Canonical Gauge- and Lorentz-Invariant Quantization of the Yang-Mills Field*

HENDRICUS G. LOOS
Douglas Advanced Research Laboratories, Huntington Beach, California 92647
and
Department of Physics, University of California at Riverside, California 92507

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The Yang-Mills field, interacting with itself but not with other fields, is quantized using canonical equal-time commutation relations that do not involve the time component of the Yang-Mills potentials or the conjugate operator. Instead of decomposing the field into physical components and gauge variables, all three spatial components of the Yang-Mills potentials are retained. This keeps the equal-time commutators simple and provides, in the Schrödinger representation, a simple solution of the problem of constraints: "Good states, i.e., states satisfying all constraints, are represented by gauge-invariant functionals of the spatial Yang-Mills potentials. A finite scalar product for good states is given as a functional integral over a "tube" in configuration space, i.e., the space of the spatial Yang-Mills potentials over all of three-dimensional space. This tube is constructed in a certain manner around a manifold Ξ of representatives for the gauge-invariant manifolds. It is shown that this scalar product does not depend on the choice of Ξ. For the purpose of proving Lorentz and gauge invariance of the theory, a modified Hamiltonian is considered which does not give rise to any constraints by itself, and which results in field equations which reduce to the Yang-Mills equations when applied to good states. The conventional primary and secondary constraints show up as conditions selecting the subspace of good states. Gauge invariance is proven for the complete theory, applied to good states. Lorentz invariance is proven for the equal-time commutation relations, using the method of Heisenberg and Pauli, for the conditions selecting the subspace of good states, and for the equations of motion applied to good states. Lorentz invariance of the scalar product of good states is shown by proving self-adjointness of the energy-momentum density operators, relative to the scalar product. The self-adjointness of these operators is assured by the particular construction of the tube in configuration space, over which the functional integral is taken. One may take the limit of the scalar product, letting the tube thickness go to zero, and arrive at a functional integral over Ξ.

I. INTRODUCTION

THE *raison d'être* for the Yang-Mills field is to bring about local isospin invariance of a local field theory of hadrons. Hence, the quantization procedure for this field must have this invariance, here called gauge invariance. Since it is hoped that the Yang-Mills field will provide a theory of strong interactions,[1] one should not rely on perturbation theory in proving gauge and Lorentz invariance or in setting up the basic quantization procedure. Among the nonperturbative quantization procedures known to the author, only the procedures of DeWitt[2] and Mandelstam[3] leave no question about gauge and Lorentz invariance. However, in DeWitt's procedure,[2] the equal-time commutators involve a Green's function which is not known explicitly in closed form, and the same Green's function occurs in the equation of motion in Mandelstam's method.[3] This complicates nonperturbative applications of the theory. Since we intend to use the Yang-Mills field in nonperturbative fashion, we look for a quantization for which the equal-time commutation relations and the equations of motion are simple, and for which Lorentz and gauge invariance is assured. Such a quantization is shown in the present paper.

The arrangement is as follows. In Sec. II we point out in detail why there is a question about gauge or Lorentz invariance in the nonperturbative Yang-Mills field quantizations known to us, with the exception of DeWitt's[2] and Mandelstam's[3] procedures. In Sec. III we show that the Yang-Mills field equations result, if a certain Hamiltonian together with canonical equal-time commutation relations are applied to a subspace of "good" states; the Hamiltonian is such that it does not give rise to constraints. The conditions selecting the subspace of good states are easily satisfied in the Schrödinger representation, where states are represented as functionals of the Yang-Mills potentials: Good states are represented by gauge-invariant functionals of the spatial Yang-Mills potentials. In Sec. IV a finite norm for good states is obtained by defining the scalar product of good states as a functional integral over a certain "tube" in the configuration space, i.e., the space of spatial Yang-Mills potentials over all of three-dimensional space. In Secs. V and VI we give a proof of gauge and Lorentz invariance of the equal-time commutation relations, the equations of motion applied to good states, and the scalar product of good states. In Sec. VII the resulting rules are summarized.

x^κ, $\kappa=0, 1, 2, 3$, are Cartesian inertial coordinates of events in the Minkowski space of special relativity. κ, λ, μ, and ν range from 0 to 3; α, β, γ, and δ range from 1 to 3. As an argument of a function, x stands for x^κ and $\mathbf{x}$ stands for x^α. The metric tensor $g_{\kappa\lambda}$ in event space is taken with signature $+---$. ∂_κ denotes ∂/∂^κ, and the summation convention is used. The operators

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[1] To our knowledge, no convincing argument has been given for or against a nonvanishing physical mass of the Yang-Mills field.
[2] B. S. DeWitt, Phys. Rev. **162**, 1195 (1967).
[3] S. Mandelstam, Phys. Rev. **175**, 1580 (1968).

$b_\kappa{}^i(x)$, $i=1, 2, 3$, are the Yang-Mills potentials; the group indices i, j, k, l, and m range from 1 to 3, and the structure constants for $SU(2)$ are denoted by $c_{ij}{}^k$. If desired, everywhere but in (4.10) and its consequences, the isospin group $SU(2)$ may be replaced by any simple and compact group, by appropriate adjustment of the range of group indices and structure constants; however, the choice $SU(2)$ is most strongly backed by experiment. The space-space components of the Yang-Mills field are defined as

$$B_{\alpha\beta}{}^i=\partial_\alpha b_\beta{}^i-\partial_\beta b_\alpha{}^i-c_{jk}{}^i b_\alpha{}^j b_\beta{}^k. \qquad (1.1)$$

We freely use the covariant derivative

$$\nabla_\kappa v^i=\partial_\kappa v^i-c_{jk}{}^i b_\kappa{}^j v^k, \qquad (1.2)$$

where v^i is any vector in the Lie-algebra space of $SU(2)$. Equation (1.2) holds for v^i numerical or operator-valued; for the latter case, the operator ordering in (1.2) is to be noted. We will use the group metric

$$g_{ij}=c_{ik}{}^l c_{jl}{}^k \qquad (1.3)$$

which is covariant constant,

$$\nabla_\kappa g_{ij}=0, \qquad (1.4)$$

because of the constancy and antisymmetry of $c_{ij}{}^k$.

II. COMMENTS ON EXISTING QUANTIZATIONS

We recall what is meant by local gauge invariance. It comes about, in the simplest setting of a local nucleon field theory of strong interactions, as a lack of physically distinguished orientation of the neutron and proton basis vectors in isospace, separately at all events x^κ (space- time points). All we can do is choose these basis vectors orthonormal at every event x^κ. Since this leaves the orientation of the basis vectors undetermined, we demand invariance of the theory under smooth but otherwise arbitrary event-dependent rotations of the orthonormal basis vectors.[4] Such a change in basis vectors is called "a local gauge transformation," or, in this paper, "a gauge transformation." It is expressed by three real numerical functions $\eta^i(x)$ which determine, at every event x^κ, the $SU(2)$ group element

$$S(x)=e^{\eta^i(x)L_i}, \qquad (2.1)$$

which affects the rotation of the isospace basis vectors; the L_i are constant matrices representing a basis for the Lie algebra of $SU(2)$. Under gauge transformations with infinitesimal $\eta^i(x)$, the Yang-Mills potentials $b_\kappa{}^i(x)$ suffer an infinitesimal change

$$\delta b_\kappa{}^i=-\nabla_\kappa\eta^i. \qquad (2.2)$$

Since the $\eta^i(x)$ are numerical, we call (2.2) a *numerical* gauge transformation if we want to emphasize the distinction with transformations (2.2) in which the $\eta^i(x)$ are operator-valued. Such transformations have been mentioned in the literature in an attempt to save local gauge invariance,[5] or to relate gauges subject to different subsidiary conditions.[6] Whatever the meaning of such operator gauge transformations, invariance of the theory under these transformations brings no relief from the requirement of invariance under numerical local gauge transformations (2.2).

In addition to the procedures of DeWitt[2] and Mandelstam,[3] nonperturbative quantizations of the Yang-Mills field known to us are the procedures of Yang and Mills,[7] Schwinger,[8–10] Arnowitt and Fickler,[6] Goto and Utiyama,[5] and Goto.[11] We will point out a lack of proof of gauge or Lorentz invariance in each of these procedures.

The subsidiary condition used by Yang and Mills[7] is in our notation

$$\partial_\kappa b^{\kappa i}|\psi\rangle=0. \qquad (2.3)$$

Under an infinitesimal gauge transformation (2.2), the operator $\partial_\kappa b^{\kappa i}$ changes by

$$\partial_\kappa\delta b^{\kappa i}=-\partial_\kappa\nabla^\kappa\eta^i=-\partial_\kappa\partial^\kappa\eta^i+c_{jk}{}^i(\partial_\kappa b^{\kappa j})\eta^k + c_{jk}{}^i b^{\kappa j}\partial_\kappa\eta^k. \qquad (2.4)$$

When applied to a state satisfying (2.3), the second term on the right-hand side does not contribute anything, leaving

$$(-\partial_\kappa\partial^\kappa\eta^i+c_{jk}{}^i b^{\kappa j}\partial_\kappa\eta^k)|\psi\rangle, \qquad (2.5)$$

which does not generally vanish. Hence, the condition (2.3) on states is not gauge-invariant.

In the first quantization procedure of Goto and Utiyama,[5] arbitrary numerical functions λ^i are used in the Hamiltonian. The result is an equation

$$\partial_0 b_0{}^i=\lambda^i, \qquad (2.6)$$

which shows $\partial_0 b_0{}^i$ to be numerical. However, $b_0{}^i$ can be related to the operator $b_\alpha{}^i$ by a Lorentz transformation, and there is a need to show that no inconsistency arises in the Heisenberg equations applied to good states, under Lorentz transformations.

