RIASSUNTO (*)

Si calcola lo spettro del fotone e dell'elettrone emessi in un decadimento muonico radiativo mediato da un bosone vettoriale. Le deviazioni dello spettro da quello dell'interazione ($V-A$) di Fermi sono, in primo ordine, inversamente proporzionali al quadrato della massa del bosone intermedio. Queste deviazioni, a certe energie dell'elettrone e del fotone, possono essere abbastanza sostanziali da permettere una determinazione approssimata della massa del bosone vettoriale.

(*) Traduzione a cura della Redazione.

IL NUOVO CIMENTO VOL. XXX, N. 5 1° Dicembre 1963

A Model for Final-State Interactions (*).

G. BONNEVAY †

Faculté des Sciences - Orsay
Faculté des Sciences - Bordeaux

(ricevuto l'1 Agosto 1963)

Résumé. — Par des méthodes de prolongement analytique, on étudie un modèle pour la désintegration d'une particule en trois particules identiques en ne tenant compte que des interactions élastiques deux à deux, dans une seule onde partielle supposée dominante. On établit ainsi une équation intégrale à noyau régulier dans la région physique. Si une résonance a lieu dans les interactions à deux corps, on montre l'importance des deux premières rediffusions. Les rediffusions d'ordre plus élevé sont sur des feuillets de Riemann de plus en plus lointains.

Introduction.

Production amplitudes $B_1+B_2 \to A_1+A_2+A_3$ are considerably simplified if Watson's conditions are valid [1], or if the final state is created via a weak coupling, or if a long-life resonance takes place between initial and final particles. In these cases the final state does not remember the way it was formed and the total amplitude depends on three variables only, s_1, s_2, s_3, the squared total energies of couples A_2A_3, A_3A_1, A_1A_2 in their respective center-of-mass

(*) The research reported in this document has been sponsored by the Office of Scientific Research, OAR, through the European Office Aerospace Research, United. States Air Force.

† Deceased in mountain accident July 28, 1963.

(1) M. WATSON: *Phys. Rev.*, 88, 1163 (1952).

systems. The total amplitude can be approximated by

$$T(s_1, s_2, s_3) = \frac{\Gamma}{S - m_M^2} F(s_1, s_2, s_3) \,, \tag{1}$$

in which m_M^2 is the mass of the resonant intermediate state M; S is the squared total energy, linearly related to the s_i

$$\sigma = s_1 + s_2 + s_3 = S + m_1^2 + m_2^2 + m_3^2 \,,$$

$F(s_1, s_2, s_3)$ is a function of the three independent variables s_i.

Now $F(s_1, s_2, s_3)$ looks like a four-leg function in which one of the external masses m is the total energy $\sqrt{S} = m$, that is to say a variable in contradistinction to the ordinary case.

It is therefore reasonable to assume that F is the analytic continuation on m, that is on σ, of the reaction amplitudes $M + A_i \to A_j + A_k$ from the region of small values of m^2 or σ, in which a Mandelstam representation can be assumed. Then, for small σ

$$F(s_1, s_2, s_3) = \sum_i \Phi_i(s_i, \sigma) + \sum_{i \neq j} \Phi_{ij}(s_i, s_j, \sigma) \,, \tag{2}$$

where Φ_i are the subtracted parts, or the Cini-Fubini terms, Φ_{ij} being the double-dispersive functions.

In a previous paper (²) we showed that the Φ_{ij} part gives an infinite number of Landau curves tangent to the physical region when σ enters the decay region $M \to A_1 + A_2 + A_3$, casting some doubt on the Cini-Fubini approximation.

Nevertheless BOUCHIAT and FLAMAND (³) have shown that if the final two-body interactions are going through a resonance the neglecting of all rescattering is not a bad approximation. This means that one could try to improve that analysis by considering a model in which only two-body forces in only one partial wave are taken into account. Then the Φ_{ij} are identically zero as it is well known (⁴) and one gets a model similar to the one of PEIERLS and TARSKI (⁴).

In the present paper we study the following well-defined mathematical problem: given the Cini-Fubini part of F in eq. (2) for small σ and the two-body unitarity condition in $M + A_i \to A_j + A_k$ channels, what is the largest

(²) G. BONNEVAY: *Proc. Roy. Soc.*, A **266**, 68 (1962), see also G. BARTON and C. KACSER: *Nuovo Cimento*, **21**, 988 (1961).
(³) C. BOUCHIAT and G. FLAMAND: *Nuovo Cimento*, **23**, 13 (1962).
(⁴) R. F. PEIERLS and J. TARSKI: *Phys. Rev.*, **129**, 981 (1963).

5884

domain of analyticity in all s_i's compatible with these equations? The final aim being to obtain workable integral equations.

In Section **1** the general method is given.

In Section **2** we define a « minimum » domain of holomorphy for Φ and its continuation on the second sheet.

Hence one sees that Φ has normal thresholds only in both variables s_i and σ. This was already proved for triangular diagrams by BARTON and KACSER (⁵) and also ANISOVICH, ANSEL'M and GRIBOV (⁶). The proof here is extended to all rescattering diagrams (Fig. 1).

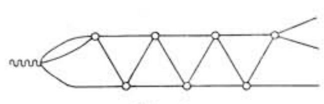

Fig. 1.

Then, in Section **3**, one gets a double-dispersion relation in s_i, σ which will be studied later on in connection with three-body unitarity.

Section **4** is devoted to the derivation of an integral equation, for fixed σ, which has a regular kernel inside the physical decay region, and which emphasizes the importance of the first two rescatterings of the resonance process. Singularities due to all rescatterings are found on successive Riemann sheets; there is a finite number of them on each sheet and one sees that their distance to the physical region on the Riemann surface increases with the order of rescattering.

Comparison with Landau singularities is given in Section **5**.

1. – Position of the problem.

We shall study our problem in a simplified version

$$M \to \pi_1 + \pi_2 + \pi_3 \,, \tag{I}$$

where an initial state M of varying mass m gives three neutral particles that we call pions of equal masses $m_\pi = 1$; has no spin; the pions interact in S-wave only. A scalar resonance can occur, we call it ρ.

We define p_1, p_2, p_3 the outgoing four-momenta of the pions, and as usual introduce

$$\begin{cases} s_i = (P - p_i)^2 = (p_j + p_k)^2 \,, \\ P = p_1 + p_2 + p_3 \,, \qquad P^2 = m^2 \,, \\ \sigma = s_1 + s_2 + s_3 = m^2 + 3 \,. \end{cases} \tag{3}$$

(⁵) G. BARTON and C. KACSER: *Nuovo Cimento*, **21**, 593 (1961).
(⁶) V. V. ANISOVICH, A. A. ANSEL'M and V. N. GRIBOV: *Žurn. Èksp. Teor. Fiz.*, **42**, 224 (1962); *Sov. Phys. J.E.T.P.*, **15**, 159 (1962).

