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Chaos and the Analysis of Experimental Data

James A. Glazier

The James Franck Institute
The University of Chicago
Chicago, IL 60637
USA

ABSTRACT: We discuss the analysis of experimental data from chaotic systems.

Quasiperiodicity in Rayleigh-Bénard Convection

I want to discuss briefly, from an experimentalist's point of view, the various ways one can analyze a chaotic time series (be it experimental or numerical). Since the emphasis of this conference has been on the theory of chaos, I will start with the simplest traditional experimental techniques, and conclude with some general questions.

There are several routes to chaos. I will describe frequency locking and quasiperiodicity in the context of hydrodynamics. However, the discussion should be equally valid for other systems. We study convection in liquid mercury, in a cell with dimensions 1.4 x .7 x .7 cm^3. We heat the fluid from below to produce two convective rolls perpendicular to the long axis of the cell, with a periodic transverse oscillation, the oscillatory instability, defining an internal frequency f_1. Applying a constant magnetic field parallel to the roll axis and injecting an AC current sheet, amplitude A_{ext}, frequency f_2, asymmetrically through the fluid produces two non-linearly coupled oscillators, one of which, f_1, can shift in frequency. We measure the temperature of the fluid, $T(t)$, at the bottom center of the cell.[1]

For low forcing amplitudes (small nonlinearities) the typical behaviors of the convection cell are frequency locking, in which f_1 shifts so that $\frac{f_1}{f_2} = \frac{p}{q}$ is rational over some range of f_2, and quasiperiodicity in which $\frac{f_1}{f_2}$ is irrational. The locked states are stable to small perturbations, the unlocked unstable, so the system is always in a locked state up to its noise level. The pattern of these locked regions agrees qualitatively with that of the one dimensional circle map:

$$\theta_{n+1} = \theta_n + \Omega - \frac{k}{2\pi}\sin(2\pi\theta_n).$$

For stronger forcing (stronger nonlinearity) new phenomena appear. Instead of quasiperiodicity, we obtain aperiodic time series which include broad band noise (chaos). For rational frequency ratios, locking persists, but cascades of period-doubling and other rational period multiplications occur as well. For still higher forcing amplitudes we also observe intermittency between locked states. For extremely strong forcing, the oscillatory instability is driven completely off resonance, resulting in a drastic decrease in its amplitude, and the system returns abruptly to periodic behavior.[1,2,3]

The details are specific to convection, but the general patterns of frequency locking are common, occurring in, mechanical and electrical oscillators, charge density waves, electrically driven germanium, binary convection in He 3/He 4 mixtures, lasers, etc...[4]

Traditional Methods of Data Analysis

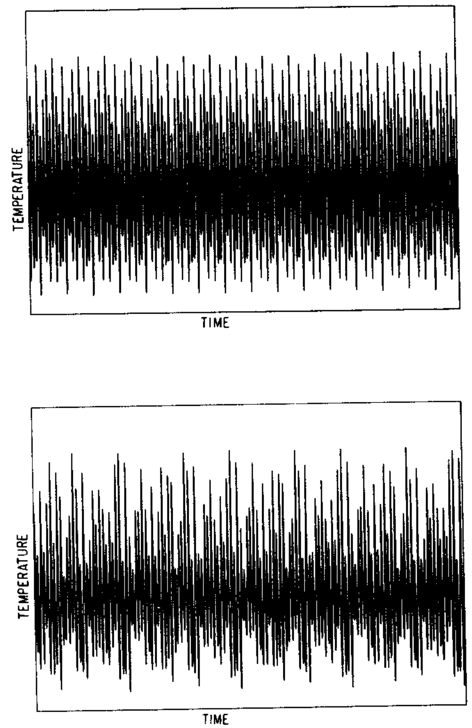
I think that it is worth beginning with the most obvious measurements, because (in conjunction with the experimenter's eye and brain) they can yield a surprising amount of information. They are also more robust than the more sophisticated techniques I will discuss later.

For weak forcing, a theorem of Takens permits us to hope that, since the flow is spatially coherent, we can reconstruct a complete description of any universal features of the experiment (generally, scalings, i.e., limits of ratios) from a single series of an arbitrarily chosen variable, in our case, temperature.[5,6] Let us begin by plotting the time series on a piece of paper and looking at it.

