respect to transverse momentum and our reasoning remains practically unaffected.

It may be amusing to note that in order to preserve the factorization of the amplitude, one has much less freedom for choosing invariants on which the τ factors may strongly depend than one may imagine. For example, if $q\cdot p_1$ and $q\cdot p_2$ are uncoupled, then arguments similar to those we have used in our calculations lead to an exponential damping of both of these variables. This is clearly nonsense. The difficulty is removed if $q\cdot p_1$ and $q\cdot p_2$ enter in τ in the product form we have used.

Note added in proof. In the case of finite masses, the problem, of course, becomes more complicated. One can, however, generalize our results by redefining what one means by a soft vector particle [see, e.g., K. E. Erikson and S. A. Yngström, Phys. Rev. **131**, 2805 (1963)]. The Low expansion is in fact an expansion with respect to the momentum transfer to the vector particles. Thus, one requires for the validity of the expansion, the smallness of certain momentum transfers, rather than the smallness of the energy of the emitted vector particles.

Our derivation of the transverse momentum distribution applies to particles carrying some charge. However, the argument can be extended to those "neutral" particles which belong to a multiplet containing a charged particle, if the multiplet becomes fully degenerate asymptotically, as can be expected. (We are indebted to Dr. C. Bouchiat for a remark on this point.)

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Dispersion Theory Model of Three-Body Production and Decay Processes

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We study three-body production and decay processes, $A+B\to a+b+c$ and $A\to a+b+c$. We assume that the amplitudes are determined in both cases solely by the final-state interactions among pairs of a, b, and c, and that the dynamics is given by single-variable dispersion terms in each two-body channel (that is, a representation of the Khuri-Treiman type). Applying a method introduced by Anisovich, we obtain a linear integral equation which gives the three-body amplitude in terms of the two-body one g. In an effective-range approximation for g, the kernel can be easily evaluated; it is the sum of the discontinuities of the triangle graph in perturbation theory, when considered as a function of an internal mass. In the S-wave case, the resulting integral equation has a unique solution depending on one arbitrary parameter (a subtraction constant). In the nonrelativistic limit, the equation coincides with Anisovich's; this in turn is analogous to the equation derived by Skornyakov and Ter-Martyrosyan for a three-body problem in potential theory, with delta-function forces. It is suggested that the present theory is therefore a relativistic analog of a potential model with short-range forces. As applications, we mention the determination of the final-state pion spectra in $K\to 3\pi$ decay, and the possibility of studying the generation of resonances in the total center-of-mass energy, induced by iterations of real particle exchange processes (the Peierls mechanism).

I. INTRODUCTION

IN this paper we shall discuss, by dispersion methods, reactions of the types

$$\text{decay:}\qquad K\to\pi_1+\pi_2+\pi_3\,, \tag{1a}$$

$$\text{production:}\quad A+B\to\pi_1+\pi_2+\pi_3\,, \tag{1b}$$

where, for simplicity, the "pions" π_i are identical spinless isoscalar particles of unit mass. Reactions (1a), (1b) are represented by Figs. 1(a), 1(b), respectively. The two reactions are of the same type if we assume that in both cases the amplitude F is determined entirely by the final-state interactions (f.s.i.) among pairs of pions; the only difference between them is then that the mass of the "decaying state" $(A+B)$ is not fixed. The aim is to find an equation for F, given the two-body π-π amplitudes.

Our starting point is the Khuri-Treiman (K-T) equa-

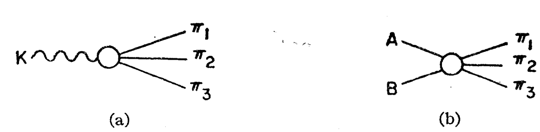

FIG. 1. (a) A decay process. (b) A production process.

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tion[1] for decay processes (Sec. IIIA). This equation has come in for extensive study recently.[2–6] Our approach is closely modeled on that of Anisovich.[6] In contrast to the authors of Refs. 2–5, all of whom work with the partial-wave projections of F, Anisovich obtained an equation leading directly to F itself. The practical advantages of his method are twofold. First, the equations for the amplitude are less singular, and so easier to handle numerically, than those for the projections. Secondly, even were the projections to be determined, a difficult integral remains to be done in order to recover the amplitude.

Anisovich treated the case in which the π-π amplitude is given by a scattering length approximation, thereby accounting for low-energy π-π effects only. For this case, the nonrelativistic kinematics he used were adequate. By summing all connected rescattering graphs, he obtained an equation similar to the K-T one. We apply his techniques directly to the K-T equation (Sec. IIIB). The result is especially simple if we assume an effective-range form for g (Sec. IV). This enables us to treat π-π resonances, as well as low-energy effects, while retaining the symmetrical invariant vairables. What emerges (Sec. V) is an integral equation of the type first studied by Skornyakov and Ter-Martyrosyan,[7] in the potential theory of the three-body problem, with δ-function forces. In the nonrelativistic limit it reduces exactly to Anisovich's equation (Sec. VI). With one subtraction, the asymptotic properties are such that the work of Danilov[8] ensures that it has a unique solution.

In Sec. VII we indicate briefly applications to the determination of

(a) final-state energy spectra in three-particle decays of resonances, and

(b) the three-particle mass distribution itself (a more obvious three-body effect).

II. KINEMATICS AND BASIC APPROACH

We give as much of the standard treatment[9] as we need to define the notation. If the four-momenta of the three pions π_i ($i=1, 2, 3$) are k_i, we introduce invariant variables s, t, u by

$$s=(k_2+k_3)^2,\quad t=(k_3+k_1)^2,\quad u=(k_1+k_2)^2,$$

where

$$s+t+u=3+m^2. \tag{2}$$

The pion mass is unity, and m is the mass of the decaying state or the c.m. energy of the three-particle system. If x is the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$, we have

$$\begin{aligned} &t=(3+m^2-s)/2-2p(s)q(s)x\,,\\ &p(s)=\{[s-(m-1)^2][s-(m+1)^2]\}^{1/2}/2s^{1/2}\,,\\ &q(s)=(s-4)^{1/2}/2\,. \end{aligned} \tag{3}$$

The physical region for the decay process (1) is then $4\leq s\leq(m-1)^2$ and $|x|\leq1$; the second condition may be written as

$$\Gamma(s,t,u)\equiv stu-(m^2-1)^2\leq0\,, \tag{4}$$

or, using Eq. (2),

$$\Gamma(s,t)\equiv st(3+m^2-s-t)-(m^2-1)^2\leq0\,. \tag{4'}$$

Γ, the Kibble cubic,[10] is drawn in Fig. 2(a) for the case $m>3$; the decay region $\mathfrak{D}$ is inside the loop, usually called the Dalitz plot. We wish to calculate the amplitudes for reaction (1) inside $\mathfrak{D}$.

The essential ideas of the present approach to the three-body processes (1), which was largely originated by Bonnevay,[2] are worth emphasizing right at the start. Consider first the decay process (1a). Evidently, it is not fundamentally different from a two-body problem, since, no matter how complicated the blob in Fig. 1(a) may be, there are finally only four external lines, as in any two-particle process, and, for m fixed, only two independent invariants [cf. Eq. (2)]. In fact the physical regions for the two-body reactions corresponding to crossed channels of (1a) are also contained in Fig. 2. For example, the region I ($s\geq(m+1)^2$, $t\leq0$) is the physical region for the s-channel reaction $\pi_2+\pi_3\to K+\pi_1$; the t and u channels, and regions II and III, are defined similarly. If m were actually less than 3, only these scattering reactions would be physically possible, and we could proceed to set up a dynamical model by postulating some form of dispersion relation, using physical two-body unitarity to evaluate the discontinuities. What we do, therefore, is to assume a simple relation for $m^2<9$, and then, after doing all the manipulations, we continue the resulting equation back in m^2 to the physical "K" value, arriving finally at an equation valid for s in the decay region $\mathfrak{D}$.[11]

Although for reaction (1a) only two of s, t, and u are independent [by Eq. (2)], we shall for symmetry always consider the amplitude F as a function of all three: $F=F(s,t,u)$. Following the procedure we have outlined, then, we shall set up equations for $F(s,t,u)$; that is, we aim to predict the final-state energy spectra in reactions of the type $K\to3\pi$.

[1] N. N. Khuri and S. B. Treiman, Phys. Rev. **119**, 1115 (1960).
[2] G. Bonnevay, Nuovo Cimento **30**, 1325 (1963).
[3] J. B. Bronzan and C. Kacser, Phys. Rev. **132**, 2703 (1963).
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[5] J. B. Bronzan, Phys. Rev. **134**, B687 (1964).
[6] V. V. Anisovich, Zh. Eksperim. i Teor. Fiz. **44**, 1593 (1963) [English transl.: Soviet Phys.—JETP **17**, 1072 (1963)]. The idea of using Anisovich's method directly on some form of the K-T equation was suggested to me by C. Kacser.
[7] G. V. Skornyakov and K. A. Ter-Martyorosyan, Zh. Eksperim. i Teor. Fiz. **31**, 775 (1956) [English transl.: Soviet Phys.—JETP **4**, 658 (1957)].
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[11] R. C. Hwa, Phys. Rev. **134**, B1086 (1964), has also considered continuing production amplitudes between disconnected regions by a similar technique, but with a rather different aim.

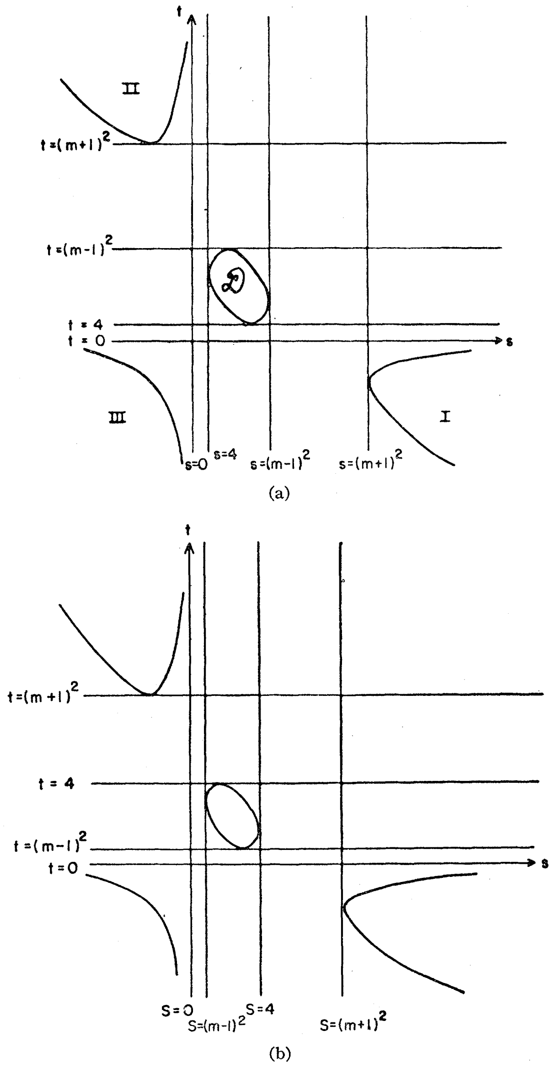

FIG. 2. (a) The curve $\Gamma=0$; the physical regions are **I**, II, III, and $\mathfrak{D}$. Case $m>3$ (b) the curve $\Gamma=0$, as (a), but case $1<m<3$.