5885

In the center-of-mass system of particles 2 and 3,

$$(4)\quad \begin{cases} s_1 = 4q^2+4 = (E_p+\omega_p)^2, \quad E_p = \sqrt{p^2+m^2}, \qquad \omega_p = \sqrt{p^2+1}, \\ s_2 = 2+2\omega_p\omega_q - 2pqx, \quad q = \dfrac{\sqrt{s-2}}{2}, \\ s_3 = 2+2\omega_p\omega_q + 2pqx, \quad p = \dfrac{\sqrt{(s-(m-1)^2(s-(m+1)^2)}}{2\sqrt{s}}; \end{cases}$$

$\boldsymbol{q}$ is the relative momentum of particles 2 and 3; $\boldsymbol{p}$ the momentum of 1: $x = \cos(\widehat{p, q})$.

Reaction (I) is then described by a function $F(s_1, s_2, s_3)$ of a point in a three-dimensional space in which there are four disconnected physical regions limited by the surface

$$(5)\qquad \mathscr{P}: \quad s_1 s_2 s_3 = (\sigma-4)^2,$$

which is obtained in writing $x^2=1$ in eqs. (4).

The four regions are:

a) Three regions of scattering

$$(\text{II})\qquad \text{M} + \pi_i \to \pi_j + \pi_k,$$

$$s_i \geqslant 4, \qquad s_j, s_k < 0.$$

b) One region for decay process (I) with

$$s_1, s_2, s_3 \geqslant 4, \qquad i.e. \qquad \sigma \geqslant 12, \quad m \geqslant 3.$$

Now for small σ, let us say $\sigma < 4$, *i.e.* $m < 1$, only reactions (II) are possible and we assume a single unsubtracted dispersion representation for $F(s_1, s_2, s_3)$:

$$(6a)\qquad F(s_1, s_2, s_3) = \sum_i \Phi(s_i, \sigma),$$

$$(6b)\qquad \Phi(s_i, \sigma) = \frac{1}{\pi}\int_4^\infty \frac{a(s', \sigma)}{s'-s}\,\mathrm{d}s',$$

with the two-body unitarity condition

$$(6c)\qquad a(s, \sigma) = q\overline{M}(s)F_0(s, \sigma),$$

$$(6d)\qquad a(s, \sigma) = q\overline{M}(s)\left\{\Phi(s, \sigma) + \int_{-1}^{+1}\Phi(s_2(s, x, \sigma), \sigma)\,\mathrm{d}x\right\},$$

where $s_2(s, x, \sigma)$ is given by (4); $\overline{M}(s)$ is the analytic continuation of π-π elastic amplitude $M(s)$ through the normal s-cut ($4 \leqslant s$, $\text{Im}\, s = 0$). On the real axis, $s > 4$, $\overline{M}(s+i\varepsilon) = M^*(s+i\varepsilon)$.

Consequently the absorptive part $a(s, \sigma)$ will in general be analytic in s. Let us rewrite (6*d*)

$$(6f)\qquad a(s, \sigma) = q\overline{M}(s)\left\{\Phi(s, \sigma) + \frac{1}{2pq}\int_{s_2^-(s,\sigma)}^{s_2^+(s,\sigma)}\Phi(s', \sigma)\,\mathrm{d}s'\right\},$$

where the new path of integration is a segment of a straight line—due to linear change of variable—joining the two points $s_2^\pm(s, \sigma) = s_2(s, \pm 1, \sigma)$ in the s'-complex plane.

Our problem is now the following: what are the analytical properties in s and σ of $a(s, \sigma)$ and $\Phi(s, \sigma)$, assuming that the only singularities are those which come from the system (6) itself? Of course one needs to know the properties of $\overline{M}(s)$. We will assume the usual ones that is to say: $\overline{M}(s)$ is meromorphic in the holomorphy domain of the S-wave amplitude $M(s)$ that is in the s-plane cut by $\{-\infty, 0\}$ and $\{4, +\infty\}$; the poles of $\overline{M}$ are the π-π resonances. For simplicity we will assume only one resonance, that is two poles $s = m_\rho^2$ and $s = m_\rho^{2*}$ corresponding to the ρ-meson (assumed scalar).

General method. – We start from an initial region $\mathscr{R}\{\text{Im}\,\sigma = \text{Im}\, s = 0,$ $\sigma < 4,\ s > 4\}$ where a and Φ are assumed analytic.

1) From this region we continue a and Φ defining a minimum domain D of holomorphy for Φ and meromorphy for a.

2) Going out of D one will find singularities in the integral term in (6*d*, *f*) for a and therefore for Φ; this will give new singularities in the integral term, for a and therefore for Φ and so on.

3) It will still be necessary to see whether or not these singularities can cancel each other.

Remark 1. – Singularities can occur in the integral term in (6*f*) in two ways: *a*) the integration contour meets some singularity of Φ; *b*) $s_2^\pm(s, \sigma)$ or $(2pq)^{-1}$ are singular. This happens when $s=0$, $s=4$, $s=(m\pm 1)^2$. The point $s=0$ will always be found singular. From eq. (4) it is clear that if $pq=0$, $s_2^+ = s_2^-$ and if such a point is reached without having to deform the path of integration, the point is not singular since the path is reduced to zero.

Remark 2. – When σ varies, as long as $a(s, \sigma)$ is analytic around the normal branch point $s=4$, Φ and F_0 can be continued through the normal

s-cut by the analytic functions $\bar{\Phi}$ and $\bar{F}_0$ defined by

(7) $$2ia(s,\sigma) = \Phi(s,\sigma) - \bar{\Phi}(s,\sigma) = F_0(s,\sigma) - \bar{F}_0(s,\sigma) .$$

From eqs. (6c) and (7), turning round $s=4$, one sees that the normal threshold is a square root branch-point for F_0 and therefore Φ; that is to say it generates two sheets only.

Remark 3. – When s turns round the normal threshold, Φ and $\bar{\Phi}$ are exchanged (see remark 2) and from eq. (7), a is multiplied by -1, so that the function a/q is regular for $s=4$.