If we are patient we can see whether the system is in a locked state, p/q, in which case the time series will be periodic with period p, or quasiperiodic (in practice this means that the period, if it exists, is longer than a few hundred (Fig. 1). In our convection cell the frequency of oscillation is 0.5 Hz, so we can measure the numerator of the locking ratio by eye about as fast as by computer. It is quite easy to tune irrational frequency ratios like the golden mean, $\sigma_G = \frac{\sqrt{5}-1}{2}$, by successively locking to their continued fraction approximants (e.g. 3/5, 5/8, 8/13,...), which are the most stable nearby rational lockings. We can also gauge roughly the degree of nonlinearity in the system by looking at the amplitude of the modulation of the main oscillation, f_1. It is difficult thus to distinguish weak chaos from quasiperiodicity or the presence of a weak third frequency, since all appear to be made of of nearly repeated blocks of rational lengths corresponding to the continued fraction sequence (e.g. for the golden mean $3,5,8,13,21...$). However, strong chaos with wildly irregular oscillations, and intermittency in which two or more periodic blocks alternate after irregular numbers of repetitions are easy to spot. Counting nearly periodic blocks also allows approximate determination of the irrational frequency ratio in quasiperiodic states. The irregularly alternating blocks of intermittency, e.g. $2/3 \leftrightarrow 3/5$, and the envelope modulationed block patterns of low order period doublings and other subharmonic cascades are also easy to pick out. Working only with a time series it is possible to map the regions of frequency locking in A_{ext} *vs* f_2 space, and to compare them quantitatively to the predictions of the one dimensional circle map.[2] A time series also exhibits immediately baseline drifts (e.g. due to problems in temperature control).

Moving up one step in complexity, we can look at the Poincaré section of the time series.[4,7] This is defined by embedding the time series in an N dimensional space, by taking multiplets of the data (e.g. $\{T(t), T(t+\tau), T(t+2*\tau), ...\}$), and recording the coordinates of intersection between this path and an $(N-1)$ dimensional surface (e.g. $(0, y, z, ...)$). This procedure reduces the apparent dimensionality of the data by 1 if the embedding corresponds to a natural periodicity in the time series. In a quasiperiodic system, there are two natural variables (the phases of the two oscillators) so we embed in three dimensions. Generally we approximate the Poincaré cut by measuring $T(t)$ in phase with the known driving frequency, f_2, to generate a stroboscopy but this approximation can result in serious problems in the presence of drift in either baseline or winding number. The fixed phase of the stroboscopy means that analog filtering, which trades amplitude errors for phase errors, is useless.[8] A better technique is to take a continuous time series and define a selection criterion, e.g. select only points with 0 time derivative. This requires storage of a much greater quantity of data but eliminates problems of phase drift.

In a two dimensional Poincaré section, a locked one dimensional state will



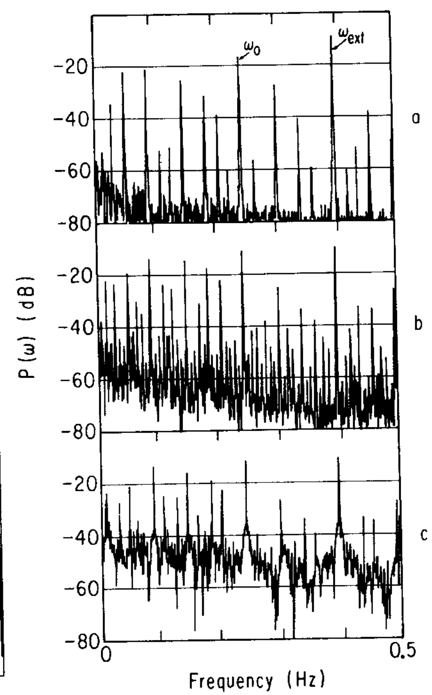


Figure 1 (left): Time Series: a) An 8/13 locked state, b) A period tripled 34/55 locked state [See Ref. 4].