Consider now the production process (1b). Here also, as mentioned in I, we asume that the amplitude is determined entirely by the final-state interactions: that is, it depends only on s, t, and u. But a general production amplitude depends on five invariants; so our assumption here amounts to ignoring the dependence on two variables (which would be of the momentum transfer, rather than energy, type). Furthermore, Eq. (2) is still true, and it implies that if we can determine the s, t, and u dependence we thereby know *some* of the m^2 dependence of the production amplitude. In fact, in our treatment, the difference between the reactions (1a) and (1b) is just simply that in the latter m is not fixed at a physical value, but is a variable, namely, the total center-of-mass energy; in our method, both amplitudes are given by the same function $F(s,t,u)$. It is precisely this sort of dependence, which is induced by "simultaneous" effects in the two-body s, t, and u channels, that is appealed to in the mechanism of Peierls[12] for the generation of higher baryon resonances, and in its applications by Pais and Nauenberg[13] and others[14] to energy peaks in multimeson systems. The variable m^2 has an obvious three-particle significance, and it is the m^2 distribution which seems likely to be the most interesting consequence of the present theory. We shall return to this point in Sec. VII.

Finally, we remark that the possibility of doing, in general, the m^2 continuation from the scattering to the decay regions is something we do not go into; in the special case we treat here, all that is required is analyticity of the triangle graph in perturbation theory as a function of an external mass, when that mass is in the upper half-plane and near the real axis; we believe that this is guaranteed by the work of Källén and Wightman.[15]

III. DERIVATION OF THE INTEGRAL EQUATION

A. The Khuri-Treiman Representation

The simple dispersion relation that we shall assume is a single-variable representation of the Khuri-Treiman (K-T) type[1]

$$F(s,t,u)=G+\Phi(s)+\Phi(t)+\Phi(u)\,, \tag{5}$$

where

$$\Phi(s)=(s-4)\frac{1}{\pi}\int\frac{g^*(s')F_0(s')ds'}{(s'-4)(s'-s-i\epsilon)}\,, \tag{6}$$

$F_0(s)$ is the s-wave projection of F in the c.m. system of pions 2 and 3, and $g(s)$ is the π-π amplitude. $g=e^{i\delta}\sin\delta$, where δ is the s-wave phase shift. We assume that one subtraction is enough for convergence (we have taken it at threshold). In what follows we shall usually omit the subtraction. The content of Eq. (5) is that in the scattering channels (cf. the $i\epsilon$) unitarity is included within the approximations:

(1) Only two-body intermediate states contribute.

(2) The two-body interactions occur in only one partial wave (in our case, for simplicity, the s wave). Notice that in this case crossing is violated.

In terms of graphs, Eqs. (5) and (6) sum up diagrams of the types shown in Fig. 3[2,4,5]; that is, anticipating the continuation in m^2 into the decay region, (a) no final-state interaction (f.s.i.) (b) f.s.i. between only 2 pions (c) f.s.k. between all three pions, interacting in pairs (pontoon graphs). We recall that the graphs of type (c) have a simplifying feature: namely, they have

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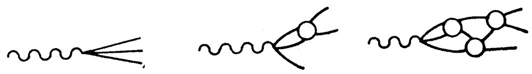


FIG. 3. (a) A constant term in the K-T representation. (b) A final-state interaction between one pair of particles. (c) Final-state interactions between several pairs of particles successively.

no complex singularities on the physical sheet in the decay region.[3,4]

In the treatment of Eq. (5) in Refs. 2–5, one obtains an integral equation for F_0 by projecting out on both sides of Eq. (5):

$$F_0(s)=G+\Phi(s)+\frac{2}{\pi}\int_4^\infty d\lambda^2 g^*(\lambda^2)F_0(\lambda^2)\times[K(s,\lambda^2-i\epsilon)-K(4,\lambda^2-i\epsilon)],\quad (7)$$

where

$$K(s,\lambda^2)=\frac{1}{2}\int_{-1}^{1}\frac{dx}{\lambda^2-t(s,x)}.\quad (8)$$

$K(s,\lambda^2)$ is very simply related to the discontinuity $\rho(s,\lambda^2)$ of the triangle graph of Fig. 4, around the singularity $s=4$. In fact,[16] for $\lambda^2\geq 4$,

$$\rho(s,\lambda^2)=(2\pi i)^2(\pi/2)[(s-4)/s]^{1/2}K(s,\lambda^2).\quad (9)$$

Hence in Eq. (7) we have an integral equation for the projection $F_0(s)$, whose kernel involves the discontinuity of the triangle graph for all internal masses $\lambda^2\geq 4$ (the limit onto the λ^2 axis being taken from below).

An examination of the kernel of Eq. (7) shows that it has singularities of two types: From (8) it follows that there are logarithmic singularities of K when $\lambda^2=t(s,\pm 1)$. The prescription for determining the correct branch of the logarithm can be obtained,[3] but a careful study shows that K develops in addition a singularity in s, of the inverse-square-root type, for $4\leq\lambda^2\leq(m+1)$. This is due to the emergence of the non-Landau singularity[16] at $s=(m-1)^2$ onto the (unphysical) edge of the physical sheet. This latter singularity further complicates the numerical resolution of Eq. (7), and even when that is done, there remains a further singular integration to recover $F(s,t,u)$ by Eqs. (5) and (6).

For these reasons we now follow Anisovich[6] and look for an equation for F itself, or rather for Φ. As has been remarked elsewhere,[17] by means of the standard Omnes transformation[18] Eq. (6) can be rewritten in the form

$$\Phi=\Phi_1+\Phi_2,\quad (10)$$

with

$$\Phi_1=\frac{G}{D}\frac{1}{\pi}\int\frac{gD}{s'-s-i\epsilon}ds',\quad \Phi_2=\frac{1}{D}\frac{1}{\pi}\int\frac{gD\bar{\Phi}}{s'-s-i\epsilon}ds',\quad (11)$$

where

$$\bar{\Phi}(s)=\int_{-1}^{1}dx\,\Phi(t(s,x)).\quad (12)$$

D is the usual denominator function[18]

$$D(s)=\exp\left\{-\frac{1}{\pi}\int\frac{\delta(s')}{s'-s-i\epsilon}ds'\right\},\quad (13)$$

where δ is the π-π phase shift: $g=e^{i\delta}\sin\delta$. In Eqs. (11) and (13) we have omitted the subtraction.

The first term of Eq. (10) is the usual solution to the problem of one f.s.i.; the second expresses the fact that there are also two interfering parallel channels. Eq. (10) may be regarded as an integral equation for Φ. However, because of the double integral in the Φ_2 term, a direct numerical attack would seem very difficult. We therefore now use the technique of Anisovich to transform this term into one involving only a single integral.

B. The Integral Equation for Φ

The first step is to rewrite Φ_2 as

$$\Phi_2=\lim_{\Lambda_2\to\infty}\frac{1}{D}\frac{1}{\pi}\int_4^{\Lambda^2}\frac{gD}{s'-s-i\epsilon}\frac{ds'}{2p(s')q(s')}\times\int_{t_-(s')}^{t_+(s')}dt\,\Phi(t),\quad (14)$$

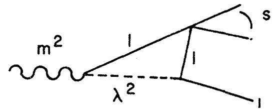
where we have used Eq. (3) to eliminate x for t, and where

$$t_\pm(s)=\frac{3+m^2-s}{2}\mp 2p(s)q(s);\quad (3')$$

$t_\pm(s)$ are thus the interactions of $\Gamma=0$ with lines of fixed $s=s'$.

Next, in accordance with our remarks in Sec. II, we assume that $\Phi(s)$ can be continued in m^2 in the upper-half m^2 plane, and consider the situation $1<m<3$. Then for $s\geq 4$ no decay is possible. But from Fig. 2(b), which shows the case $1<m<3$, we see at once that for $4\leq s\leq\Lambda^2$, $t_\pm(s)$ vary in such a way that they are either both complex or both real but negative; also $|t_\pm(s)|<\Lambda^2$. The one exceptional point is $s'=4$, but there the t-integration path τ shrinks to zero. In Fig. 5 we show the evolution of τ as s' increases from 4, for $1<m<3$.

FIG. 4. Triangle graph in perturbation theory.



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[17] I. J. R. Aitchison, Nuovo Cimento (to be published).
[18] R. Omnes, Nuovo Cimento 8, 316 (1958).

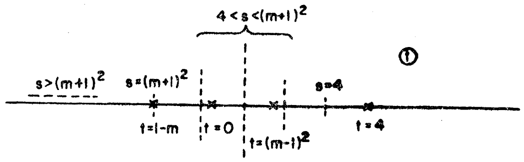

FIG. 5. The t plane, showing the contour of integration $(t_-(s),t_+(s))$ dashed, for various values of s.

Noting now that $\Phi(s)$ itself has a branch point at $s=4$ only, we can rewrite Eq. (14) as

$$\Phi_2=\lim_{\Lambda^2\to\infty}\frac{1}{D}\frac{1}{\pi}\int_4^{\Lambda^2}\frac{gD}{s'-s-i\epsilon}\frac{ds'}{2pq}\times\int_{t_-(s')}^{t+(s')}dt\frac{1}{2\pi i}\int_C\frac{\Phi(\lambda^2)d\lambda^2}{\lambda^2-t(s',x)},\quad(15)$$

where the contour C is shown in Fig. 6; the integration over the large circle proceeds in such a way that $|\lambda^2|>\Lambda^2$. The point of the previous maneuver is now clear: for $m>3$, the points $t_\pm(s')$ overlap the λ^2 contour C along the real axis [cf. Fig. 2(a)]. As we want to invert the orders of integration in Eq. (15), we would then have the tricky problem of determining the proper branch of the various integrands. This task is much easier when the contours do not intersect, as is the case for $m<3$; and in our particular case it is mainly for this reason that we have wanted to assume the possibility of continuation in m^2.

