the prediction falls in the range of the estimated experimental data with $F_K/F_\pi=1.05$ to 1.20.

As a result of these considerations, the following remarks may be made:

(a) From Fig. 3 we can safely exclude $\delta<-1$, though we cannot exclude $\delta>0$ for large F_K/F_π.

(b) From Fig. 1 we conclude that $\delta_\varphi\lesssim-0.21$. If the ω-φ mixing effect does not "break" the parameters badly, we may well exclude the possibility of $\epsilon_\rho\simeq\delta>0$.

The conclusions (a) and (b) confirm the result obtained by Schnitzer and Weinberg.[1-4]

(c) Figures 1–4 show that the value of F_K/F_π cannot be larger than 1.25 without badly spoiling one of the predictions for $\Gamma_{\varphi\to K+\bar{K}}$, $\Gamma_{K_A\to K+\omega}$, or $\Gamma_{K_A\to K+\rho}$. The most likely value of F_K/F_π seems to be in the range 1.10–1.15.

A more conclusive test of the validity of the extension of Schnitzer and Weinberg's approach to the $SU(3)\times SU(3)$ algebra will require an accurate determination of the ratio F_K/F_π.

Note added in proof. After this paper was submitted, the authors acknowledged an unpublished report by S. Fenster and F. Hussain which is closely related to this work.[25]

ACKNOWLEDGMENT

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Kinematic Singularities and Threshold Relations for Helicity Amplitudes*

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The kinematic singularities of two-body helicity amplitudes at thresholds and the concomitant relations among these amplitudes are discussed in a direct and elementary way, without recourse to the singularity structure of the crossing matrix. The tools are those of nonrelativistic quantum mechanics, as befits a situation where $p\to 0$, with spins combined into channel spins $\mathbf{S}$ and Russell-Saunders coupling of $\mathbf{L}+\mathbf{S}=\mathbf{J}$. The kinematic singularities are shown to follow from a mismatch between J and L for each term in the partial-wave series. The method is applicable at pseudothresholds $(m_1-m_2)^2$ as well as normal thresholds $(m_1+m_2)^2$ with two formal changes involving an intrinsic parity and a helicity-dependent phase. The relations among the different helicity amplitudes at the thresholds are shown to result from the presence at threshold of fewer Russell-Saunders amplitudes than there are independent helicity amplitudes. The use of invariant amplitudes is shown to be an alternative which automatically yields the kinematic singularities and also the threshold relations among the helicity amplitudes. A discussion is given of dynamical exceptions to the threshold constraints, resulting from less singular than standard behavior at a threshold. The threshold relations are important constraints on the amplitudes, and must be satisfied by any realistic model. In the use of t-channel amplitudes for peripheral processes in the s channel, the explicit imposition of all the relations at the t-channel thresholds is necessary in order to assure a differential cross section without spurious, polelike singularities in t whose variation could in some circumstances completely control the t dependence. The reactions $\pi N\to KY$ and $\pi N\to\pi'\Delta$ are used as illustrations. The latter process is especially illuminating because its t-channel amplitudes have a pole (rather than a simple inverse-square-root singularity) at the $\bar{N}\Delta$ pseudothreshold, $t=0.09$ (GeV/c)^2. The proximity of this point to the physical region of the s channel means that the threshold relations there are of crucial importance. The consequences of these constraints on the cross-section and decay density matrix of the Δ are discussed within the framework of the Regge-pole model. Comparison with experiment implies that the dynamics make the amplitudes for $\pi\pi'\to\bar{N}\Delta$ have less than the standard kinematic singularity at $\bar{N}\Delta$ pseudothreshold and so avoid almost all the threshold constraints. Examples are cited from the literature where use of Regge-pole formulas possessing the spurious kinematic factors has led to incorrect inferences concerning the dynamic behavior of Regge residues.

I. INTRODUCTION

THE question of kinematic singularities of S-matrix elements, that is, singularities associated with the threshold values of s, t, and u, and so depending on the external masses, has received considerable attention in the past few years. Historically, the use of invariant

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amplitudes in combination with explicit kinematic factors made up from the momenta and the spin or Dirac operators automatically took into account the kinematic singularities of the problem. Classic examples are the A and B amplitudes in pion-nucleon scattering and the four invariant amplitudes $A_1, \cdots, A_4$ in pion photoproduction.[1] The existence and construction of invariant amplitudes free of kinematic singularities for a general process has been discussed by Hepp,[2] Williams,[3] and more recently by Fox.[4] But with the consideration of processes involving particles of arbitrary spin, the use of helicity amplitudes[5] became prevalent, chiefly because (a) the formalism is completely general, (b) the angular momentum and parity expansions are straightforward, and finally (c) the helicity amplitudes satisfy elegant crossing relations.[6,7] The work of Hara[8] and Wang[9] solved, apart from a few details, the problem of determining the kinematic singularities of helicity amplitudes. Wang made extensive use of the crossing matrix, while Hara used partial-wave threshold behavior and the crossing matrix. Since then, other discussions of the kinematic-singularity structure of helicity amplitudes have been given from other points of view.[10,11]

In peripheral reactions, the t-channel amplitudes often possess kinematic singularities that are sometimes close to the physical region of large s and small (negative) t. For example, in the process $ab \to cd$, illustrated in Fig. 1, the t-channel helicity amplitudes may have inverse-square-root (or worse) singularities at one or more of the points $t=(m_a+m_c)^2$, $(m_b+m_d)^2$ and $t=(m_a-m_c)^2$, $(m_b-m_d)^2$, the normal thresholds and pseudothresholds, respectively. The pseudothresholds can lie considerably closer to the physical s-channel region than dynamic singularities, such as t-channel poles. Consequently, it seems important to take proper account of such kinematic singularities in a theoretical model that is to be confronted with experiment. An attempt was made for the Regge-pole model to do this by exhibiting in the s-channel cross section all the t-channel kinematic singularities, leaving supposedly smoothly varying residue functions for phenomenological fitting.[12] This compendium of formulas for many different reactions was then to be viewed as the ultimate in Regge-pole phenomenology. Some analysis of data on the basis of these formulas has already been done.[13,14]

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[13] M. Krammer and U. Maor, Nuovo Cimento **50A**, 963 (1967).
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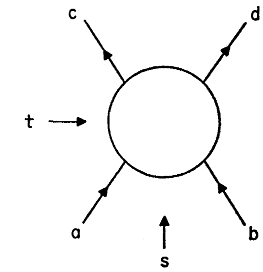

FIG. 1. Diagram for the process $a+b \to c+d$.

But the structure of the formulas of Ref. 12 has been questioned, with special reference to the point $t=0$ by Lin[15] and on general grounds by Stack.[16]

Another aspect of this general problem, recognized during the past year, is the existence of relationships between various helicity amplitudes at the kinematic thresholds. These *threshold conditions* or *kinematic constraints* are discussed by Jones[17] in terms of partial-wave expansions and orbital angular momentum for the normal thresholds, by Diu and LeBellac[18] in terms of the connection between invariant and helicity amplitudes, with special emphasis on $t=0$, and also by Cohen-Tannoudji, Morel and Navelet,[10] and Fox.[19] In Regge-pole theory with two particles of equal mass (e.g., $N\bar{N} \to \pi\rho$), the appropriate pseudothreshold moves to $t=0$. There the problem of kinematic constraints is solved by "conspiracy" or "evasion,"[20] depending on whether or not a given trajectory needs the assistance of another trajectory in order to satisfy the conditions in a nontrivial fashion.

The main purposes of the present paper are (1) to present a unified and straightforward treatment of the kinematic singularities and threshold conditions for helicity amplitudes using orbital angular momentum, and (2) to show within the framework of the Regge-pole model how to incorporate properly the kinematic structure into the cross sections and density matrices. We show that the general results of Refs. 9 and 10 are obtainable by considerations of the thresholds alone, without reference to the crossing matrix. Our use of orbital angular momentum parallels the original work of Hara,[8] but we are careful to distinguish between normal thresholds and pseudothresholds. Frautschi and Jones[14] have also used orbital angular momentum arguments to verify and interpret the singularity structure in a number of specific examples.

The end results of the proper incorporation of the kinematic structure into the cross section and density matrices are phenomenological formulas very different from those of Wang[12] in that they conform to the requirements of Lin[15] and Stack[16] and possess no t-channel kinematic-singularity factors. The somewhat

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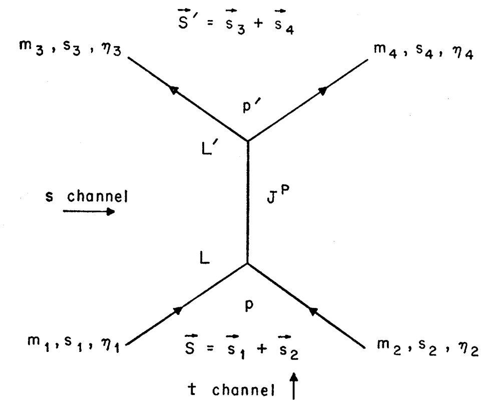

FIG. 2. Diagram defining notation for Russell-Saunders coupling. The t-channel process is $m_1+m_2 \to m_3+m_4$, where the ith particle has mass, spin, and intrinsic parity m_i, s_i, and η_i, respectively. The initial and final momenta in the center of mass are p and p', respectively, while the channel spins are $\mathbf{S}=\mathbf{s}_1+\mathbf{s}_2$ and $\mathbf{S}'=\mathbf{s}_3+\mathbf{s}_4$, and the orbital angular momenta are L and L'.

confusing and even subtle aspects of these problems are hopefully illuminated by parallel treatment of some examples in terms of Feynman perturbation theory and the use of invariant amplitudes.

II. NOTATION AND BASIC CONCEPTS

The present discussion of kinematic singularities is based entirely on the use of orbital angular momentum and the standard centrifugal-barrier factors of non-relativistic quantum mechanics with no consideration of crossing relations. That nonrelativistic concepts should be suitable at thresholds is not surprising. But in spite of the use of orbital angular momentum arguments for some aspects of these problems,[8,14,17,21] it does not seem to be recognized that a consistent discussion of the whole question can be given in those terms alone.

Our interest ultimately is in peripheral processes and the Regge-pole model. Consequently, the t-channel amplitudes and their singularities are emphasized in the choice of notation; the treatment is readily transcribed to other channels. We consider for the most part amplitudes with all four external masses different in order to separate the normal and pseudothreshold points from $t=0$.

