Energy landscape of 2D fluid foams

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Abstract: We study the equilibrium energies of 2D non-coarsening fluid foams which consist of bubbles with fixed areas. The equilibrium states correspond to local minima of the total perimeter. (i) We find an approximate value of the global minimum perimeter, and a marker to determine directly from an image how far a foam is from its ground state. (ii) For (small) area disorder, small bubbles tend to sort inwards and large bubbles sort outwards. (iii) Topological charges of the same signs 'repel' while charges of the opposite signs 'attract'. (iv) We also discuss : boundary conditions; uniqueness of pattern when topology is prescribed; extensions to 3D.

I. INTRODUCTION

A 2D fluid foam or a cellular fluid consists of a collection of bubbles separated by a continuous phase which tends to minimize its perimeter energy under the bubble area constraints. Our motivations to study fluid foams are threefold.

Mathematically, how to determine the minimal perimeter enclosing a cluster of N bubbles with known areas [2]? Here we discuss cases that have thus far escaped rigorous study, including large N, real boundary conditions and area dispersity. We estimate the value of the perimeter minimum, and conjecture the corresponding patterns. We hope to

provide insight for future rigorous mathematical proofs. Physically, most studies on foam structure have focused on special consequences of the energy minimization [6,7], such as topology. Here we derive general consequences of the energy minimization, to address open questions such as: Given an image, can we determine whether the foam is stressed and deformed? How regular are bubble shapes at equilibrium? How do topology (number of neighbors of a bubble), pressure and energy relate? What variables best describe the foam on a mesoscopic scale? Why do pentagons and heptagons tend to cluster in pairs in 2D foams? More practically, understanding foam structure is an important step towards predicting foam's mechanical properties, *e.g.* the quasistatic stress-strain relationship.

To capture the essential features of energy minimization, we make the following restrictions, which we can relax later. (i) 2D foam: we hope our progresses will clarify 3D foams. (ii) "Dry foam" limit: fluid fraction $\phi \ll 1$. (iii) Weakly disordered foams "close to a honeycomb structure": both the area and the edge number distributions have small variances. (iv) No coarsening: each bubble's area is constant. In fact, foams are almost always at quasi-equilibrium because the bubble walls equilibrate on a very short time-scale after a mechanical perturbation. The processes that cause area changes, such as coarsening, wall breakage, cell division, and cell nucleation, all act on a much slower time-scale.

For a foam with N bubbles with given areas $\{A_i,\ i = 1, \ldots, N\}$ (*e.g.* Fig. 3a), and a line tension γ, the energy is simply the sum of edge lengths ℓ_{ij} between bubbles i and j :

$$H = \gamma \sum_{0 \le i < j \le N} \ell_{ij}, \tag{1}$$

where $i = 0$ denotes the medium that surrounds the foam.

At equilibrium, *i.e.* in a local energy minimum, the foam obeys the Plateau rules [6,7]: bubble edges are circular arcs which meet in triples at $2\pi/3$ angles [4]. According to Laplace's law their algebraic curvatures ($\kappa_{ij} = -\kappa_{ji} > 0$ when bubble i is convex compared with bubble j) are related to the 2D pressure P_i inside bubble i: $\kappa_{ij} = \frac{P_i - P_j}{\gamma}$. Thus the algebraic curvatures of the three edges that meet at the same vertex must add to zero [5]:

$$\kappa_{ij} + \kappa_{jk} + \kappa_{ki} = 0. \quad (2)$$

Equation (2) extends to any closed contour crossing more edges; and to 3D, with κ the mean curvature of a bubble face.