Figure 2 (right): Golden Mean Spectra: a) Subcritical, b) Critical, c) Chaotic. [See Ref. 1].

show up as discrete clumps of points. In the case of a stroboscopy, the number of clumps gives the denominator, q, of the locking ratio. A quasiperiodic time series will be a continuous, possibly tangled, smooth, closed one dimensional curve.[1] Increasing the forcing amplitude at a fixed frequency ratio, cusps first appear on the curve at the transition to chaos, then the curve breaks up into a haze of points around its original position. We do not yet know whether taking a true Poincaré section will restore the one dimensional curve. Period multiplication by n splits each clump into n nearby sub-clumps.[3] A weak third frequency braids the section. Higher dimensional and truly aperiodic systems, have space filling sections without apparent structure.[8] In a stroboscopy an intermittent state looks like a strongly chaotic state, in a true phase section, intermittency appears as two or more well defined sets of clumps, irregularly visited. The section can also provide information about system dynamics. A gradually locking or unlocking state shows up as a narrow streak leading to a clump. Phase or amplitude drifts smear out the section, phase drifts by rotation, amplitude drifts by diagonal translation. In both cases, large amplitude drifts render the section essentially unusable over long time periods. If the period q is known, however, high frequency noise can be eliminated by averaging known corresponding points to arrive at an "ideal" Poincaré section.

Any more sophisticated method of data analysis starts with either a continuous time series or a Poincaré section, and therefore inherits its typical advantages and disadvantages. The two most common techniques are Fourier analysis and autocorrelation. Power spectra can be used to measure frequency ratios, but for locked states they are less convenient than direct counting from the time series. For irrational frequency ratios all combination peaks of form $nf_1 \pm mf_2$ appear, the amplitude of higher order combinations decreasing with increasing m and n and increasing with the forcing strength. For a well behaved irrational like the golden mean, the lowest order combinations are all powers of the frequency ratio (Fig. 2b). The degree to which they are equally spaced on a log frequency plot gives a rough estimate of how well tuned the ratio is. At criticality, at the golden mean, these primary peaks have amplitudes proportional to the square of their frequency.[2] Period multiplications by n show up in the presence of nth subharmonics. Within period doubling cascades, the amplitude of the subharmonic peaks gives a rough indication of the approach to chaos. Chaos itself shows up clearly as broad-band noise which gradually swallows the combination peaks and comes to dominate the spectrum (Fig. 2c).

Autocorrelations have the advantage of being more sensitive to long time behaviors, but are chiefly useful for studying locked states. They are able to pick out really high order multiplicities in locked states, and to measure very long period locking. Like power-spectra, they are slow, and occasionally ambiguous.

Multifractal Scaling Techniques of Data Analysis

The techniques mentioned above are quite general, and help us understand the physics of our experiment. Let us next turn our attention to methods specifically designed to establish exact correspondences between experiments and universality classes of low dimensional maps (actually proving equivalence is usually impossible). That these measurements answer different types of questions should always be kept in mind.

If the Poincaré section of the time series is low dimensional (i.e. one or two dimensional) the local scaling, α_i, at each point, $\mathbf{x}_i$, of the section is well defined. We can define in a simple way the section's multifractal spectrum, $f(\alpha)$, the dimension, f, of that subset of points with scaling exponent α.[9,10]

For our experimental signal a naive box counting algorithm requires too many points to be useful and is too sensitive to drift. One way to improve the experimental accuracy is to average the position of points which are known to correspond, for example every 13th point in an 8/13 locked state. We have used this method successfully to obtain the $f(\alpha)$ spectrum for a period doubling cascade of the 8/13 locked state (Fig. 3).[3]

However, in the quasiperiodic case long term averaging is impossible since there is no exact correspondence between points. We can get around this problem by using the known frequency ratio of the system to establish a list of nearest neighbors.[8] If the frequency ratio is σ with continued fraction approximants $\frac{\{p_n}{q_n\}}$ then each q_n is an approximate period, and we may approximate the section by selecting q_n consecutive points. If we treat the section as a tangled circle and look at the order of nearest neighbor points on a circle $\{\theta_{i+1} = \theta_i + \frac{p_n}{q_n}\}$ with winding number $\frac{p_n}{q_n}$, we can establish a corresponding sequence on the section, $\{\mathbf{T}_{r(m)}\}$. The probability of visiting a given segment between nearest neighbor points of the section is inversely proportional to the segment's length, $l_m = |\mathbf{T}_{r(m)} - \mathbf{T}_{r(m+1)}|$. Then the generalized dimension is $d_q = (q-1)\tau_q$ where τ_q satisfies

$$1 = \Gamma_n = \sum (p_m^q)/(l_m^{\tau_q}) = q_n{}^{-q} \sum l_m{}^{-\tau_q}.$$