A. Notation

The general labeling of the variables is indicated in Fig. 2. The t-channel process is

$$1+2 \to 3+4, \tag{1}$$

where the ith particle has mass m_i, spin s_i, and intrinsic parity η_i. The initial and final center-of-mass momenta are p and p', respectively, and are given by

$$\begin{aligned} p^2&=[t-(m_1+m_2)^2][t-(m_1-m_2)^2]/4t, \\ p'^2&=[t-(m_3+m_4)^2][t-(m_3-m_4)^2]/4t. \end{aligned} \tag{2}$$

It is convenient to introduce separate terminology for the square roots of the brackets in Eq. (2). Thus we define

$$\begin{aligned} T_N&=[t-(m_1+m_2)^2]^{1/2}, \\ T_P&=[t-(m_1-m_2)^2]^{1/2}, \\ T_N'&=[t-(m_3+m_4)^2]^{1/2}, \\ T_P'&=[t-(m_3-m_4)^2]^{1/2}, \end{aligned} \tag{3}$$

where the subscripts N and P stand for normal threshold and pseudothreshold and the prime or lack of it is associated with p' and p. From Eqs. (2) and (3) we have

$$\begin{aligned} T_N T_P&=2(\sqrt{t})p, \\ T_N' T_P'&=2(\sqrt{t})p'. \end{aligned} \tag{4}$$

The t-channel helicity amplitudes are functions of t and $\cos\theta_t$. For discussion of analytic properties we will make considerable use of the expression for $\cos\theta_t$ in terms of s, t, and u:

$$\cos\theta_t=\frac{1}{4pp'}\left[s-u+\frac{(m_1^2-m_2^2)(m_3^2-m_4^2)}{t}\right]. \tag{5}$$

Evidently, then, $(tpp'\cos\theta_t)$ is a polynomial in s, t, and u, possessing no threshold or other singularities. Another convenient relation is that between the Kibble boundary function $\varphi(s,t,u)$,[22]

$$\begin{aligned} \varphi(s,t,u)=stu&-s(m_1^2m_3^2+m_2^2m_4^2) \\ &-t(m_1^2m_2^2+m_3^2m_4^2)-u(m_1^2m_4^2+m_2^2m_3^2) \\ &+2m_1^2m_2^2m_3^2m_4^2\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}+\frac{1}{m_3^2}+\frac{1}{m_4^2}\right), \end{aligned} \tag{6}$$

and $\sin\theta_t$:

$$\varphi=4tp^2p'^2\sin^2\theta_t. \tag{7}$$

The virtue of Eq. (7) is that it tells one what powers of momenta and energy to associate with $\sin^2\theta_t$ in order to obtain a polynomial in s, t, and u of impeccable analytic properties.[23]

B. No Spins

The existence of kinematic singularities is a complication entirely caused by the presence of particles with spin. Without spin, the threshold behavior of partial-wave amplitudes provides just the necessary powers of momenta to combine with the corresponding Legendre polynomials to give expressions manifestly free of

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[22] T. Kibble, Phys. Rev. **117**, 1159 (1959).

[23] As is well known, the symmetry of φ between the three channels allows one to infer that φ also satisfies Eq. (7) with the corresponding s- or u-channel quantities on the right-hand side.

kinematic singularities. Consider a spinless process with invariant amplitude A expanded in a partial-wave series:

$$A(t,\theta_t)=\sum_L (2L+1)A_L(t)P_L(\cos\theta_t). \tag{8}$$

We focus our attention on one of the thresholds, say $p\to 0$. The behavior of the partial-wave amplitude A_L in this limit is as p^L. This can be taken as a law of nature, or can be derived from the Froissart-Gribov formula and the Mandelstam representation (see, for example, Ref. 21). Similarly, at $p'\to 0$, $A_L\sim p'^L$. Thus $A_L(t)$ can be written as $A_L(t)=(pp')^L\tilde{A}_L(t)$, where $\tilde{A}_L(t)$ is an analytic function of t in the neighborhood of either threshold. The Legendre polynomials are finite series of all even or all odd powers of $\cos\theta_t$, with the highest power being $(\cos\theta_t)^L$. This means that the expansion (8) can be written as

$$A(t,\theta_t)=\sum_L (2L+1)\tilde{A}_L(t)(pp'\cos\theta_t)^L \times\left[1+O\left(\frac{1}{\cos^2\theta_t}\right)\right], \tag{9}$$

where the square bracket represents a finite series in powers of $(\cos\theta_t)^{-2}$. From the definition of $\cos\theta_t$, Eq. (5), we see that the combination $(pp'\cos\theta_t)$ is analytic in s and t except perhaps at $t=0$. Furthermore, the square bracket in (9) is also well behaved, since $(\cos\theta_t)^{-2}=p^2p'^2/(pp'\cos\theta_t)^2$. Evidently, then, if the partial-wave expansion converges, we have demonstrated that the amplitude $A(t,\theta_t)$ has no kinematic singularities. The amplitude can, of course, be defined outside the domain of convergence of the partial-wave series by analytic continuation.

C. Outline of the Method

When spins are present, the situation is complicated by a mismatch between the total angular momentum J and the orbital angular momentum L. It is the latter which governs the centrifugal-barrier factors p^L while it is the former that determines the power of $\cos\theta_t$. Obviously, the difference $(J-L)$ will specify the power of p and/or p' left over, and so the specific kinematic singularity for the amplitude in question. In detail, care must be taken to distinguish between normal thresholds and pseudothresholds, so that various powers of T_N, T_P, T_N', and T_P' will occur, rather than simple powers of p and p'. This is spelled out in detail in Sec. III C. But the basic approach is to use the concepts of nonrelativistic nuclear physics, to combine the spins of the particles into channel spins, $\mathbf{S}=\mathbf{s}_1+\mathbf{s}_2$ and $\mathbf{S}'=\mathbf{s}_3+\mathbf{s}_4$, to add orbital angular momenta $\mathbf{L}$, $\mathbf{L}'$ to give $\mathbf{J}=\mathbf{L}+\mathbf{S}$, $\mathbf{J}=\mathbf{L}'+\mathbf{S}'$ with due account of parity. The maximum difference for $(J-L)$ is then determined and the kinematic-singularity structure established for each term in the partial-wave series and hence for the full amplitude. In practice, this is elementary and quick to do for any specific case, as shown in Sec. III A and Appendix B, and also simple for the general case (Sec. III C). But before proceeding to the relatively trivial task just described it is necessary to discuss some of the basic formulas concerning helicity amplitudes and the differences between normal thresholds and pseudothresholds.

D. Helicity Amplitudes and Their Partial-Wave Expansions

In the helicity representation the invariant amplitude M yields helicity amplitudes $f_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ with partial-wave expansions[5,24]

$$f_{\lambda_3\lambda_4;\lambda_1\lambda_2}(t,\theta_t)=\sum_J (J+\tfrac{1}{2})\langle\lambda_3\lambda_4|F^J(t)|\lambda_1\lambda_2\rangle d_{\lambda\mu}{}^J(\theta_t), \tag{10}$$

where $\lambda=\lambda_1-\lambda_2$, $\mu=\lambda_3-\lambda_4$. The properties of the Wigner d functions in (10) are well known.[5,25,26] For our immediate purposes we note that

$$d_{\lambda\mu}{}^J(\theta)=(\cos\tfrac{1}{2}\theta)^{|\lambda+\mu|}(\sin\tfrac{1}{2}\theta)^{|\lambda-\mu|}P_{\lambda\mu}{}^J(\cos\theta), \tag{11}$$

where $P_{\lambda\mu}{}^J(z)$ is a polynomial in z whose highest power is z^{J-m} and m is the larger of $|\lambda|$, $|\mu|$.

It will be convenient to consider parity-conserving amplitudes.[9,26] For each J, parity-conserving amplitudes are given by linear combinations of the helicity partial-wave amplitudes in Eq. (10):

$$F_{\lambda_3\lambda_4;\lambda_1\lambda_2}{}^{J,P}=\langle\lambda_3\lambda_4|F^J|\lambda_1\lambda_2\rangle +P\eta_1\eta_2(-1)^{J-s_1-s_2}\langle\lambda_3\lambda_4|F^J|-\lambda_1-\lambda_2\rangle, \tag{12}$$

where P is the parity eigenvalue. Following Gell-Mann *et al.*,[26] we choose the parity-conserving states in anticipation of Regge trajectories and their equivalence to a superposition of states in a definite spin-parity sequence. Thus we introduce a parity factor $\eta=\pm 1$ such that the parity of a given J state is

$$P=\eta(-1)^{J-v}, \tag{13}$$

where $v=0$ for integral J and $v=\frac{1}{2}$ for odd half-integral J. For integral J, the natural-parity trajectories $(\rho,\omega,A_2,\cdots)$ have $\eta=+1$, while the unnatural-parity trajectories $(\pi,A_1,\cdots)$ have $\eta=-1$. For odd half-integral J, the N trajectory $(\frac{1}{2}^+,\frac{5}{2}^+,\cdots)$ has $\eta=+1$, while the Δ trajectory $(\frac{3}{2}^+,\frac{7}{2}^+,\cdots)$ has $\eta=-1$, etc. In terms of η, the parity-conserving partial-wave amplitudes, Eq. (12), are

$$F_{\lambda_3\lambda_4;\lambda_1\lambda_2}{}^{J\eta}=\langle\lambda_3\lambda_4|F^J|\lambda_1\lambda_2\rangle +\eta\eta_1\eta_2(-1)^{s_1+s_2-v}\langle\lambda_3\lambda_4|F^J|-\lambda_1-\lambda_2\rangle. \tag{14}$$

[24] The expansion (10) is formally the same as Jacob and Wick's Eq. (31), but their and our f differ by a factor of $-8\pi(tp/p')^{1/2}$, with a corresponding difference in T^J and F^J. For elastic scattering of spinless particles our $F^J=-16\pi(\sqrt{t})e^{i\delta_J}\sin\delta_J/p$.

[25] F. Calogero and J. M. Charap, Ann. Phys. (N.Y.) **26**, 44 (1964); F. Calogero, J. M. Charap, and E. J. Squires, *ibid.* **25**, 325 (1963).

[26] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964).

Parity-conserving helicity amplitudes based on (10) can be constructed after extracting the half-angle dependence of the Wigner d functions shown in (11). A detailed discussion is given in Ref. 26. We only quote the key results. The so-called parity-conserving amplitudes are defined by

$$F_{\lambda_3\lambda_4;\lambda_1\lambda_2}{}^{\eta}(t,z) = \xi(\theta_t) f_{\lambda_3\lambda_4;\lambda_1\lambda_2}(t,\theta_t) + \eta(-1)^{m-\mu}\eta_1\eta_2(-1)^{s_1+s_2-v} \times \xi(\pi-\theta_t) f_{\lambda_3\lambda_4;-\lambda_1-\lambda_2}(t,\theta_t), \quad (15)$$

where $z=\cos\theta_t$ and

$$\xi(\theta_t) = [\sqrt{2}\cos\tfrac{1}{2}\theta_t]^{-|\lambda+\mu|} \times [\sqrt{2}\sin\tfrac{1}{2}\theta_t]^{-|\lambda-\mu|}. \quad (16)$$

The amplitudes F^{η} are almost the same as the $\bar{f}^{(\pm)}$ of Ref. 9. But care must be taken in relating $\eta=\pm1$ to the $(\pm)$ superscript of $\bar{f}$.[27] The partial-wave expansion of (15) in terms of the amplitudes (14) is

$$F_{\lambda_3\lambda_4;\lambda_1\lambda_2}{}^{\eta}(t,z) = \sum_J (J+\tfrac{1}{2})[e_{\lambda\mu}{}^{J+}(z) F_{\lambda_3\lambda_4;\lambda_1\lambda_2}{}^{J\eta} + e_{\lambda\mu}{}^{J-}(z) F_{\lambda_3\lambda_4;\lambda_1\lambda_2}{}^{J,-\eta}]. \quad (17)$$

The functions $e_{\lambda\mu}{}^{J\pm}(z)$ are defined in Ref. 26, Eq. (2.8) and Appendix A.[28] For our purposes the essential facts about the $e_{\lambda\mu}{}^{J\pm}(z)$ are that they are finite polynomials in z of either all odd or all even positive powers. Specifically,

$$\begin{aligned} e_{\lambda\mu}{}^{J+}(z) &= z^{J-m}\left(A+\frac{B}{z^2}+\cdots\right), \\ e_{\lambda\mu}{}^{J-}(z) &= z^{J-1-m}\left(A'+\frac{B'}{z^2}+\cdots\right). \end{aligned} \quad (18)$$

The original helicity amplitude (10) can be recovered by adding together the amplitudes (17) for $\eta=+1$ and $\eta=-1$ and dividing the result by 2ξ. The virtue of (17) is twofold: firstly, the half-angle dependence ξ has been removed so that the resulting t-channel amplitudes have only dynamical singularities in s,[9,10] and secondly, the dependence on J and parity is explicitly exhibited with coefficients which are polynomials in z. The problem with spins has thus been reduced as far as possible towards the example of no spins.