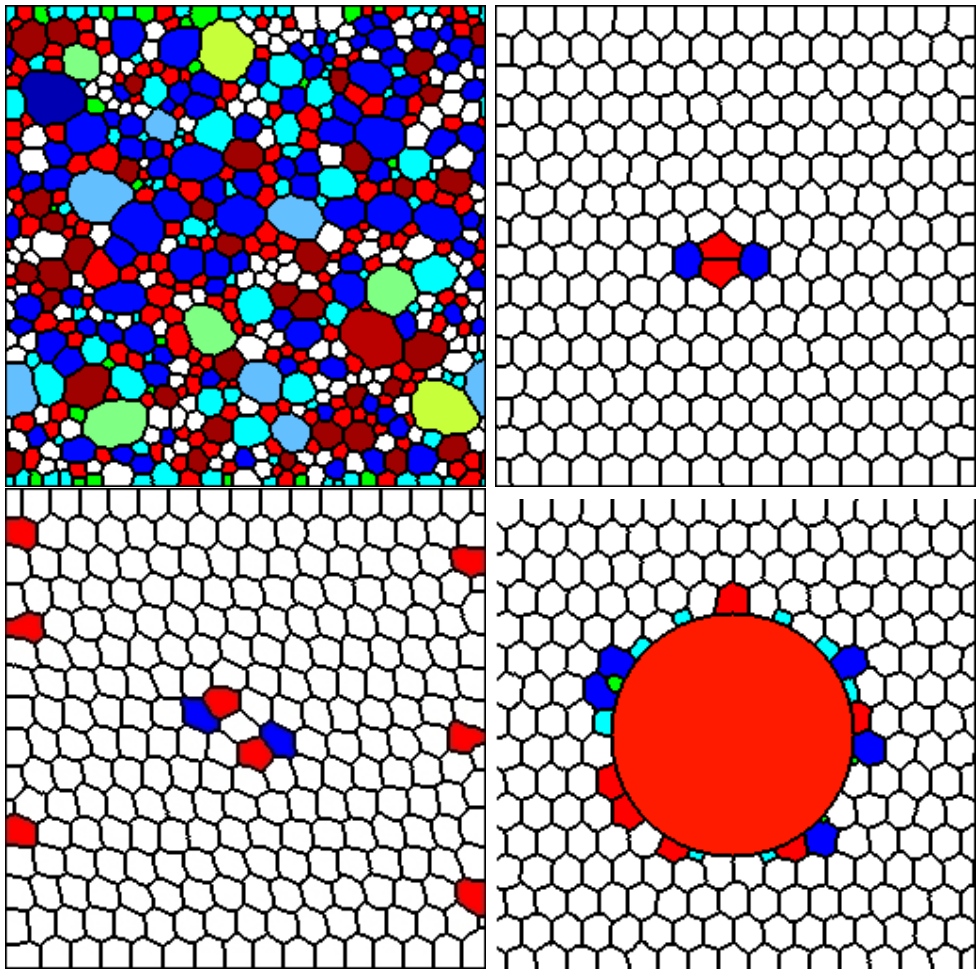

FIG. 1. *Simulated foams with fixed areas: (a) A typical configuration of a polydispersed foam at equilibrium. The top and bottom boundaries are fixed, the lateral boundaries periodic. Shades of grey encode the topology. (b) A pentagon-heptagon-pentagon-heptagon cluster artificially constructed in a regular foam with equal areas and periodic boundaries. (c) Two dipoles (pentagon-heptagon pairs) result in a curvature field in the hexagons around them. (d) A circular obstacle in the center of a hexagonal foam induces a topological charge distribution.*

II. ZEROTH-ORDER ESTIMATE OF THE GLOBAL MINIMUM

If all the bubble areas A_i are known but their topology is free to vary, the minimum value $min(H)$ for the foam energy (*i.e.* the ground state) exists [3]. However, its value,

and the corresponding pattern(s), is an open problem. We conjecture [1] that the energy of a natural, polydisperse, random foam is at least the energy of a collection of regular hexagons with the same area A (*i.e.* perimeter $3.72\ \sqrt{A}$):

$$H_h \sim 3.72\ \frac{\gamma}{2} \sum_{i=1}^{N} \sqrt{A_i}; \tag{3}$$

$min(H)$ is close to, but larger than H_h. This estimate of the global minimum H_h depends only on the area distribution, not the pattern. Thus given an image, we can simultaneously measure H, through the actual edge lengths, and H_h, through the areas. The ratio H/H_h is a global marker of the energy stored in the foam, or how far the foam is from its global minimum at prescribed areas.

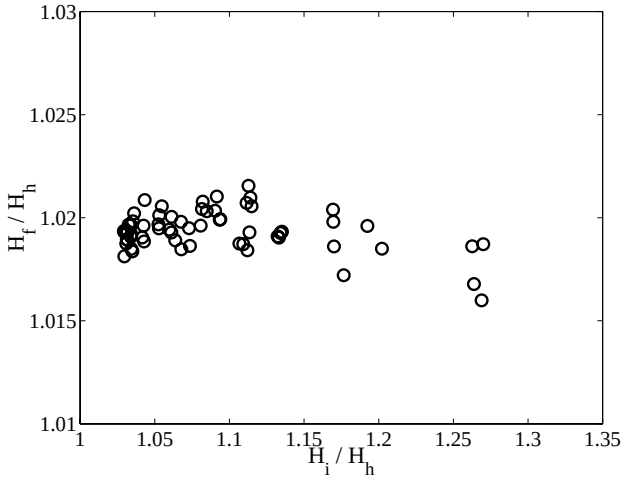

FIG. 2. *Relaxed energy of a random foam (Fig. 1a). The final rescaled equilibrium energies, H_f/H_h, after long relaxations (10^6 MCS) are plotted against their initial energy, H_i/H_h. Note the difference in horizontal and vertical scales.*

