Spatially Coherent States in Fractally Coupled Map Lattices

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We study coupled map lattices with a scaling form of connectivity and show that the dynamics of these systems exhibit a transition from spatial disorder to spatially uniform, temporal chaos as the scaling is varied. We numerically investigate the eigenvalue spectrum of the random matrix characterizing fluctuations from spatial uniformity, and find that the spectrum is real, bounded, and has a gap between the largest eigenvalue (corresponding to the uniform solution) and the remaining $N - 1$ eigenvalues (nonuniform solutions). The width of this gap depends on the scaling exponent. We associate the transition to the coherent state with the appearance of this gap.

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Animal brains are complex networks of elements that can perform a broad variety of information processing tasks. Specialized areas of the brain, such as the motor and visual cortex, are wired in specific, identifiable ways to carry out their functions [1]. It is much harder, however, to determine the structure of the network in higher brain regions, which often seem rather amorphous, but are able to perform different types of computations without any significant change in synaptic coupling. Not only is the computational activity in these higher brain regions localized for a given task [2], regions allotted to different computations, such as hippocampal place fields, often overlap [3]. This dynamic partitioning of nodes into domains of fixed point, oscillating, and chaotic domains has been observed as a generic behavior in simulations of neural networks, coupled oscillators, and coupled maps.

This segregation of activity in networks is strongly dependent on the way the network is connected [4]. Most existing studies employ either coupling restricted to some neighborhood of a site or global coupling, which is a mean field extension of local models. However, most dendritic branching is fractal [5]. The omnipresence of fractally connected neurons in biological neural networks suggests that fractal coupling may be necessary for greater efficiency in performing higher order computations.

Coupled map lattices (CMLs) are simple, computationally tractable dynamical networks that display behavior qualitatively similar to that of more complicated models [6–10]. Chaotic maps like the logistic map capture some essential features of neuronal dynamics, such as fixed point, oscillatory, or chaotic behavior [11], depending upon the applied stimulus, and have been used to study clustering phenomena and coding in neural ensembles [12]. We use these networks as simplified models of neural ensembles to study the effect of connection architecture on network behavior.

We consider a CML with local dynamics described by the logistic map

$$x_{n+1}(i) = f(x_n(i)) = 1 - a x_n^2(i)\,, \qquad (1)$$

where the index n denotes the time and i the position on a linear chain on N sites. Each pair of sites, i and j, are connected according to the probability distribution

$$\rho(C_{ij}) = p(ij)\delta(C_{ij} - 1) + [1 - p(ij)]\delta(C_{ij})\,, \qquad (2)$$

where $p(ij)$ is given by a simple scaling assumption for the connection neighborhood

$$p(ij) = |\mathbf{r}_i - \mathbf{r}_j|^{-\alpha}, \qquad j = \pm 1, \pm 2, \ldots, \qquad (3)$$

where $\mathbf{r}_i$ and $\mathbf{r}_j$ are the coordinates of the ith and jth sites, respectively. This form of connectivity is different from random connectivity models with a fixed value of p_{ij}. The probability that two sites are connected depends on the intersite distance and number of connections in a d-dimensional sphere of radius R will scale as $R^{d-\alpha}$ (for large enough lattices), which we define as fractal connectivity. The coupled lattice is thus given by

$$x_{n+1}(i) = \frac{1}{A_i + 1}\left[f(x_n(i)) + \sum_{j \in \text{conn}} f(x_n(j)) \right], \qquad (4)$$

where A_i is the number of connections at the ith site, and the sum over j runs over all the sites connected to site i. We have two parameters at our disposal: a, which is like a forcing parameter for a reaction-diffusion equation, and α, which defines the connection neighborhood as well as the strength of the interaction between sites.

We impose periodic boundary conditions on the lattice, i.e., $x(1) = x(N + 1)$. The form of the probability distribution for site-site coupling allows us to smoothly vary the interaction neighborhood of a site. Thus, $\alpha \to \infty$ is the nearest-neighbor coupling limit, and $\alpha \to 0$ is the global coupling limit.

For the limiting cases of $\alpha \longrightarrow \infty$ and $\alpha \longrightarrow 0$, the lattice exhibits a fairly well understood range of behavior, including pattern selection, intermittency, and spatiotemporal chaos [13]. In this Letter, we are mainly concerned

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with the study of the lattice dynamics for α values of $O(1)$. This implies that the coupling between sites is no longer strictly short range or infinite range. This kind of scaling of the connectivity occurs in animal brains [14], and the particular range of values of α results in rather dilute connectivity which is biologically realistic.