E. Normal Thresholds and Pseudothresholds

The process illustrated in Fig. 2 has four t-channel thresholds, $t=(m_1+m_2)^2$, $(m_3+m_4)^2$, $(m_1-m_2)^2$, $(m_3-m_4)^2$. Since we will consider the initial and final states separately, it will suffice to discuss only the two thresholds of the initial state. The first one [$t=(m_1+m_2)^2$] is the normal threshold, while the second [$t=(m_1-m_2)^2$] is called the pseudothreshold. In nonrelativistic quantum theory we are familiar with only the normal threshold, but p vanishes at both. It is almost unnecessary to say that at the normal threshold such ideas of nonrelativistic quantum mechanics as the vector addition of angular momenta and the straightforward application of parity conservation can be utilized without further thought. But care must be taken at the pseudothreshold.

At the normal threshold the particles are at rest with $E_1=m_1$ and $E_2=m_2$. Inspection of the energy expression for each particle shows that at the pseudothreshold the particles are again at rest, but $E_1=-m_1$, while $E_2=m_2$, where the particles have been labelled so that $m_1<m_2$. The change in sign of the energy of the lighter particle in going from the normal threshold to the pseudothreshold has two consequences.[19] The single-particle state $|p,E,\lambda\rangle$, where the momentum is along the z axis, can be obtained from the state at rest (normal threshold) $|0,m,\lambda\rangle$ by application of a "boost" operator

$$|p,E,\lambda\rangle = e^{-i\zeta K_3}|0,m,\lambda\rangle,$$

where $p=m\sinh\zeta$, $E=m\cosh\zeta$. The unphysical complex "boost" which transforms the particle from a state at rest with $E=m$ (normal threshold) to a state at rest with $E=-m$ (pseudothreshold) has $\zeta=i\pi$. For the irreducible representations of the Lorentz group used to describe particles of definite spin S [labeled $(S,0)$ or $(0,S)$], $\mathbf{K}$ has the same representations as $i\mathbf{J}$. Thus we see that the transition from normal threshold to pseudothreshold for particle 1 gives back the normal threshold state multiplied by a phase factor $\exp(i\pi\lambda_1)$. As a consequence, if we consider the helicity amplitude (10) near the pseudothreshold, the spin structure of the right-hand side will be as if the particles were "normal" particles at a normal threshold, except that there will be a phase factor $\exp(i\pi\lambda_1)$. We can therefore define new amplitudes, called pseudoamplitudes,

$$f_{\lambda_3\lambda_4;\lambda_1\lambda_2}{}^{P} = (-1)^{s_1-\lambda_1} f_{\lambda_3\lambda_4;\lambda_1\lambda_2}, \quad (19)$$

to which the ordinary laws of addition of spin and orbital angular momenta can be applied at the pseudothreshold. Consideration of the parity transformation [Eq. (44) of Ref. 5] applied to the pseudoamplitudes shows that there is an additional factor of $\exp(2\pi\lambda_1 i) = (-1)^{2s_1}$ in the connection between $f_{\lambda_3\lambda_4;\lambda_1\lambda_2}{}^{P}$ and $f_{-\lambda_3-\lambda_4;-\lambda_1-\lambda_2}{}^{P}$ over what appears for the f's. This can be interpreted as an effective change of the intrinsic parity of particle 1 from η_1 to $\eta_1(-1)^{2s_1}$. The change in intrinsic parity for the lighter fermion at the pseudothreshold is familiar for spin $\frac{1}{2}$ in the connection between negative-energy states and antiparticles, as is, in fact, the phase factor. This change in the effective intrinsic parity of a fermion whose energy is $E=-m$ at the pseudothreshold has already been pointed out by

[27] If we define η_W to be equal to the $\pm$ sign in L. L. Wang's definition of $\bar{f}^{(\pm)}$, then the connection is

$$\eta_W = \eta(-1)^{\lambda+m}\eta_3\eta_4(-1)^{s_3+s_4-v}.$$

[28] They should not be confused with the functions $e_{\lambda\mu}{}^{J}(z)$, defined, for example, by Andrews and Gunson [M. Andrews and J. Gunson, J. Math. Phys. 5, 1391 (1964)]. The latter are related to $d_{\lambda\mu}{}^{J}$ as Q_l is to P_l, while the present functions are linear combinations of $d_{\lambda\mu}{}^{J}$ and $d_{\lambda,-\mu}{}^{J}$ with the half-angle factors extracted.

Frautschi and Jones,[14] and also by Franklin in his erratum.[21]

In summary, the normal thresholds and pseudothresholds can be handled on an equal footing with nonrelativistic quantum mechanics provided that at a pseudothreshold two modifications are made: (1) that the intrinsic parity η_l of the lighter particle is replaced by $(-1)^{2s_l}\eta_l$, and (2) that the pseudoamplitudes (19) are considered rather than the regular amplitudes (10). The first modification has the consequence of giving different kinematic singularities at the normal thresholds and pseudothresholds if the lighter particle is a fermion. The second alteration is important for the kinematic constraints or threshold relations among the amplitudes, independently of whether the lighter particle has integral or half-integral spin.

The remaining point is specification of the threshold behavior of the partial-wave amplitudes (14). Suppose that the smallest allowed values of orbital angular momentum in the initial (final) state at the normal (pseudo) thresholds are L_N (L_P) $[L_N'$ $(L_P')]$. Then we will assume that $F^{J\eta}$ can be written

$$F_{\lambda_3\lambda_4;\lambda_1\lambda_2}{}^{J\eta}=(T_N)^{L_N}(T_P)^{L_P}(T_N')^{L_N'}(T_P')^{L_P'}\times\bar{F}_{\lambda_3\lambda_4;\lambda_1\lambda_2}{}^{J\eta},\quad(20)$$

where T_N, etc., are the threshold factors (3), proportional to the nonrelativistic momenta at the respective thresholds, and $\bar{F}^{J\eta}$ is a reduced partial-wave amplitude, free of threshold singularities. The partial-wave threshold behavior contained in (20) seems to be contrary to that obtained for $\pi N\to\pi N$ by Frye and Warnock[29] at the s-channel pseudothreshold $s=(m-\mu)^2$. As discussed by Franklin,[21] their result hinges on the coincidence of the u-channel normal threshold and the s-channel pseudothreshold, and does not apply for unequal masses. Franklin also argues that the *kinematic* threshold singularities are always given by (20); other behavior can be viewed as dynamical.

III. KINEMATIC SINGULARITIES

A. $\pi\pi'\to\bar{N}\Delta$

The basic tools for the analysis have been described in Secs. II C and II E. Before proceeding to the general case it is instructive to consider a specific example, namely, the s-channel process of isobar production,

$$(0^-)+(\tfrac{1}{2}^+)\to(0^-)+(\tfrac{3}{2}^+),$$

examples of which are $\pi^+p\to\pi^0\Delta^{++}$ and $K^-p\to\pi^-Y_1^{*+}$. We will, for convenience, write the t-channel reaction as

$$\overset{\pi}{(1)}+\overset{\pi'}{(2)}\to\overset{\bar{N}}{(3)}+\overset{\Delta}{(4)},$$

with the first-mentioned process in mind, but the results apply to any reaction with the same spins and parities.

[29] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963).

TABLE I. Minimum L and L' values (and associated channel spins S') for $\pi\pi'\to\bar{N}\Delta$ and the corresponding threshold behavior of F^{J+}.

J^P	L values Normal	L values Pseudo	L' values Normal	L' values Pseudo	Threshold behavior
0^+	0	0	1 ($S'=1$)	2 ($S'=2$)	$T_N'T_P'^2$
1^-	1	1	0 ($S'=1$)	1 ($S'=1,2$	$T_NT_PT_P'=\dfrac{4tpp'}{T_N'}$
2^+	2	2	1 ($S'=1, 2$)	0 ($S'=2$)	$(T_NT_P)^2T_N'=\dfrac{(4tpp')^2}{T_N'T_P'^2}$
3^-	3	3	2 ($S'=1, 2$)	1 ($S'=2$)	$(T_NT_P)^3T_N'^2T_P'=\dfrac{(4tpp')^3}{T_N'T_P'^2}$

The process $\pi\pi'\to\bar{N}\Delta$ is an especially good one because it is relatively simple, with only four independent amplitudes and spinless particles in the initial state, but it still has a relatively complicated kinematic singularity structure because of the spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ baryons in the final state. Because of the zero-spin particles of the same intrinsic parity initially, the allowed angular momentum parity states belong to the natural parity sequence $(\eta=+1)$. Thus only F^{J+} in (14) is different from zero and just the first term in (17) occurs. Another way of saying it is that for this reaction the basic helicity amplitude (10) is already a parity-conserving amplitude in the sense of Ref. 26.

We now proceed to construct the channel spins and parities. For the initial state we obviously have

$$S_N=0^+,\quad S_P=0^+,$$

where the subscripts N and P denote normal thresholds and pseudothresholds, respectively. For the final state of $\bar{N}(\frac{1}{2}^-)$ and $\Delta(\frac{3}{2}^+)$ we have

$$S_N'=1^-,\,2^-,$$
$$S_P'=1^+,\,2^+.$$

Note that for the pseudothreshold the $\bar{N}$ parity has been formally reversed according to the rules of Sec. II E. The laws of addition of angular momentum and parity conservation are applied to the orbital angular momentum $\mathbf{L}$ and the channel spin $\mathbf{S}$ to yield a total angular momentum $\mathbf{J}$ and parity $(-1)^J$. The results of this elementary calculation are tabulated in Table I. Where more than one L or L' value is possible, only the smallest one is tabulated, because that is the one which governs the threshold behavior of the amplitude in (20).