We use the extended large-Q Potts model, which allows large numbers of bubbles, $N \gg 1$, no fluid fraction $\phi = 0$, fixed bubble areas, a large range of area distributions, quick equilibration [8] (Fig. 1a). By biasing the Monte Carlo process, we can apply a steady shear [8] to prepare a distorted foam: a higher shear rate results in more distorted bubbles, thus higher initial energy H_i. We let the distorted foams relax towards equilibrium, i.e. a local energy minimum. Fig. (2) shows that whatever their initial energy, the relaxed foams all have final energies H_f 2-3% above H_h. We now write the energy of the foam

as the ground state H_h plus three corrections, which for simplicity we want to treat separately in the next three sections. This applies to weakly disordered foams with small area and topology variations.

III. AREA DISORDER

The edges of a regular hexagon of area A have a length $L = 3.72\sqrt{A}/6$. If two bubbles of different areas $A_i > A_j$ share a common edge, its length ℓ_{ij} obviously cannot be simultaneously equal to both L_i and L_j. There is an energy cost associated to the area mismatch, $\epsilon = (A_i - A_j)/(A_i + A_j)$, which vanishes only for $\epsilon = 0$ [1]. If the area disorder is small enough [11], the foam reduces its energy when the topological disorder is small: each bubble is surrounded by neighbors of nearly the same areas and reaches a nearly regular shape, *i.e. bubbles sort according to their sizes.* For a foam with free boundaries and no external force field, we expect the smaller bubbles to sort inwards, the larger ones outwards.

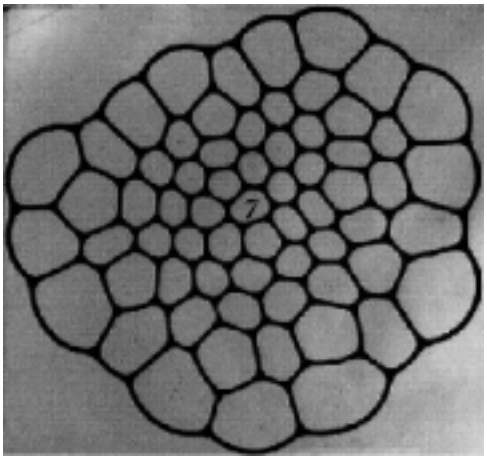

FIG. 3. *Photograph of a ferro-fluid foam. The number 7 indicates the only heptagon.*

This is exactly what we observe in an "annealed" foam. We prepare a ferrofluid foam between two Plexiglas plates [10]. We then tilt the plates from the horizontal plane to an angle of 0.1°, inducing a low effective gravity field. Large bubbles drift upwards, small bubbles downwards, resulting in vertical sorting according to size [12]. We then bring the plates back to horizontal, and the bubbles slowly drift back and settle. This

procedure allows the bubbles to rearrange and explore the energy space to find a lower energy configuration. The final stable pattern (Fig. 3) displays rounded bubbles and radial sorting according to size, larger bubbles surrounding smaller ones. We expect this size sorting to occur also in 3D.

IV. TOPOLOGICAL DISORDER AND ELECTROSTATIC ANALOGY

The "topological charge" quantifies the deviation from a hexagonal lattice: an n-sided bubble has a charge $q = (6 - n)$ (*i.e.* a dislocation $(6 - n)\pi/3$). Charge is additive: the charge of a collection of bubbles is the sum of their individual charges (*topological* Gauss-Bonnet theorem [9]).

We then have introduced the following *geometrical* Gauss-Bonnet theorem [1]. If the fluid fraction ϕ is small enough that we can interpolate P within the Plateau borders, the gradient of pressure, $-\vec{\nabla}P$ is proportional to edge curvatures (Laplace's law) and its circulation along any closed contour is zero.