Goto and Utiyama's second procedure[5] employs the subsidiary condition (2.3) for good states which, as shown above, is not gauge-invariant; introduction of an operator gauge transformation[5] does not relieve the requirement of invariance under numerical gauge transformations.

Goto's work[11] on the separation of redundant variables involves Eq. (2.6), which, as discussed above,

[4] There is a further condition that the rotation must vanish at spatial infinity, but this does not affect the considerations of this section.

[5] T. Goto and R. Utiyama, Progr. Theoret. Phys. (Kyoto) Suppls. **37** and **38**, 322 (1966).
[6] R. L. Arnowitt and S. I. Fickler, Phys. Rev. **127**, 1821 (1962).
[7] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).
[8] J. Schwinger, Phys. Rev. **125**, 1043 (1962).
[9] J. Schwinger, Phys. Rev. **130**, 402 (1963).
[10] J. Schwinger, Nuovo Cimento **30**, 278 (1963).
[11] T. Goto, Progr. Theoret. Phys. (Kyoto) **36**, 1283 (1966).

raises a question of Lorentz invariance. Goto[11] offers an argument intending to show Lorentz invariance, utilizing the Lorentz invariance of the second Goto-Utiyama quantization.[5] However, for that procedure, gauge invariance is in question, as indicated above. Hence, invariance under Lorentz and gauge transformations has not been shown satisfactorily.

Schwinger[8] and Arnowitt and Fickler[6] use a gauge for which

$$\partial_\alpha b^{\alpha i}=0. \tag{2.7}$$

Suppose there exists such a gauge; we take it as an initial gauge, perform an infinitesimal gauge transformation, and show an inconsistency by calculating $\delta\partial_\alpha b^{\alpha i}$ in two different ways. One way is to simply substitute (2.2):

$$\delta\partial_\alpha b^{\alpha i}=-\partial_\alpha\nabla^\alpha\eta^i=-\partial_\alpha\partial^\alpha\eta^i+c_{jk}{}^ib^{\alpha j}\partial_\alpha\eta^k, \tag{2.8}$$

where use has been made of (2.7). The other way of calculating $\delta\partial_\alpha b^{\alpha i}$ is by similarity transformation of the operator by $I+B$, which represents the gauge transformation in state space:

$$\delta\partial_\alpha b^{\alpha i}=-[B,\partial_\alpha b^{\alpha i}]=0, \tag{2.9}$$

where (2.7) has been used. Consistency of (2.8) and (2.9) would demand that

$$-\partial_\alpha\partial^\alpha\eta^i+c_{jk}{}^ib^{\alpha j}\partial_\alpha\eta^k=0. \tag{2.10}$$

Since the $\eta^i(x)$ are numerical, and the $b_\alpha{}^j$ are operators, the only solutions of (2.10) are $\eta^i(x)$ with

$$\partial_\alpha\eta^i(x)=0, \tag{2.11}$$

in contradiction with the stipulation that the $\eta^i(x)$ in (2.2) can be taken as any set of smooth functions. The same argument, but with α replaced by κ, holds for Schwinger's Lorentz gauge.[9]

A similar contradiction arises if it is assumed that there exists a gauge with[6]

$$b_3{}^i=0. \tag{2.12}$$

Starting with this gauge, the two ways of calculating $\delta b_3{}^i$ due to an infinitesimal gauge transformation give $-\partial_3\eta^i$ and 0.

Schwinger's quantization using group parameters[10] employs a decomposition of the Yang-Mills potentials $b_\alpha{}^i(x)$, and the conjugate operators $\Pi_\alpha{}^i(x)$, into transverse and longitudinal parts:

$$b_\alpha{}^i=b_\alpha{}^{iT}+\partial_\alpha\lambda^i, \tag{2.13}$$

$$\Pi_\alpha{}^i=\Pi_\alpha{}^{iT}-\partial_\alpha\psi, \tag{2.14}$$

$$\partial_\alpha b^{\alpha iT}=0, \tag{2.15}$$

$$\partial_\alpha\Pi^{\alpha iT}=0. \tag{2.16}$$

The decompositions (2.13) and (2.14) are not gauge-invariant, since the transverse and longitudinal parts mix. Hence, the procedure is not manifestly gauge-invariant, and a separate proof of gauge invariance is required. To point out that a gauge transformation is a similarity transformation[12] of the operators, and thus leaves all commutation relations invariant, is not sufficient; as we have seen above, inconsistencies may show up if the result of the similarity transformation is compared with the result of making the change (2.2). Until this comparison has been satisfactorily worked out for Schwinger's procedure,[10] there remains a question of gauge invariance.

III. HAMILTONIANS, COMMUTATORS, AND CONSTRAINTS

The time component of the conventional Yang-Mills field equations is inconsistent with the canonical equal-time commutation relations. We deal here with this problem by modifying the Hamiltonian such that no constraints arise from it, and such that the new field equations reduce to the Yang-Mills equations, when applied to a subspace of "good" states. An adequate modified Hamiltonian is

$$H'=\int d^3x\big(\tfrac{1}{2}\Pi_\alpha{}^i\Pi^\alpha{}_i-\tfrac{1}{4}B_{\alpha\beta}{}^iB^{\alpha\beta}{}_i-b_{0i}\nabla_\alpha\Pi^{\alpha i}-(\partial_\alpha b^{\alpha i})\Pi_{0i}-u^i\Pi_{0i}+\tfrac{1}{2}\Pi_0{}^i\Pi_{0i}\big), \tag{3.1}$$

where the operators $\Pi^\kappa{}_i(x)$ are canonically conjugate to the operators $b_\kappa{}^i(x)$, and

$$u^i(x)=f^i(b_\kappa{}^j(x))+c^i(x), \tag{3.2}$$

where the f^i are arbitrary functions of the $b_\kappa{}^i(x)$ not involving their derivatives, and the $c^i(x)$ are arbitrary real numerical functions of x^κ. The equal-time commutation relations are canonical:

$$[\Pi^{\kappa i}(\mathbf{x}),b_{\lambda j}(\mathbf{y})]=-i\delta_\lambda{}^\kappa\delta_j{}^i\delta^3(\mathbf{x}-\mathbf{y}), \tag{3.3}$$

$$[b_\kappa{}^i(\mathbf{x}),b_\lambda{}^j(\mathbf{y})]=0, \tag{3.4}$$

$$[\Pi_\kappa{}^i(\mathbf{x}),\Pi_\lambda{}^j(\mathbf{y})]=0. \tag{3.5}$$

From the Hamiltonian (3.1) and the commutation relations we calculate, in the Heisenberg picture, the time derivative of the operators $b_\alpha{}^i(x)$, $b_0{}^i(x)$, $\Pi_\alpha{}^i(x)$, and $\Pi_0{}^i(x)$; the results may be written

$$\Pi_\alpha{}^i=B_{0\alpha}{}^i, \tag{3.6}$$

$$\Pi_0{}^i=\partial_\kappa b^{\kappa i}+u^i, \tag{3.7}$$

$$\nabla_\kappa B^\kappa{}_\lambda{}^i=-\partial_\lambda\Pi_0{}^i+(\partial f^j/\partial b^\lambda{}_i)\Pi_{0j}, \tag{3.8}$$

where

$$B_{0\alpha}{}^i=\partial_0b_\alpha{}^i-\partial_\alpha b_0{}^i-c_{jk}{}^ib_0{}^jb_\alpha{}^k. \tag{3.9}$$

Equations (3.6) and (3.7) relate the generalized momentum densities $\Pi_\kappa{}^i(x)$ to the generalized velocities $\partial_0b_\kappa{}^i(x)$. Equation (3.8) is a dynamical statement expressing $\partial_0\Pi_\lambda{}^i$; there is no inconsistency with the equal-

[12] To mention unitarity of the gauge transformation in the context of Schwinger's paper (Ref. 10) would require further elaboration, because his scalar product involves only the transverse variables, while the generators $\nabla_\alpha\Pi^{\alpha i}(\mathbf{x})$ of gauge transformations have a longitudinal as well as a transverse part.

time commutation relations. The field equations (3.8) applied to "good" states, i.e., states $|\psi\rangle$ satisfying

$$\Pi^{0i}(x)|\psi\rangle=0\,, \tag{3.10}$$

$$\nabla_\alpha\Pi^{\alpha i}(x)|\psi\rangle=0\,, \tag{3.11}$$

reduce to the conventional Yang-Mills equations, applied to $|\psi\rangle$:

$$\nabla_\kappa B^\kappa{}_\lambda{}^i|\psi\rangle=0\,. \tag{3.12}$$

If (3.10) and (3.11) are satisfied at one time t_0, they are satisfied at all times. This can be seen as follows. Let (3.10) and (3.11) be satisfied at time t_0. Then, (3.6) and (3.8) give, at t_0,

$$\nabla_\kappa B^\kappa{}_\alpha{}^i|\psi\rangle=0\,, \tag{3.13}$$

$$\partial_0\Pi^{0i}|\psi\rangle=0\,. \tag{3.14}$$

Taking the time derivative of (3.8), and using (3.14) gives, at t_0,

$$\partial_0\nabla_\kappa B^\kappa{}_\alpha{}^i|\psi\rangle=0\,. \tag{3.15}$$

Equation (3.13) may be written, at t_0,

$$\nabla_0 B^{0\alpha i}|\psi\rangle+\nabla_\beta B^{\beta\alpha i}|\psi\rangle=0\,. \tag{3.16}$$