The final column in Table I exhibits the threshold behavior of $F^{J\eta}$ according to Eq. (20) for successive partial waves. While the first two J values show abnormalities which are of interest in understanding differences that arise between the general results and the singularities found for specific Feynman diagrams (see Sec. IV C), a pattern establishes itself for $J\geqslant 2$. For the Jth partial wave $(J\geqslant 2)$ the threshold behavior is

$$F^{J+}=[(4tpp')^J/T_N'T_P'^2]\tilde{F}^{J+}, \qquad (21)$$

where $\tilde{F}$ does not have singularities at the four thresholds. From (17), (18), and (5) it is seen that the combination of z^J with the numerator in (21) yields an analytic structure free from kinematic singularities for each J value, at least for $m=|\mu|=0$.[30] Thus the class of helicity amplitudes with $\lambda_3=\lambda_4$ can be written

$$f_{\lambda\lambda;00}=\tfrac{1}{2}F_{\lambda\lambda;00}{}^{+}=A_{\lambda\lambda}(s,t)/T_N'T_P'^2, \qquad (22)$$

where $A_{\lambda\lambda}(s,t)$ is free of kinetic (threshold) singularities. This result is in agreement with the general results given by Wang[9] and others.

B. Behavior at the Physical Boundary: Powers of t

For λ, $\mu\neq 0$ the discussion of the previous section is incomplete. The threshold singularities are determined correctly, but the behavior of the helicity amplitudes (10) at the boundary of the physical region and at $t=0$ was not considered. For all four masses unequal there are no singularities in the helicity amplitudes at $t=0$.[10,11] Indeed, for $(m_1^2-m_2^2)(m_3^2-m_4^2)<0$ the point $t=0$ lies inside the s-channel physical region where, from the crossing relations, it is clear that the continued t-channel amplitudes cannot have singularities.

The singularity at the physical boundary arises from the half-angle factors in the connection between the helicity amplitudes f [Eq. (10)] and the parity-conserving amplitudes F^η [Eq. (15)]. The latter are functions of $z=\cos\theta_t$ and so have no singularities at $\varphi=0$. But $\sin\frac{1}{2}\theta_t$ is proportional to $\sqrt{\varphi}$ near $\cos\theta_t=1$ and $\cos\frac{1}{2}\theta_t$ is proportional to $\sqrt{\varphi}$ near $\cos\theta_t=-1$. Hence the helicity amplitude (10) will behave as

$$\begin{aligned} f_{\lambda_3\lambda_4;\lambda_1\lambda_2} &\propto (\sqrt{\varphi})^{|\lambda-\mu|} \quad \text{for} \quad \cos\theta_t\approx 1 \\ &\propto (\sqrt{\varphi})^{|\lambda+\mu|} \quad \text{for} \quad \cos\theta_t\approx -1. \end{aligned} \qquad (23)$$

The behavior at the two ends of the physical region can be written as $(\sqrt{\varphi})^{|M|}$, where M is the difference between the initial and final z components of total angular momentum.[4,11]

The parity-conserving amplitudes F^η have singularities at $t=0$ as a consequence of the half-angle factor ξ [Eq. (16)], even though the f amplitudes do not. From (5) it follows that near $t=0$, $\cos\theta_t=\epsilon+O(t)$, where $\epsilon=\pm 1$ for $(m_1^2-m_2^2)(m_3^2-m_4^2)\gtrless 0$. This means that

$$\xi(\theta_t)\propto(\sqrt{t})^{-|\lambda-\epsilon\mu|},$$
$$\xi(\pi-\theta_t)\propto(\sqrt{t})^{-|\lambda+\epsilon\mu|},$$

near $t=0$. Consequently, the $t=0$ singularity of F^η is

$$F_{\lambda_3\lambda_4;\lambda_1\lambda_2}{}^{\eta}\propto(1/\sqrt{t})^{|\lambda|+|\mu|}. \qquad (24)$$

[30] The $J=0$, 1 partial waves differ from (21) by positive *even* powers of T_N' and T_P', and so have the same singularity structure as $J\geqslant 2$.

For $m\neq 0$ this added singularity must be included along with the threshold factors such as (21).

For the example of $\pi\pi'\to\bar{N}\Delta$, where $\lambda=0$, insertion of a factor $(\sqrt{t})^{-|\mu|}$ into (21) leads to helicity amplitudes of the form

$$f_{\lambda_3\lambda_4;00}=\frac{(\sqrt{\varphi})^{|\mu|}A_{\lambda_3\lambda_4}(s,t)}{[t-(m_\Delta-m_N)^2][t-(m_\Delta+m_N)^2]^{1/2}}, \qquad (25)$$

where $A_{\lambda_3\lambda_4}(s,t)$ is free of singularities at the kinematic thresholds, on the physical boundary, and at $t=0$. Equation (25) exhibits the threshold singularities explicitly, as well as the known behavior at the boundary of the physical region.

C. General Result

The example discussed in Sec. III A can be generalized in an obvious way. Combining (20) with (17), taking cognizance of (18), and including the $t=0$ behavior [Eq. (24)], we see that we can write

$$\begin{aligned} F_{\lambda_3\lambda_4;\lambda_1\lambda_2}{}^{\eta}=[(T_N)^{\alpha_N}(T_P)^{\alpha_P}(T_N')^{\beta_N}(T_P')^{\beta_P}(\sqrt{t})^{|\lambda|+|\mu|}]^{-1} \\ \times A_{\lambda_3\lambda_4;\lambda_1\lambda_2}{}^{\eta}(s,t), \end{aligned} \qquad (26)$$

where $A(s,t)$ has only dynamical singularities, and the exponents are $\alpha_i=J-L_i-m$ and $\beta_i=J-L_i'-m$. The differences $(J-L)$ are the "mismatches" discussed in Sec. II C.[31] There is one question, namely, whether or not $(J-L)$ is independent of J. Inspection of Table I shows the general behavior. For small J there are differences caused by the channel spins being larger than J. But for $J\geqslant S_{\max}$, the difference $(J-L)$ is independent of J. Physically, this occurs because the minimum orbital angular momentum demands the maximum channel spin. Then $(J-L)$ is equal to $S_{\max}$ or $(S_{\max}-1)$, depending upon the parities involved. The only slight problem for the general case is exhibiting the switch that gives $S_{\max}$ or $S_{\max}-1$ in a compact manner. The erratic behavior of the L and L' values for small J is of no consequence, because the threshold behavior for those partial waves differs from the standard pattern by positive even powers of the T's (see Ref. 30).

We consider first the initial state, with $S_{\max}=s_1+s_2$. At the normal threshold the intrinsic parity associated with the channel spin is $\eta_1\eta_2$. For a state with total angular momentum J and parity $\eta(-1)^{J-v}$, the minimum allowed orbital angular momentum is

$$L=J-(s_1+s_2)+\begin{Bmatrix}0\\1\end{Bmatrix} \quad \text{if} \quad \eta\eta_1\eta_2(-1)^{s_1+s_2-v}=\begin{Bmatrix}+1\\-1\end{Bmatrix}.$$

This can be written in the form

$$L=J-(s_1+s_2)+\tfrac{1}{2}[1-\eta\eta_1\eta_2(-1)^{s_1+s_2-v}]. \qquad (27)$$

[31] The second term in (17) seems to give rise to one more power of $(4tpp')$, but this is compensated by a difference of one unit in L, L' in the threshold behavior because of its opposite parity.

Equation (27) is the basic result that gives the α's and β's in (26). The only caution to be observed is at the pseudothresholds where the effective intrinsic parity of the channel is $\eta_1\eta_2(-1)^{2s_l}$. The four exponents are

$$\begin{aligned}\alpha_N&=s_1+s_2-m-\tfrac{1}{2}[1-\eta\eta_1\eta_2(-1)^{s_1+s_2-v}],\\ \alpha_P&=s_1+s_2-m-\tfrac{1}{2}[1-\eta\eta_1\eta_2(-1)^{s_2-s_1-v}],\\ \beta_N&=s_3+s_4-m-\tfrac{1}{2}[1-\eta\eta_3\eta_4(-1)^{s_3+s_4-v}],\\ \beta_P&=s_3+s_4-m-\tfrac{1}{2}[1-\eta\eta_3\eta_4(-1)^{s_4-s_3-v}].\end{aligned}\tag{28}$$

For definiteness in (28), it has been assumed that $m_1<m_2$, $m_3<m_4$. Note that for integral J (i.e., $v=0$) the choice of which particles are lighter is immaterial.

The results contained in (26) and (28) can be shown to be exactly equivalent to those of Wang[9] and Cohen-Tannoudji, Morel, and Navelet[10] for the case of all unequal masses, although some care must be taken in correlating correctly the parities.[32,33] The specialization to equality of various masses has been discussed in Ref. 9 and much more thoroughly in Ref. 10, where the analyticity in $\sqrt{t}$ for half-integral J is also treated (see also Hara[8]). The above results hold for positive values of $\sqrt{t}$ if J is odd half-integral.

IV. THRESHOLD RELATIONS

The various helicity amplitudes for a given process are in general related only by dynamical assumptions. But at certain regions in the (s,t) plane there are connections among them. One type of relation occurs at the boundary of the physical region ($z=\pm1$), where amplitudes with $(\lambda\mp\mu)\neq0$ must vanish by conservation of angular momentum. The crossing relations[6,7] then imply that certain linear combinations of the crossed-channel amplitudes vanish there. This type of constraint, first noted by Goldberger, Grisaru, MacDowell, and Wong,[34] is discussed in general by Abers and Teplitz.[35] Our interest is in another kind of relation between helicity amplitudes, one that occurs at the kinematic thresholds. For unequal masses these thresholds are distinct from any boundary of the physical region.

Threshold relations between amplitudes of different helicities are almost trivial if one restricts oneself to the physical region of $\cos\theta$. Thus, for a physical process at threshold, only those amplitudes with orbital angular momentum $L=0$ will be nonvanishing. Typically, only one term in each partial-wave expansion, that with $J=S$, will survive, and the different helicities will be related by an ordinary angular momentum Clebsch-Gordan coefficient $\langle s_as_b\lambda_a-\lambda_b|S(\lambda_a-\lambda_b)\rangle$. This was pointed out by Jones,[17] who discussed the normal thresholds, but applies equally to the pseudothresholds with the modifications discussed in Sec. II E.

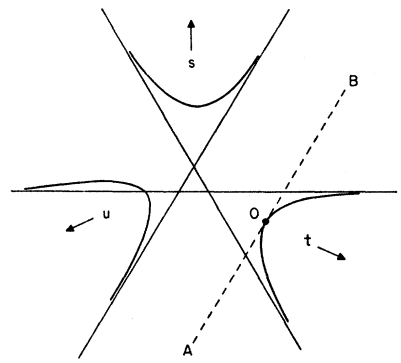

FIG. 3. Schematic Mandelstam diagram showing the physical regions of s, t, and u. The dashed line AB represents the normal t-channel threshold, $t=(m_3+m_4)^2$ or $t=(m_1+m_2)^2$. The point O is the physical threshold in the t channel, where $|\cos\theta_t|\leqslant1$.