Equation (15) therefore becomes

$$\Phi_2(s)=\frac{1}{D(s)}\frac{1}{2\pi i}\int_C d\lambda^2\Phi(\lambda^2)2\phi(s,\lambda^2),\quad(16)$$

where

$$\phi(s,\lambda^2)=\frac{1}{\pi}\lim_{\Lambda^2\to\infty}\int_4^{\Lambda^2}\frac{gD}{s'-s-i\epsilon}ds'\frac{1}{2}\int_{-1}^{1}\frac{dx}{\lambda^2-t(s',x)}\quad(17)$$

$$=\frac{1}{\pi}\lim_{\Lambda^2\to\infty}\int_4^{\Lambda^2}\frac{gDK(s',\lambda^2)}{s'-s-i\epsilon}ds'\quad(18)$$

using Eq. (8).

The third step in the transformation of Φ_2 is to shrink the contour C onto the real axis, thus picking up contributions from the discontinuities around the λ^2 singularities of $\phi(s,\lambda^2)$. Thus we obtain

$$\Phi(s)=\Phi_1(s)+\frac{1}{D(s)}\sum_i\frac{1}{\pi}\int_{C_i(m^2)}d\lambda^2 2d_i(s,\lambda^2;m^2)\Phi(\lambda^2),\quad(19)$$

where $C_i(m^2)$ is a contour running along the cut associated with the ith λ^2 singularity of $\phi(s,\lambda^2)$, and $d_i(s,\lambda^2;m^2)$ is the discontinuity of $\phi(s,\lambda^2)$ across that cut; the summation is over all singularities i. We indicate explicitly that both C_i and d_i depend on the value of m^2.

Equation (19) is a linear integral equation for Φ, and the final step in Anisovich's method is to continue it in m^2 back into the decay region $m>3$. We thus get finally

$$\Phi(s)=\Phi_1(s)+\frac{2}{D(s)}\sum_i\frac{1}{\pi}\int_{C_i}d\lambda^2d_i(s,\lambda^2)\Phi(\lambda^2),\quad(20)$$

the d_i and C_i being evaluated at the physical mass $m>3$.

Once g is given, we can calculate D and hence Φ_1, and, in principle, find the singularities of $\phi(s,\lambda^2)$ and the discontinuities across them, d_i. Thus Eqs. (20) and (5) give $F(s,t,u)$ directly in terms of the solution of a single linear integral equation depending on the two-body parameters. In order to investigate the structure of Eq. (20) further, we will now make some assumptions about g.

IV. A SIMPLIFYING CHOICE FOR g

Let us write

$$gD=N\rho,\quad(21)$$

where, as usual,[19] ρ is the phase-space factor $[(s-4)/s]^{1/2}$ and N is the numerator function with singularities only for $s\leq0$. While we could proceed by making only general assumptions about N, it is immediately clear from Eq. (8), and the remarks following it, that our task becomes routine if we take an effective range type of approximation for N. For consider the two cases:

$$N=a,$$
$$N=b/(s+s_0),$$

where a, b and s_0 are constants $(s_0<0)$. In the first case, $\phi(s,\lambda^2)$ reduces simply to [cf. Eq. (9)].

$$\phi(s,\lambda^2)=\frac{a}{\pi}\lim_{\Lambda^2\to\infty}\int_4^{\Lambda^2}ds'\left(\frac{s'-4}{s'}\right)^{1/2}\frac{K(s',\lambda^2)}{s'-s-i\epsilon}=\frac{2a}{\pi(2\pi i)^2}f(s,\lambda^2),\quad(22)$$

where $f(s,\lambda^2)$ is the triangle graph amplitude (Fig. 4). In the second case, similarly, $\phi(s,\lambda^2)$ becomes

$$\phi(s,\lambda^2)=\frac{2b}{\pi(2\pi i)^2}\frac{1}{s+s_0}[f(s,\lambda^2)-f(-s_0,\lambda^2)].\quad(22')$$

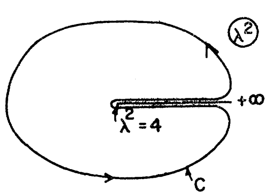

FIG. 6. The λ^2 plane, showing the contour C.

[19] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961).

Thus, in both cases, the determination of the λ^2 singularities and associated discontinuities of $\phi(s,\lambda^2)$ reduces simply to the corresponding analysis for the triangle graph $f(s,\lambda^2)$, and this is well known.[3,16]

Before proceeding with that analysis, however, we first recall the consequences of these two choices for N. As Jackson and Kane[20] have shown, the choice $N=$ constant leads to a scattering length approximation. Define a function H by

$$H=[q/(q^2+1)^{1/2}]\cot\delta=[q/(q^2+1)^{1/2}](g^{-1}+i).$$

From the once-subtracted dispersion relation for H we obtain

$$H(q^2)=a^{-1}+\frac{2}{\pi}[q/(q^2+1)^{1/2}]\ln[q+(q^2+1)^{1/2}],$$

where a is a constant, while

$$D=a[H-iq/(q^2+1)^{1/2}],$$

which can also be seen directly by integration of the standard equation for D (including a subtraction at $s=4$).

Then Eq. (11) for Φ_1 becomes

$$\Phi_1(s)=G\frac{\{-(2/\pi)[q^2/(q^2+1)]^{1/2}\ln[q+(q^2+1)^{1/2}]+iq/(q^2+1)^{1/2}\}}{\{a^{-1}+(2/\pi)[q^2/(q^2+1)]^{1/2}\ln[q+(q^2+1)^{1/2}]-iq/(q^2+1)^{1/2}\}}$$

and for Φ we get, using Eq. (22),

$$\Phi=\Phi_1+2\{a^{-1}+(2/\pi)[q/(q^2+1)^{1/2}]\ln[q+(q^2+1)^{1/2}]-iq/(q^2+1)^{1/2}\}^{-1}\sum_i\frac{1}{\pi}\int_{C_i}[\Delta_i(s,\lambda^2)-\Delta_i(4,\lambda^2)]\Phi(\lambda^2)d\lambda^2. \quad (23)$$

The $\Delta_i(s,\lambda^2)$ are the discontinuities around the λ^2 branch points of $f(s,\lambda^2)$, multiplied by the factor $(2/\pi)(2\pi i)^{-2}$ [cf. Eq. (22)]. They will be evaluated below (Sec. V).

To compare with Anisovich, we neglect the logarithmic terms and set $(q^2+1)^{1/2}=1$, obtaining the nonrelativistic scattering-length equation

$$\Phi(s)=Giqa/(1-iqa)+[2a/(1-iqa)]\sum_i\frac{1}{\pi}\int_{C_i}[\Delta_i(s,\lambda^2)-\Delta_i(4,\lambda^2)]\Phi(\lambda^2)d\lambda^2. \quad (23')$$

This scattering-length approximation—the s-wave Chew-Mandelstam solution[19]—will not give resonance behavior. To obtain a resonance, it is conventionally thought necessary to go to p waves. However, to keep our model simple, we can suppose that an s-wave π-π resonance occurs. As in the work of Frazer and Fulco,[21] this will result from the second choice for N,

$$N=b/(s+s_0).$$

In this case no subtraction is necessary, and an exactly analogous calculation shows that

$$D=1+bq^2[h(q^2)-h(-q_0^2)]/(q^2+q_0^2)-ibq/[4(q^2+1)^{1/2}(q^2+q_0^2)], \quad (24)$$

where

$$h(q^2)=[H(q^2)-a^{-1}]/4q^2,$$
$$s_0=4+4q_0^2,$$

leading to the solution for Φ_1, representing a resonance,

$$\Phi_1(s)=G\frac{4q_\gamma^2+4\gamma q^2/b}{\{4q_r^2-4q^2[1-\gamma h(q^2)]-i\theta(q^2)\gamma q/(q^2+1)^{1/2}\}}, \quad (25)$$

where

$$q_r^2=\gamma q_0^2/b,$$

and

$$\gamma=b/[bh(-q_0^2)-1]. \quad (26)$$

Using Eq. (22′), Eq. (20) for Φ now reduces to

$$\Phi(s)=\Phi_1(s)+2\gamma\{4q_r^2-4q^2[1-\gamma h(q^2)]-i\gamma q/(q^2+1)^{1/2}\}^{-1}\sum_i\frac{1}{\pi}\int_{C_i}[\Delta_i(s,\lambda^2)-\Delta_i(-s_0,\lambda^2)]\Phi(\lambda^2)d\lambda^2, \quad (27)$$

exactly analogous to (23).

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[20] J. F. Jackson and G. L. Kane, Nuovo Cimento 23, 444 (1962).
[21] W. R. Frazer and J. R. Fulco, Phys. Rev. 117, 1609 (1960).

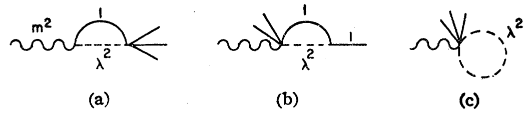

FIG. 7. (a) Reduced diagram giving the singularity $\lambda^2=(m-1)^2$ (square root). (b) Reduced diagram giving the singularity $\lambda^2=0$ (square root). (c) Reduced diagram giving the singularity $\lambda^2=0$ (logarithm).

We now have to evaluate the Δ_i and specify the contours C_i.

V. EVALUATION OF THE DISCONTINUITIES $\Delta_i(s,\lambda^2)$

Cutkosky has given rules for calculating the discontinuities around all singularities of any Feynman graph[16]; the only novel feature here is that we are considering singularities of $f(s,\lambda^2)$ in the internal mass λ^2, rather than an external one such as s.

In fact, an analysis of the λ^2 singularities of $f(s,\lambda^2)$ was made in some detail by Bronzan and Kacser[3] for fixed real $s\geq 4$, and extended to all s, λ^2 by Kacser and the present author.[22] We first of all list the results for the physical sheet singularities of $f(s,\lambda^2)$, labeled by the index i.

$i=1$, $\lambda^2=(m-1)^2$. This arises from the normal threshold in the m^2 channel, i.e., the reduced Landau diagram of Fig. 7(a). It is a square-root branch point.

$i=2$, $\lambda^2=0$. This is a square-root branch point associated with the reduced graph of Fig. 7(b). In general the singularity would not be at zero, but rather, analogously to the case $i=1$, at the point $\lambda^2=(M_1-M_2)^2$, where M_1 and M_2 are the internal and external masses at the vertex from which particle π_1 leaves. It is at zero only accidentally, because we have chosen an equal-mass example.