Remark 4. – Function $a(s,\sigma)$ and therefore Φ and F_0 on the second sheet have the same ρ resonance than $M(s)$ due to the factor $\bar{M}(s)$ in eq. (6f, d). Let us write

(8) $$\frac{a(s,\sigma)}{q} = \frac{R^*(\sigma)}{2i(s-m_\rho^*)} - \frac{R(\sigma)}{2i(s-m_\rho)} + \frac{a'(s,\sigma)}{q} .$$

Substitution of (8) into eq. (6b) gives

(9) $$\Phi(s,\sigma) = \frac{R^*(\sigma)}{8(q+q_\rho^*)} - \frac{R(\sigma)}{8(q-q_\rho)} + \Phi'(s,\sigma) ,$$

with

(10) $$q_\rho = -\tfrac{1}{2}\sqrt{m_\rho^2-4} = \bar{q}_\rho - i\gamma/4 ,$$

$m_\rho^2 = \bar{m}_\rho^2 - 2i\gamma\bar{q}_\rho$ (by definition $\operatorname{Im}\sqrt{s-4}>0$); γ is the $\rho\to 2\pi$ partial width.

The first two terms in eq. (9) give a Breit-Wigner formula

(11) $$\Phi_{BW} = \frac{R(\sigma)\bar{q}_\rho}{\bar{m}_\rho^2 - s - 2i\gamma q} ,$$

when q_ρ is substituted into the numerator and γ^2 neglected.

One defines the partial width $\Gamma(\sigma)$ for $M\to\rho+\pi$ by

$$\left|\frac{\sqrt{\gamma\Gamma(\sigma)}}{-2i\gamma\bar{q}_\rho}\right|^2 = |(\Phi_{BW}(\bar{q}_\rho)|^2 .$$

Hence, from unitarity condition eq. (6g) (see Section 3)

(12) $$R(\sigma) = 2i\frac{2\gamma}{\bar{q}_\rho}\, q_\rho \tilde{F}_0(-q_\rho,\sigma) ,$$

one gets

(13) $$\Gamma(\sigma) = 16\gamma\,|q_\rho \tilde{F}_0(-q_\rho,\sigma)|^2 .$$

The complex number $2iq_\rho\tilde{F}_0(-q_\rho,\sigma) = f_{M\pi\rho}(\sigma)$ is the M-π-ρ coupling constant. If $\sigma<12$ and γ negligible, $f_{M\pi\rho}$ becomes real.

2. – The minimum domain of analyticity.

In the initial region $\mathscr{R}$, Φ is assumed analytic in σ; therefore $a(s,\sigma)$ is also analytic, since the path of integration is on the real negative axis where $\Phi(s',\sigma)$ is analytic in s'. When σ and s move and go out of $\mathscr{R}$, the path of integration in (6f) moves in the s'-plane. Using remark 1 and analytical properties of Φ, eq. (6b), one sees that the integral terms will certainly be analytic as long as the integration contour does not cross the normal s'-cut of $\Phi(s',\sigma)$ neither meets the only singularity of Φ, $s'=4$.

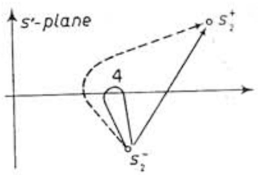

Fig. 2.

This defines the «minimum domain of meromorphy», D for a/q.

The frontiers of D are given by

(14) $$\begin{cases} \mathscr{F}: & s_2^{\pm}(s,\sigma) = s' , & \operatorname{Re} s' > 4 , & \operatorname{Im} s' = 0 , \\ \mathscr{G}: & s_2(s,x,\sigma) = 4 , & -1 \leqslant \operatorname{Re} x \leqslant 1 , & \operatorname{Im} x = 0 . \end{cases}$$

When the point (s,σ) crosses $\mathscr{F}$ the integration contour enters the 2nd sheet of Φ, when (s,σ) crosses $\mathscr{G}$, the integration contour has to be deformed (see dashed curve in Fig. 2).

Remark 5 (7). – Inside D all eqs. (6) are valid. If (s,σ) crosses $\mathscr{G}$ without crossing $\mathscr{F}$, eqs. (6) are still valid except (6c) which must be replaced by

(6g) $$a(s,\sigma) = q\,\bar{M}(s)\,\tilde{F}_0(s,\sigma) ,$$

where $\tilde{F}_0$ is not the partial wave, but the analytic continuation of it

(6h) $$\tilde{F}_0(s,\sigma) = F_0(s,\sigma) + \frac{1}{2pq}\int_{s_2^-(s,\sigma)}^{4} a(s',\sigma)\,\mathrm{d}s' .$$

(7) This important remark is due to Bessis and Pham, Saclay preprint, 1963. Inside $\bar{D}$, but outside D, the partial-wave series does not converge if one uses $\tilde{F}_l$.

$F_0(s, \sigma)$ being the true partial wave in this region, that is the integral of F along a rectilinear path from s_2^- to s_2^+ (see Fig. 2). The enlarged domain obtained by suppressing frontier $\mathscr{G}$ will be called $\bar{D}$.

Let us call D_σ, $\bar{D}_\sigma$ the restriction of D, $\bar{D}$ to the s-plane for fixed σ, and correspondingly D_s, $\bar{D}_s$ to the σ-plane for fixed s, with frontiers $\mathscr{F}_\sigma$, $\mathscr{G}_\sigma$ and $\mathscr{F}_s$, $\mathscr{G}_s$, respectively.

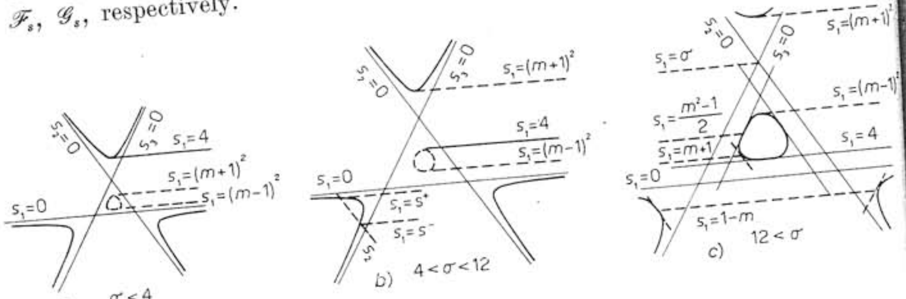

Fig. 3.

$\mathscr{F}_\sigma$ and $\mathscr{F}_s$ are easy to construct graphically for real σ and s, respectively. For instance to get $\mathscr{F}_\sigma$ one considers the intersection of the surface $\mathscr{P}$ eq. (5) by the real plane $s_1+s_2+s_3=\sigma$; one gets the Mandelstam diagram Fig. 3. Given s' means a line parallel to the $s_2=0$ axis; this line intersects curve $\mathscr{P}$ in two points $s_1=s^\pm$; when s' runs from 4 to infinity, $s^\pm$ describes $\mathscr{F}_\sigma$. Note that $\mathscr{F}_\sigma$ is just what is usually called the left-hand cut for the partial wave F_0.