The covariant divergence of (3.16) is, at t_0,

$$\nabla_\alpha\nabla_0 B^{0\alpha i}|\psi\rangle+\nabla_\alpha\nabla_\beta B^{\beta\alpha i}|\psi\rangle=0\,. \tag{3.17}$$

We use the Ricci identity[13]

$$2\nabla_{[\kappa}\nabla_{\lambda]}v^i=-c_{jk}{}^i B_{\kappa\lambda}{}^j v^k\,, \tag{3.18}$$

which holds for any v^i which changes under an infinitesimal gauge transformation (2.2) by

$$\delta v^i=-c_{jk}{}^i\eta^j v^k\,. \tag{3.19}$$

The identity (3.18) is well known[14] for numerical $b_\kappa{}^i$ and v^i, and it remains valid for operator-valued $b_\kappa{}^i$ and v^i, because of the commutivity of the $b_\kappa{}^i$ at equal times. For operator-valued $b_\kappa{}^i$ and v^i, the factor ordering in (3.18) should be noted. Using (3.18) and the antisymmetry of the structure constants, (3.17) may be written, at t_0,

$$\nabla_0\nabla_\alpha B^{0\alpha i}|\psi\rangle=0 \tag{3.20}$$

or

$$\partial_0\nabla_\alpha B^{0\alpha i}|\psi\rangle-c_{jk}{}^i b_0{}^j\nabla_\alpha B^{0\alpha k}|\psi\rangle=0\,. \tag{3.21}$$

The last term in (3.21) vanishes because of (3.11). Hence, (3.21) amounts to, at t_0,

$$\partial_0\nabla_\alpha B^{0\alpha i}|\psi\rangle=0\,. \tag{3.22}$$

Equations (3.14) and (3.22) show that (3.10) and (3.11), if satisfied at one time, are satisfied at all times.

The Hamiltonian (3.1) on good states $|\psi\rangle$ gives the same result as $H|\psi\rangle$, where

$$H=\int d^3x(\tfrac{1}{2}\Pi_\alpha{}^i\Pi^\alpha{}_i-\tfrac{1}{4}B_{\alpha\beta}{}^iB^{\alpha\beta}{}_i)\,. \tag{3.23}$$

H does not involve the time component $b_0{}^i$ of the Yang-Mills potentials.

We now pass to the Schrödinger picture, and use the Schrödinger representation in which states are represented by functionals Ψ of the real functions $b_\kappa{}^i(x)$ at a fixed time [the $b_\kappa{}^i(x)$ at a fixed time are written here as $b_\kappa{}^i(\mathbf{x})$], $b_\kappa{}^i(\mathbf{x})$ operating on Ψ amounts to multiplication of Ψ with the *real*[15] function $b_\kappa{}^i(\mathbf{x})$, and the operators $\Pi^\kappa{}_i(\mathbf{x})$ are represented by functional derivatives,

$$\Pi^\kappa{}_i(\mathbf{x})=-i\delta/\delta b_\kappa{}^i(\mathbf{x})\,. \tag{3.24}$$

The Schrödinger representation provides a very simple solution of the constraints (3.10) and (3.11). The primary constraint (3.10) is satisfied by restricting the state functionals Ψ to be functionals of the *spatial* Yang-Mills potentials $b_\alpha{}^i(\mathbf{x})$ only. The secondary constraint (3.11) is satisfied by the additional restriction to *gauge-invariant* state functionals of the $b_\alpha{}^i(\mathbf{x})$. To show this, we perform an infinitesimal gauge transformation (2.2), and calculate the change in a state functional $\Psi[b_\alpha]$:

$$\delta\Psi=-\int d^3x\frac{\delta\Psi}{\delta b_\alpha{}^i}\nabla_\alpha\eta^i$$
$$=-i\int d^2x\,\Pi^n{}_i\Psi\eta^i+i\int d^3x(\nabla_\alpha\Pi^\alpha{}_i)\Psi\eta^i\,, \tag{3.25}$$

where use has been made of (3.24), partial integration, and Gauss's theorem. $\Pi^n{}_i$ is the component of $\Pi^\alpha{}_i$ normal to the surface of integration at spatial infinity. Suppose Ψ is invariant under all gauge transformations (2.2) for which

$$\eta^i(x)\to 0\quad\text{for}\quad|\mathbf{x}|\to\infty\,, \tag{3.26}$$

in such a manner that the surface integral in (3.25) vanishes; then one has from (3.25),

$$0=\int d^3x(\nabla_\alpha\Pi^\alpha{}_i)\Psi\eta^i\,, \tag{3.27}$$

and (3.11) follows. From here on, unless specified differently, a gauge transformation will have $\eta^i(x)$ restricted by the condition (3.26).

In judging the gauge invariance of a state functional, it is easiest not to think of the gauge transformation in state space, but to simply replace $\Psi[b_\alpha]$ by $\Psi[b_\alpha+\delta b_\alpha]$, where δb_α is given by (2.2). Whether or not a state functional is invariant under this substitution is easily seen by inspection. Also, a gauge-invariant state results from operating on a gauge-invariant state with a gauge-

[13] Square brackets around indices denote alternation:

$a_{[\alpha\beta]}=\tfrac{1}{2}(a_{\alpha\beta}-a_{\beta\alpha})$,

$a_{[\alpha\beta\gamma]}=(1/3!)(a_{\alpha\beta\gamma}+a_{\gamma\alpha\beta}+a_{\beta\gamma\alpha}-a_{\beta\alpha\gamma}-a_{\alpha\gamma\beta}-a_{\gamma\beta\alpha})$;

see J. A. Schouten, *Ricci Calculus* (Springer Verlag, Berlin, 1954), p. 14.

[14] H. G. Loos, J. Math. Phys. 8, 2114 (1967).

[15] Reality is demanded here to secure self-adjointness of certain operators on good states; see Sec. VI.

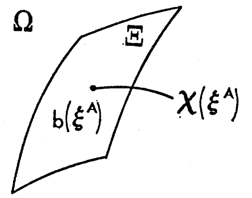

FIG. 1. Gauge-invariant manifolds $\chi(\xi^A)$ and manifold Ξ of representatives.

invariant operator, and gauge invariance of an operator can be judged easily. It should be noted that the gauge invariance of "physical" state functionals already follows from the interpretation of the state functional $\Psi[b_\alpha{}^i(\mathbf{x})]$ as the amplitude for finding, upon measuring a complete commuting set of gauge-invariant operators, results described by the eigenvalues $b_\alpha{}^i(\mathbf{x})$; this amplitude is the same as that for finding the result of the measurement described by $b'_\alpha{}^i(\mathbf{x})$, related to the $b_\alpha{}^i(\mathbf{x})$ by a gauge transformation.

Since good state functionals do not depend on $b_0{}^i$, and since the Hamiltonian (3.23) applied to good states does not involve $b_0{}^i$, we can dispense with $b_0{}^i$ altogether, and work only with the spatial Yang-Mills potentials $b_\alpha{}^i$. The only equal-time commutators needed are the spatial components, $\kappa=\alpha$, $\lambda=\beta$, of (3.3)–(3.5). These commutators originally were meant to apply to the full state space of all functionals $\Psi[b_\kappa]$. Since $b_\alpha{}^i$ and $-i(\delta/\delta b_\alpha{}^i)$ applied to a functional of the spatial Yang-Mills potentials gives again a functional of the spatial Yang-Mills potentials, we may restrict the $\kappa=\alpha$, $\lambda=\beta$ components of the equal-time commutators to act on the space of the $\Psi[b_\alpha]$, not restricted to be gauge-invariant. Anyway, the operators $b_\alpha{}^i$ and $\Pi^\alpha{}_i$ turn a nood state into a bad state, because these operators are got gauge-invariant. We could project the resulting state back on to the good state space, but the resulting commutators would be complicated.

Our method of dealing with the constraints is similar to the method used by Wheeler[16] and DeWitt[17] for the gravitational field,[18] which method was previously given by Peres[19] in the context of the Hamilton-Jacobi theory for the gravitational field. For the Yang-Mills field, it was noted by Schwinger[9] that gauge-invariant states satisfy the secondary constraint, but no use of this fact has been made in the Schrödinger representation, which makes it possible to judge gauge invariance of a state by inspection.

The spatial components of the equal-time commutation relations (3.3)–(3.5) are the same as the commutation relations for the extended operators used by Schwinger.[10] These extended operators, which were constructed using a decomposition of potentials into physical variables and gauge variables, are just the $b_\alpha{}^i$ and $B^{0\alpha}{}_i$ of the present paper. The decompositions used by Schwinger[10] make it difficult to solve the constraint conditions.