The argument of the preceding paragraph is correct for amplitudes at threshold with $|\cos\theta_t|\leqslant1$. This corresponds to the determination of relations among the amplitudes at the point O in Fig. 3. But it is desirable to know what constraints occur all along the line AB, that is, for fixed threshold values of t, but arbitrary s. Equation (5) shows that, for arbitrary s, $\cos\theta_t$ becomes infinite at the t-channel thresholds; the point O in Fig. 3 is the only exception. This means that all J values in the partial-wave expansion contribute, not just those corresponding to $L=0$.[36] It turns out that, while this complicates the analysis somewhat, it is still possible to exhibit systematically the threshold relations among the amplitudes. The method of Jones[17] sometimes yields the same connections as the present approach, but his development is really only valid at the threshold in the physical region. Pion-nucleon scattering is one example. This is discussed in Appendix B from several points of view.

An alternative, but entirely equivalent, way of establishing these relations at threshold is the use of invariant amplitudes. This approach has been employed by Abers and Teplitz[35] in their Appendix for scalar-meson–vector-meson scattering, and by Diu and LeBellac[18] with special attention to $t=0$ (for $\bar{N}N\to\pi\sigma$, $\bar{N}N\to\pi\gamma$). We discuss the use of invariant amplitudes in Sec. IV B. Another equivalent method, discussed by Cohen-Tannoudji, Morel, and Navelet,[10] utilizes transversity amplitudes and the singularity structure of their crossing matrix. This yields linear combinations of helicity amplitudes (and sometimes of derivatives of helicity amplitudes) that vanish at threshold.

[32] Actually the results of Wang are in error at the pseudothresholds for half-integral J (boson-fermion scattering) when the boson is heavier than the fermion. But for the important cases of πN or KN scattering they are correct.

[33] In comparisons with Wang (Ref. 9), the parity correspondence is given in Ref. 27. The other essential ingredient is the identity $\max(\eta)$ of $[N]=N-v-\frac{1}{2}(1-\eta(-1)^{N-v})$, where the left-hand side is a notation of Wang. $v=0$ if N is integral and $v=\frac{1}{2}$ if N is odd half-integral. With Ref. 10 further care is necessary because the authors do not use Jacob and Wick's phases for particles 2 and 4.

[34] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960).

[35] E. Abers and V. Teplitz, Phys. Rev. **158**, 1365 (1967).

[36] This point was emphasized to the authors by R. D. Mathews.

A. $\pi\pi' \to \bar{N}\Delta$

To illustrate the threshold relations we consider our previous example, $\pi\pi' \to \bar{N}\Delta$. Only the $\bar{N}\Delta$ thresholds are relevant here. We first examine the normal threshold, $t=(m_3+m_4)^2$. The helicity amplitudes (10) can, in this case, be written

$$f_{\lambda_3\lambda_4;00}=\sum_J (J+\tfrac{1}{2})\langle\lambda_3\lambda_4|F^J|00\rangle d_{0\mu}{}^J(\theta_t)\,, \tag{29}$$

where $\mu=\lambda_3-\lambda_4$. At threshold it is appropriate to introduce a Russell-Saunders coupling expansion for the partial-wave amplitude $\langle\lambda_3\lambda_4|F^J|00\rangle$:

$$\langle\lambda_3\lambda_4|F^J|00\rangle=\sum_{L',S'}\langle s_3\,s_4\,\lambda_3\ -\lambda_4|S'\mu\rangle \times\langle L'S'0\mu|J\mu\rangle F_N{}^J(L',S')\,, \tag{30}$$

where the sum is over $S'=1$, 2 and all allowed L' values. $F^J(L',S')$ is the reduced matrix element, and is a function of t as well as its visible indices. Near threshold, $F^J(L',S')$ has a behavior $(T_N)^{L'}$. Thus, for fixed S' we need only keep the minimum L' value, as given in Table I, and also only keep one S' value if that happens to have a smaller L' than the other. At the normal threshold, the minimum L' value is $L_N'=J-1$ (apart from the first two J values—see Ref. 30). Both values of channel spin can occur. For elegance in later formulas we extract not only the $\bar{N}\Delta$ normal-threshold behavior, but also the factors occurring in Eq. (20) for the other thresholds as well. Thus we write

$$F_N{}^J(L'=J-1,\,S')=(T_NT_P)^J(T_N')^{J-1}(T_P')^{J-2} \times\tilde{F}_N{}^J(L'=J-1,\,S')\,, \tag{31}$$

where $\tilde{F}_N$ has no singularities at threshold. The factors other than $(T_N')^{J-1}$ can be thought of as merely constants. Multiplying both sides of (29) by $T_N'T_P'^2$ and making use of (30) and (31) with $L'=J-1$, we obtain the nonvanishing part of the numerator of Eq. (25) near $t=(m_3+m_4)^2$:

$$\begin{aligned}T_N'T_P'^2f_{\lambda_3\lambda_4;00}&=\sum_{S'}\langle s_3\,s_4\,\lambda_3\ -\lambda_4|S'\mu\rangle\sum_J (J+\tfrac{1}{2})\tilde{F}_N{}^J(J-1,\,S')\\&\quad\times\langle(J-1)S'0\mu|J\mu\rangle d_{0\mu}{}^J(\theta_t)(4tpp')^J\,.\end{aligned} \tag{32}$$

It is clear that only the highest power of z in the d function will survive at threshold. It is shown in Appendix A that the product of Clebsch-Gordan coefficient and the leading term of the d function appearing in the sum over J can be written [Eqs. (A6) and (A7)]

$$\langle(J-1)S'0\mu|J\mu\rangle d_{0\mu}{}^J(\theta_t) =\mu^{S'-1}a_{S'}(J)z^{J-S'}d_{0\mu}{}^{S'}(\theta_t)\,, \tag{33}$$

where $S'=1$, 2 and $a_{S'}(J)$ is given by (A9) and (A10). Thus the right-hand side of (32) can be expressed as

$$\begin{aligned}T_N'T_P'^2f_{\lambda_3\lambda_4;00}&=\sum_{S'}\langle s_3\,s_4\,\lambda_3\ -\lambda_4|S'\mu\rangle\mu^{S'-1}(4tpp')^{S'}d_{0\mu}{}^{S'}(\theta_t)\\&\quad\times\sum_J (J+\tfrac{1}{2})\tilde{F}_N{}^J(J-1,\,S')a_{S'}(J)(4tpp'z)^{J-S'}\,.\end{aligned} \tag{34}$$

The use of (33) is the key step in the development because it causes the separation of the helicity dependence from J. The two terms in the S' sum of (34) have explicit dependence on λ_3, λ_4, and μ times a partial-wave series that depends only on S' and on the polynomial

$$4tpp'z=t(s-u)+(m_1^2-m_2^2)(m_3^2-m_4^2)\,.$$

The partial-wave expansions can be assumed to converge, at least if the s values are on the line AB in the neighborhood of O in Fig. 3. Labeling these J sums in (34) as $y_{S'}(s)$, we have, at $t=(m_3+m_4)^2$,

$$\begin{aligned}T_N'T_P'^2f_{\lambda_3\lambda_4;00}=y_1(s)\langle\tfrac{3}{2}\,\tfrac{1}{2}\,\lambda_3\ -\lambda_4|1\mu\rangle(4tpp')d_{0\mu}{}^1(\theta_t)\\+\mu y_2(s)\langle\tfrac{3}{2}\,\tfrac{1}{2}\,\lambda_3\ -\lambda_4|2\mu\rangle(4tpp')^2d_{0\mu}{}^2(\theta_t)\,.\end{aligned} \tag{35}$$

Equation (35) shows that the four helicity amplitudes are related at $t=(m_3+m_4)^2$, actually depending on only two dynamical functions of s. We note in passing that, if $\cos\theta_t$ is held in the physical domain as threshold is approached, only the first term in (35) survives. Thus at the point O in Fig. 3 the helicity amplitudes are determined by a single constant, the $J=1$, $L=0$, $S=1$ amplitude, as can be seen from Table I. Away from O, the relations written out in detail are [remember $t=(m_3+m_4)^2$]

$$\begin{aligned}T_N'T_P'^2f_{\frac{1}{2}\frac{1}{2};00}&=(1/\sqrt{2})(4tpp'z)y_1\,,\\T_N'T_P'^2f_{\frac{1}{2},-\frac{1}{2};00}&=-(1/\sqrt{8})[y_1-3(4tpp'z)y_2]\\&\qquad\times(4tpp'\sin\theta_t)\,,\\T_N'T_P'^2f_{\frac{3}{2}\frac{1}{2};00}&=(\sqrt{\tfrac{3}{8}})[y_1+(4tpp'z)y_2];\\&\qquad\times(4tpp'\sin\theta_t)\,,\\T_N'T_P'^2f_{\frac{3}{2},-\frac{1}{2};00}&=(\sqrt{\tfrac{3}{2}})y_2(4tpp'\sin\theta_t)^2\,.\end{aligned} \tag{36}$$

Here we have identified particles 3 and 4 as Δ and $\bar{N}$, respectively. It is easy to translate (36) into expressions for the singularity-free amplitudes $A_{\lambda_3\lambda_4}(s,t)$ in (25).

We now consider the pseudothreshold $t=(m_3-m_4)^2$. The arguments are the same, with the modification of the intrinsic parity of $\bar{N}$ and the use of pseudoamplitudes (19). Table I shows that the minimum orbital angular momentum is now $L'=J-2$, and only $S'=2$ need be considered.[37] Thus the sum over L' and S' in (30) reduces to a single term, and the equivalent of

[37] For $S'=1^+$, only one L' value occurs for each J, namely $L'=J$.