Thus for $\sigma<4$ and for $4<\sigma<12$ one gets frontiers drawn on Fig. 4a, b. $\mathscr{F}_\sigma$ is represented by a solid line, $\mathscr{G}_\sigma$ by a dashed line.

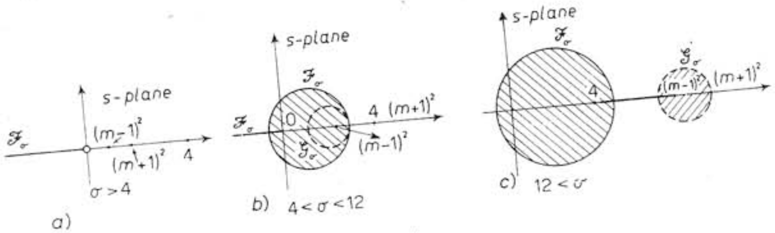

Fig. 4.

If $\sigma<4$, $D_\sigma\equiv\bar{D}_\sigma$ is the whole s-plane cut by $\{\operatorname{Re} s\leqslant 0,\ \operatorname{Im} s=0\}$. If ... increases, $\sigma>4$, a «forbidden» loop appears.

One sees that $s=0$ is always outside D_σ. In the first case $s=4$, $s=(m\pm$ ... When $\sigma>4$, the poi... are inside D_σ and are not singular (see remark 1). When ... $s=(m-1)^2$ goes out of D_σ.

For $\sigma<12$ the normal s-cut $\{4, +\infty\}$ is entirely inside D_σ so that formulae (6) and remarks 1 to 4 are valid. This gives the way to construct the complete domains D and $\bar{D}$: for each σ real or complex, the normal s-cut must be inside D_σ, that is to say, conversely, σ must be in D_s for any s running from 4 to infinity.

For real s greater than 4, $\mathscr{F}_s$ is easy to construct and one finds the real axis from σ_0 to infinity with

$$\sigma_0=s+4+2\sqrt{s}\,,\qquad s\geqslant 4\,,\qquad \operatorname{Im} s=0$$

or solving in s

$$s=(m-1)^2\,.$$

The frontier $\mathscr{G}_s$ is a closed loop (see Fig. 5).

There are no singular points on $\mathscr{G}_s$ except, may be, the point $\sigma=2s+4$ in which the extremity s_2^+ coincides with the threshold $s'=4$. This point corresponds to the leading Landau singularity for the triangular diagram. In fact, this singularity does not exist as was proved by BARTON and KACSER (5).

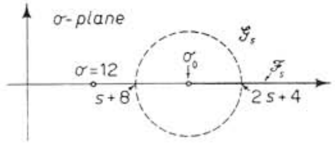

Fig. 5.

Now the point $\sigma=\sigma_0$ inside $\mathscr{G}_s$ is singular since when σ approaches σ_0, the contour of integration is deformed and does not tend to zero but to $2[(m+1)-4]$, while $(2pq)^{-1}$ tends to infinity. Then, in this point, a behaves like the square root term $(\sigma-\sigma_0)^{-\frac{1}{2}}$.

When s runs from 4 to infinity σ goes from 12 to infinity, that is m from 3 to infinity, so that the domain $\bar{D}$ is defined by the condition: σ is in the plane cut by the normal σ-cut $\{12, +\infty\}$ and s in $\bar{D}_\sigma$.

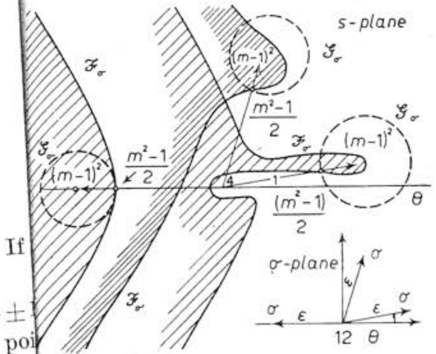

Fig. 6.

When σ is real greater than 12 (on the normal σ-cut) the domain $\bar{D}_\sigma$ seems to exclude a part of the normal s-cut (see Fig. 4c); but σ real is only a limiting case. To see what happens we study $\bar{D}_\sigma$ for complex σ in the vicinity of 12: $\sigma-12=\varepsilon\exp[i\theta]$ where θ varies from π to zero. Figure 6 shows continuous variation of $\mathscr{F}_\sigma$ from Fig. 4b to 4c for infinitesimal ε, that is to say, in the non-relativistic limit. The loop $\mathscr{F}_\sigma$ which was a branch of a cubic becomes now a branch of an hyperbola for $\theta=\pi$, and a branch of an hyperbola plus a straight line covering the whole physical region when $\theta=0$. When θ is slightly different from zero, one explicitly sees that the normal cut is completely inside $\bar{D}_\sigma$, that is the nonshaded region in Fig. 6.

Furthermore we get the important conclusion that in the limit of real $\sigma>12$, $\mathscr{F}_\sigma$ has a real part (see Fig. 4c) above the real axis if σ tends to its real value with a positive vanishing imaginary part; in other words the physical region for a/q is *below* that part of the left-hand cut of Fig. 4c.

3. – The double-dispersion relation.

In $\bar{D}$, eq. (6b) is valid, $a(s,\sigma)$ is analytic in σ so that if one assumes a and Φ well-behaved at infinity, one gets the double-dispersion representation in s and σ.

$$a(s,\sigma)=\frac{1}{\pi}\int_{12}^{\infty}\frac{\varphi(s,\sigma')}{\sigma'-\sigma}\,\mathrm{d}s'\,,\qquad 4\leqslant s\,,\qquad \operatorname{Im} s=0\,,\tag{15a}$$

$$\Phi(s,\sigma)=\frac{1}{\pi^2}\int_4^{\infty}\mathrm{d}s'\int_{12}^{\infty}\mathrm{d}\sigma'\,\frac{\varphi(s',\sigma')}{(s'-s)(\sigma'-\sigma)}\,.\tag{15b}$$

We saw that $\sigma=\sigma_0(s)$ which is on the frontier of $\bar{D}$ for s real greater than 4, is a branch point for a. Then we can apply the second part of our general method of Section **1**, that is, Φ has the singularity $\sigma=\sigma_0(s)$ corresponding to $s=(m-1)^2$, then by end-point singularity we find the possible singular point for the integral term and therefore for a and Φ, in writing

$$s_2^{\pm}(s,\sigma)=(m-1)^2\,,$$

which gives $s=(m+1)$, and by iterating

$$s_2^{\pm}(s,\sigma)=m+1\,,$$

which gives $s=m+1$ and $s=(m-1)^2$, and so on.