It may be of interest to note that the Hamiltonian (3.23) may be written in bilinear form:

$$H=\frac{1}{2}\int d^3x\, a^{(+)\alpha}{}_i a^{(-)}{}_\alpha{}^i=\frac{1}{2}\int d^3x\, a^{(-)}{}_\alpha{}^i a^{(+)\alpha}{}_i, \quad (3.28)$$

where

$$a^{(\pm)\alpha}{}_j=\mp iB^{0\alpha}{}_j+\tfrac{1}{2}\epsilon^{\alpha\beta\gamma}B_{\beta\gamma j}, \quad (3.29)$$

and $\epsilon^{\alpha\beta\gamma}$ is the totally antisymmetric tensor density of unit weight which has $\epsilon^{123}=1$. The functionals

$$\Psi_{(\pm)}=e^{\pm U}, \quad (3.30)$$

where

$$U=\frac{1}{2}\int d^3x\,\epsilon^{\alpha\beta\gamma}(b_\alpha{}^i\partial_\beta b_{\gamma i}-\tfrac{1}{3}c_{ijk}b_\alpha{}^i b_\beta{}^j b_\gamma{}^k), \quad (3.31)$$

satisfy

$$a^{(\pm)}{}_\alpha{}^i\Psi_{(\pm)}=0 \quad (3.32)$$

and are gauge-invariant; they have zero energy, momentum, angular momentum, and electric charge, and have even parity. However, their behavior at certain indefinitely increasing Yang-Mills potentials disqualifies them as physical state functionals. The functional

$$\Psi_0=e^{-|U|} \quad (3.33)$$

falls off appropriately with increasing $b_\alpha{}^i(\mathbf{x})$. If we would make the rule to omit configurations $b_\alpha{}^i(\mathbf{x})$ for which $U=0$ from the domain of definition of all functional differential operators, then Ψ_0 would have all the properties required of the vacuum. However, if one puts the structure constants to zero, Ψ_0 does not reduce to the electromagnetic vacuum state functional given by Katz.[20]

IV. SCALAR PRODUCT OF GOOD STATES

A conspicuous feature of the present method is the redundancy of description of the Yang-Mills field configuration by means of all three spatial components $b_\alpha{}^i(\mathbf{x})$ of the Yang-Mills potentials; two such components would suffice. Since the simplicity of the equal-time commutation relations and the simplicity of the solution of the constraint problem depends on this method of retaining all three components $b_\alpha{}^i(\mathbf{x})$, we do not wish to perform the fashionable decomposition of the potentials into physical variables and gauge variables. However, if we naively define a scalar product of states as a functional integral over all $b_\alpha{}^i(\mathbf{x})$, an infinite norm results.[9,10] For electrodynamics, we get a finite and otherwise satisfactory scalar product by integrating only over the transverse potentials.[10,20] However, for the Yang-Mills field, the situation is com-

[16] J. A. Wheeler, *Relativity, Groups, and Topology, 1963 Les Houches Lectures* (Gordon and Breach, Science Publishers, Inc., New York, 1964).
[17] B. S. DeWitt, Phys. Rev. **160**, 1113 (1967).
[18] This similarity was pointed out to the author by L. E. Thomas.
[19] A. Peres, Nuovo Cimento **26**, 53 (1962).
[20] A. Katz, Nuovo Cimento **37**, 342 (1965).

plicated by the fact that, under gauge transformations, the transverse and longitudinal parts of the potentials mix.

We define a scalar product for good states which is finite and Lorentz- and gauge-invariant. Consider the configuration space Ω for the Yang-Mills field, i.e., the space of all numerical real-valued functions $b_\alpha{}^i(\mathbf{x})$, possibly restricted by certain conditions at spatial infinity. We begin by assigning a metric to this space. We notice that such a metric is already manifest in the Lagrangian L of the Yang-Mills field; the term of L which is quadratic in the generalized velocities is

$$\frac{1}{2}\int d^3x(\partial_0 b_\alpha{}^i)\partial_0 b^\alpha{}_i=\frac{1}{2(dt)^2}\int d^3x(db_\alpha{}^i)db^\alpha{}_i, \quad (4.1)$$

showing a metric proportional to

$$g_{\alpha\beta}g_{ij}\delta^3(\mathbf{x}-\mathbf{y}). \quad (4.2)$$

We simply take the metric (4.2) as the metric for the configuration space Ω, although we do not know whether this is necessary. This makes Ω a Eucledian space with Cartesian coordinates $b_\alpha{}^i(\mathbf{x})$; the distance $\|\Delta b_\alpha{}^i(\mathbf{x})\|$ between any two points with coordinate difference $\Delta b_\alpha{}^i(\mathbf{x})$ is given by

$$\|\Delta b_\alpha{}^i(\mathbf{x})\|^2=\int d^3x\, \Delta b_\alpha{}^i \Delta b^\alpha{}_i. \quad (4.3)$$

This distance is gauge-invariant, since $\Delta b_\alpha{}^i \Delta b^\alpha{}_i$ is gauge-invariant.

Consider a point $b_\alpha{}^i(\mathbf{x})$ of Ω. A gauge transformation maps $b_\alpha{}^i$ into $b'_\alpha{}^i$. Performing all possible gauge transformations on $b_\alpha{}^i$ gives a manifold χ of points $b'_\alpha{}^i$; it is easy to see that χ is gauge-invariant. Calling two points of Ω equivalent if and only if there exists a gauge transformation which maps one point into the other, the set of all gauge-invariant manifolds χ forms a set of equivalence classes. Each equivalence class may be represented by one of its points. We choose these representatives such that they form a manifold Ξ. Let Ξ be coordinatized by coordinates ξ^A, where A denotes a continuous label. Then, the ξ^A may be used to label the gauge-invariant manifolds; this is indicated by the notation $\chi(\xi^A)$. Figure 1 shows a finite-dimensional analog of the situation. Since it turns out that Ξ has "twice" as many dimensions as χ, the manifold Ξ is shown as a surface and χ as a curve. It can be shown that χ is not flat. Since no two points of Ξ can be related by a gauge transformation, the coordinates ξ^A in Ξ may be interpreted as physical variables. For example, Ξ may be taken as the set of transverse potentials, and ξ^A as their Fourier components. It can be shown that for the Yang-Mills field there does not exist a manifold Ξ which intersects all $\chi(\xi^A)$ orthogonally; for the electromagnetic field, such $\Xi\perp\chi(\xi^A)$ does exist, and it turns out to be the manifold of transverse potentials.

After this preparation, we define the scalar product of

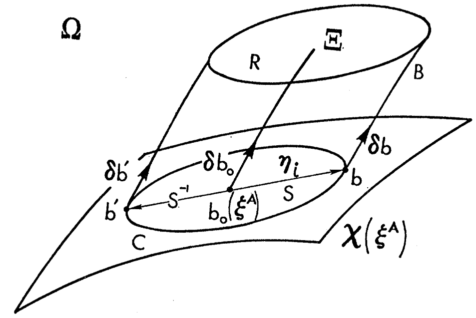

FIG. 2. Tubelike region R of Ω.

good states $|\phi\rangle$ and $|\psi\rangle$ as the functional integral

$$\langle\phi|\psi\rangle=N\int_R \delta b\, \Phi^*[b]\Psi[b], \quad (4.4)$$

where the measure is taken as the one belonging to the metric (4.2); this measure is uniform in the coordinates $b_\alpha{}^i(\mathbf{x})$. R is a tubelike region of Ω, which will be discussed in detail, and which is indicated in Fig. 2. In Fig. 2, the relative dimensions of Ξ and χ are chosen incorrectly, in order to show most clearly the construction of the tube R. For the purpose of this construction, we first label points $b_\alpha{}^i(\mathbf{x})$ of Ω by the coordinates $(\xi^A,\eta^i(\mathbf{x}))$, where ξ^A are the coordinates in Ξ of a point $b_{0\alpha}{}^i(\mathbf{x})\epsilon\Xi$ which lies in the same gauge-invariant manifold χ as $b_\alpha{}^i(\mathbf{x})$, and $\eta^i(\mathbf{x})$ are the parameters of a gauge transformation (2.1) which maps $b_{0\alpha}{}^i(\mathbf{x})$ into $b_\alpha{}^i(\mathbf{x})$. The details of this map can be found from

$$b_\alpha{}^i L_i=S^{-1}(b_{0\alpha}{}^i L_i S-\partial_\alpha S), \quad (4.5)$$

where $S(\mathbf{x})$ is given by (2.1); multiplying (4.5) on the right-hand side with L_j and taking the matrix trace gives

$$b_\alpha{}^j(\mathbf{x})=a_i{}^j(\mathbf{x})b_{0\alpha}{}^i(\mathbf{x})-a_\alpha{}^j(\mathbf{x}), \quad (4.6)$$

where

$$a_i{}^j(\mathbf{x})=(1/a)\,\mathrm{Tr}S^{-1}(\mathbf{x})L_iS(\mathbf{x})L^j, \quad (4.7)$$

$$a_\alpha{}^j(\mathbf{x})=(1/a)\,\mathrm{Tr}L^jS^{-1}(\mathbf{x})\partial_\alpha S(\mathbf{x}), \quad (4.8)$$

and a is given by

$$\mathrm{Tr}L_iL_j=ag_{ij}, \quad (4.9)$$

where g_{ij} is the group metric (1.3). For the specification of the tube R to be used in (4.4), we specialize to the two-dimensional representation of $SU(2)$, so that S of (2.1) is a 2×2 unitary matrix. The tube R is taken as the point set $\{(\xi^A,\eta^i(\mathbf{x}))\}$, for which the ξ^A range over all of Ξ, and for which the $\eta^i(\mathbf{x})$ satisfy the condition

$$\int d^3x[2-\mathrm{Tr}S(\mathbf{x})]\le\epsilon, \quad (4.10)$$

where ϵ is a positive number; of the $\eta^i(\mathbf{x})$ which satisfy (4.10) we only use the set which is connected to the element $\eta^i(\mathbf{x})=0$. It will be shown in Secs. V and VI

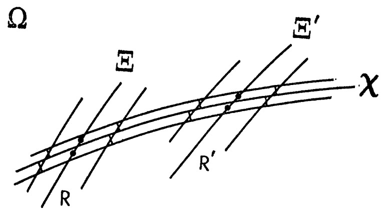

FIG. 3. Slices cut off R and R' by χ's.

that this choice of functional integration region R assures gauge and Lorentz invariance of the scalar product (4.4). The normalization constant N in (4.4) depends on the choice of ϵ in (4.10).