The multifractal spectrum is the Legendre transform of τ_q, given by

$$\alpha = \frac{d\tau}{dq}$$
$$f(\alpha) = \alpha q - \tau(q).$$

This method converges rapidly in n for positive q, so the left hand side of the spectrum can be determined with as few as 50 points. The right hand (negative q) side converges slowly, and is more sensitive to noise (Fig. 3). If an inappropriate or too large q_n is chosen, the noise amplitude will be comparable to the point spacing, and the curve will widen abruptly and erratically. High frequency noise affects the high density side and low frequency noise the low density side. Subcritical curves are symmetrically narrowed, while supercriticality or the presence of a third frequency in a critical time series, narrows more on the low density side.[8]

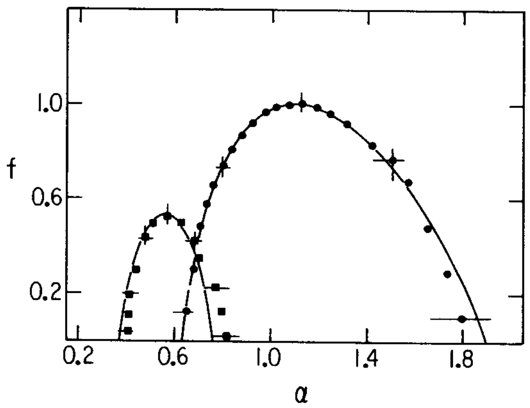


Figure 3: $f(\alpha)$ curves for the transition to chaos of a period doubling cascade in the 8/13 locked state (left) and for a golden mean frequency ratio (right). Solid lines are theoretical, dots experimental [See Ref. 4].

Arneodo has shown that for nearly critical quasiperiodic time series (which have a true $f(\alpha)$ curve consisting of a single point at (1,1), the approximate $f(\alpha)_n$ curve narrows in a consistent way as n increases.[11] Therefore the strength of the non-linearity k can be directly determined from the time series for $k \approx 1$. This correspondence is particularly valuable, because experimental control parameters often do not correspond simply to the parameters of iterated maps.[8]

The $f(\alpha)$ curve identifies fairly unambiguously the universality class of Poincaré sections, but spatially averages over the whole section. The trajectory scaling function (tsf), a smoothed version of the local scaling as a function of position, preserves

this information.[12] We generate an ordered Poincaré section as described above and look at the hierarchy of ratios between successive nearest neighbor distances as we increase n in our approximation.[13] Since the result again depends on ratios between small nearest neighbor distances it is sensitive to noise and requires clump-wise averaging. In the period doubling case, everything is simple. For a period n orbit (where $N = 2^n$), skipping a few subtleties, we define the tsf, σ_j :

$$\sigma_j = \frac{|\mathbf{x}_j - \mathbf{x}_{j+N/4}|}{|\mathbf{x}_j - \mathbf{x}_{j+N/2}|} \quad if \quad 0 < j \le N/4$$
$$= \frac{|\mathbf{x}_j - \mathbf{x}_{j-N/4}|}{|\mathbf{x}_j - \mathbf{x}_{j+N/2}|}. \quad if \quad N/4 < j \le N/2$$

To examine quasiperiodic states, we work with locked states whose frequency ratio approximates the desired quasiperiod and define the tsf for an F_n cycle approximating the quasiperiod to be:

$$\sigma_j = \frac{|\mathbf{x}_j - \mathbf{x}_{j+F_{n-2}}|}{|\mathbf{x}_j - \mathbf{x}_{j+F_{n-1}}|} \quad if \ \ 0 < j \le F_{n-2}$$
$$= \frac{|\mathbf{x}_j - \mathbf{x}_{j+F_{n-2}}|}{|\mathbf{x}_j - \mathbf{x}_{j-F_{n-2}}|} \quad if F_{n-2} < j \le F_{n-1}.$$

In both cases our results for the Rayleigh-Bénard convection experiment agree with the one dimensional circle map, though we have too few data points and too large an experimental error for the agreement to be conclusive (Fig. 4). However, we feel the results are good enough to make it worth while to refine the technique further, especially since the tsf is universal, directly calculable from the fixed point renormalization group, and contains all scaling information about the time series.