We show the temporal evolution of the lattice in Fig. 1 for a lattice of size 100 with the value of the map parameter set to the band-periodic region of the logistic map ($a = 1.44$). We see that the lattice splits into almost periodic domains for large α (local limit) with the selection of some characteristic stable wavelength. The domain boundaries do not move with time. As α is decreased, the number of domains decreases and the motion within a domain becomes more chaotic. As α is lowered further, the lattice trajectories are almost band periodic and there is little spatial order within domains, suggestive of phase turbulence, where a slow chaotic mode prevents the system from locking. Past a critical value of α, the lattice becomes spatially uniform. The uniform state is stable for all parameter (a) values of the uncoupled logistic map (fixed point, periodic, semiperiodic, and chaotic) though the critical value of α shifts slightly depending upon a. Scaling forms of connectivity might result in domains with fractal structure, which we have not explored in this Letter.

In order to locate the critical value of α at which the uniform state becomes stable, we sweep α, while holding a steady, and record the size of the coherent domain. The size of the coherent domain is defined as the number of sites such that $|x_i - x_{i+1}| < \delta$, with δ taken to be 10^{-11}. We average over 500 different initial conditions for each α and 100 time steps for each initial condition after discarding transients. We show the average coherent domain size for lattice sizes from $2^6 - 2^{10}$ for two parameter values ($a = 1.44$ and $a = 1.9$) of the logistic map in Fig. 2.

We see that for different values of a, the critical value for α at which the size of the coherent domain diverges, shifts, depending on whether the single logistic map is band periodic or chaotic. The uniform state appears to be attracting and globally stable past the critical value of α. The transition from an initial uncorrelated state to a spatially uniform one becomes sharper as the system size increases. This behavior resembles a phase transition, where continuous variation in a parameter, α, results in a discontinuous jump in an order parameter, here, the size of the coherent domain.

The emergence of coherent structures in CMLs has been studied by stability analysis and statistical mechanics [15–19]. While, in general, temporally chaotic states in systems with short range interactions lead to a loss of spatial coherence with exponentially decaying correlations in space [20], in models with asymmetric coupling and/or open boundary conditions, a stable uniform state can be observed [15,16]. We show below that chaotic temporal states of systems with probabilistic, long range interactions, which are statistically symmetric, can exhibit long range spatial order with temporal chaos.

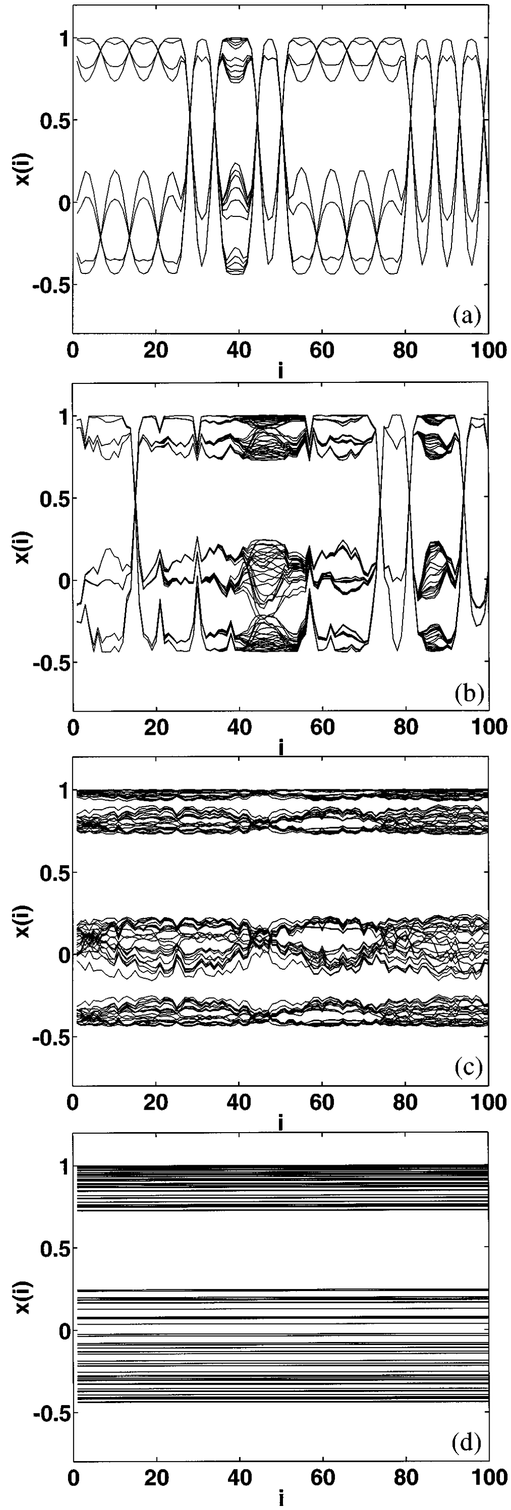

FIG. 1. Space-time diagram for fractal CML with $a = 1.44$ and (a) $\alpha = 15$, (b) $\alpha = 2.0$, (c) $\alpha = 1.3$, and (d) $\alpha = 1.1$ for random initial conditions.