$i=3$, $\lambda^2=0$. This comes from the reduced graph of Fig. 7(c). A logarithmic singularity, it is the one unusual feature of the present analysis.[23]

$i=4$, $\lambda^2=\lambda_-^2(s)$. This is also logarithmic in nature, and comes from the uncontracted diagram Fig. 4 itself. $\lambda_-^2(s)$ is one root of $\Gamma=0$ [cf. Eq. (4)]. This singularity appears on the (unphysical) boundary of the physical sheet for $s\geq\frac{1}{2}(m^2-1)$.

Following Cutkosky's rules, it is simple to calculate the discontinuities Δ_1 and Δ_2. We find

$$\Delta_1(s,\lambda^2)=k^{-1}(s,m^2,1)\ \ln[(R-U^{1/2})/(R+U^{1/2})], \quad (28)$$

where

$$R=-m^4+m^2(s+\lambda^2)+(\lambda^2-1)(s-1),$$
$$U^{1/2}=k(s,m^2,1)k(\lambda^2,m^2,1),$$

$$k(a^2,b^2,c^2)=\{[a^2-(b-c)^2][a^2-(b+c)^2]\}^{1/2},$$
symmetric in a, b, and c,

and

$$\Delta_2(s,\lambda^2)=k^{-1}(s,m^2,1)\ \ln[(R'-U'^{1/2})/(R'+U'^{1/2})], \quad (29)$$

where

$$R'=-2+m^2+s+\lambda^2-(\lambda^2-1)(m^2-s)$$
$$U'^{1/2}=k(s,m^2,1)k(\lambda^2,1,1).$$

The calculation of Δ_3 is not quite so straightforward, and is discussed in Appendix A. The result, Eq. (A8), is rather complicated and will not be repeated here. Finally, Δ_4 is just the discontinuity across a singularity of the logarithm in Eq. (28), i.e., $\pm 2\pi i/k(s,m^2,1)$. In Appendix B, we show that Δ_4 in fact need not enter explicitly into the final equation.

To each singularity $\lambda^2=(m-1)^2$, $\lambda^2=0$ (twice) we attach a cut going along the real λ^2 axis to $-\infty$. The physical boundary of $f(s,\lambda^2$)is the limit onto the real λ^2 axis from below [cf. the remarks after Eq. (9)], so that the contours C_i are conveniently taken as lines running just under the real axis, as shown in Fig. 8. Only C_1 depends on m^2; it is the singularity $\lambda^2=(m-1)^2$ which, as we increase m^2 to the decay region $m^2>9$, overlaps the right-hand cut of Φ. To resolve the ambiguity, we give m^2 a small positive imaginary part, according to our analyticity assumptions; this shifts the cut C_1 to lie just about the real λ^2 axis (Fig. 8).

Returning now to our basic equation, (23) or (27), we recall that the physical region in s is $4\leq s\leq(m-1)^2$. The left-hand sides of these equations express $\Phi(s)$ in this range as integrals over λ^2 of $\Phi(\lambda^2)$ again, but now for all $\lambda^2\leq(m-1)^2$; with the choice of contours C_i of Fig. 8, only real values of λ^2 enter. Thus in order to be able to solve these equations, we must specify the branches of the various logarithms in the kernels $\Delta_i(s,\lambda^2)$, for all real s, $\lambda^2\leq(m-1)^2$ (not just for s in $4\leq s\leq(m-1)^2$).[24]

The details of this specification are relegated to Appendix B. We give here only a brief discussion of the answer. We start by defining the logarithms to be on their principal branch when λ^2 is large and negative, and when s is in the range $4\leq s\leq(m-1)^2$. They leave their principal branches at singularities of the $\Delta_i(s,\lambda^2)$; and these we know in general[16] lie only at the Landau singularities of the triangle graph amplitude $f(s,\lambda^2)$. As an illustration of this, we see that the logarithm in $\Delta_1(s,\lambda^2)$

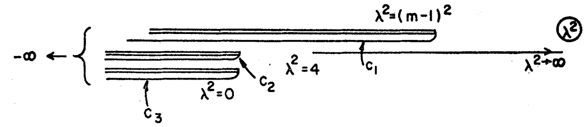

FIG. 8. The λ^2 plane, showing the cuts (double lines) and contours (single lines).

[22] I. J. R. Aitchison and C. Kacser, Phys. Rev. 133, B1239 (1964).

[23] I take this opportunity to apologize to those readers of Ref. 22 who penetrated into Appendix B of it and were puzzled. There are actually two singularities superimposed at $\lambda^2=0$: one a square root, the other a logarithm. The coincidence in position is accidental, as explained above. Hence any crossing of the real λ^2 axis below $\lambda^2=0$ would involve crossing *three* cuts.

[24] I am grateful to Dr. C. Kacser for pointing this out to me.

is singular when

$$R=\pm(U)^{1/2}, \quad \text{or} \quad R^2=U. \tag{30}$$

Equation (30) can be rewritten as

$$4m^2\Gamma(s,\lambda^2)=0 \tag{30$'$}$$

[see Eq. (4′)]. Thus the branch points of Δ_1 are at the points $\lambda^2=\lambda_{\pm}{}^2(s)$, the intersections of $\Gamma=0$ with a line of fixed real s:

$$\lambda_{\pm}{}^2(s)=(3+m^2-s)/2\pm(2s)^{-1}k(s,m^2,1)k(s,1,1). \tag{31}$$

$\Gamma=0$ is exactly the Landau surface of singularities corresponding to the uncontracted triangle graph.[3]

The integration contour C_1 being chosen along the real λ^2 axis, $\lambda^2\leq(m-1)^2$, only real intersections $\lambda_{\pm}{}^2(s)$ interest us. They can be immediately read off Fig. 2(a): they lie in regions III and $\mathfrak{D}$ only. In Appendix B we show that the correct definition of the logarithm in Δ_1 is that it acquires an $i\pi$ for s, λ^2 in $\mathfrak{D}$, and a $-i\pi$ for s, λ^2 in III. That is,

$$\begin{aligned}\Delta_1=\bar{\Delta}_1&=k^{-1}(s,m^2,1)\ln|(R-U^{1/2})/(R+U^{1/2})|, \\ &\qquad\text{for } s,\ \lambda^2 \text{ not in } \mathfrak{D} \text{ or III} \\ &=\bar{\Delta}_1+i\pi k^{-1}(s,m^2,1) \quad \text{for } s,\ \lambda^2 \text{ in } \mathfrak{D} \\ &=\bar{\Delta}_1-i\pi k^{-1}(s,m^2,1) \quad \text{for } s,\ \lambda^2 \text{ in III}.\end{aligned} \tag{32}$$

In the same way, the branch points of the logarithm in Δ_2 occur when

$$R'=\pm(U'^{1/2}), \quad \text{or} \quad 4\Gamma=0.$$

This time, however, the contour C_2 is over real $\lambda^2\leq0$; thus only the intersections $\lambda_{\pm}{}^2(s)$ in region III are relevant. In Appendix B, we show that Δ_2 acquires an imaginary part $+i\pi k^{-1}(s,m^2,1)$ in III; hence in III the imaginary parts of Δ_1 and Δ_2 just cancel. Finally, we find that Δ_3 has an imaginary part $+i\pi k^{-1}(s,m^2,1)$ in III.

Assembling these results from Appendix B, we obtain finally, for all real $s\leq(m-1)^2$ and $m^2\geq9$:

$$\begin{aligned}\Phi(s)=\Phi_1(s)&+\frac{2}{\pi}k^{-1}(s,m^2,1)D^{-1}(s)\left\{\int_{-\infty}^{(m-1)^2}d\lambda^2\Phi(\lambda^2)\ln|(R-U^{1/2})/(R+U^{1/2})|\right.\\ &\left.+\int_{-\infty}^{0}d\lambda^2\Phi(\lambda^2)\ln|(R'-U'^{1/2})/(R'+U'^{1/2})|+\int_{-\infty}^{0}d\lambda^2\Phi(\lambda^2)|\bar{\Delta}_3|\right\}\\ &+\begin{cases}\dfrac{2}{\pi}k^{-1}(s,m^2,1)D^{-1}(s)\displaystyle\int_{\lambda_-{}^2(s)}^{\lambda_+{}^2(s)}d\lambda^2 i\pi\Phi(\lambda^2) & \text{for } s,\ \lambda^2 \text{ in } \mathfrak{D}\\ \dfrac{2}{\pi}k^{-1}(s,m^2,1)D^{-1}(s)\displaystyle\int_{-\infty}^{\lambda_-{}^2(s)}d\lambda^2 i\pi\Phi(\lambda^2) & \text{for } s,\ \lambda^2 \text{ in III}\end{cases}.\end{aligned} \tag{33}$$

Equation (33) is the main result of this paper, and we make the following comments on it.

Remark 1. The extra $i\pi$ terms enter precisely in regions which have a physical significance: $\mathfrak{D}$ is the physical region for the actual decay under consideration, and III is that for the crossed reaction of scattering in the u channel. The $i\pi$'s are dictated by the $i\epsilon$ attached to s—that is, by the unitarity prescription in the s channel. It may be verified that the answer is the same if we allow the small imaginary part of m^2 to dominate: that is, if we invoked unitarity in the "three-particle" m^2 channel. This equivalence of the two prescriptions was first noted by Bronzan and Kacser[3] for the triangle amplitude $f(s,\lambda^2)$ itself; it has recently been discussed by Hwa,[11] for a situation rather similar to ours, but with reference to the principle of maximum analyticity.

Remark 2. The three integrals in braces in Eq. (33) have singular kernels. We have defined how to pass these branch points, but clearly their presence will complicate the numerical inversion of Eq. (33). Nonetheless, we have no second-type singularity, which makes the corresponding equation for $F_0(s)$ even worse (see above, IIB). In fact, Danilov,[8] and Danilov and Lebedev,[25] have shown that an equation of the form (33) has a unique solution, thanks to its asymptotic behavior. We shall take up this point again in Sec. VI when we consider the nonrelativistic limit of Eq. (33).