The scalar product (4.4) does not depend on the choice for the manifold Ξ of representatives for the gauge-invariant manifolds. We will prove this by considering two arbitrary choices, Ξ and Ξ', as shown in Fig. 3. Ξ and Ξ' need not be related by a gauge transformation. The functional integral (4.4) is the sum of contributions from slices cut off R by gauge-invariant manifolds $\chi(\xi^A)$. The distance between neighboring χ's is constant, since gauge transformations are isometries in Ω. Therefore, the slices cut off R and R' by two neighboring χ's have equal thickness. If we could show that these slices also have equal area in a χ, then it would follow that (4.4) has the same value, using Ξ or Ξ', since $\Phi[b]$ and $\Psi[b]$ in (4.4) are constant along a χ, because of the gauge invariance of these functionals. The proof that the slices, cut off R and R' by a χ, have equal areas goes as follows and refers to Fig. 4. We denote by b_0 the intersection of Ξ and χ, by b_0' the intersection of Ξ' and χ, by $T(\mathbf{x})$ the gauge transformation which maps b_0 into b_0', by C and C', respectively, the intersections of the boundaries of R and R' with χ, by $S(\mathbf{x})$ the gauge transformation which maps b_0 into the point b on C, and by b' the result of applying the gauge transformation $T(\mathbf{x})$ to b. Since b lies on C, we have, from (4.10),

$$\int d^3x[2-\mathrm{Tr}S(\mathbf{x})]=\epsilon. \tag{4.11}$$

b' is the result of applying the gauge transformation $T(\mathbf{x})S(\mathbf{x})$ to b_0. From the consistency of the map from b_0 to b to b' and the map from b_0 to b_0' to b', and the arbitrariness present in the choice of points, it follows that we must have

$$T(\mathbf{x})S(\mathbf{x})=U(\mathbf{x})T(\mathbf{x}) \tag{4.12}$$

or

$$U(\mathbf{x})=T(\mathbf{x})S(\mathbf{x})T^{-1}(\mathbf{x}). \tag{4.13}$$

From (4.13) and (4.11) it follows that

$$\int d^3x[2-\mathrm{Tr}U(\mathbf{x})]=\epsilon, \tag{4.14}$$

so that b' lies on C'. Considering the arbitrariness of the choice of b on C it follows that, under the gauge transformation $T(\mathbf{x})$, C maps into C'. Since gauge transformations in Ω are isometries, C and C' have equal areas. This concludes the proof that the scalar product (4.4) is independent of the choice of Ξ.

Of course, the gauge-invariant state functionals must be restricted such that the functional integral (4.4) is convergent. This places a restriction on the behavior of the state functionals at the open ends of the tube R, which is analogous to the condition on wave functions $\Psi(\mathbf{x})$ for $|\mathbf{x}|\to\infty$.

One may take the limit of (4.4) as $\epsilon\to 0$, and obtain a functional integral over Ξ, involving a measure which may be calculated from the limit process. A practical choice for Ξ would be one for which this measure is relatively simple. It may turn out that some of the complications of the manifestly covariant formalism,[2,3] which we tried to avoid by choosing the canonical formalism, crop up in the measure over Ξ to be used in the limit $\epsilon\to 0$. In any case, the canonical formalism outlined here is expected to be simpler for dealing with physical problems not involving the scalar product, such as eigenvalue problems.

V. GAUGE INVARIANCE

We take the $\Pi_\kappa{}^i$ to transform as a vector in the Lie algebra space of $SU(2)$, i.e., under the infinitesimal gauge transformation (2.2), $\Pi_\kappa{}^i$ is to suffer a change

$$\delta\Pi_\kappa{}^i=-c_{jk}{}^i\eta^j\Pi_\kappa{}^k. \tag{5.1}$$

This assignment implies gauge invariance of the equal-time commutation relations (3.3)–(3.5), of the relation (3.6) between the spatial generalized momenta and velocities, and of the constraint conditions (3.10) and (3.11). The operators u^i in the relation (3.7) between the timelike generalized momenta and velocities are restricted to belong to the class (3.2). We attempt an assignment of transformation of u^i under gauge transformations (2.2), such that the momentum-velocity relation (3.7) becomes an invariant statement and u^i stays in the class (3.2).

Let δf^i and δc^i be the infinitesimal changes in f^i and c^i under the gauge transformation (2.2), which make the momentum-velocity relation (3.7) a gauge-invariant statement. From (5.1) for $\kappa=0$, one then has

$$-c_{jk}{}^i\eta^j\Pi_0{}^k=\partial_\kappa(-\nabla^\kappa\eta^i)+(\partial f^i/\partial b_\lambda{}^j)(-\nabla_\lambda\eta^j)+\delta f^i(b_\lambda{}^j)+\delta c^i(x). \tag{5.2}$$

From (5.2) and (3.7) it follows that one must take

$$\delta c^i=\partial_\kappa\partial^\kappa\eta^i-c_{jk}{}^i\eta^jc^k \tag{5.3}$$

and

$$\delta f^i=-c_{jk}{}^i\eta^jf^k-c_{jk}{}^ib^{\kappa j}\partial_\kappa\eta^k+\frac{\partial f^i}{\partial b_\lambda{}^j}\partial_\lambda\eta^j-\frac{\partial f^i}{\partial b_\lambda{}^j}c_{kl}{}^jb_\lambda{}^k\eta^l, \tag{5.4}$$

showing that $f^i+\delta f^i$ are functions of the operators $b_\lambda{}^j(x)$, not involving their derivatives. Hence, under gauge transformations, the u^i remain in the class (3.2).

With the transformation assignments (5.1), (5.3), and (5.4), the equations of motion (3.8) are not gauge-invariant. The difference between the changes of the left- and right-hand side of (3.8) under the gauge transformation (2.2) is

$$-c_{jk}{}^i\eta^j\nabla_\kappa B^\kappa{}_\lambda{}^k-c_{jk}{}^i\eta^j\partial_\lambda\Pi^{0k}-c_{jk}{}^i(\partial_\lambda\eta^j)\Pi^{0k}-\delta\left(\frac{\partial f^j}{\partial b^\lambda{}_i}\right)\Pi^0{}_j+c_{jkl}\frac{\partial f^j}{\partial b^\lambda{}_i}\eta^k\Pi_0{}^l, \quad (5.5)$$

where $\delta(\partial f^i/\partial b^\lambda{}_l)$ is the change suffered by $\partial f^i/\partial b^\lambda{}_l$, which could be calculated from (5.4). Write

$$\delta\left(\frac{\partial f^j}{\partial b^\lambda{}_i}\right)=-c_{lk}{}^j\eta^l\frac{\partial f^k}{\partial b^\lambda{}_i}-c_{lk}{}^i\eta^l\frac{\partial f^j}{\partial b^\lambda{}_k}+\delta^*\left(\frac{\partial f^j}{\partial b^\lambda{}_i}\right); \quad (5.6)$$

then (5.5) may be written

$$-c_{jk}{}^i(\partial_\lambda\eta^j)\Pi^{0k}-\delta^*(\partial f^j/\partial b^\lambda{}_i)\Pi^0{}_j, \quad (5.7)$$

showing the gauge variance of the equations of motion (3.8). But, since the operator (5.7) applied to good states gives zero, the operator equations of motion (3.8) applied to good states give gauge-invariant results.

Turning to the scalar product (4.4), we note that the measure in Ω is gauge-invariant, and that $\Phi^*[b]$ and $\Psi[b]$ are gauge-invariant as well. That makes the scalar product gauge-invariant, if the gauge transformation in Ω is taken as a coordinate transformation. However, from the proof that (4.4) is independent of the choice for Ξ, we can see that the scalar product (4.4) is also invariant under point gauge transformations in Ω.

VI. LORENTZ INVARIANCE

We investigate the invariance of the theory under proper Lorentz transformations. The Hamiltonian (3.1) may be derived from the Lagrangian density

$$\mathfrak{L}'=\tfrac{1}{4}B_{\kappa\lambda}{}^iB^{\kappa\lambda}{}_i+\tfrac{1}{2}(\partial_\kappa b^{\kappa i}+u^i)(\partial_\lambda b^\lambda{}_i+u_i). \quad (6.1)$$

The Yang-Mills potentials $b_\kappa{}^i$ are taken to be a vector under Lorentz transformations. That makes the $B_{\kappa\lambda}{}^i$ of (1.1) and (3.9) tensors. In order to make $\mathfrak{L}'$ a scalar, we take the u^i to be scalars. The notation $\Pi_\alpha{}^i$ used in Sec. III obscures the dependence of these operators on the time direction. The matter is clarified by calculating

$$\Pi^{\kappa\lambda}{}_i=\partial\mathfrak{L}'/\partial\partial_\kappa b_\lambda{}^i. \quad (6.2)$$

From (6.1) one finds that

$$\Pi^{\kappa\lambda}{}_i=B^{\kappa\lambda}{}_i+g^{\kappa\lambda}\Pi^0{}_i, \quad (6.3)$$

showing that, for a fixed time direction, one has

$$\Pi^0{}_i=\Pi^{00}{}_i, \quad (6.4)$$

$$\Pi^\alpha{}_i=\Pi^{0\alpha}{}_i. \quad (6.5)$$

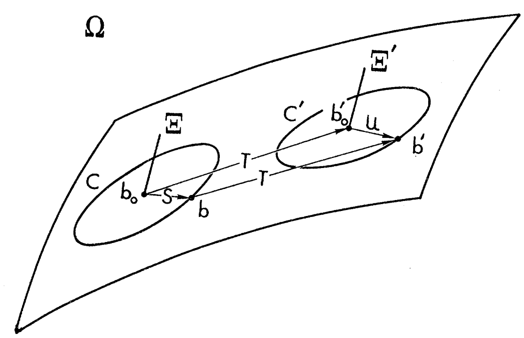

FIG. 4. Mapping in χ pertaining to different choices Ξ and Ξ'.