Nevertheless one can see that the first iteration $s=m+1$ is inside D so that it cannot be singular as long as s remains in D, the iteration procedure stops, and the second iteration cannot destroy (8) the first singularity discovered. In Section

(8) In fact, all these singular points are in different Riemann sheets. In Section we find the first iteration $s=m+1$, after s has turned round the $s=(m-1)^2$ branch point.

$s=(m-1)^2$ in the border of $\bar{D}$. The other iteration procedure for singularity is to remark that from eq. (15b), Φ having the normal branch point $\sigma=12$, $a(s,\sigma)$ must have also the same branch point (see eq. (6d, f)); hence, in eq. (15a), φ is not zero for $12\leqslant\sigma\leqslant\sigma_0(s)$.

In fact $\varphi(s,\sigma)$ which is a real function of real variables can be split into two parts which have supports indicated in Fig. 7, and are limits on the real s-σ-plane of two analytic functions φ_1, φ_2 of s and σ

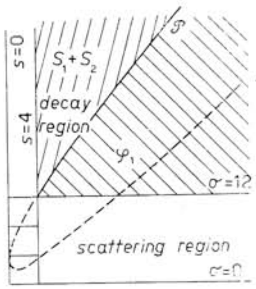

Fig. 7.

$$\varphi(s,\sigma)=\left[\varphi_1(s^+,\sigma^+)\,\theta(\sigma-12)+\varphi_2(s^+,\sigma^-)\,\theta(\sigma-\sigma_0(s))\right],\tag{16}$$

where φ_1 and φ_2 are the « periods » of $a(s,\sigma)$ when σ turns round the branch points $\sigma=12$ and $\sigma=\sigma_0(s)$, respectively. One finds

$$\varphi_1(s,\sigma)=\frac{1}{2i}[a(s,\sigma)-\bar{a}_{12}(s,\sigma)]\quad=q\bar{M}(s)\left\{b(s,\sigma)+\frac{1}{2pq}\oint_{s_2^-(s,\sigma)}^{s_2^+(s,\sigma)}b(s',\sigma)\,\mathrm{d}s'\right\},\tag{17a}$$

$$\varphi_2(s,\sigma)=\frac{1}{2i}[a(s,\sigma)-\bar{a}_{\sigma_0}(s,\sigma)]\quad=q\bar{M}(s)\,\frac{1}{2pq}\oint_{s_2^-(s,\sigma)}^{s_2^+(s,\sigma)}a(s',\sigma)\,\mathrm{d}s'\,,\tag{17b}$$

where

$$b(s,\sigma)=\frac{1}{2i}[\Phi(s,\sigma)-\bar{\Phi}_{\sigma=12}(s,\sigma)]=\frac{1}{\pi}\int_4^{\infty}\frac{\varphi(s',\sigma)}{s'-s}\,\mathrm{d}s'\tag{18}$$

is the period (9) of Φ around $\sigma=12$, and will be called the absorptive part of $\Phi(s,\sigma)$ in the σ channel; the sign $\oint$ is to remember that one has to deform the integration contour when σ crosses $\mathscr{G}_s$.

Note that in eq. (16) one has to write $\varphi_2(s^+,\sigma^-)$, that is the value of the analytic function φ_2 when σ becomes real from below the real axis.

When $12<\sigma<\sigma_0(s)$, $\varphi=\varphi_1$ and one finds (17a) easily. When $\sigma>\sigma_0(s)$, one gets (16) and (17b) writing

$$2i\varphi(s,\sigma)=[a(s^+,\sigma^+)-a(s^+,\sigma^-)]=$$
$$=q\bar{M}(s)\left\{\Phi(s^+,\sigma^+)-\Phi(s^+,\sigma^-)+\frac{1}{2pq}\int_{s_2^-}^{s_2^+}[\Phi(s'^+,\sigma^+)-\Phi(s'^-,\sigma^-)]\,\mathrm{d}s'\right\},$$

(9) The notation $\bar{a}_{12}(s,\sigma)$ means the value of a after σ has turned round the branch point $\sigma=12$ counterclockwise etc. Note that $\bar{a}_{\sigma_0(s)}(s,\sigma)=\bar{a}_{(m-1)^2}(s,\sigma)$; that is to say if σ turns round $\sigma_0(s)$ or s turns round $(m-1)^2$ one gets the same value of a: this is because the tangent to the curve $\sigma=\sigma_0(s)$ is never parallel to $s=0$ or $\sigma=0$.

since when σ turns round $\sigma_0(s)$, the path of integration goes from above to below the normal cut in the s'-plane. Then one writes

$$\Phi(s^+, \sigma^+) - \Phi(s^-, \sigma^-) = \Phi(s^+, \sigma^+) - \Phi(s^-, \sigma^+) + \Phi(s^-, \sigma^+) - \Phi(s^-, \sigma^-) \,,$$
$$= 2i[b(s^+, \sigma) + a(s, \sigma^-)] = 2i[b(s^-, \sigma) + a(s, \sigma^+)] \,.$$

Clearly the absorptive part $b(s, \sigma)$ is connected with unitarity in the σ channel, that is, the three-body unitarity. We shall study this point in a future paper and show that identification of b with that part of three-body unitarity which comes from iterated two-body forces, gives an integral equation on b which in principle determines the σ-dependence.

4. – Integral equation for the absorptive part.

If one is not interested in the behaviour in the total energy, the system (6) gives an integral equation for fixed σ in Φ, F_0 or a. For example in F_0 one gets the usual Muskelishvili-Omnès equation for the partial wave, the solution of which is defined up to an arbitrary function of σ in the best case, *i.e.* zero or only one subtraction constant.

Unfortunately when $\sigma > 12$, in the decay region, the path of integration, that is the left-hand cut $\mathscr{F}_\sigma$, becomes complex and a real part of it covers the physical region $4 \leqslant s \leqslant (m-1)^2$ as is shown in Fig. 4*c*, and the usual iteration method becomes doubtful.

To avoid this difficulty we will study the analytical properties of $a(s, \sigma)/q$, continued through the frontier $\mathscr{F}_\sigma$, and find another integral equation with a regular kernel when s is in the physical region.

As we have seen at the end of Section **1**, the physical region must be reached from *below* the real axis for the function a/q, which has no right-hand cut (see remark 3). We shall make all continuations from this region.

When s crosses the $\mathscr{F}_\sigma$ frontier, the path of integration in eq. (6*f*) enters the second sheet of Φ where it can meet singularities of $\overline{\Phi}$, then one finds new singularities for a, that is for $\overline{\Phi}$ and so on. A first example was the point $s = (m-1)^2$.