Remark 3. Since Eq. (33) is evidently fairly difficult to resolve numerically, we might ask if any important properties of the solution can be read off immediately. In particular, we may look for the second-sheet singularities of $\Phi(s)$, near the physical region. Returning to Eq. (16) with $\phi(s,\lambda^2)$ replaced by $f(s,\lambda^2)$, it is clear that the question is simply answered if we know the λ^2 properties of $f(s,\lambda^2)$ as s is continued across the $s\geq4$ cut, down into the second sheet These have been given elsewhere.[22] As s is continued into sheet two in the region $4\leq s\leq(m+1)$, a λ^2 singularity of $f(s,\lambda^2)$ appears [in fact, $\lambda_+{}^2(s)$] for $\lambda^2\geq(m+1)^2$, forcing a distortion of the λ^2 contour C in Eq. (16) downward into the lower half-λ^2 plane. If, then, $\Phi(\lambda^2)$ has a second-sheet pole at

[25] G. S. Danilov and V. I. Lebedev, Zh. Eksperim. i Teor. Fiz. 44, 1509 (1963) [English transl.: Soviet Phys.—JETP 11, 1015 (1963)].

$\lambda^2=I^2$, say, (a resonance of mass I) as in the model of Eq. (25), we will find a singularity in s when $\lambda_+^2(s)$ hits I^2, provided I^2 is suitably placed[26]: The contribution to the amplitude near the singularity will then be proportional to

$$D^{-1}(s)f(s,I^2),$$

where $f(s,I^2)$ is the triangle graph with a complex internal mass. Such graphs have been studied,[5,27] and found to give, characteristically, effects near certain low-energy boundaries of the Dalitz plot. Their effect seems likely to be small. Since this is the nearest singularity, it would appear that the final-state energy spectra are likely to be fitted pretty well by the usual Breit-Wigner terms $D^{-1}(s)$ alone, in agreement with the analysis of Bouchiat and Flamand.[28]

The situation may be quite different, however, for the m^2 spectrum. We are unable to real off nearby singularities in m^2, and it could well be that many apparently unimportant effects in s, t, u could conspire to give a big effect in m^2. A numerical resolution of Eq. (33) seems well worthwhile, therefore, in view of the evident richness of three-particle systems now being found experimentally.

VI. THE EQUATION IN THE NONRELATIVISTIC LIMIT

In order to compare our result with the work of Anisovich, and with potential theory, we now examine the nonrelativistic limit of Eq. (33). We write

$$s=4+4q^2,\quad \lambda^2=4+4p^2,\quad m=3+\kappa^2,$$

and take first-order terms in q^2, p^2, and k^2. In this limit the integrals beginning at $\lambda^2=0$ or $p^2=-1$ may be neglected. The nonrelativistic limit of $\Delta_1(s,\lambda^2)$ is then $-\Delta(q^2,p^2)$, where

$$\Delta(q^2,p^2)=[4(3)^{1/2}(\kappa^2-q^2)^{1/2}]^{-1} \times\ln\left\{\frac{[\frac{1}{2}(\kappa^2-p^2)^{1/2}-(\kappa^2-q^2)^{1/2}]^2-\frac{3}{4}p^2}{[\frac{1}{2}(\kappa^2-p^2)^{1/2}+(\kappa^2-q^2)^{1/2}]^2-\frac{3}{4}p^2}\right\}.$$

For the scattering-length model of Eq. (23), we thus obtain for Φ

$$\Phi(q^2)=Giqa/(1-iqa)+2a/(1-iqa)\pi \times\int_{\kappa^2}^{-\infty}4dp^2[\Delta(q^2,p^2)-\Delta(0,p^2)]\Phi(p^2).\quad (34)$$

Equation (34) is identical with Eqs. (20) and (23) of Anisovich.

The nonrelativistic limit of $\Gamma=0$ is

$$q^4+p^2q^2+p^4-\tfrac{3}{2}p^2\kappa^2-\tfrac{3}{2}q^2\kappa^2+(9/16)\kappa^4=0$$

which gives an ellipse in the (q^2,p^2) plane for the Dalitz region $\mathfrak{D}$. The definition of $\Delta(q^2,p^2)$ is, as before, that the logarithm acquires an imaginary part of π inside $\mathfrak{D}$; otherwise Δ is real.

What appears to be the most interesting feature of Eq. (34) is that its kernel is of exactly the same type as arises in the potential theory of three-particle scattering, using zero range (δ-function) forces. This theory was developed by Skornyakov and Ter-Martyrosyan,[7] and applied to the neutron-deuteron problem. It turned out that the equation analogous to Eq. (34) converged without a subtraction in the quartet spin case, so that no information additional to the two-body parameters was needed to calculate, for example, the scattering length $a_{3/2}$; and the agreement with experiment was good. For the doublet case, however, the calculation was not so successful. A careful investigation by Danilov[8] showed that the reason was that a subtraction was needed in this case, due essentially to the possibility of a bound state (the triton). Introducing the triton binding energy as an extra parameter, Danilov and Lebedev[25] were able to get satisfactory agreement for $a_{1/2}$ too.

This analogy with potential theory encourages us to hope that the dispersion model, Eq. (33), may be a good one for the relativistic three-body problem.

VII. CONCLUSION

First we summarize what has been done. We treated three-particle production and decay processes in an identical way, by assuming that in both cases the amplitude was determined entirely by the final state interactions (f.s.i.) among pairs of the three particles. The dynamical model chosen was a single variable dispersion representation of the Khuri-Treiman[1] type; this satisfies two-body unitarity in each two-body channel, within the approximation that only one partial wave is dominant. However, it violates crossing symmetry. The "right-hand cut" of the resulting equation was first factored out by the Omnes transformation. The equation then involved two parts, of which one, representing f.s.i. in only one channel, was immediately solved. The other represented the effects of f.s.i. in two or three parallel channels. This second term was then transformed by the technique of Anisovich,[6] and we obtained finally a linear, single variable, integral equation which determined the amplitude. The kernel of this equation involved only the two-body amplitude g, and assumed a particularly simple form if we took an effective range type of approximation for g. This approximation, however, still enables us to treat both low-energy (effective-range) effects, and phenomena associated with several final-state resonances. In both cases, the final equation bears a close resemblance to an equation derived by Skornyakov and Ter-Martyrosyan,[7] for the three-body problem in potential theory, using short-range forces. It would thus appear to be a relativistic analog of their potential-theory model.

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[26] The possibility of such a nearby singularity, and the conditions under which it occurs, were first pointed out by G. Bonnevay, Ref. 2. See also Refs. 5, 22, and 27.

[27] I. J. R. Aitchison, Phys. Rev. 133, B257 (1964).

[28] C. Bouchiat and G. Elamand, Nuovo Cimento 23, 13 (1962).

Let us now turn to applications. The most obvious application of Eq. (33) would be to try to explain the pion spectra in $K \to 3\pi$ decay, in terms of the π-π data. To do this one has to include isotopic spin, obtaining then two coupled equations for the $J=T=0$, $J=0$, $T=2$ amplitudes.[6] (There are in general, as many amplitudes as there are different channels.) But, while we may reasonably assume that the $T=0$ amplitude is dominant, so that the equations decouple, the experimental situation at present is such that unless a $T=J=0$ π-π resonance occurs, no test of the theory is possible. The reason is that, with no such resonance, two subtractions will be needed in Eq. (33). It follows that only the quadratic dependence on s will be free of unknown constants, whereas, unfortunately, only linear terms can be measured with any accuracy at the moment.[29] Nonetheless, there is a fair amount of evidence in favor of such a $T=J=0$ π-π resonance (the σ meson),[30] and, in any case, the problem of investigating the effect on the density of points in the Dalitz plot due to two or three simultaneous final-state interactions is well worth investigating numerically, using Eq. (33) as a model.

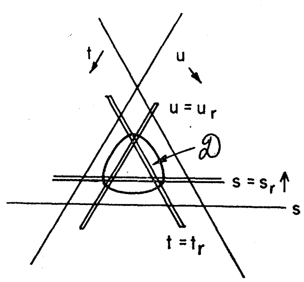

FIG. 9. The decay region $\mathfrak{D}$ plotted in triangular coordinates, showing resonance bands at s_r, t_r, and u_r.

As remarked in Sec. II, though, the dependence on the final-state energies is not all that can be got from Eq. (33). We first recall Eq. (2):

$$s+t+u=3+m^2\,, \tag{2}$$

where m is the total c.m. energy of the three-particle system. The three variables s, t, u can be exhibited more symmetrically than we have done in Figs. 2(a) and 2(b) by using triangular coordinates.[10] The Dalitz plot, or decay region $\mathfrak{D}$, then takes the form shown in Fig. 9. Suppose now that there are strong resonances in the s and u channels: that is, most of the events fall along the bands $s=s_r$, $u=u_r$. Then, for most events, Eq. (2) becomes

$$t+s_r+u_r=3+m^2\,. \tag{2'}$$

Consider now the effect of a third resonance, in t, at $t=t_r$. By Eq. (2′), this will appear as a resonance in m^2, at $m^2=m_r^2=(t_r+s_r+u_r-3)$. And, vice versa, a resonance in m^2 at m_r^2, will appear as one in t, at t_r.[31] Thus, *in so far as the m^2 dependence is determined entirely by the final-state interactions*, three simultaneous (i.e., all physically realizable) resonances in the final state may lead to an apparent resonance in the total c.m. energy variable (and vice versa).

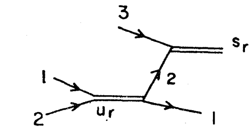

FIG. 10. A real particle exchange process.

But for such an effect it is perhaps not necessary to have *three* final-state resonances at once; two may, in themselves, be enough to induce a nearby singularity in m^2. This phenomenon was first discussed in a rather different context by Peierls.[12] To see how it comes about, let us fix s at the resonance value s_r, and consider the resonance pole in u, $(u-u_r)^{-1}$. This term is represented by Fig. 10. The integrated effect of this pole on the amplitude is then, using Eqs. (2), (3), and (3′),

$$\int_{t_-(s_r)}^{t_+(s_r)} dt(3+m^2-s_r-u_r-t)^{-1}.$$

Thus it will lead to logarithmic singularities in m^2 at the points

$$m_\pm^2=s_r+u_r-3+t_\pm(s_r)\,. \tag{35}$$

Recalling that $t_\pm(s)$ are the intersections of $\mathfrak{D}$ with a line of fixed s, we see that (35) implies an effect in m^2 whenever the bands $u=u_r$, $s=s_r$ cross on the boundary of $\mathfrak{D}$.[27]

Now, of course, Fig. 10 is not what happens experimentally; rather, one has to couple it to an initial state consisting of less than three particles, and here is where the doubts about this enhancement mechanism start. The simplest graph involving Fig. 11 and a two-particle initial state, Fig. 11, can be shown to lead to an effect in m^2 only in very special cases; even then, it is unlikely to be very striking.[17,27,32] A discussion of some possible examples where the effect of Fig. 11 might be seen has been given by Chang and Tuan,[33] and by Kacser,[34] for strange particle reactions, and by Kacser and the author for some nuclear physics reactions.[35]

The possibility remains, however, that repeated iterations of Fig. 10, of the type implied in Fig. 3(c) and

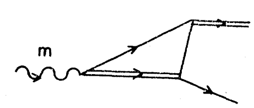

FIG. 11. Triangle graph involving resonance internally.