Equation (3.7) shows the $\Pi_0{}^i$ to be scalars, and then, the $\Pi^{\kappa\lambda}{}_i$ given by (6.3) are tensors. The primary constraint (3.10) is an invariant condition on states. With (3.6), the secondary constraint (3.11) together with the equations of motion (3.8) $(\lambda=\alpha)$ applied to good states may be written in the form (3.12), which is an invariant statement.

Lorentz invariance of the $\kappa=\alpha$, $\lambda=\beta$ components of the equal-time commutators (3.3)–(3.5), subject to the Hamiltonian (3.23), has been shown by Schwinger,[10] using his method involving the equal-time commutator of energy densities.[21] We give here an independent proof of Lorentz invariance of the equal-time commutation relations, and we will do this for the complete system (3.3)–(3.5), subject to the Hamiltonian (3.1). We follow the method of Heisenberg and Pauli[22]; unfortunately, their results cannot be immediately applied to our case, because H' of (3.1) involves the spatial derivatives of the momentum densities $\Pi^{\alpha i}$.

The commutation relation (3.3) may be expressed as

$$\int_\sigma df_\kappa\, h(x)[\Pi^{\kappa\lambda i}(x),b_{\mu j}(y)]=-i\delta_\mu{}^\lambda\delta_j{}^ih(y), \quad (6.6)$$

where σ is the equal-time surface through y^κ, df_κ is an element of σ at x^κ, and $h(x)$ is any operator-valued or numerical function which is good in the sense of Lighthill.[23] Equation (6.6) is invariant under spatial rotations. An infinitesimal Lorentz transformation of the observer has two effects: The equal-time surface σ of integration in (6.6) is tilted, and the coordinate system to which (6.6) refers is changed. The latter change leaves (6.6) invariant, since $\Pi^{\kappa\lambda i}$ of (6.3) is a tensor, and df_κ and $b_{\mu j}$ are vectors. A pure infinitesimal Lorentz transformation which tilts σ around y^κ gives a displacement of x^κ,

$$\delta x^\kappa=v^\kappa{}_\lambda(x^\lambda-y^\lambda), \quad (6.7)$$

where $v_{\kappa\lambda}$ is some infinitesimal antisymmetric tensor

[21] J. Schwinger, Phys. Rev. **127**, 324 (1962).
[22] W. Heisenberg and W. Pauli, Z. Physik **56**, 1 (1929).
[23] M. J. Lighthill, *Introduction to Fourier Analysis and Generalized Functions* (Cambridge University Press, Cambridge, England, 1964).

with $v_{\alpha\beta}=0$; the surface element df_κ changes by

$$\delta df_\kappa = -v^\lambda{}_\kappa df_\lambda. \tag{6.8}$$

Under the infinitesimal tilt, the left-hand side of (6.6) changes by[24]

$$\int df_\kappa(\partial_0 h)v^0{}_\alpha(x^\alpha-y^\alpha) \times[\Pi^{\kappa\lambda i},b_{\mu j}(y)]-\int v^0{}_\kappa df_0 h[\Pi^{\kappa\lambda i},b_{\mu j}(y)] +\int df_0 h[v^0{}_\alpha(x^\alpha-y^\alpha)\partial_0\Pi^{0\lambda i},b_{\mu j}(y)]. \tag{6.9}$$

Using (6.6), the first integral of (6.9) is seen to vanish, leaving

$$\int df_0 h[-v^{0\lambda}\Pi^{0i}-v^0{}_\alpha B^{\alpha\lambda i} +v^0{}_\alpha(x^\alpha-y^\alpha)\partial_0\Pi^{0\lambda i},\, b_{\mu j}(y)], \tag{6.10}$$

where use has been made of (6.3). Separate calculation of (6.10) for the four cases obtained by taking λ as 0 or α, and μ as 0 or β, gives zero, using (3.3)–(3.5) and (3.8). Hence, an infinitesimal Lorentz transformation does not change the left-hand side of (6.6). The right-hand side also remains unchanged. Hence, (6.6) is Lorentz-invariant, and since (6.6) is nothing but a transcription of (3.3), the commutation relation (3.3) is Lorentz-invariant.

The commutation relation (3.4) is invariant under coordinate transformations. Under the infinitesimal point transformation (6.7), this commutator changes by

$$v^0{}_\alpha(x^\alpha-y^\alpha)[\partial_0 b^{\kappa i},b_{\lambda j}(y)], \tag{6.11}$$

and this expression is found to be zero for all four cases obtained by taking κ equal to 0 or γ and λ equal to 0 or β. Hence, (3.4) is Lorentz-invariant.

The commutator (3.5) may be expressed as

$$\int_\sigma df_\kappa h[\Pi^{\kappa\lambda i},v_\mu\Pi^{\mu\nu j}(y)]=0, \tag{6.12}$$

where v_μ is a constant vector for which

$$v_\mu df_\kappa - v_\kappa df_\mu = 0. \tag{6.13}$$

The statements (6.12) and (6.13) are invariant under coordinate transformations. For Cartesian coordinates with $x^0=0$ on σ we have $df_\alpha=0$, $v_\alpha=0$; (6.12) thus reduces to (3.5), in view of (6.4). Under an infinitesimal tilt $\sigma\rightarrow\sigma'$ described by (6.7) and (6.8), v_μ suffers the change

$$\delta v_\mu = -v^\lambda{}_\mu v_\lambda, \tag{6.14}$$

[24] From here on in this section, any function of event coordinates shown without specification of the argument is meant to be taken at x.

and the left-hand side of (6.12) changes by

$$\int_\sigma df_0 h\{[-v^0{}_\alpha\Pi^{\alpha\lambda i}+v^0{}_\alpha(x^\alpha-y^\alpha)\partial_0\Pi^{0\lambda i},\, v_0\Pi^{0\nu j}(y)] -v^0{}_\alpha v_0[\Pi^{0\lambda i},\Pi^{\alpha\nu j}(y)]\}. \tag{6.15}$$

We calculate (6.15) separately for λ, ν equal to 0 or β. The calculation is straightforward for $\lambda=0$, $\nu=0$, for $\lambda=0$, $\nu=\beta$, and for $\lambda=\beta$, $\nu=0$. For $\lambda=\beta$, $\nu=\gamma$, the calculation is rather involved, and we show some of the steps. For that case, the expression within the curly brackets in (6.15) is

$$[-v^0{}_\alpha B^{\alpha\beta i}+v^0{}_\alpha(x^\alpha-y^\alpha)\partial_0 B^{0\beta i},v_0 B^{0\gamma j}(y)] -v_0 v^0{}_\alpha[B^{0\beta i},B^{\alpha\gamma j}(y)]. \tag{6.16}$$

From (3.3) it can be shown that[13]

$$[B^{\alpha\beta i},B^{0\gamma j}(y)]=2i\nabla^{[\alpha}g^{\beta]\gamma}g^{ij}\delta^3(\mathbf{x}-\mathbf{y}), \tag{6.17}$$

where ∇^α is taken with respect to $\mathbf{x}$ and operates only on i. Using (6.17), expression (6.16) may be written

$$-2iv_0v^0{}_\alpha\nabla^{[\alpha}g^{\beta]\gamma}g^{ij}\delta^3(\mathbf{x}-\mathbf{y}) -2iv_0v^0{}_\alpha\nabla^{[\alpha}g^{\gamma]\beta}g^{ij}\delta^3(\mathbf{x}-\mathbf{y})+v_0v^0{}_\alpha(x^\alpha-y^\alpha) \times[\nabla_0 B^{0\beta i},B^{0\gamma j}(y)], \tag{6.18}$$

where use has been made of (1.4). With (3.8), (6.18) may be expressed as

$$-2iv_0v^0{}_\alpha\{\nabla^{[\alpha}g^{\beta]\gamma}g^{ij}\delta^3(\mathbf{x}-\mathbf{y})+\nabla^{[\alpha}g^{\gamma]\beta}g^{ij}\delta^3(\mathbf{x}-\mathbf{y}) +(x^\alpha-y^\alpha)\nabla_\delta\nabla^{[\delta}g^{\beta]\gamma}g^{ij}\delta^3(\mathbf{x}-\mathbf{y})\}. \tag{6.19}$$

The last term in the curly brackets of (6.19) is

$$\begin{aligned}&\nabla_\delta\{(x^\alpha-y^\alpha)\nabla^{[\delta}g^{\beta]\gamma}g^{ij}\delta^3(\mathbf{x}-\mathbf{y})\}\\ &\quad-(\nabla_\delta(x^\alpha-y^\alpha))\nabla^{[\delta}g^{\beta]\gamma}g^{ij}\delta^3(\mathbf{x}-\mathbf{y})\\ &=\nabla_\delta\nabla^{[\delta}(x^\alpha-y^\alpha)g^{\beta]\gamma}g^{ij}\delta^3(\mathbf{x}-\mathbf{y})\\ &\quad-\nabla_\delta(\nabla^{[\delta}(x^\alpha-y^\alpha))g^{\beta]\gamma}g^{ij}\delta^3(\mathbf{x}-\mathbf{y})\\ &\quad-\nabla^{[\alpha}g^{\beta]\gamma}g^{ij}\delta^3(\mathbf{x}-\mathbf{y})\\ &=-\nabla^{[\alpha}g^{\gamma]\beta}g^{ij}\delta^3(\mathbf{x}-\mathbf{y})-\nabla^{[\alpha}g^{\beta]\gamma}g^{ij}\delta^3(\mathbf{x}-\mathbf{y}).\end{aligned} \tag{6.20}$$

Hence, the expression (6.19) vanishes. This completes the proof of Lorentz invariance of the equal-time commutators (3.3)–(3.5), subject to the Hamiltonian (3.1).