Open Problems

At this conference, in the papers of Barnsley, on multiple affine map techniques, and Arneodo, on wavelet transforms, we have heard some more sophisticated ways to analyze higher dimensional data.[14,15] In their robustness, and in the general type of information they provide, these techniques seem closer to the "physical" methods of analysis we have discussed. The powerful method of singular value decomposition is also in this class. I hope they will serve similarly as jumping off points for "chaotic" techniques as well.

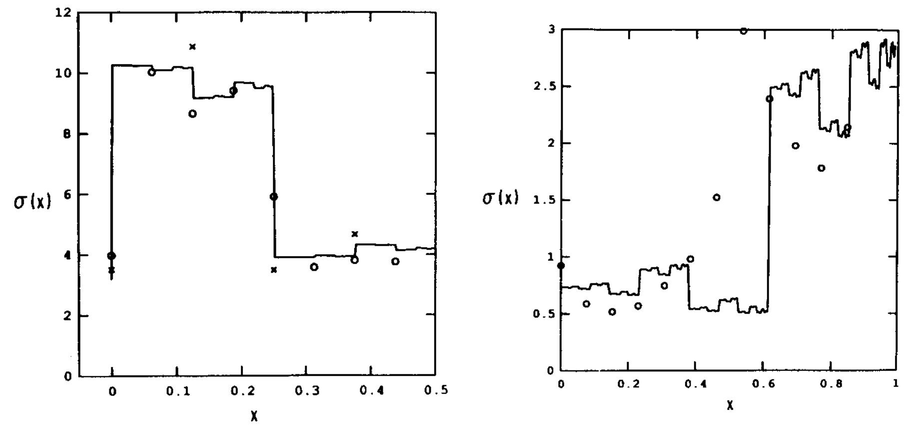


Figure 4: Trajectory Scaling Functions at the transition to chaos: (left) Period doubling. Circles are from a nonlinear resonator circuit, x's from experiment. (right) Golden mean quasiperiodicity. Circles show experimental data [See Ref. 13].

We still lack well developed techniques for "chaotic" analyses of strongly chaotic time series. Searching for the ghosts of unstable orbits is one promising technique, which we are currently attempting to use to examine strongly chaotic time series from our convection experiment.[16,17] But this method can work for us only because the intermittency is low dimensional, and well described by a two dimensional extension of the circle map.

What would an experimentalist like to have? It would be nice to have more refined black boxes to calculate standard quantities like $f(\alpha)$ curves or trajectory scaling functions. Current techniques have too many free parameters and too much sensitivity to details of the calculation (like choosing window sizes). Error analyses are either rudimentary or non-existent, though Farmer and Sidorowich's work on singular value decomposition, and Barnsley's on distances between patterns are important advances.[14,18,19] We need to know how much drift in winding number, how much high frequency noise we can have, and still calculate an $f(\alpha)$ curve with a given certainty.

More seriously, the scope of "chaotic" techniques is still far from obvious. In general chaos is too complicated to understand throughout a parameter space:

our techniques only work at particular, well behaved values of the frequency ratio, which must be known as an external parameter. The few methods that do have general applicability (e.g. looking for positive Lyaponov exponents) do not provide enough information to be really useful, though Eckmann has made some progress on this point.[20] In hydrodynamics, even medium aspect ratio convection has dimension greater than two. Techniques for high-dimensional analysis are almost non-existent. Weak intermittency is currently intractable. The three frequency problem is poorly understood.[21,22] It would be nice to know at least what can be known.

While "chaotic" methods allow us to prove rigorously the correspondence between an experiment and a known class of iterated maps, they do not in usually suggest the correspondence to look for. We established that Rayleigh-Bénard convection has circle map frequency locking by looking at the time series, not by calculating an $f(\alpha)$ curve. It is not yet clear how much more chaos methods can tell us than simpler traditional methods. Only if chaos theorists take direct interest in experiment, and the needs of experimentalists, will chaotic techniques become a useful part of physics.

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