[29] This remark is due to Dr. C. Kacser, to whom I am indebted for discussions on this point.

[30] L. M. Brown and P. Singer, Phys. Rev. Letters **8**, 460 (1962); Phys. Rev. **133**, B812 (1964). Also A. N. Mitra and S. Ray, *ibid.* **135**, B146 (1964), and A. O. Barut and W. S. Au, Phys. Rev. Letters **13**, 165 (1964), and experimental references cited therein.

[31] Indeed it appears that the major part of the effect in the $K\bar{K}\pi$ system reported by R. Armenteros, *et al.* at the Sienna International Conference, 1963 (unpublished), could be the result of three strong interactions: K^*, $\bar{K}^*$, and $(K\bar{K})$.

[32] C. J. Goebel, Phys. Rev. Letters **13**, 143 (1964).

[33] Y. F. Chang and S. F. Tuan, Purdue University report, 1964 (unpublished).

[34] C. Kacser, Phys. Letters **12**, 269 (1964).

[35] I. J. R. Aitchison and C. Kacser (to be published).

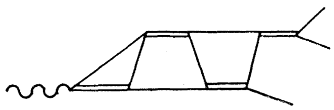

FIG. 12. Iterations of the real particle exchange process.

illustrated in Fig. 12, could lead to a more pronounced effect in m^2. These are processes in which the exchanged particles may be real. In this connection, it is encouraging that such a mechanism appears capable of generating a three-body resonance pole in the Lee model.[36] Certainly the experimental results on peaks in three-meson systems, involving two-meson resonances,[37] are sufficiently interesting to warrant the investigation of this possibility by numerical solution of Eq. (33).[38]

ACKNOWLEDGMENTS

I am very happy to thank Dr. C. Kacser, who suggested to me the application of Anisovich's technique to the Khuri-Treiman representation, for stimulating discussions and correspondence; I am especially indebted to him for some very helpful and pertinent criticisms of the original draft. I also gratefully acknowledge several patient explanations from Dr. F. Pham of the method of calculating discontinuities of Feynman graphs in general, and of its application to the calculation of Δ_3 in particular. Finally, it is a pleasure to thank Professor A. Messiah and the members of his group at Saclay for their kind hospitality.

APPENDIX A

We give here the calculation of $\Delta_3(s,\lambda^2)$, the discontinuity around the branch point $\lambda^2=0$ associated with the reduced graph Fig. 7(c). Following Cutkosky[16] we have

$$\Delta_3(s,\lambda^2)=\frac{2(2\pi i)}{\pi(2\pi i)^2}\int\frac{d^4k\;\delta(k_0{}^2-\mathbf{k}^2-\lambda^2)}{(p_1{}^2-2p_1\cdot k+k^2-1)(p_2{}^2-2p_2\cdot k+k^2-M^2)}\,, \tag{A1}$$

where $k=(k_0,\mathbf{k})$, p_1, p_2 are the four-momenta involved in the uncontracted graph (Fig. 9). In Eq. (A1) we have introduced the mass M of the internal particle at the vertex 1 in order to avoid confusion between the (normal threshold) branch point at $\lambda^2=(M-1)^2$ (cf. Sec. V; this is the singularity $i=2$), and the present singularity $\lambda^2=0$. We shall take $M\to 1$ later. Since Eq. (A1) is invariant we may evaluate it in the frame $p_1=(m,0)$, obtaining

$$\Delta_3(s,\lambda^2)=\frac{-i}{\pi p}\int\frac{|\mathbf{k}|d|\mathbf{k}|dk_0\;\delta(k_0{}^2-\mathbf{k}^2-\lambda^2)}{(m^2-2mk_0+k_0{}^2-\mathbf{k}^2-1)}\ln\left\{\frac{M^2-1+k_0{}^2-\mathbf{k}^2-2p_0k_0+2p|\mathbf{k}|}{M^2-1+k_0{}^2-\mathbf{k}^2-2p_0k_0-2p|\mathbf{k}|}\right\}, \tag{A2}$$

where $p=|\mathbf{p}_2|$, $p_0=p_{20}$. The two-dimensional integration has to be done subject to the constraint $k_0{}^2-\mathbf{k}^2=\lambda^2$, expressed by the δ function. In the $(|\mathbf{k}|,k_0)$ plane this is a hyperbola, Fig. 14. We see at this point that Eq. (A2) is not a complete definition of Δ_3; we need to specify more precisely the area of integration. The correct prescription for obtaining discontinuities from Cutkosky's rules in general[39] is to integrate over some closed cycle that vanishes as we approach the Landau singularity in question. In the present case, the singularity being $\lambda^2=0$, it is clear from Fig. 14 that we have to integrate over the complex part of the hyperbola, shown dashed in Fig. 14; or, putting $|\mathbf{k}|=i\alpha$, over the circle $k_0{}^2+\alpha^2=\lambda^2$. Introducing polar variables

$$\alpha=r\sin\theta,\quad k_0=r\cos\theta,\quad d\alpha dk_0=rdrd\cos\theta,$$

Eq. (A2) becomes

$$\Delta_3(s,\lambda^2)=\frac{i}{2\pi p}\int_0^{2\pi}\frac{\sin\theta d\theta dr^2\delta(r^2-\lambda^2)}{(m^2-2mr\cos\theta+r^2-1)}\ln\left\{\frac{r^2-2rp_{20}\cos\theta+2ip\,r\sin\theta+M^2-1}{r^2-2rp_{20}\cos\theta-2ip\,r\sin\theta+M^2-1}\right\}. \tag{A3}$$

The r^2 integration is done by means of the δ function, leaving

$$\Delta_3(s,\lambda^2)=\tilde{\Delta}_3(s,\lambda)+\tilde{\Delta}_3(s,-\lambda)\,, \tag{A4}$$

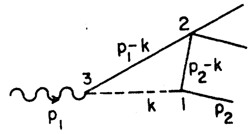

FIG. 13. Triangle graph in perturbation theory.

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[36] I. J. R. Aitchison, Nuovo Cimento **34**, 508 (1964).
[37] See references cited in Refs. 13 and 14.
[38] We remark that our treatment refers only to inelastic reactions; the coupling back to the initial state, necessary for describing an elastic reaction in our terms, is something we have not dealt with at all. It was actually this case that was first discussed by Peierls (Ref. 11).
[39] F. Pham (private communication).

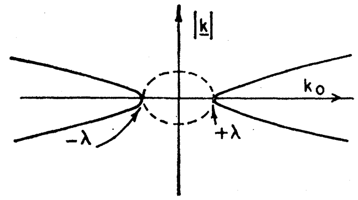


FIG. 14. The $(k_0,|k|)$ plane, showing the hyperbola $_0{}^2-|k|^2=\lambda^2$; the complex part is dashed.

with

$$\tilde{\Delta}_3(s,\lambda)=\frac{i\lambda}{2\pi p}\int_0^{\pi}\frac{\sin\theta d\theta}{(m^2-2m\lambda\cos\theta+\lambda^2-1)}\ln\left\{\frac{\lambda^2-2\lambda p_0\cos\theta+2ip\lambda\sin\theta+M^2-1}{\lambda^2-2\lambda p_0\cos\theta-2ip\lambda\sin\theta+M^2-1}\right\}. \quad \text{(A5)}$$

Making the substitution $t=\tan\frac{1}{2}\theta$ leads to

$$\tilde{\Delta}_3(s,\lambda)=-\frac{i\lambda}{\pi p}\int_{-\infty}^{\infty}\frac{tdt}{(t^2+1)(t^2+t_p{}^2)[(m+\lambda)^2-1]}\ln\left\{\frac{t^2(M^2-1+\lambda^2+2p_0\lambda)-4ip\lambda t+(M^2-1+\lambda^2-2p_0\lambda)}{t^2(M^2-1+\lambda^2+2p_0\lambda)+4ip\lambda t+(M^2-1+\lambda^2-2p_0\lambda)}\right\}, \quad \text{(A6)}$$

where

$$t_p=i\left[\frac{(m-\lambda)^2-1}{(m+\lambda)^2-1}\right]^{1/2}$$

and we have used the fact that the integrand is an even function of t. The logarithm has branch points at

$$t_{1,2}=(M^2-1+\lambda^2+2p_0\lambda)^{-1}\{2pi\lambda\pm[\{(M+1)^2-\lambda^2\}\{(M-1)^2-\lambda^2\}]^{1/2}\} \quad \text{for the numerator}$$

and

$$t_{3,4}=(M^2-1+\lambda^2+2p_0\lambda)^{-1}\{-2pi\lambda\pm[\{(M+1)^2-\lambda^2\}\{(M-1)^2-\lambda^2\}]^{1/2}\} \quad \text{for the denominator}. \quad \text{(A7)}$$

It is clear from (A7) that in order to decide in which half-plane the t_i are, it is necessary to choose a determination of the various square roots which enter. The roots $\lambda^2=(M\pm1)^2$ correspond, as we mentioned earlier, to thresholds at the vertex 1; the definition which is appropriate is that $\{[(M+1)^2-\lambda^2][(M-1)^2-\lambda^2]\}^{1/2}$ is positive for $\lambda^2<(M-1)^2$ [Fig. 17(b)]. The other definition we need is that of λ itself, since Δ_3 is to be evaluated for $\lambda^2\leq0$. The definition chosen is illustrated in Fig. 17(c). To make the task simpler, however, we know that we need only evaluate Δ_3 in the neighborhood of $\lambda^2=0$; since there are no other singularities below $\lambda^2=0$, the answer will be true for all $\lambda^2\leq0$. In that case, t_1 and t_3 are located in the upper half-λ^2 plane, while t_2 and t_4 are in the lower. The integral in Eq. (A6) can now be done, by splitting the denominator up into a sum of poles, and by closing the λ^2 contour in the half-plane opposite each t_i, thereby picking up only the pole contributions. The locations of the poles and branch points are shown in Fig. 15. We find, then,

$$\tilde{\Delta}_3(s,\lambda)=\frac{1}{2pm}\left\{\ln\left[\frac{(i+t_1)(t_p-t_2)}{(i-t_2)(t_p+t_1)}\right]\right\}.$$

Eventually, after some algebra, this leads, taking the limit $M\to1$, to

$$\Delta_3(s,\lambda^2)=k^{-1}(s,m^2,1)\ln\left\{\frac{[(-\lambda^2)^{1/2}-(4-\lambda^2)^{1/2}]^2+4(p_0+p)^2}{[(-\lambda^2)^{1/2}-(4-\lambda^2)^{1/2}]^2+4(p_0-p^2)}\,\frac{R_1k(\lambda^2,m^2,1)+R_2k(s,m^2,1)+R_3k(\lambda^2,1,1)}{R_1k(\lambda^2,m^2,1)-R_2k(s,m^2,1)+R_3k(\lambda^2,1,1)}\right\}, \quad \text{(A8)}$$

where

$$R_1=4p_0{}^2-\lambda^2,\quad R_2=2[\lambda^2-(p_0/m)(m^2+\lambda^2-1)],\quad R_3=3+m^2-2s-\lambda^2.$$

In the frame $\mathbf{p}_1=0$, we have

$$p=k(s,m^2,1)/2m,\quad p_0=(m^2+1-s)/2m.$$

We observe that (A8) is an even function of p, or $k(s,m^2,1)$, as it should be from Eq. (A2).

