed, and that the time T to absorption has the expectation $E(T) = N^2 - m^2$.

Solve the corresponding problem when the barrier at $x = -N$ is made a reflecting one (the barrier at $x = N$ remaining an absorbing barrier).

5 B.) A particle is initially at the origin. At time n ($= 1, 2, 3, \ldots$) it receives an impulse which sends it to the right one unit with probability $\frac{1}{2}a^n$, where $a > 0$, and to the left one unit with probability 2^{-n}, and with probability $1 - \frac{1}{2}a^n - 2^{-n}$ does not move it; the movements at different times are independent. Find for different values of a the probability that the particle eventually disappears to infinity.

6 B.) A population consists of individuals of two types, fertile and very fertile, which "reproduce" at time $t = 0, 1, 2, \ldots$. At each time a fertile individual remains unchanged with probability $\frac{1}{3}$, and becomes very fertile with probability $\frac{1}{3}$; a very fertile individual remains unchanged with probability $\frac{1}{4}$, splits into two fertile individuals with probability $\frac{1}{4}$, and dies with probability $\frac{1}{4}$. All individuals behave quite independently of each other, and are unaffected by those past. Obtain the probability q, that the population becomes extinct, given that it starts from 1 fertile, 1 very fertile individual respectively.

5 B.) Let X, Y be two independent random variables whose distributions are negative exponential with the same parameter a. Show that

$$P\{X \geqslant t + t \,|\, X \geqslant t\} = e^{-at} = P\{X \geqslant t\},$$

and the distribution of $\min (X, Y)$.

Three persons A, B, C enter a Post Office with two counters. A and B begin their service immediately, and C begins his service as soon as either A or B completes his service to a counter free. The service times are independent, and all have negative exponential distributions with the same parameter. Find (i) the probability that C is the last of the three to complete his service, and (ii) the distribution of the time spent by C in the Post Office.

6 B.) For a certain athletic competition there are 4 judges who each award a mark independently which may be considered uniformly distributed over the interval $(0, 1)$. To obtain a single overall mark the largest and smallest of the judges' 4 marks are discarded, and the remaining two, X and $Y > X$, are averaged. Find the joint distribution of X and Y, and hence compare the mean and variance of the result of this procedure with those of the arithmetic mean of all the 4 original four marks.

6 $\boxed{1}$ At each trial of a sequence of independent trials a phenomenon A can occur or not, with probability p and $1 - p$ respectively. Find the distribution of N, the number of the trial at which A first occurs.

In a certain shop the times required by customers for service are independent random variables $X_n (n \geqslant 0)$, each having the same (absolutely continuous) distribution function F with density f. Find the joint distribution function of N and X_N, where N is the smallest $n \geqslant 1$ for which $X_n > X_0$, and deduce the distribution and mean of N.

6 $\boxed{2}$ A random walk in continuous time is defined as follows: the steps are independent random variables taking the values $+1$ and -1 with probabilities p and $q = 1 - p$ respectively, and the instants at which they occur form a Poisson process (which is independent of the steps themselves). The rate of the Poisson process is unity. Show that the probability $p_r(t)$ of being at position r at time t is

$$p_r(t) = (p/q)^{\frac{1}{2}r} e^{-t} I_r(2(pq)^{\frac{1}{2}}t) \qquad (r = 0, \pm 1, \pm 2, \ldots),$$

where, for $n \geqslant 0$,

$$I_n(x) = I_{-n}(x) = \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!} \left|\frac{x}{2}\right|^{2k+n}.$$

Hence derive an expression for the generating function

$$g(u, x) = \sum_{-\infty}^{\infty} u^n I_n(x).$$

5 $\boxed{C}$ Alice and Belinda are the finalists in a beauty contest in which there are three judges, each of whom independently awards to each finalist a mark which may be considered as being uniformly distributed in $(0, 1)$. Each finalist must decide in advance of the marking whether she wishes to be credited with the sum of her worst two marks or with her best mark; Alice opts for the first alternative and Belinda for the second. Prove that

(i) Alice and Belinda have the same expected mark;

(ii) there is a chance $\frac{1}{2}$ that Alice's best mark will exceed the sum of her worst two marks;

(iii) their chances of winning are not equal.

6 $\boxed{C}$ A certain house is haunted. In a short time interval dt any of the following events happen with the probabilities stated, the first and second of them being independent:

(i) If no trained observer is present, a ghost may appear, with probability $2dt$.

(ii) If no trained observer is present, a trained observer may enter the house, with probability $2dt$.

(iii) If a trained observer is present, he may become bored and leave the house, with probability $3dt$.

A ghost will not appear while a trained observer is present. Initially the house is empty. What is the probability that a ghost appears before time T?

9 C State the strong and weak laws of large numbers, and prove the weak law in the case when the distribution has finite variance.

Show that the conclusions of the strong and weak laws do not hold for a random variable which has the Cauchy distribution

$$p(x) = \frac{1}{\pi(1+x^2)} \quad \text{for } -\infty < x < \infty.$$

Why does this not contradict the laws!

9 ? A random real variable has probability density function $f(x)$. A sample of $N = 3n + 1$ independent values of the variable is taken; find in terms of $f(x)$ the probability density functions of the largest member of the sample, and of the median of the sample.

Find the mean value of the largest member, the median and the least member of a sample of three values, in the case

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases}$$

9 C A man is trying to unlock a door in the dark. He has a bunch of n keys, just one of which will fit the lock. Determine the mean number of trials he must make in order to unlock the door, under each of the three following hypotheses:

(i) After an unsuccessful trial, he drops the bunch of keys; thus for the next trial he selects a key at random.

(ii) After an unsuccessful trial, he selects a key at random from among the $(n-1)$ keys other than which he last tried.

(iii) He never tries a key more than once.

[If you assume that the mean number of trials is finite, some justification should be...]

9 C State the two laws of large numbers and the central limit theorem, all for independent identically distributed variables. Prove the weak law of large numbers in the case when the distribution has finite variance.

The random variable X is said to have Cauchy's distribution if its probability density function is

$$\frac{1}{\pi(1+x^2)}$$

in $(-\infty, \infty)$. If the independent variables X_1 and X_2 each have Cauchy's distribution, and if c_1, c_2 are positive constants such that

$$c_1 + c_2 = 1,$$

prove that $Y = c_1 X_1 + c_2 X_2$ also has Cauchy's distribution.

Deduce that the conclusions of the two laws of large numbers do not hold for the Cauchy distribution, and explain why these laws are not applicable to this case.

27. Bernstein's Good Idea

Let ψ_i be independent random variables taking the value 0 and 1 with probability $1 - x$ and x respectively. Let

$$Z_n = 1/n \sum_{i=1}^{n} \psi_i$$

and $B_n(f, x) = \langle f(Z_n) \rangle$ where $f(Z)$ is any continuous function of Z in $[0,1]$.

(i) Derive an expression for $B_n(f, x)$ in terms of $f(k/n)$, $k = 0 \ldots n$. Show it is a polynomial in x of degree n.

(ii) Use the central limit theorem to show $B_n(f, x) \to f(x)$ as $n \to \infty$ and comment on the error.

(iii) Sketch how this can be applied to prove Weierstrass approximation theorem for continuous functions in terms of polynomials.