To see what happens one needs to know in what regions $s_2^\pm$ goes when s moves. The interesting regions are separated by the image of $\mathscr{F}_\sigma$ through the $s_2^\pm(s_1, \sigma) = s_2$ mapping, that is, finally, the image of the whole real axis, since $\mathscr{F}_\sigma$ is already the mapping of a part of it (the s-cut).

Figure 8 shows this correspondence: the s-plane is divided into six regions I, II, III and their symmetrical I*, II*, III*. When s is in I, $s_2^+(s, \sigma)$ is in I$^+$, $s_2^-(s, \sigma)$ in I$^-$, and so on.

When s follows the dashed curve 1-2-3, the path takes positions 1-2-3 shown in Fig. 8.

In I and II the path is entirely in the first sheet of Φ. There are no singular points on the frontier between II and III since when s is on this curve s_2^- is on the border of the second sheet but comes above the real axis, while the only singularity for $\Phi(s', \sigma)$ on the real axis is $s' = (m-1)^2$ if s' reaches it from *below* (see Fig. 6, at the limit of real σ, the singularity $s = (m-1)^2$ appears on the border of the physical sheet *below* the real axis).

Thus s can freely enter region III. In that region, s_2^- comes into the second sheet in region III$^-$, while s_2^+ remains on the first sheet. Now III$^-$ is identical to I and inside D, where $\overline{\Phi}$ is meromorphic. Two cases are then possible:

a) Either the assumed $s = \alpha_0 = m_\rho^2$ pole is not in I, then region III is entirely free of singularity and there remains a real cut for $a(s, \sigma)/q$ from $-\infty$ to $(m-1)^2$, plus the two resonance poles m_ρ^2 and m_ρ^{2*}.

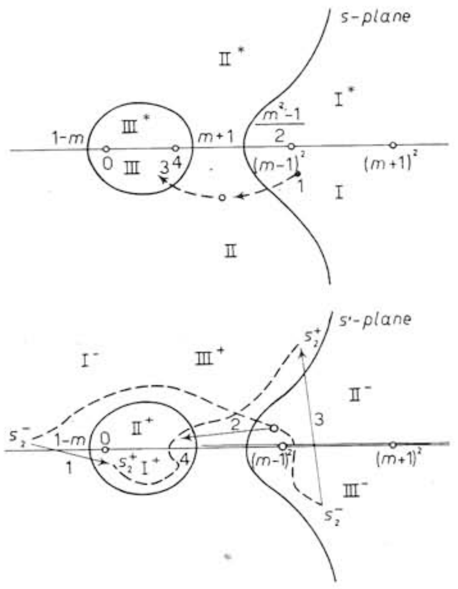

Fig. 8.

b) $s = \alpha_0 = m_\rho^2$ is in I. In that case one finds a logarithmically singular point inside region III, $s = \alpha_1$, given by (end-point singularity) $s_2^-(\alpha_1, \sigma) = \alpha_0$.

When s turns round α_1, $s_2^-(s, \sigma)$ turns round α_0 and the integral term in eq. (6*f*) is increased by $-2i\pi$ times the residue of Φ at the pole $\alpha_0 = m_\rho^2$, that is, $-2i\pi q_\rho^2 R(\sigma)$ from eqs. (8) and (9). Hence one gets the period of a/q around α_1,

$$\frac{1}{2iq}\left(a(s, \sigma) - \bar{a}_{\alpha_1}(s, \sigma)\right) = \pi q_\rho R(\sigma) \frac{\overline{M}(s)}{2pq} \,. \tag{19}$$

Similarly if s reaches the point α_1^* in III* passing at the right of $s = (m-1)^2$,

$$\frac{1}{2iq}\left(a(s, \sigma) - \bar{a}_{\alpha_1^*}(s, \sigma)\right) = \pi q_\rho R^*(\sigma) \frac{\overline{M}(s)}{2pq} \,. \tag{19*}$$

Drawing complex cuts from α_1 and α_1^* to infinity, using Eqs. (19) and (19*), and remembering (⁹) that $\varphi_2(s, \sigma)$ in eq. (17b) gives the discontinuity of a/q through the cut from $-\infty$ to $(m-1)^2$, one can write a dispersion relation for $a(s, \sigma)$ (see Fig. 9).

But a part of the cut still covers the physical decay region, from 4 to $(m-1)^2$. Therefore we shall try to continue a/q through the real axis, from below, at the left of $(m-1)^2$, in order to push the cut starting at $(m-1)^2$ towards the positive real axis.

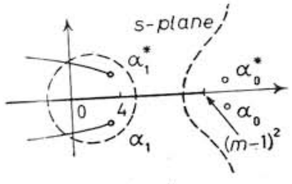

Fig. 9.

Let us assume we are from now on in case b).

First we see that the only singular points on the real axis when s is coming from below are $s=(m-1)^2$ and $s=0$ (in that case the integration contour is infinite), so that one can continue a/q freely between 0 and $(m-1)^2$.

When s goes from region III to III*, s_2^- comes back to the first sheet, s_1^- goes to the second sheet in III⁺, that is, region II which is free of any singularity. Hence a/q is holomorphic in III*.

When s goes into region II*, the path of integration comes entirely into the second sheet (s_2^+ into II⁺*, that is III; s_2^- into II⁻*, that is I). Thus when s reaches the point α_2 defined by

$$s_2^+(\alpha_2, \sigma) = \alpha_1 ,$$

$s_2^+(s, \sigma)$ reaches α_1, while $s_2^-(s, \sigma)$ reaches α_0, and $s=\alpha_2$ is again a logarithmic point.

We apply our general method (Section **1**) of iterating singularities and we should obtain the infinite set of singularities $s=\alpha_n$ given by

$$s_2^\pm(\alpha_{n+1}, \sigma) = \alpha_n , \tag{20}$$

starting from the pole $\alpha_0 = m_\rho^2$.

In the case of equal masses that we discuss here all these points are apparently superposed, since $\alpha_{3p}=\alpha_0$, $\alpha_{3p+1}=\alpha_1$, $\alpha_{3p+2}=\alpha_2$. But in fact they are in different Riemann sheets, and therefore cannot destroy each other.

As a matter of fact, if one draws a cut from $(m-1)^2$ to $-\infty$, one meets only two of them, α_0 and α_1; if one pushes this cut to the right of the real axis, one meets α_2; if s, coming from the physical region below the real axis, turns round the point $(m-1)^2$ clockwise, one meets α_3 since for $s=\alpha_0=\alpha_3$, $s_2^-=\alpha_1$, $s_2^+=\alpha_2$, and the integration path is now in the second sheet where these points are singular; we notice that $s=m+1$ and $s=4$ are now singular. One finds α_4 in the same way. To find α_5, s must reach α_2 after crossing the real axis between $s=4$ and $s=m+1$, otherwise the path of integration would reappear on the first sheet and one would find no singularity in $\alpha_5=\alpha_2$.