We examine the stability of the uniform state by means of a linear stability analysis about the uniform solution.

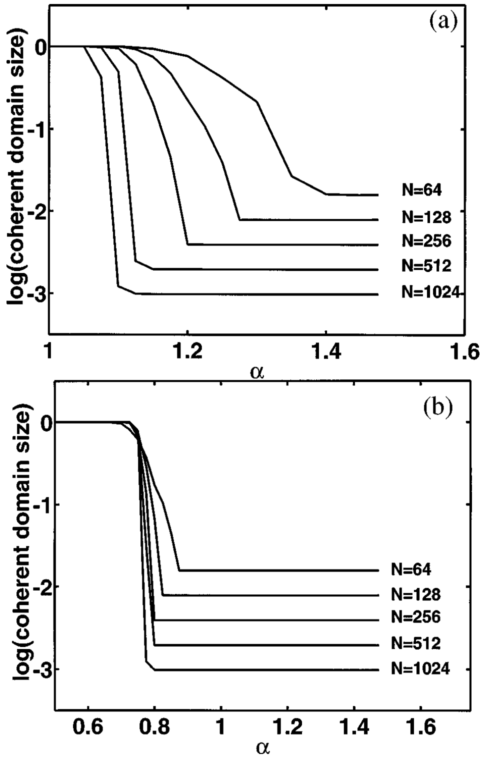

FIG. 2. Fraction of coherent sites. Results are plotted on a log scale, averaged over 500 random initial conditions after discarding 10 000 transient steps. (a) $a = 1.44$ (band-periodic region of logistic map), and (b) $a = 1.90$ (chaotic region of logistic map).

To linear order in the deviations, $e_n(i) = x_n(i) - x_n$, we have

$$e_{n+1}(i) = \frac{f'(x(n))}{A_i + 1}\left[e_n(i) + \sum_{j\in \mathrm{conn}} e_n(j)\right], \qquad (5)$$

where the connection neighborhood is chosen according to Eq. (3).

The trivial solution for Eq. (5) is $e_n(x) = 0$, which implies that if the lattice starts from a uniform state $x_0(j) = x_0 \forall j$, the sites stay uniform. We are interested in the long term behavior of a system perturbed infinitesimally from the uniform state.

We can rewrite Eq. (5) as a matrix equation,

$$\mathbf{e_{n+1}} = f'(x_n)\mathbf{M}\mathbf{e_n}. \qquad (6)$$

If $\mathbf{M}$ is diagonalizable, $e_n \propto (\rho_0\lambda)^n_{\max}$, where $\lambda_{\max}$ is the largest eigenvalue of **M** and $\ln(\rho_0)$ is the Lyapunov exponent of the map where ρ_0 is defined as

$$\rho_0 = \lim_{n\to\infty}\left(\prod_l |f'(x_l)|\right)^{1/n}. \qquad (7)$$

M is an asymmetric matrix with entries picked randomly according to the probability distribution of Eq. (3). The conditions imposed on **M** are (a) the entries in every row sum to 1 because the lattice dynamics has to map the interval onto itself, (b) $M_{i,i}$, $M_{i,i+1}$, and $M_{i,i-1}$ are positive, since the connection neighborhood always includes the nearest neighbors, and (c) $M_{ij} \geq 0 \quad \forall\, i,j$. Condition b ensures that there exists an integer k such that $\mathbf{M}^k$ is strictly positive. Thus **M** is irreducible and primitive [21] and it has a nondegenerate, real positive eigenvalue $\lambda_{\max}$ such that

$$\lambda_{\max} > |\lambda_i|, \qquad i = 1,\ldots,N-1, \qquad (8)$$

where λ_i is any eigenvalue of **M**. Condition a ensures that $\lambda_{\max} = 1$ [21] and the corresponding eigenvector is uniform. The matrix **M** belongs to the general class of matrices known as stochastic matrices.