(32) at $t=(m_3-m_4)^2$ is

$$T_N'T_P'^2f_{\lambda_3\lambda_4;00}=(-1)^{\frac{1}{2}-\lambda_4}\langle\tfrac{3}{2}\,\tfrac{1}{2}\,\lambda_3-\lambda_4|2\,\mu\rangle\sum_J(J+\tfrac{1}{2})\tilde{F}_P{}^J(J-2,2)\times\langle(J-2)\,2\,0\,\mu|J\mu\rangle d_{0\mu}{}^J(\theta_t)(4tpp')^J. \quad (37)$$

Use of Appendix A, Eq. (A8), allows separation of the helicity dependence from J. Then the J series can be formally summed, and (37) becomes

$$T_N'T_P'^2f_{\lambda_3\lambda_4;00}=y_3(s)(-1)^{\frac{1}{2}-\lambda_4}\langle\tfrac{3}{2}\,\tfrac{1}{2}\,\lambda_3-\lambda_4|2\,\mu\rangle\times(4tpp')^2d_{0\mu}{}^2(\theta_t), \quad (38)$$

showing that the amplitudes are related to a single function $y_3(s)$ at the pseudothreshold. Here it happens that the method of Jones,[17] based on only the $L'=0$ contribution, gives the same result, as can be seen from Table I. The equivalent of the four equations in (36) are $[t=(m_3-m_4)^2]$

$$\begin{aligned}
T_N'T_P'^2f_{\frac{1}{2}\frac{1}{2};00}&=(3/\sqrt{8})(4tpp'z)^2y_3\,,\\
T_N'T_P'^2f_{\frac{1}{2},-\frac{1}{2};00}&=-\left(\frac{\sin\theta_t}{\cos\theta_t}\right)T_N'T_P'^2f_{\frac{1}{2}\frac{1}{2};00}\,,\\
T_N'T_P'^2f_{\frac{3}{2}\frac{1}{2};00}&=\frac{1}{\sqrt{3}}\left(\frac{\sin\theta_t}{\cos\theta_t}\right)T_N'T_P'^2f_{\frac{1}{2}\frac{1}{2};00}\,,\\
T_N'T_P'^2f_{\frac{3}{2},-\frac{1}{2};00}&=-\frac{1}{\sqrt{3}}\left(\frac{\sin\theta_t}{\cos\theta_t}\right)^2T_N'T_P'^2f_{\frac{1}{2}\frac{1}{2};00}.
\end{aligned} \quad (39)$$

Here the ratio $(\sin\theta_t/\cos\theta_t)$ is equal to $\pm i$ at threshold, but is better given a meaning through the boundary function as

$$(\sin\theta_t/\cos\theta_t)=(\sqrt{\varphi})/[2(m_3-m_4)pp'z].$$

Equations (36) and (39) do not exhaust the relations among the amplitudes at threshold. The results derived so far concern the most singular parts of each amplitude. If we imagine expansions of the amplitudes in powers of $T_N'^2$ around the normal threshold or $T_P'^2$ around the pseudothreshold, we can ask whether or not there are relations among the coefficients of higher powers of T'^2, i.e., among the derivatives of the amplitudes with respect to t, for fixed s. In order to obtain such relationships, if any exist, it is necessary to retain more than the lowest L' value for each J in the Russell-Saunders expansions and lower powers of z than the highest in the expansion of the d functions. A discussion of the present example is given in Appendix C. It is not difficult to show that at the normal threshold no relations beyond those contained in (36) occur. The basic reason is that the first-order terms in $T_N'^2$ receive contributions from $L'=J-1$ and $L'=J+1$ for both $S'=1$ and $S'=2$. Four unknown functions of s, analogous to y_1 and y_2 in (36), are thus introduced. Since there are only four independent helicity amplitudes, no relation among the derivatives arises.

At the pseudothreshold, however, the first-order terms in $T_P'^2$ involve $L'=J$ for $S'=1$ and $L'=J-2$ and $L'=J$ for $S'=2$. Only three unknown functions are present; there is thus one relation, (C5), among the first derivatives. With the $T_P'^4$ terms the final L' value, $L'=J+2$ for $S'=2$, enters and there are as many unknown functions as there are amplitudes. The one derivative relation (C5) is written here for convenience as

$$\left[\frac{\tilde{f}_{\frac{1}{2}\frac{1}{2};00}+\tilde{f}_{\frac{1}{2},-\frac{1}{2};00}-(1/\sqrt{3})(\tilde{f}_{\frac{3}{2}\frac{1}{2};00}+\tilde{f}_{\frac{3}{2},-\frac{1}{2};00})}{t-(m_3-m_4)^2}\right]^{(0)}=\frac{\tilde{f}_{\frac{1}{2}\frac{1}{2};00}{}^{(0)}}{3(m_3-m_4)^2}\left[1-\frac{(m_3-m_4)^2}{(T_P'z)^2}\right]. \quad (40)$$

The tilde amplitudes, defined by (C2), are such as to remove the powers of $(\sin\theta_t/\cos\theta_t)$ in (39). The superscript zero on both sides indicates the value at pseudothreshold.

In a Regge-pole model the threshold relations embodied in (36), (39), and (40) impose constraints on the residue functions $\beta_{\lambda_3\lambda_4}(t)$ at $t=(m_3\pm m_4)^2$. These constraints are as important as the kinematic singularities. Indeed, the two are different aspects of the same kinematic phenomenon. In practice the relations at the pseudothresholds are most important in peripheral processes because of the proximity of these thresholds to the physical s-channel region. For the reactions $\pi^+p\to\pi^0\Delta^{++}$ and $K^+p\to K^0\Delta^{++}$, the $\bar{N}\Delta$ pseudothreshold is at $t=0.09$ $(\text{GeV}/c)^2$. Its closeness to $t\simeq0$ and the degree of kinematic singularity there [see Eq. (25)] demands careful attention to the constraints contained in (39) and (40), as we shall discuss in detail in Sec. V B.

B. Invariant Amplitudes for $\pi\pi'\to\bar{N}\Delta$

A method of incorporating the kinematic-singularity structure and the threshold relations is the use of invariant amplitudes, as was mentioned in the Introduction. We examine the amplitudes for $\pi\pi'\to\bar{N}\Delta$ within this framework in order to make comparisons with Secs. III and IV A. The Feynman amplitude for the s-channel reaction, $\pi N\to\pi'\Delta$, has the general form

$$M=\bar{U}_\mu(p')O_\mu\gamma_5u(p)\,,$$

where $U_\mu(p')$ is the Rarita-Schwinger wave function[38] for the Δ, $u(p)$ is a Dirac spinor for the N, and O_μ is a polar vector made up from the available momenta and Dirac γ matrices.[39] It is not difficult to convince oneself

[38] W. Rarita and J. Schwinger, Phys. Rev. **60**, 61 (1941); S. Kusaka, *ibid.* **60**, 61 (1941).

[39] The Dirac notation used here is that described by J. D. Jackson and H. Pilkuhn, Nuovo Cimento **33**, 906 (1964), Appendix A. See also the present Appendix B, Sec. 3.

that the most general structure for the amplitude is[40]

$$M=\bar{U}_\mu(p')\{[-A_1+i\gamma\cdot\tfrac{1}{2}(q+q')B_1](q+q')_\mu+[-A_2+i\gamma\cdot\tfrac{1}{2}(q+q')B_2](q-q')_\mu\}\gamma_5 u(p), \quad (41)$$

where q and q' are the 4-momenta of π and π', respectively, and the A_i and B_i are four arbitrary scalar functions of s and t. The notation is chosen in analogy with πN scattering.

The t-channel amplitude is obtained from (41) by substituting $q\to p_2$, $q'\to -p_1$, $p\to -p_4$, $p'\to p_3$, and $u(p)\to v(p_4)$. The Jacob-Wick helicity amplitudes are constructed by choosing definite helicities for $\bar{N}$ and Δ in the center-of-momentum frame. The reduction is straightforward and yields

$$\begin{aligned}
f_{\frac{1}{2}\frac{1}{2};00}={}&(\surd\tfrac{2}{3})\frac{(t+m_3^2-m_4^2)}{m_3T_N'}(2pp'z)A_1+\frac{1}{\surd 6}\frac{T_N'T_P'^2}{m_3}A_2\\
&+(\surd\tfrac{2}{3})\frac{1}{T_N'T_P'^2}\Bigg[\left(\frac{m_3+m_4}{m_3}\right)T_P'^2(2pp'z)^2\\
&-12t(pp'z)^2+4tp^2p'^2\Bigg]B_1\\
&-(\surd\tfrac{2}{3})\left(\frac{m_3-m_4}{m_3}\right)T_N'(pp'z)B_2,\\
f_{\frac{1}{2},-\frac{1}{2};00}={}&\frac{\surd\varphi}{T_N'}\Bigg[-(\surd\tfrac{2}{3})A_1\\
&+(\surd\tfrac{2}{3})\frac{(3m_3(m_3-m_4)+T_P'^2)}{m_3T_P'^2}(2pp'z)B_1\\
&+\frac{1}{\surd 6}\frac{T_N'^2}{m_3}B_2\Bigg],\\
f_{\frac{3}{2},\frac{1}{2};00}={}&\frac{\sqrt{2}\surd\varphi}{T_N'}\left[A_1-(m_3-m_4)\frac{(2pp'z)}{T_P'^2}B_1\right],\\
f_{\frac{3}{2},-\frac{1}{2};00}={}&\sqrt{2}\frac{\varphi}{T_N'T_P'^2}B_1.
\end{aligned} \quad (42)$$

In writing (42), we have assumed that π and π' have the same mass in order to simplify somewhat the kinematics.

The kinematic singularities established in (25) are evident in the expressions (42). The operator structure of (41) is such that the kinematic singularities are built in; the invariant amplitudes are then free of such singularities. Similarly, the threshold conditions (36) and (39) are satisfied automatically. At the normal threshold only terms in (42) involving A_1 and B_1 survive. The four amplitudes are thus given in terms of two, just as in (36). The specific connections between y_1, y_2 and A_1, B_1 at $t=(m_3+m_4)^2$ are

$$y_1=\frac{2}{\sqrt{3}(m_3+m_4)}[4m_3m_4A_1-(3m_3-m_4)pp'zB_1],$$

$$y_2=\frac{B_1}{2\sqrt{3}(m_3+m_4)^2}.$$

At the pseudothreshold, all the amplitudes are proportional to B_1 and the relations (39) are obtained.

The derivative relation (40) follows directly from the kinematic structure in (42). That such a relation exists is evident from the fact that the invariant amplitude A_2 has as its coefficient $T_P'^4$, relative to the most singular terms. This means that near $T_P'=0$, the first-order terms in $T_P'^2$ will involve only A_1, B_1, and B_2 and a derivative relation will occur.

The use of invariant amplitudes has its obvious virtues in handling the kinematic singularities and threshold relations in an automatic way. The only difficulty for a process involving high spins is the establishment of a set of invariant amplitudes. This has been solved, in principle at least, by Hepp[2] and by Williams,[3] and worked out for a number of cases by Fox,[4] from the starting point of M functions. A discussion of kinematic-threshold constraints from the point of view of M functions, with special reference to $\pi\pi'\to\bar{N}\Delta$, has been given by Stack.[41]

C. Dynamical Exceptions

The results of Secs. IV A and IV B give a description of the singularities of and relations among amplitudes resulting solely from kinematics. The assumption of specific dynamical mechanisms may cause departures from the purely kinematic results. Two reasons for such departures are (1) the presence of only a limited number of low J values, and (2) the absence of some values of channel spin.

Anomalous behavior for small J values is illustrated for $\pi\pi'\to\bar{N}\Delta$ in Table I. For $J=0^+$, 1^- the standard singularity behavior of Eq. (21) does not occur. If $J=0^+$ were the only state contributing, the amplitudes would vanish as $T_N'T_P'^2$ at the thresholds, rather than being infinite there. For $J=1^-$ the amplitudes have the standard behavior at the normal threshold, but are finite at the pseudothreshold rather than varying as $(T_P')^{-2}$. Going along with the less than standard singularity behavior is a departure from threshold relations such as (36), (39), and (40). In general, these anomalies of the first kind will occur in dynamical models that involve angular momentum states with $J<(s_1+s_2)$, (s_3+s_4). For such states the minimum orbital angular momentum for each J cannot be physically realized; a higher L value is necessary, with correspondingly less singular behavior at the threshold.

[40] S. Mandelstam, J. E. Paton, R. F. Peierls, and A. Q. Sarker, Ann. Phys. (N. Y.) **18**, 198 (1962).

[41] J. D. Stack (unpublished).