We see how the square-root branch points $(m-1)^2$ and $m+1$ generate an infinite number of Riemann sheets. On each of these sheets a finite number of α_n is found.

The important point is that, on the Riemann surface, the distance to the physical region of the successive α_n is incraasing with n, so that it is reasonable to « disguise » their influence by an integral on a cut from $(m-1)^2$ to $+\infty$, taking into account the nearest points only (¹⁰), that is α_0, α_0^*, α_1, α_2.

As for α_1, one can easily calculate the period of a/q around α_2. There are now two contributions due to coincidences of the two extremities of the integration contour with the two singular points α_0 and α_1:

$$\frac{1}{2iq}\left(a(s,\sigma)-\bar{a}_2(s,\sigma)\right) = -q_\rho R(\sigma)\frac{\overline{M}(s)}{2pq} + 2iq_\rho R(\sigma)\frac{\overline{M}(s)}{2pq}\oint_{s_2^-}^{\alpha_1}\frac{\overline{M}(s')}{2p'}\,\mathrm{d}s' . \tag{21}$$

Now we are in a position to write a dispersion relation for a/q, taking into account the two poles α_0, α_0^*; the two logarithmic branch points α_1, α_2, and the two remaining cuts, $\{-\infty, 0\}$ and $\{(m-1)^2, +\infty\}$ (see Fig. 10).

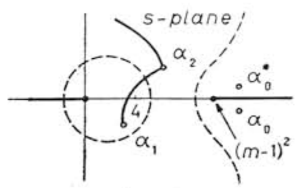

Fig. 10.

To get an integral equation we use eq. (8), which gives the pole terms; eqs. (19), (21), which give the weight function on α_1 and α_2-cuts; and finally we must know the jump of a/q across the two other cuts. For the $(m-1)^2$-cut this jump is just $\varphi_2(s, \sigma)$, given by eq. (17b) but continued in a different region. In particular, when s goes from $(m-1)^2$ to $+\infty$ above the real axis, that is in region I*, the path of integration is entirely inside the second sheet; s_2^+, s_2^- are in I⁺*, I⁻*, that is, in III* and II, respectively, in which we know Φ is holomorphic; but the integration contour crosses region III and has to be deformed to avoid singularities $s=0$ and $s=\alpha_1$ (see Fig. 11).

The jump through the $\{-\infty, 0\}$ cut is calculated in a similar way.

Finally one gets the integral equation

$$\begin{aligned} a(s,\sigma) = a_{\mathrm{BW}}(s,\sigma) + qq_\rho R(\sigma)\frac{1}{\pi}\int_{\alpha_1}^{\alpha_2}\frac{\overline{M}(s')}{2p'q'}\frac{\mathrm{d}s'}{s'-s} - \\ - qq_\rho R(\sigma)2i\int_{\alpha_2}^{\infty}\frac{\overline{M}(s')}{2p'q'}\frac{\mathrm{d}s'}{s'-s}\int_{s_2^-(s',\sigma)}^{\alpha_1}\frac{\overline{M}(s'')}{2p''}\,\mathrm{d}s'' + \frac{q}{\pi}\int_{(m-1)^2}^{\infty}\frac{\varphi_2(s',\sigma)}{s'-s}\,\mathrm{d}s' + \frac{q}{\pi}\int_{-\infty}^{0}\frac{\varphi_3(s',\sigma)}{s'-s}\,\mathrm{d}s' , \end{aligned} \tag{22}$$

(¹⁰) Notice that α_1^* has disappeared. One could find it and all the set α_n^* by turning round $(m-1)^2$ and $m+1$ counterclockwise. But the distance of α_n^* from the physical region is greater than the distance of α_n for the same n.

where $a_{BW}(s,\sigma)$ is the pole terms in eq. (8); $\varphi_2(s',\sigma)$ is given by eq. (17.b); and

$$\varphi_3(s,\sigma)=\frac{1}{2i}\big(a(s,\sigma)-a_0(s,\sigma)\big)=q\,\frac{M(s)}{2pq}\oint\limits_4^{s_2^+(s,\sigma)} a(s',\sigma)\,ds'\,, \tag{23}$$

where again the contour has to be deformed to avoid α_1 in the second sheet.

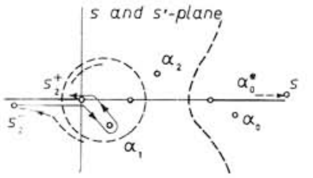

Fig. 11.

In conclusion, we have derived an integral eq. (22) for $a(s,\sigma)$ which has a regular kernel on the physical region. Then one can iterate the inhomogeneous term which already takes into account the nearest singularities α_0, $\overset{*}{\alpha_0}$, α_1, α_2. Furthermore, the solution depends linearly on an arbitrary parameter $R(\sigma)$, which is a function of the total energy σ, directly related to the M-ρ-π and ρ-π-π coupling constants by eq. (12). Of course if the two-body interactions take place in a P-wave state, as is the case in a more realistic model, $\omega\to 3\pi$ for example, one would have another subtraction constant.

We are now going to give an interpretation of the inhomogeneous term in eq. (22).

5. – Comparison with Landau singularities.

If everything but a_{BW} is neglected in eq. (22) one gets the usual Watson's approximation, that is, the sum of three Breit-Wigner formulae for $F(s_1, s_2, s_3)$ of eq. (6a). This corresponds to reaction $M\to\rho+\pi\to 3\pi$ described by graph (0) of Fig. 12.

The $s=0$ singularity is the usual « pseudo-normal » threshold given by Landau's rules. The point $s=(m-1)^2$ is the non-Landauian singularity for triangular diagrams discovered by CUTKOSKY [11].

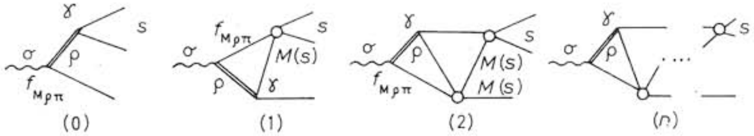

Fig. 12.

[11] R. E. CUTKOSKY: *Journ. Math. Phys.*, 1, 429 (1960).

5898

As can be seen by a dual diagram method, for example, the set (20) of logarithmic points α_n, with $\alpha_0=m^2$, is the set of leading Landau singularities of Feynman amplitudes $\Phi_r^{(n)}$ corresponding to the graphs of Fig. 12, and which are n-th order rescattering of the resonance process (0).