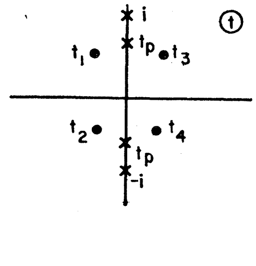


FIG. 15. The t plane, showing the poles at $t=\pm i,\pm t_p$ and the branch points at $t=t_1$, t_2, t_3 and t_4.

APPENDIX B

Here we explain the choice of the branch of the logarithm in $\Delta_1(s,\lambda^2)$, Eq. (32) of the text. From Cutkosky's rules,[16] we find at once that the discontinuity around the $\lambda^2=(m-1)^2$ branch point is

$$\Delta_1(s,\lambda^2)=k^{-1}(s,m^2,1)\int_{-1}^{1}\frac{dz}{z-R/U^{1/2}}, \qquad \text{(B1)}$$

where

$$R=-m^4+m^2(s+\lambda^2)+(\lambda^2-1)(s-1), \qquad \text{(B2)}$$

$$U^{1/2}=k(s,m^2,1)k(\lambda^2,m^2,1).$$

This discontinuity is just the same as that around the identical, but more familiar, branch point $m^2=(\lambda+1)^2$, We now follow the method of Bronzan and Kacser[3] in determining the phase of the integral in (B1). Write

$$L=\int_{-1}^{1}\frac{dz}{z-R/U^{1/2}}=\ln\left|\frac{R-U^{1/2}}{R+U^{1/2}}\right|+i(\text{phase of } L).$$

It is clear that L has branch points in λ^2 for fixed s when $R=\pm U^{1/2}$ or $R^2=U$. This can be written as

$$4m^2\Gamma=0,$$

which has roots

$$\lambda^2=\lambda_\pm^2(s)=(3+m^2-s)/2\pm(1/2s)k(s,m^2,1)k(s,1,1).$$

$\Gamma=0$ is drawn in Fig. 2(a); as explained in Sec. V, we are interested only in the regions III and $\mathfrak{D}$. $\lambda_\pm^2(s)$ are the intersections of λ^2 with a line of fixed s, the real s axis being approached, in the physical limit, from above. Let us first get the positions of $\lambda_\pm^2(s)$ clear. The regions III and $\mathfrak{D}$ are drawn again in Fig. 16(a), with the two arcs corresponding to the two roots $\lambda_+^2(s)$, $\lambda_-^2(s)$ indicated separately; in Fig. 16(b) we draw the motion of $\lambda_\pm(s)$ in the λ^2 plane, as s increases from $-\infty$ to $+\infty$, for the case $m>3$. This latter figure is obtained from Fig. 2(a) by considering its intersections with lines of fixed s, with small positive imaginary part [cf. Eq. (6)].

From each of $\lambda_+^2(s)$, $\lambda_-^2(s)$ we draw a cut running along the real λ^2 axis to $-\infty$. We define L to be on its principal branch when λ^2 is large and negative, and s is in $4\leq s\leq(m-1)^2$. Finally, we define $U^{1/2}$ to be positive for $4\leq\lambda^2\leq(m-1)^2$ and $s\leq(m-1)^2$; this choice for $U^{1/2}$ is illustrated in Fig. 17(a).

It remains to continue L by increasing λ^2 and decreasing s. As we do this we eventually come upon the branch points $\lambda_\pm^2(s)$. To understand what happens then, we first plot R in the $s-\lambda^2$ plane, Fig. 18. We have shaded the integration region, which is the real λ^2 axis for $\lambda^2\leq(m-1)^2$, and the region in s in which Δ_1 must be defined, $-\infty\leq s\leq(m-1)^2$.

In the λ^2 plane, for a given s, R is a straight line. As we change s, this line changes its slope and position. The points $\lambda_\pm^2(s)$ are the intersections of R with the curves $\pm U^{1/2}$ [the sign of the square roots being determined from Fig. 17(a)]. From Fig. 18 we see that three distinct situations arise.

A. $s\leq 1-m^2$

R vanishes for $\lambda^2\leq 1-m^2$, on the integration path. A schematic picture of R and $U^{1/2}$ for this case is given in Fig. 19(a). It is clear that there is a root λ_-^2 in region III, corresponding to $R=-U^{1/2}$. Hence we may rewrite L as (compare Ref. 27).

$$L=2\ln(R-U^{1/2})-\ln(R^2-U),$$

where $R-U^{1/2}$ is regular at $\lambda_-^2(s)$; or using Eq. (30′) as

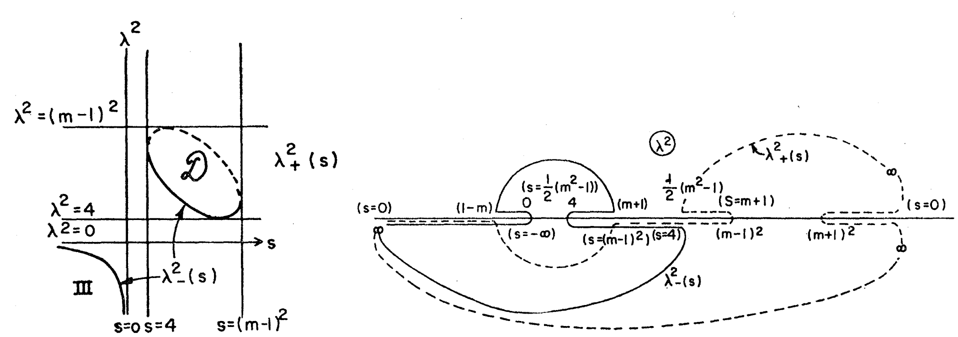
$$L=\text{regular function}-\ln[\lambda^2-\lambda_-^2(s)]. \qquad \text{(B3)}$$

Referring to Fig. 16(b) where the roots are displaced from the real axis according to the small positive imaginary part of s, we see that for $s\leq 1-m^2$ the phase of L is $-\pi$ for s, λ^2 in III and zero otherwise.

As s increases from $1-m^2$ to 4, R now vanishes for $\lambda^2\geq(m-1)^2$ outside the integration contours. However Fig. 19(a) remains substantially unchanged; all that happens is that the line R passes through zero slope at $s=1-m^2$ to positive slope for $s>1-m^2$; the phase of L is still $-\pi$ for s,λ^2 in III.

B. $4\leq s\leq(m+1)$

R vanishes for $(m-1)^2\leq\lambda^2\leq(m-1)^2$, outside the integration region. In Fig. 19(b) we plot R and $U^{1/2}$ for fixed s. It is clear that only the root $R=-U^{1/2}$ is



(a) (b)

FIG. 16. (a) The regions $\mathfrak{D}$ and III of $\Gamma=0$. (b) The motion of the roots $\lambda^2_\pm(s)$ in the λ^2 plane as s varies. The dashed curve is $\lambda^2_+(s)$, the solid $\lambda^2_-(s)$. The values of s are in brackets.

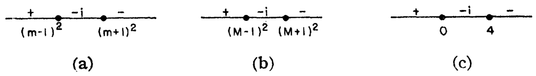


FIG. 17. (a) The definition of $[(\lambda^2-(m-1)^2)(\lambda^2-(m+1)^2)]^{1/2}$. (b) The definition of $[(\lambda^2-(M-1)^2)(\lambda^2-(M+1)^2)]^{1/2}$. (c) The definition of $[\lambda^2(\lambda^2-4)]^{1/2}$.

possible, for both $\lambda_+^2(s)$ and $\lambda_-^2(s)$. Hence we may rewrite L as

$$L=2\ln(R-U^{1/2})-\ln(R^2-U),$$

where $R-U^{1/2}$ is regular at $\lambda_\pm^2(s)$; or using Eq. (30′) as

$$L=\text{regular function}-\ln[(\lambda-{}^2\lambda_+^2(s))(\lambda-{}^2\lambda_-^2(s))]. \quad \text{(B4)}$$

Referring to Fig. 16(b), where the roots are displaced from the real axis according to the positive imaginary part of s, we see at once that for $4\leq s\leq(m+1)$, the phase of L is π for $\lambda_-^2(s)\leq\lambda^2\leq\lambda_+^2(s)$ and zero otherwise.

C. $(m+1)\leq s\leq(m-1)^2$

Now R vanishes for $(m+1)\leq\lambda^2\leq(m-1)^2$ on the integration contour, and the plot of R and $U^{1/2}$ looks like Fig. 20. We see that $\lambda_-^2(s)$ still corresponds to $R=-U^{1/2}$, but $\lambda_+^2(s)$ is now reached when $R=+U^{1/2}$. Thus near $\lambda_+^2(s)$ we write

$$\begin{aligned}L&=-2\ln(R+U^{1/2})+\ln(R^2-U)\\&=\text{regular function}+\ln(\lambda^2-\lambda_+^2(s)),\end{aligned}$$

while near $\lambda_-^2(s)$ we write Eq. (B2) as before.

Referring now to Fig. 16(b), we see that two further cases arise.

1. $(m+1)\leq s\leq\frac{1}{2}(m^2-1)$. For this range in s, $\lambda_-^2(s)$ remains (infinitesimally) below the real λ^2 axis, and the phase of L can be immediately read off; it is π in $\lambda_-^2(s)\leq\lambda^2\leq\lambda_+^2(s)$ as before, and zero otherwise.