The example of pion exchange ($J^P=0^-$) in the t-channel process $\bar{N}\Delta\to\pi\rho$ ($s_1+s_2=2$, $s_3+s_4=1$) has been discussed by Frautschi and Jones.[14]

A well-known model of the process $\pi\pi'\to\bar{N}\Delta$ is that of vector-meson exchange in perturbation theory. The $V\bar{N}\Delta$ vertex involves three coupling constants or vertex form factors G_1, G_2, and G_3.[39] A simple computation shows that for this model the invariant amplitudes in (41) are

$$A_1=\frac{gG_1}{m_V^2-t},\quad A_2=\frac{gG_3}{2(m_V^2-t)}\frac{(s-u)}{(m_3+m_4)^2},$$
$$B_1=0,\quad B_2=-\frac{2gG_2}{(m_3+m_4)(m_V^2-t)}.$$

Here g is the $\pi\pi'V$ coupling constant and m_V is the mass of the vector meson. The kinematics have again been simplified by taking the π and π' masses equal. The notable point about these amplitudes is that $B_1=0$ ($J=1$ cannot have $\mu=2$). The remaining amplitudes (42) are finite at $T_P'=0$ and the interconnection (39) becomes empty. At the normal threshold, $y_2=0$ in (36) and the first three amplitudes are all proportional to one another, while the fourth vanishes, as befits a situation where $J=1$ and $L'=0$ (see Table I). The derivative relation (40) holds in a degenerate form with $\tilde{f}_{\frac{3}{2},-\frac{1}{2};00}=0$ on the left- and the right-hand side equal to zero because the tilde amplitudes (C2) vanish at the pseudothreshold.

For specific couplings, even the remaining relation at the normal threshold may disappear. The Stodolsky-Sakurai model,[42] for example, with its purely magnetic dipole coupling (no electric or longitudinal multipoles) corresponds to

$$G_1=-\frac{T_{N}'^2}{2m_3(m_3+m_4)}G_2,\quad G_3=\left(\frac{m_3+m_4}{m_3}\right)G_2.$$

Now A_1 vanishes at the normal threshold. The helicity amplitudes are proportional to T_N' instead of its reciprocal, and (36) becomes an empty statement. In fact, the four amplitudes reduce to two at all values of t:

$$\begin{aligned} f_{\frac{1}{2}\frac{1}{2};00}{}^{SS}&=f_{\frac{3}{2},-\frac{1}{2};00}{}^{SS}=0,\\ f_{\frac{3}{2}\frac{1}{2};00}{}^{SS}&=\sqrt{3}f_{\frac{1}{2},-\frac{1}{2};00}{}^{SS}.\end{aligned}\tag{43}$$

The one independent amplitude vanishes at the normal threshold and is finite at the pseudothreshold. The derivative relation (40) is satisfied trivially.

The second cause of anomalous behavior, absence of one or more values of channel spin, can be seen from Table I, or equivalently (37) or (42). If, for example, the dynamics are such that channel spin $S'=2$ is not present, i.e., $B_1(s,t)=0$, the amplitudes will have less than the standard singularity at the pseudothreshold. Consequently, (39) will reduce to $0=0$. Furthermore, at the normal threshold $y_2=0$ in (36), so that the three nonzero amplitudes are proportional. The derivative relation (40) will hold in the degenerate form of its right-hand side equal to zero. The left-hand side of (40) now actually represents the less singular amplitudes themselves, rather than their derivatives, and its vanishing is the only relation remaining among the three amplitudes at the pseudothreshold when $B_1=0$. A model with $B_1=0$ identically contains the Stodolsky-Sakurai model as a special case, but is considerably more general. An even less restrictive variant is a model in which B_1 vanishes only at the pseudothreshold. All the remarks of this paragraph still apply, with only slight modification.

The two examples just discussed illustrate causes of less singular behavior at thresholds than standard, with its consequences of failure or modification of the threshold relations. The avoidance of constraints such as (39) and (40) can have important consequences in the nearby s-channel physical region. It is well known that the Stodolsky-Sakurai amplitudes [Eq. (43)] give density matrix elements for the Δ decay of the form $\rho_{\frac{1}{2}\frac{1}{2}}=\frac{1}{8}$, $\rho_{\frac{3}{2},-\frac{1}{2}}=\sqrt{3}/8$, $\rho_{\frac{3}{2}\frac{1}{2}}=0$. The constraint equation (38), with its equivalence to $J=2$ exchange, gives a quite different set of density matrix elements. We discuss this point further in Sec. V B.

D. s-Channel Threshold Relations for $\pi N\to\pi'\Delta$

The singularity structure of the s-channel amplitudes and the relations between them at the thresholds can be treated analogously to the t-channel amplitudes. The example of $\pi N\to KY$ is given in Appendix B 5. Because of the presence of spin in both initial and final state and the occurrence of both parity sequences it is convenient to use parity-conserving amplitudes[26] as discussed in Sec. II D. To avoid confusion with the previous sections on the t channel, we redefine the masses and helicity labels as follows: $m_\pi=\mu$, $m_{\pi'}=\mu'$, $m_N=m$, $m_\Delta=m'$, and $\lambda_N=\lambda$, $\lambda_\Delta=\lambda'$. The helicity amplitudes in the s channel are denoted by $g_{\lambda';\lambda}$, while the parity-conserving amplitudes (15) are written as $F_{\lambda';\lambda}{}^\eta(s,z_s)$. The initial and final center-of-mass momenta are p_s and p_s', respectively. We assume $\sqrt{s}>0$.

We need only consider the four amplitudes with helicity indices $\lambda=\frac{1}{2}$, $-\frac{1}{2}$ and $\lambda'=\frac{1}{2}$, $\frac{3}{2}$. The inverse of (15) gives these amplitudes as

$$\begin{aligned} g_{\frac{1}{2};\frac{1}{2}}&=\frac{1}{\sqrt{2}}\cos\tfrac{1}{2}\theta_s\,(F_{\frac{1}{2};\frac{1}{2}}{}^-+F_{\frac{1}{2};\frac{1}{2}}{}^+),\\ g_{\frac{1}{2};-\frac{1}{2}}&=\frac{1}{\sqrt{2}}\sin\tfrac{1}{2}\theta_s\,(F_{\frac{1}{2};\frac{1}{2}}{}^- - F_{\frac{1}{2};\frac{1}{2}}{}^+),\\ g_{\frac{3}{2};\frac{1}{2}}&=\frac{1}{\sqrt{2}}\cos\tfrac{1}{2}\theta_s\sin\theta_s\,(F_{\frac{3}{2};\frac{1}{2}}{}^-+F_{\frac{3}{2};\frac{1}{2}}{}^+),\\ g_{\frac{3}{2};-\frac{1}{2}}&=\frac{1}{\sqrt{2}}\sin\tfrac{1}{2}\theta_s\sin\theta_s\,(F_{\frac{3}{2};\frac{1}{2}}{}^- - F_{\frac{3}{2};\frac{1}{2}}{}^+).\end{aligned}\tag{44}$$

[42] L. Stodolsky and J. J. Sakurai, Phys. Rev. Letters 11, 90 (1963); L. Stodolsky, Phys. Rev. 134, B1099 (1964).

The kinematic singularities can be read off from Eqs. (26) and (28). The results are, assuming $\mu<m$, $\mu'<m'$,

$$
\begin{aligned}
F_{\frac{1}{2};\frac{1}{2}}{}^{+}&=\frac{p_s}{(\sqrt{s})p_s'}A_{\frac{1}{2};\frac{1}{2}}{}^{+}(s,t)\,,\\
F_{\frac{1}{2};\frac{1}{2}}{}^{-}&=\frac{1}{(\sqrt{s})}A_{\frac{1}{2};\frac{1}{2}}{}^{-}(s,t)\,,\\
F_{\frac{3}{2};\frac{1}{2}}{}^{+}&=2p_s{}^2A_{\frac{3}{2};\frac{1}{2}}{}^{+}(s,t)\,,\\
F_{\frac{3}{2};\frac{1}{2}}{}^{-}&=2p_sp_s'A_{\frac{3}{2};\frac{1}{2}}{}^{-}(s,t)\,,
\end{aligned}
\tag{45}
$$

where the A's are all free of kinematic singularities. The amplitudes (44) can thus be written with their singularity structure exhibited explicitly:

$$
\begin{aligned}
g_{\frac{1}{2};\frac{1}{2}}&=\frac{1}{\sqrt{2}}(\cos\tfrac{1}{2}\theta_s)\frac{1}{\sqrt{s}}\left(A_{\frac{1}{2};\frac{1}{2}}{}^{-}+\frac{p_s}{p_s'}A_{\frac{1}{2};\frac{1}{2}}{}^{+}\right),\\
g_{\frac{1}{2};-\frac{1}{2}}&=\frac{1}{\sqrt{2}}(\sin\tfrac{1}{2}\theta_s)\frac{1}{\sqrt{s}}\left(A_{\frac{1}{2};\frac{1}{2}}{}^{-}-\frac{p_s}{p_s'}A_{\frac{1}{2};\frac{1}{2}}{}^{+}\right),\\
g_{\frac{3}{2};\frac{1}{2}}&=\frac{1}{\sqrt{2}}(\cos\tfrac{1}{2}\theta_s)\frac{\sqrt{\varphi}}{\sqrt{s}}\left(A_{\frac{3}{2};\frac{1}{2}}{}^{-}+\frac{p_s}{p_s'}A_{\frac{3}{2};\frac{1}{2}}{}^{+}\right),\\
g_{\frac{3}{2};-\frac{1}{2}}&=\frac{1}{\sqrt{2}}(\sin\tfrac{1}{2}\theta_s)\frac{\sqrt{\varphi}}{\sqrt{s}}\left(A_{\frac{3}{2};\frac{1}{2}}{}^{-}-\frac{p_s}{p_s'}A_{\frac{3}{2};\frac{1}{2}}{}^{+}\right).
\end{aligned}
\tag{46}
$$

The remaining half-angle factors appearing in (46), as compared with (25), for example, are a consequence of the half-integral spin in the s channel. The $(\sqrt{s})^{-1}$ singularity in all four equations is only apparent. In the discussion of Sec. III B above Eq. (24), it is noted than $\sin\frac{1}{2}\theta_s\propto\sqrt{s}$ near $s=0$, while $\cos\frac{1}{2}\theta_s$ is well behaved. Thus the factor of $(\sqrt{s})^{-1}$ in the second and fourth equations compensates for the behavior of $\sin\frac{1}{2}\theta_s$. But in the first and third, the functions A^- and A^+ are related at $s=0$ in such a way as to remove the square-root infinity, just as in Eq. (B14) for the example of $\pi N\to KY$.