The absorptive parts of the $\Phi_r^{(n)}(s,\sigma)$ are given by Cutkosky's rules, that is, unitarity for graphs in the s-channel:

$$\left\{\begin{aligned} a_r^{(1)}&=q_\rho R\,\frac{M(s)}{2p}\int\limits_{s_2^-(s,\sigma)}^{s_2^+(s,\sigma)}\frac{ds'}{s'-m_\rho^2}\,,\\ a_r^{(n)}&=q_\rho R\,\frac{M(s)}{2p}\int\limits_{s_2^-(s,\sigma)}^{s_2^+(s,\sigma)} ds'\,\Phi_r^{(n-1)}(s',\sigma)\,,\\ \Phi_r^{(n)}&=\frac{1}{\pi}\int\limits_4^{\infty}\frac{a_r^{(n)}(s',\sigma)}{s'-s}\,ds'\,. \end{aligned}\right. \tag{24}$$

This is exactly what we should get by iterating the pole term a_{BW} of eq. (8) in the system (6b, f).

Hence one sees that the first two terms $a_r^{(1)}$ and $a_r^{(2)}$ have the logarithmic branch points α_1, α_2 with periods given by eq. (19) and the first term of eq. (21) for $a_r^{(1)}$, the second term of eq. (21) for $a_r^{(2)}$.

Therefore the function

$$\big(a(s,\sigma)-a_r^{(1)}(s,\sigma)-a_r^{(2)}(s,\sigma)\big)/q$$

is now regular in α_1, α_2.

Nevertheless, we have reintroduced new singularities by this procedure, due to the factor $\overline{M}(s)$. For example, the new function has the normal threshold $s=4$ now. Hence, eq. (23) can be replaced by

$$\begin{aligned} a(s,\sigma)&=a_{BW}(s,\sigma)+a_r^{(1)}(s,\sigma)+\\ &+a_r^{(2)}(s,\sigma)+\frac{q}{\pi}\int\limits_4^{+\infty}\frac{a_M(s')}{s'-s}\,\frac{q_\rho R}{2p'}\,ds'\int\limits_{s_2(s',\sigma)}^{s_2^+(s',\sigma)}\frac{ds''}{s''-m^2}+\frac{q}{\pi}\int\limits_4^{\infty}\frac{\varphi_{r_2}^{(2)}(s',\sigma)}{s'-s}\,ds'+\\ &+\frac{q}{\pi}\int\limits_{-\infty}^{0}ds'\,\frac{\varphi_{r_2}^{(1)}(s',\sigma)+\varphi_{r_2}^{(2)}(s',\sigma)}{s'-s}+\frac{q}{\pi}\int\limits_{(m-1)^2}^{+\infty}\frac{\varphi_2(s',\sigma)}{s'-s}\,ds'+\frac{q}{\pi}\int\limits_{-\infty}^{0}\frac{\varphi_3(s',\sigma)}{s'-s}\,ds'\,, \end{aligned} \tag{25}$$

where $a_M(s')$ is the absorptive part of M; $\varphi_{r_2}^{(2)}(s',\sigma)$, $\varphi_{r_2}^{(1,2)}(s',\sigma)$ being the dis-

5899

continuities of $a_r^{(1,2)}$ across the $\{-\infty, 0\}$ and $\{4, +\infty\}$ cuts that can easily be calculated from eqs. (24).

Thus it is clear that the inhomogeneous term in eq. (22) or eq. (25) cannot be reduced to the Feynman rescattering terms only; one has to add other contributions, three integral terms in (25), which cancel the supplementary singularities coming from the Feynman amplitudes. Our analysis just proved that thanks to these additional terms the iteration procedure can converge, since now the kernel is regular inside the physical region, in contradistinction to usual integral eqs. (6b, f).

6. – Concluding remarks.

In the present paper we have emphasized the importance of the logarithmic singularities due to the first two rescatterings when the two-body interaction takes place through a resonance.

If, for instance, the resonance pole is in region I of Fig. 8 with $(m^2-1)/2 \leqslant \overline{m}_\rho^2 \leqslant (m-1)^2$ one has $4 \leqslant \operatorname{Re}\alpha_1 \leqslant m+1$, $\operatorname{Im}\alpha_1 \simeq -\gamma < 0$ and the distance of the first logarithmic point α_1 from the physical region is very small and of the order of the width γ. The point α_2 is further away since $\operatorname{Im}\alpha_2 > 0$: to reach it one has to turn round the point $s=4$, crossing the normal cut. Figure 13 shows different paths to reach α_0, α_1, α_2, $(m-1)^2$, $\alpha_3=\alpha_0$ from a physical point s.

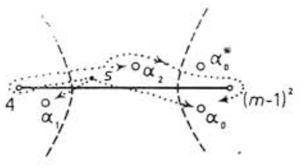

Fig. 13.

Thus for practical purposes one could start with the inhomogeneous term of eq. (22) only. The following approximation should be done to evaluate the $\{(m-1)^2, +\infty\}$ integral by iteration, forgetting about $\{-\infty, 0\}$ which is far away.

When $\overline{m}_\rho^2 > (m-1)^2$, α_1 and α_2 move in the complex plane on the loop $\mathscr{F}_\sigma$ if $\gamma \to 0$. When $\overline{m}_\rho^2 < (m^2-1)/2$, $\operatorname{Im}\alpha_1 > 0$, the distance to α_1 increases. When $\overline{m}_\rho^2 < m+1$, α_2 disappears through the $\{(m-1)^2, +\infty\}$ cut.

Let us add the final remark that the previous discussion applies when $4 < \sigma < 12$, that is, in the ordinary case of the N/D method. Then, of course, it is not useful to push the left-hand cut to the right side. But one can suppress the loop which appears in Fig. 4b and discover the logarithmic points α_1 and α_1^*. The present analysis is then a justification of the isobar approach, in which only Born terms, resonance poles and their crossed terms are taken into account. These crossed terms give the logarithmic points α_1, α_1^* in the partial waves. Nevertheless one should correct this approximation by terms which

would cancel the supplementary singularities introduced by the crossed diagrams.

* * *

I wish to thank C. BOUCHIAT, G. FLAMAND, D. BESSIS, M. FROISSART and F. PHAM for interesting discussions and critics.

RIASSUNTO (*)

Con metodi di continuazione analitica, si studia un modello per la disintegrazione di una particella in tre particelle, non tenendo conto che delle loro interazioni elastiche a due a due, in una sola onda parziale supposta predominante. Si stabilisce così un'equazione integrale a nocciolo regolare nella regione fisica. Si mostra l'importanza delle due prime ridiffusioni, quando si manifesta una risonanza nelle interazioni a due corpi. Le ridiffusioni di ordine più elevato si trovano su foglietti di Riemann sempre più lontani.

(*) *Traduzione a cura della Redazione.*