For the case $\alpha \to \infty$, each site is connected only to its nearest neighbors and the matrix becomes symmetric and tridiagonal, with additional entries in the upper right and lower left corners due to the periodic boundary conditions. In this case the eigenvalue spectrum of the matrix is

$$\lambda(i) = \frac{1}{A+1}\left[1 + 2\cos\left(\frac{2i\pi}{N}\right)\right], \qquad i = 0,1,\ldots. \qquad (9)$$

For the eigenvalue spectrum of Eq. (9), the eigenvector corresponding to the largest eigenvalue represents the state of uniform fluctuations which is unstable if $\rho_0 > 1$. Since the eigenvalue spectrum is continuous, there is a band of unstable nonuniform modes which destroys the spatial coherence of the system.

For cases where α is of $O(1)$, we calculate the eigenvalues of the stability matrix **M** numerically for a large number of matrices with fixed values of α, in order to make general statements about the nature of the uniform state. We find that the matrices have *all real* eigenvalues and the largest eigenvalue is 1 corresponding to the uniform solution, as expected. The eigenvalue spectrum has a gap which separates the largest eigenvalue from a continuous band of $N - 1$ eigenvalues corresponding to nonuniform solutions (Fig. 3), which is a surprising result. The size of the gap is determined by the value of α. For $\alpha > 2$, the gap goes smoothly to zero and the eigenvalue spectrum approaches that of Eq. (9) as α increases. For $\alpha < 1$, the gap is large. Intermediate values of α result in gaps which depend upon the system size. In the thermodynamic limit of an infinite system, we expect the gap to appear discontinuously for $\alpha < \alpha_{\mathrm{crit}}$ which is reflected in the divergence of coherent domain size. While the size of the gap is related to the system size, the actual appearance of the gap depends only on the particular form of the coupling. The origin of the gap as a result of the particular matrix structure is not entirely clear but is an interesting unsolved problem from the standpoint of random matrix theory.

Thus, we can relate the stability of the spatially uniform but temporally chaotic states to the connectivity exponent α. If we consider a gap of width g, then for states such that $\rho_0 > 1$ and $\rho_0(1 - g) < 1$, the uniform mode is spatially stable (in the sense that any small initial

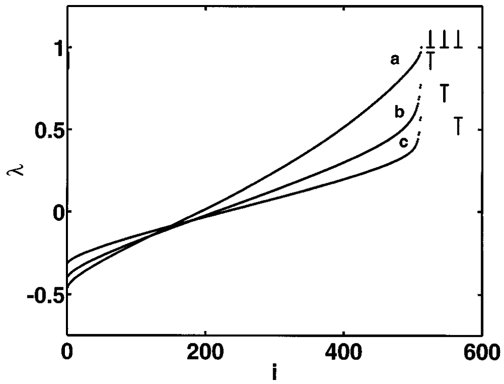

FIG. 3. Eigenvalue spectrum of connectivity matrix **M** of size 512×512; (a) $\alpha = 1.5$, (b) $\alpha = 1.0$, and (c) $\alpha = 0.75$. Bars show the gap in the eigenvalue spectrum.

nonuniformity will lie in the gap and will die out) and temporally chaotic. The remaining $N - 1$ nonuniform modes are stable since their eigenvalues are less than 1. The instability to uniform fluctuations remains but does not destroy the spatial coherence of the system. A similar gap in the eigenvalue spectrum was observed in Ref. [16] for a nearest-neighbor coupling CML with open boundary conditions, which was responsible for a stable uniform state. For temporally periodic states ($\rho_0 < 1$), the uniform solution is always stable for all values of α, and any infinitesimal perturbation from an initial uniform state dies out, which we observe numerically. However, for random initial conditions, the temporally periodic, spatially uniform state does not appear for $\alpha > \alpha_{\text{crit}}$, implying a lack of global stability above α_{crit}, which would not be indicated by a local, linear stability analysis. The above results for stability of the uniform state also hold qualitatively if we choose $p(ij)$ in Eq. (2) to be a constant (C/N), with $p(i, i \pm 1) = 1$ and vary C instead of α. However, this probability distribution is not independent of system size, resulting in finite size effects which are not present when using the scaling form of the distribution.

We can also interpret the dynamic behavior of the fractal CML in light of the relationship between structure and function of a network, in particular, the effect of network connectivity on stability [22]. The stability of a dynamical system with a given Jacobian matrix with randomly chosen elements is usually addressed in the framework of the Wigner-May theorem [23,24] which states that for a random connection matrix of zero mean, the system is almost surely unstable if the connectivity exceeds a threshold. However, the chaotic fractal CML with all non-negative elements in the Jacobian matrix is unstable for low values of connectivity (and therefore interaction strength) but is stable for connectivity larger than some critical value. We will study extensions of the structure-stability relationship to neuronal populations with probabilistic connections to determine the effect on network performance during associative recall, coding, and computation.

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