2. $\frac{1}{2}(m^2-1)\leq s\leq(m-1)^2$. As soon as s gets larger than $\frac{1}{2}(m^2-1)$, though, the root $\lambda_-^2(s)$ crosses the λ^2 axis; that is, the small positive imaginary part of s instructs us to pass *under* the branch point $\lambda_-^2(s)$. But we have to remember that the contour of integration itself, C_1, runs just below the real axis (see Fig. 8). Hence as $\lambda_-^2(s)$ tries to cross the λ^2 axis, we have to deform C_1 upwards out of its way, to make the analytic continuation (see Fig. 21). What has happened, in fact, is that the singularity $\lambda_-^2(s)$ has emerged onto the physical sheet of the triangle graph (although on the unphysical edge of the λ^2 cut). Thus when, at Eq. (19) of the text, we shrink the large λ^2 contour C back onto the real axis, we have to include $\lambda_-^2(s)$ as one of the singularities of $f(s,\lambda^2)$, and we can draw the resulting

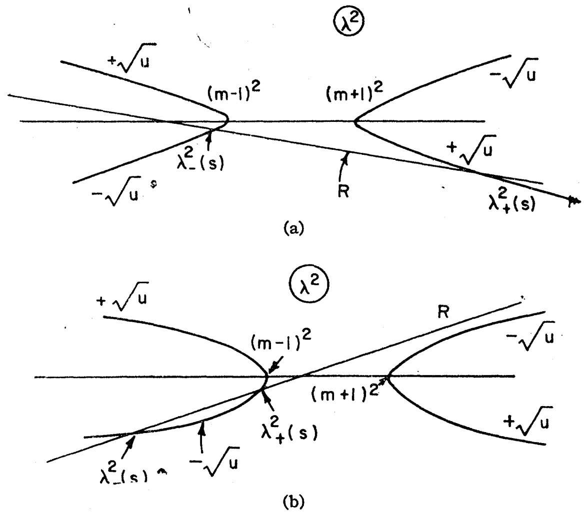


FIG. 19. (a) R and $U^{1/2}$ in the λ^2 plane for $s<1-m^2$. (b) R and $U^{1/2}$ in the λ^2 plane for $4<s<(m+1)$.

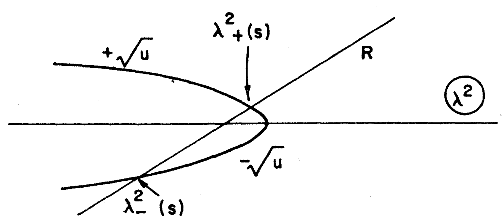


FIG. 20. R and $U^{1/2}$ in the λ^2 plane for $(M+1)<s<(m-1)^2$.

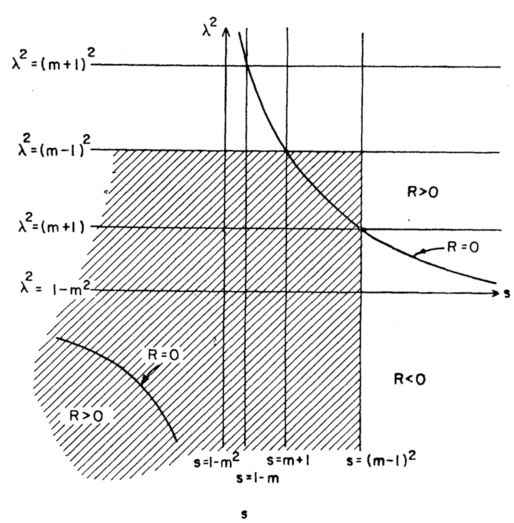


FIG. 18. The function R in the (s,λ^2) plane.

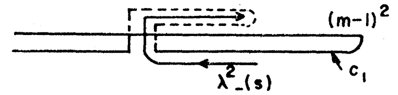


FIG. 21. The motion of the root $\lambda^2{}_-(s)$ for $\frac{1}{2}(m^2-1)\leqslant s\leqslant(m-1)^2$ causing a deformation of the contour C_1 upwards onto the second sheet (dashed) with respect to the square root cut attached to $\lambda^2=(m-1)^2$.

part of C as in Fig. 21. This singularity is just what we called $i=4$ in III, and our result at this point coincides with that of Bronzan and Kacser.[3] In Fig. 21 the distorted part of C_1 is shown dashed to indicate that it is on the second sheet of the branch cut drawn from $\lambda^2=(m-1)^2$.

Assembling these results, we see that for $\frac{1}{2}(m^2-1)\leq s\leq(m-1)^2$,

$$\int_{-\infty}^{(m-1)^2} d\lambda^2\Delta_1(s,\lambda^2)\Phi(\lambda^2)$$

$$=\left\{\int_{\lambda_+^2(s)}^{(m-1)^2} d\lambda^2\bar{\Delta}_1(s,\lambda^2)+\int_{\lambda_-^2(s)}^{\lambda_+^2(s)} d\lambda^2\left[\bar{\Delta}_1(s,\lambda^2)+\frac{i\pi}{k(s,m^2,1)}\right]+\int_{\lambda_-^2(s)}^{4} d\lambda^2\left[\bar{\Delta}_1(s,\lambda^2)+\frac{2\pi i}{k(s,m^2,1)}\right]\right.$$
$$\left.+\int_{4}^{\lambda_-^2(s)} d\lambda^2\left[\bar{\Delta}_1(s,\lambda^2)+\frac{2\pi i}{k(s,m^2,1)}\right]+\int_{4}^{\lambda_-^2(s)} d\lambda^2\bar{\Delta}_1(s,\lambda^2)+\int_{-\infty}^{4} d\lambda^2\bar{\Delta}_1(s,\lambda^2)\right\}\Phi(\lambda^2) \quad \text{(B5)}$$

$$=\left\{\int_{\lambda_+^2(s)}^{(m-1)^2} d\lambda^2\bar{\Delta}_1(s,\lambda^2)+\int_{\lambda_-^2(s)}^{\lambda_+^2(s)} d\lambda^2\left[\bar{\Delta}_1(s,\lambda^2)+\frac{i\pi}{k(s,m^2,1)}\right]+\int_{-\infty}^{\lambda_-^2(s)} d\lambda^2\bar{\Delta}_1(s,\lambda^2)\right\}\Phi(\lambda^2)\,, \quad \text{(B6)}$$

where

$$\bar{\Delta}_1(s,\lambda^2)=k^{-1}(s,m^2,1)L\,.$$

With regard to Eq. (B5), we notice that two of the integrals along the contour between $\lambda_-^2(s)$ and 4 cancel; this is because the first and second sheets of $U^{1/2}$ are actually the same in this region [Fig. 17(a)]. Secondly, as we move around $\lambda_-^2(s)$ and return to $\lambda^2=4$, the phase of L decreases by $2\pi i$ from Eq. (B2), so that for $\lambda^2<\lambda_-^2(s)$ the phase is zero. This discontinuity of L around $\lambda_-^2(s)$ is just Δ_4, the jump across the logarithm. We see that it only serves to cancel the accrued phase of L, and never appears explicitly in the final equation.

Another, perhaps simpler, way of seeing this result is to note that for $\frac{1}{2}(m^2-1)\leq s\leq(m-1)^2$, the branch point $\lambda_-^2(s)$ crosses the λ^2 contour C_1. The phase of all points to the left of $\lambda_-^2(s)$ changes by $2\pi i$; hence to make the continuation analytic we have to subtract $2\pi i$ from the logarithm at these points. Since the phase of the logarithm is already $2\pi i$, it therefore returns to zero.

An exactly similar analysis, which we do not need to give, can be made for $\Delta_2(s,\lambda^2)$, Eq. (29) of the text. Now, though, only region III matters, since the λ^2 integration stops at $\lambda^2=0$, and there are no other real intersections of $\Gamma=0$ for real s. We find that in III Δ_2 develops an imaginary part $i\pi/k(s,m^2,1)$; elsewhere it is real. It follows that, as stated in Sec. V, the imaginary parts of Δ_1 and Δ_2 in region III just cancel.

It remains to define the branches of Δ_3, Eq. (A8). First, the definition of $k(\lambda^2,1,1)=[\lambda^2(\lambda^2-4)]^{1/2}$ is shown in Fig. 17(c). It is clear that the only branch points of the logarithm in Eq. (A8) are at

$$R_1k(\lambda^2,m^2,1)+R_3(\lambda^2,1,1)=\pm R_2k(s,m^2,1)\,.$$

It may be verified that the roots of this equation are again $\lambda^2=\lambda_\pm^2(s)$. As for Δ_2, only region III interests us. We easily see that in this region R_1 and R_3 are always positive; R_2, on the other hand, is just $R(s,\lambda^2)/m^2$, where R is given by Eq. (B2), and drawn in Fig. 18. A discussion similar to that for Δ_1 then shows that in III, Δ_3 acquires an imaginary part $i\pi/k(s,m^2,1)$.

Note added in proof. The link with the Skornyakov-Ter-Martyrosyan[7] (STM) equation has been pointed out in the text (Sec. VI). There is, in fact, a wider connection with more general potential models, and with the model of R. D. Amado [Phys. Rev. **132**, 485 (1963)].

It is easy to see that Eq. (23) and Eq. (27) may both be written in a form in which the integral term is similar to that appearing in Amado's equation (*loc. cit.*) for the scattering of a particle from the bound state of two other similar particles. The inhomogeneous terms are naturally different, simply because the initial states are: In one case, as in $K\to3\pi$ decay, a single particle; in the other, a state of particle+two-body bound state. Our equations have the advantage of not using nonrelativistic variables from the start, but are less general in at least two respects: Firstly, the amplitude $\Phi(s)$ depends on only one variable (because we treated the s-wave case). Secondly, the form factors in Amado's work are reduced to constants. With these simplifications, the integral terms are actually identical in the two models, being both the same as in the STM equation.

Amado's model has been derived from potential theory by L. Rosenberg [Phys. Rev. **134**, B937 (1964)], and the connection with the STM equation was also pointed out by him. Recently, Amado and co-workers have published very encouraging results of numerical calculations using his model, both for the spinless case [R. Aaron, R. D. Amado, and Y. Y. Yam, Phys. Rev. **136**, B650 (1964)], and for a more realistic three nucleon case [R. Aaron, R. D. Amado, and Y. Y. Yam, Phys. Rev. **13**, 574 (1964)].

At the time this article was written, the author was insufficiently aware of the above work, and he regrets that no reference was made to it in the text.