The Lorentz invariance of the scalar product (4.4) may be proved by showing that the generators of Lorentz transformations in state space are self-adjoint with respect to this scalar product, so that Lorentz transformations are unitary in the space of good states. The generators of pure Lorentz transformations in state space, found from the Lagrangian (6.1), are

$$M'^{0\alpha}=\int d^3x\{[(\partial_0 b_\gamma{}^i)B^{0\gamma}{}_i+(\partial_0 b_0{}^i)\Pi^{00}{}_i-\mathcal{L}']x^\alpha -[(\partial^\alpha b_0{}^i)\Pi^{00}{}_i+(\partial^\alpha b_\gamma{}^i)B^{0\gamma}{}_i]x^0 -b^{\alpha i}\Pi^{00}{}_i+b^{0i}B^{0\alpha}{}_i\}. \tag{6.21}$$

These generators satisfy the mapping equations for the Heisenberg picture:

$$\begin{aligned}i[M'^{0\alpha},b_\beta{}^j(x)]&=x^\alpha\partial_0 b_\beta{}^j-x^0\partial^\alpha b_\beta{}^j+\delta_\beta{}^\alpha b^{0j},\\ i[M'^{0\alpha},b_0{}^j(x)]&=x^\alpha\partial_0 b_0{}^j-x^0\partial^\alpha b_0{}^j-b^{\alpha j}.\end{aligned} \tag{6.22}$$

The operators (6.21) involve $b_0{}^i$ and are not gauge-invariant; therefore, they will turn a good state into a bad state. Fortunately, $M'^{0\alpha}$ operating on sufficiently localized good states may be expressed as a gauge-invariant operator, not involving $b_0{}^i$, operating on these states. To do this, we first note that the operator

$$X^{0\alpha}=\int d^3x\,\partial_\beta(b^{\alpha i}B^{0\beta}{}_i x^0-b^{0i}B^{0\beta}{}_i x^\alpha) \qquad (6.23)$$

vanishes[25] on states which are sufficiently localized. Therefore, on good states with this property, we may as well take as the generators of pure Lorentz transformations

$$M^{0\alpha}=M'^{0\alpha}+X^{0\alpha}=\int d^3x\{x^0(B_\beta{}^{\alpha i}B^{0\beta}{}_i+b^{\alpha k}\nabla_\beta B^{0\beta}{}_k)$$
$$+x^\alpha(-b_0{}^j\nabla_\beta B^{0\beta}{}_j+B_{0\beta}{}^iB^{0\beta}{}_i-\mathcal{L}')\}. \quad (6.24)$$

Equation (6.24) may be simplified by dropping the terms with $\nabla_\beta B^{0\beta}{}_k$, since they give zero on good states; the result is, on good localized states,

$$M^{0\alpha}=\int d^3x(T^{00}x^\alpha-T^{0\alpha}x^0), \qquad (6.25)$$

where

$$T^{\kappa\lambda}=-(B^\lambda{}_\mu{}^iB^{\mu\kappa}{}_i+\tfrac{1}{4}g^{\kappa\lambda}B_{\mu\nu}{}^iB^{\mu\nu}{}_i) \qquad (6.26)$$

is the gauge-invariant energy-momentum tensor acting on good states. Since the operator (6.25) is gauge-invariant and does not involve b^{0i}, it turns a good state into a good state.

We now proceed to prove Lorentz invariance of the scalar product (4.4) by showing that $M^{0\alpha}$ of (6.25) is self-adjoint with respect to (4.4). The question of self-adjointness of $M^{0\alpha}$ is complicated by the circumstance that we cannot specify the operators $b_\alpha{}^i(\mathbf{x})$ to be self-adjoint since a scalar product is here only defined for good states, and the non-gauge-invariant operators $b_\alpha{}^i$ turn a good state into a bad state. What can be done, and what turns out to be sufficient, is to specify that in the Schrödinger representation the $b_\alpha{}^i(\mathbf{x})$ are to be represented by *real* numerical functions $b_\alpha{}^i(\mathbf{x})$, and the states are to be represented by functionals of the *real* functions $b_\alpha{}^i(\mathbf{x})$. Then T^{00} occurring in (6.25) is self-adjoint, if the functional differential operator

$$\delta^2/\delta b_\alpha{}^i(\mathbf{x})\delta b^\alpha{}_i(\mathbf{x}) \qquad (6.27)$$

is self-adjoint; this can be seen from (6.26), (3.6), and (3.24). From the identity

$$\int_R \delta b\Phi^*\frac{\delta^2\Psi}{\delta b_\alpha{}^i\delta b^\alpha{}_i}-\int_R\delta b\left(\frac{\delta^2\Phi^*}{\delta b_\alpha{}^i\delta b^\alpha{}_i}\right)\Psi$$
$$=\int_R\delta b\frac{\delta}{\delta b_\alpha{}^i}\left(\Phi^*\frac{\delta\Psi}{\delta b^\alpha{}_i}-\left(\frac{\delta\Phi^*}{\delta b^\alpha{}_i}\right)\Psi\right), \quad (6.28)$$

[25] Since $X^{0\alpha}$ is not gauge-invariant, the norm of $X^{0\alpha}|\psi\rangle$ is not defined; we mean here component-wise vanishing of the resulting state vector.

it follows that the operator (6.27) is self-adjoint with respect to (4.4) if

$$\int_R\delta b\frac{\delta}{\delta b_\alpha{}^i(\mathbf{x})}v_\alpha{}^i(\mathbf{x})=0, \qquad (6.29)$$

where

$$v_\alpha{}^i(\mathbf{x})=\Phi^*\frac{\delta\Psi}{\delta b^\alpha{}_i(\mathbf{x})}-\left(\frac{\delta\Phi^*}{\delta b^\alpha{}_i(\mathbf{x})}\right)\Psi. \qquad (6.30)$$

Equation (6.29) is satisfied if

$$\int d^3x\,f(\mathbf{x})\int_R\delta b\frac{\delta}{\delta b_\alpha{}^i(\mathbf{x})}v_\alpha{}^i(\mathbf{x})=0 \qquad (6.31)$$

for all $f(\mathbf{x})$ subject to some condition at spatial infinity. Calling

$$f(\mathbf{x})v_\alpha{}^i(\mathbf{x})=w_\alpha{}^i(\mathbf{x}), \qquad (6.32)$$

(6.31) may be written

$$\int_R\delta b\frac{\delta}{\delta b_{\alpha i\mathbf{x}}}w_{\alpha i\mathbf{x}}=0, \qquad (6.33)$$

where writing $\mathbf{x}$ as a repeated index implies integration over all $\mathbf{x}$, by an obvious extension of the summation convention. Using Gauss's theorem, (6.33) may be expressed as

$$\int_B\delta b^{\alpha i\mathbf{x}}w_{\alpha i\mathbf{x}}=0, \qquad (6.34)$$

where B is the surface of the tube R as shown in Fig. 2. Equation (6.33) reduces to (6.34), provided that the state functionals $\Phi(\xi^A)$, $\Psi(\xi^A)$ fall off rapidly enough as $|\xi^A|\to\infty$, such that the surfaces at $|\xi^A|=\infty$ do not contribute to the integral. We will show that (6.34) is satisfied on account of the gauge invariance of the state functionals $\Phi[b]$ and $\Psi[b]$, and the special construction of the surface B. This is done as follows.

Let $b(\xi^A,\eta^i(\mathbf{x}))$ be a point of the intersection C of the surface B and the gauge-invariant manifold X, as shown in Fig. 2. The gauge transformation S of (2.1) maps the point $b_0(\xi^A,0)$ into the point b. Since b lies on B, $S(\mathbf{x})$ satisfies (4.11). The unitary matrix S is similar to a diagonal matrix with $e^{i\alpha}$ and $e^{-i\alpha}$ on the diagonal, $\alpha(\mathbf{x})$ being a real function. Therefore, we have

$$\mathrm{Tr}S^{-1}(\mathbf{x})=\mathrm{Tr}S(\mathbf{x}), \qquad (6.35)$$

and it follows with (4.11) that the point b' with coordinates ξ^A, $-\eta^i(\mathbf{x})$ lies on C. Any infinitesimal vector δb_0 in Ξ at b_0 is mapped by $S(\mathbf{x})$ into a vector δb in B at b, and is mapped by $S^{-1}(\mathbf{x})$ into a vector $\delta b'$ in B at b'. Also, $\delta b'$ is the image of δb under the map $S^{-2}(\mathbf{x})$.