If a closed contour C (with $\hat{n}$ its normal) encloses bubbles with charges q_i, then the outwards flux of $-\vec{\nabla}P$ across C is determined by the total charge $Q = \sum q_i$:

$$\oint_C \vec{\nabla}P \cdot \hat{n}\, d\ell \propto Q, \tag{4}$$

There is a complete analogy with 2D electrostatics (Table I). It extends to topological "dipoles" or "quadrupoles", and the distorsion they induce in a regular lattice (Fig. 1b,1c).

Generalization to 3D cannot be exact, since the pressure and topology correlate with mean and Gaussian curvatures, respectively, which in general are independent quantities. However, numerical and experimental observations of a 3D growth law [13] suggest a correlation between the mean and Gaussian curvatures, which suggests an approximate analogy may hold in 3D.

The analogy even extends to the expression of energy [1]. For instance, a single topological charge is seldom observed in a real foam, because its energy cost diverges (logarithmically) with the foam size. Conversely, two charges of opposite sign lower their

energy when they get closer. This explains the origin of the "effective interaction" [6] assumed to explain the high occurrence of heptagon-pentagon pairs. In Fig. (3), the heptagon has two pentagonal neighbours.

V. BOUNDARY CONDITIONS

The possible boundary conditions for foam are: periodic, Fig. (1b); free, if the foam is surrounded by a fluid medium, Fig. (3); fixed, if it touches a solid box, Fig. (1d); or some combinations of these three, Fig. (1a).

For a free foam, the outer fluid fixes the pressures at the foam boundary. A fixed boundary, on the other hand, requires that the pressure gradient $-\vec{\nabla}P$ (perpendicular to each bubble edge, itself perpendicular to the boundary) is parallel to the boundary. Thus in these two cases, the topological charges determine the pressure in the foam as a Dirichlet or a Neumann problem, respectively. Therefore the pressure field should be *unique* for fixed topology [1].

Periodic boundary conditions guarantee that the total charge $Q = 0$ (Euler theorem). For all other boundary conditions, we can apply the topological Gauss-Bonnet theorem [9] to the N_b bubbles at the foam's boundary: the total charge of a foam is $Q = (N_b + 6)$. Introducing a modified definition $\tilde{q} = (5 - n)$ instead of $(6 - n)$ for the N_b bubbles at the boundary is more convenient: the total topological charge of a foam becomes simply $\tilde{Q} = \sum_i \tilde{q}_i = 6$ (*i.e.* a dislocation of $6\pi/3 = 2\pi$).

The shape of the solid box determines the distribution of these 2π among the bubbles touching the boundary. If the solid box has all corner angles a multiple of $\pi/3$, all bubble edges can simultaneously be straight (isobaric foam). In general, however, the solid box results in curved bubble edges. The same applies to a concave boundary, for instance an obstacle placed in the middle of the foam, which introduces a dislocation -2π, as can be visually checked in Fig. 1d.

VI. SUMMARY

Equilibrium energy of 2D foams with given bubble areas helps understanding foam structure in general. The zeroth-order estimate of the ground energy, H_h, is a function of the area distribution only, independent of the topology. The ratio H/H_h provides a global marker of how far a foam is from its ground state. We have similarly defined a local marker for each edge length, opening the way towards an intrinsic definition of strain in a foam [1].

The present analysis physically explains the different contributions to a foam's energy: area mismatch, topology, and boundaries, predicts the foam configurations corresponding to their ground state. The analysis does not depend on the characteristic size and energy scales. It is also valid for foams in which bubble areas vary slowly. Deriving approximate results for 3D seems possible.

The pressure, edge curvature and topological charge together form a set of good variables to characterize a foam. They present a profound analogy with 2D electrostatic potential, field and charge, respectively, through the geometrical Gauss-Bonnet theorem (eq. 4). The topology determines a foam's energy and pattern. This explains the origin of topological and geometrical correlations in foams: The bubbles sort according to their size; topological charges of same signs tend to separate and opposite signs tend to aggregate.

Acknowledgments: We would like to thank F. Morgan, N. Kern and P. Swart. YJ is supported in part by DOE under contract W-7405-ENG-36.

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	potential	field	charge
$2D$ electrostatics	potential V	electric field $-\vec{\nabla}V$	electric charge e
$2D$ foams	pressure P	curvature $\propto -\vec{\nabla}P$	topological charge $\propto (6-n)$

TABLE I. Proposed analogy between foams and electrostatics in two dimensions.