So far, we have discussed vectors δb in B at b, which are the gauge transforms of vectors δb_0 in Ξ at b_0. Next, we consider infinitesimal vectors δb in B at b, which lie in X, as shown in Fig. 5. Any point of X may simply be

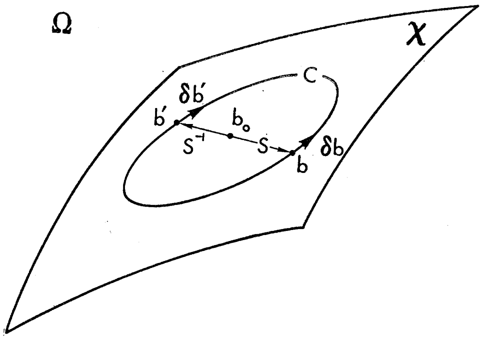

FIG. 5. Vectors δb in χ, tangent to C.

labeled by the gauge transformation $S(\mathbf{x})$ which produces the point from b_0. This is done in Fig. 6; the point b_0 is labeled by the identity-matrix function $I(\mathbf{x})$. Let $u(\mathbf{x})$ be an infinitesimal anti-Hermitian matrix. The point $[1+u(\mathbf{x})]S(\mathbf{x})$ lies on the curve C, if

$$\int d^3x\{2-\mathrm{Tr}[1+u(\mathbf{x})]S(\mathbf{x})\}=\epsilon; \tag{6.36}$$

with (4.11) this implies that

$$\int d^3x\,\mathrm{Tr}u(\mathbf{x})S(\mathbf{x})=0. \tag{6.37}$$

In order to see whether the point

$$T=S^{-2}(\mathbf{x})[1+u(\mathbf{x})]S(\mathbf{x}) \tag{6.38}$$

lies on C, we calculate

$$\int d^3x(2-\mathrm{Tr}T)=\int d^3x[2-\mathrm{Tr}S^{-2}(1+u)S]$$
$$=-\int d^3x\,\mathrm{Tr}uS^{-1}, \tag{6.39}$$

using the previous result that S^{-1} lies on C. Since S is unitary and u is anti-Hermitian, the integral on the right-hand side of (6.39) is

$$-\int d^3x\,\mathrm{Tr}uS^{-1}=\int d^3x\,\mathrm{Tr}u^\dagger S^\dagger$$
$$=\left(\int d^3x\,\mathrm{Tr}uS\right)^*=0, \tag{6.40}$$

by (6.37). Hence, T lies on C. It follows that the transform by S^{-2} of any infinitesimal vector δb in C at b of Fig. 5 is a vector $\delta b'$ in C at b'. Combining this result with our findings concerning vectors δb of Fig. 2 at b, we see that the gauge transformation $S^{-2}(\mathbf{x})$, which maps a point b of B into a point b' of B, also maps *any* infinitesimal vector δb in B at b into a vector $\delta b'$ in B at b'. Therefore, we have

$$[\delta b^{\alpha i}(\mathbf{x})L_i]_{b'}=S^2(\mathbf{x})[\delta b^{\alpha i}(\mathbf{x})L_i]_b S^{-2}(\mathbf{x}), \tag{6.41}$$

for the surface elements $\delta b^{\alpha ix}$ of B in (6.34), at "opposite" points b and b' of the surface B.

Next, we compare $w_{\alpha ix}$ at opposite points b and b' of B. First, we investigate how the functional derivative $\delta\Psi/\delta b_\alpha{}^i(\mathbf{x})$ of a gauge-invariant functional $\Psi[b_\alpha]$ transforms under an infinitesimal gauge transformation (2.2). Changing the argument b of the functional by an arbitrary infinitesimal amount Db produces the change

$$D\Psi=\int d^3x\frac{\delta\Psi}{\delta b_\alpha{}^i(\mathbf{x})}Db_\alpha{}^i(\mathbf{x}). \tag{6.42}$$

Now we perform an infinitesimal gauge transformation (2.2). Since this leaves $D\Psi$ invariant, we have, from (6.24),

$$0=\int d^3x\left(\delta\frac{\delta\Psi}{\delta b_\alpha{}^i}Db_\alpha{}^i+\frac{\delta\Psi}{\delta b_\alpha{}^i}\delta Db_\alpha{}^i\right), \tag{6.43}$$

where

$$\delta Db_\alpha{}^i=-c_{jk}{}^i\eta^j Db_\alpha{}^k, \tag{6.44}$$

as follows from (2.2). Substitution of (6.44) in (6.43) gives

$$0=\int d^3x\left(\delta\frac{\delta\Psi}{\delta b_\alpha{}^k}-c_{jk}{}^i\eta^j\frac{\delta\Psi}{\delta b_\alpha{}^i}\right)Db_\alpha{}^k \tag{6.45}$$

for arbitrary $Db_\alpha{}^k$; hence, we must have

$$\delta\frac{\delta\Psi}{\delta b_\alpha{}^k}=c_{jk}{}^i\eta^j\frac{\delta\Psi}{\delta b_\alpha{}^i}. \tag{6.46}$$

From (6.30) and (6.32) it follows that $w_{\alpha i}(\mathbf{x})$ has the same transformation law under infinitesimal gauge transformations as $\delta\Psi/\delta b^{\alpha i}$. Therefore, one has

$$[w_{\alpha j}(\mathbf{x})L^j]_{b'}=S^2(\mathbf{x})[w_{\alpha j}(\mathbf{x})L^j]_b S^{-2}(\mathbf{x}) \tag{6.47}$$

for opposite points b and b' of B. Writing, with the help of (4.9), the expression under the integral of (6.34)

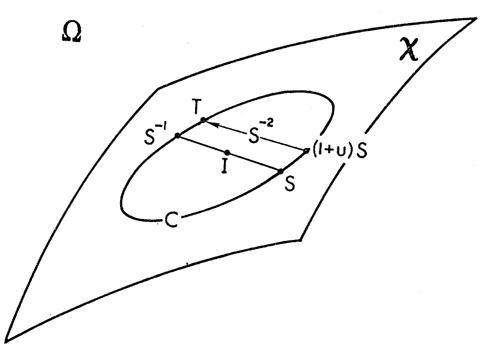

FIG. 6. Mapping pertaining to opposite points of C.

as

$$\frac{1}{a}\int d^3x\, \mathrm{Tr}\delta b^{\alpha i}(\mathbf{x})L_i w_{\alpha j}(\mathbf{x})L^j, \tag{6.48}$$

we see from (6.41) and (6.47) that this infinitesimal flux has equal values at opposite points b and b' of B. Hence, the total flux (6.34) through B vanishes, completing the proof of self-adjointness of the operator T^{00}, with respect to the scalar product (4.4).

The proof of self-adjointness of the remaining term of (6.25), involving $T^{0\alpha}$, follows similar lines, once it is shown that

$$\int d^3x\, T^0{}_\alpha = \int d^3x\, B_{\alpha\beta}{}^i B^{0\beta}{}_i \tag{6.49}$$

is independent of the operator ordering. Because of the antisymmetry of the structure constants, (6.17) gives

$$[B_{\alpha\beta}{}^i(\mathbf{x}), B^{0\beta}{}_i(\mathbf{y})] = 6i\partial_\alpha \delta^3(\mathbf{x}-\mathbf{y}). \tag{6.50}$$

Hence, the difference between the expressions (6.49) with different operator orderings is

$$6i\int d^3x \lim_{\mathbf{y}\to\mathbf{x}} \partial_\alpha \delta^3(\mathbf{x}-\mathbf{y}), \tag{6.51}$$

which vanishes if defined as the limit of integrals like (6.51), with $\delta^3(\mathbf{x}-\mathbf{y})$ replaced by a Gaussian which is more and more peaked.[26] This concludes the proof of self-adjointness of (6.25) with respect to the scalar product (4.4) and, therefore, of the Lorentz invariance of this scalar product. The self-adjointness of $T^{0\alpha}$ also shows (4.4) to be invariant under spatial rotations, since for localized good states, the angular momentum operator may be expressed as

$$M^{\alpha\beta} = \int d^3x (T^{0\alpha}x^\beta - T^{0\beta}x^\alpha), \tag{6.52}$$

i.e., the orbital expression; the spin part has been removed by a manipulation like the one shown in (6.23) and (6.24).

[26] This consideration falls somewhat outside the theory of tempered distributions, but it is consistent with a well-known method of dealing with δ functions (Ref. 23).

VII. RESULTING RULES

We now collect our results. In the Schrödinger picture, and using the Schrödinger representation, the Yang-Mills field, interacting with itself but not with other fields, is quantized according to the following rules.

States are represented by functionals $\Psi[b_\alpha]$ of the *spatial* Yang-Mills potentials $b_\alpha{}^i(\mathbf{x})$, $\alpha = 1, 2, 3$, which, in turn, are real functions over all of three-dimensional space. The time-space components of the Yang-Mills field are represented by functional derivatives,

$$B^{0\alpha}{}_i(\mathbf{x}) = -i\delta/\delta b_\alpha{}^i(\mathbf{x}), \tag{7.1}$$

acting on the state functionals. Good states are represented by gauge-invariant functionals $\Psi[b_\alpha]$, i.e., functionals which are invariant under all gauge transformations (2.2), subject to condition (3.26) at spatial infinity.

The Hamiltonian on good states is

$$H = \int d^3x (\tfrac{1}{2}B^{0\alpha}{}_i B_{0\alpha}{}^i - \tfrac{1}{4}B_{\alpha\beta}{}^i B^{\alpha\beta}{}_i), \tag{7.2}$$

where the $B_{\alpha\beta}{}^i$ are given by (1.1).

The scalar product of good states is given by the functional integral (4.4) over the tubelike region R of configuration space, R being defined by (4.10), in which $S(\mathbf{x})$ is the gauge transformation which maps b_0 (which lies on Ξ) into b. The scalar product has been expressed this way in order to prove invariances. In practice, one would contract the tube R to zero radius; in the limit, the scalar product becomes a functional integral over the manifold Ξ, involving a measure which may be calculated from the limit process. The resulting scalar product does not depend on the choice of Ξ. Observables are represented by gauge-invariant functional differential operators; their matrix elements between good states are scalar products between good states, and are expressible as functional integrals over the manifold Ξ.

These rules give Lorentz- and gauge-invariant results.

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