Some of the threshold relations can be obtained by inspection of (46). At both normal and pseudothresholds for the πN channel ($p_s\to 0$, $p_s'\neq 0$) only the A^- terms survive. Thus, in the limits $s=(m+\mu)^2$ and $s=(m-\mu)^2$,

$$g_{\frac{1}{2};\frac{1}{2}}/\cos\tfrac{1}{2}\theta_s=g_{\frac{1}{2};-\frac{1}{2}}/\sin\tfrac{1}{2}\theta_s\,,\tag{47}$$

$$g_{\frac{3}{2};\frac{1}{2}}/\cos\tfrac{1}{2}\theta_s=g_{\frac{3}{2};-\frac{1}{2}}/\sin\tfrac{1}{2}\theta_s\,.\tag{48}$$

Similarly, at the thresholds,

$$s=(m'+\mu')^2\quad\text{and}\quad s=(m'-\mu')^2\,,$$

the A^+ terms dominate, and the amplitudes are related as follows:

$$g_{\frac{1}{2};\frac{1}{2}}/\cos\tfrac{1}{2}\theta_s=-g_{\frac{1}{2};-\frac{1}{2}}/\sin\tfrac{1}{2}\theta_s\,,\tag{49}$$

$$g_{\frac{3}{2};\frac{1}{2}}/\cos\tfrac{1}{2}\theta_s=-g_{\frac{3}{2},-\frac{1}{2}}/\sin\tfrac{1}{2}\theta_s\,.\tag{50}$$

The relations (47)–(50) are akin to those obtained by Jones[17] and Trueman[43] for πN scattering [see (B.15)]. They allow the creation of linear combinations of amplitudes with more rapidly convergent asymptotic behavior for $\sqrt{s}\to\infty$.

The question now arises as to whether there are more threshold relations. In particular, can the relations involving $\lambda'=\frac{3}{2}$, (48) and (50), be connected to their counterparts, (47) and (49), with $\lambda'=\frac{1}{2}$? The answer is that (49) and (50) can be related, but (47) and (48) cannot. To demonstrate this we use arguments on orbital angular momentum. Consider the partial-wave expansion (17) for $F_{\lambda';\lambda}{}^{\eta}$ and a Russell-Saunders expansion similar to (30) for $F_{\lambda';\lambda}{}^{J\eta}$. Now there will be an expansion over L, S and over L', S'. Actually, S and S' are fixed at $S=\frac{1}{2}$, $S'=\frac{3}{2}$, and only one L value occurs for each J. But L' takes on two values. For $\eta=+1$, we have $L=J+\frac{1}{2}$, $L'=J-\frac{3}{2}$, $J+\frac{1}{2}$, and for $\eta=-1$, $L=J-\frac{1}{2}$, $L'=J-\frac{1}{2}$, $J+\frac{3}{2}$.

At the πN thresholds ($p_s\to 0$, $p_s'\neq 0$), evidently the $\eta=-1$ sequence dominates because it has the smaller L value, but both L' values will be present. This means that the dependence on initial-state helicity is determined, as in (47) and (48), but the different final-state helicities cannot be connected. A development parallel to that from (30) to (36) gives explicit demonstration of the fact.

At the $\pi'\Delta$ thresholds, on the other hand, only the smallest L' value, namely $L'=J-\frac{3}{2}$ in the $\eta=+1$ sequence, survives. We now have only one L and one L' effective in the partial-wave expansion. All four helicity amplitudes are related. The derivation of the connections is exactly as in Sec. IV A, with the leading powers in z_s of $e_{\lambda\mu}{}^{J\pm}$ being given in Appendix A, by Eqs. (A12) and (A13) and the Clebsch-Gordan coefficients by (A14) and (A15). The result is that, for $s\to(m'\pm\mu')^2$,

$$F_{\frac{1}{2};\frac{1}{2}}{}^{+}=\sqrt{3}z_sF_{\frac{3}{2};\frac{1}{2}}{}^{+}$$

or

$$A_{\frac{1}{2};\frac{1}{2}}{}^{+}=\sqrt{3}[2(\sqrt{s})p_sp_s'z_s]A_{\frac{3}{2};\frac{1}{2}}{}^{+}.\tag{51}$$

The two relations, (49) and (50), can thus be combined,

$$
\begin{aligned}
\frac{g_{\frac{1}{2};\frac{1}{2}}}{\cos\frac{1}{2}\theta_s}=-\frac{g_{\frac{1}{2};-\frac{1}{2}}}{\sin\frac{1}{2}\theta_s}&=\sqrt{3}\cot\theta_s\frac{g_{\frac{3}{2};\frac{1}{2}}}{\cos\frac{1}{2}\theta_s}\\
&=-\sqrt{3}\cot\theta_s\frac{g_{\frac{3}{2};-\frac{1}{2}}}{\sin\frac{1}{2}\theta_s}\,,
\end{aligned}
\tag{52}
$$

at the $\pi'\Delta$ thresholds $s=(m'\pm\mu')^2$.

E. Invariant Amplitudes in the s-Channel

It is by now obvious that the relations of the previous section will all appear automatically when the amplitudes are expressed in terms of the invariant amplitudes A_1, A_2, B_1, B_2 of Eq. (41). For completeness we list the s-channel analogs of (42), or rather, the invariant-

[43] T. L. Trueman, Phys. Rev. Letters **17**, 1198 (1966), Ref. 7.

amplitude equivalents of (45):

$$
\begin{aligned}
F_{\frac{1}{2};\frac{1}{2}}{}^{+}={}&\frac{1}{\sqrt{3}}F_{\frac{3}{2};\frac{1}{2}}{}^{-}+\frac{1}{\sqrt{3}}F_{\frac{3}{2};\frac{1}{2}}{}^{+}\left[\left(\frac{2E'}{m'}+1\right)z_s-\frac{2p_s'E}{m'p_s}\right]\\
&-\frac{4}{\sqrt{3}}\left(\frac{E'+m'}{E+m}\right)^{1/2}\frac{(\sqrt{s})p_sp_s'}{m'}\\
&\times\left\{A_1+B_1\left[\sqrt{s}-\left(\frac{m'-m}{2}\right)\right]\right\},\\
F_{\frac{1}{2};\frac{1}{2}}{}^{-}={}&-\frac{1}{\sqrt{3}}F_{\frac{3}{2};\frac{1}{2}}{}^{+}+\frac{1}{\sqrt{3}}F_{\frac{3}{2};\frac{1}{2}}{}^{-}\left[\left(\frac{2E'}{m'}-1\right)z_s-\frac{2p_s'E}{m'p_s}\right]\\
&+\frac{4}{\sqrt{3}}\left(\frac{E+m}{E'+m'}\right)^{1/2}\frac{(\sqrt{s})p_s'^2}{m'}\\
&\times\left\{A_1-B_1\left[\sqrt{s}+\left(\frac{m'-m}{2}\right)\right]\right\},\\
F_{\frac{3}{2};\frac{1}{2}}{}^{+}={}&-p_s^2\left(\frac{E'+m'}{E+m}\right)^{1/2}\\
&\times\left\{A_1+A_2+(B_1+B_2)\left[\sqrt{s}-\left(\frac{m'-m}{2}\right)\right]\right\},\\
F_{\frac{3}{2};\frac{1}{2}}{}^{-}={}&p_sp_s'\left(\frac{E+m}{E'+m'}\right)^{1/2}\\
&\times\left\{A_1+A_2-(B_1+B_2)\left[\sqrt{s}+\left(\frac{m'-m}{2}\right)\right]\right\}.
\end{aligned}
\tag{53}
$$

Inspection of these expressions shows that (a) the singularity structure of (45) is present, and (b) that the threshold relation (51) emerges as $p_s'\to 0$.

F. General Remarks

The examples of $\pi N\to\pi'\Delta$ and $\pi N\to KY$ (Appendix B) in both the t and s channels illustrate the methods of determining the kinematic-threshold relations, if any, between the various helicity amplitudes. The general pattern of how many relations exist at a given threshold is also evident. We summarize the general situation in a list of comments to follow. For simplicity we will speak of the relations between amplitudes at the thresholds in the initial state. But the words initial and final can be interchanged. The notation is that of Fig. 2, with initial orbital angular momentum L and final, L', etc. Unequal masses are assumed unless otherwise stated.

(1) For the initial and final states, determine the allowed values of channel spin S and S'.

(2) For given η, determine the allowed values of L and L' with the intrinsic parities appropriate to the threshold in question. The kinematic singularities at that threshold are given by (26) with the "mismatch" exponents found from the smallest values of L and L'.

(3) At the initial-state threshold, consider only the minimum L sequence. This will correspond to one of the values of η. If the minimum L can occur for only one value of channel spin, S, the various helicity amplitudes with definite final helicity, but different initial helicities, are related. Examples are the s-channel amplitudes for $\pi N\to\pi'\Delta$ at the πN thresholds, as shown in (47) and (48), and $\pi\pi'\to\bar{N}\Delta$ at the $\bar{N}\Delta$ pseudothreshold, as given by (38) or (39).

(4) The minimum L sequence may occur for two S values, namely $S=s_1+s_2$ and $S=s_1+s_2-1$. Then the amplitudes with different initial helicities (but fixed final helicity) are given in terms of two independent functions. An example is $\pi\pi'\to\bar{N}\Delta$ at the normal threshold, as shown in (36). Note that, no matter how many different channel spins there are, no more than the largest and next to largest contribute to the minimum L or L' sequence. Hence the dependence at threshold involves no more than two undetermined functions for each set of the final helicity values.

(5) For unequal masses, the final state is not at threshold when the initial state is. This means that all possible L' and S' combinations can occur, and while amplitudes of different initial helicity may be related, there will be no connections for different final helicities. An example is $\pi N\to\pi'\Delta$ at the πN thresholds, (47) and (48), where the final helicities $\lambda'=\frac{1}{2}$ and $\lambda'=\frac{3}{2}$ are unrelated.

(6) If the final channel happens to have only one L', S' combination, then the dependence on that helicity index is also determined. This occurs trivially for $\pi\pi'\to\bar{N}\Delta$, but less trivially for $\pi N\to\pi'\Delta$ at the $\pi'\Delta$ thresholds, as exhibited in (52).

(7) For masses such that the initial- and final-state thresholds occur at the same energy, the threshold behavior in both channels must be considered simultaneously. This normally occurs only if the particles are the same in the initial and final states. Roughly speaking, one then gets the square of the formulas discussed here. For example, in the t-channel process $N\bar{\Delta}\to\bar{N}\Delta$ (corresponding to $\bar{p}p\to\bar{\Delta}\Delta$ in the s channel), the relation between the various helicity amplitudes with $\eta=+1$ at the $\bar{N}\Delta$ $(=N\bar{\Delta})$ pseudothreshold has the appearance of (38), but with another Clebsch-Gordan coefficient for the initial-state helicities, and $d_{\lambda\mu}{}^2$, instead of $d_{0\mu}{}^2$. In $\pi N\to\pi N$, the merging of the thresholds gives (B15) as the relation at the normal threshold and pseudothresholds.

(8) To determine the nonderivative threshold relations explicitly, consider the partial-wave expansion (17) with the F^J given by keeping only the lowest allowed L or L' value in the Russell-Saunders expansion. Then use the leading powers of z in $e_{\lambda\mu}{}^{J\pm}$ [given by (A12) and (A13)] in combination with the Clebsch-

Gordan coefficients [see (A14) and (A15)] to separate the sum over J values from the helicities and a possible sum over channel spin.

(9) The existence of derivative relations can be established by considering how the higher L values enter successively in an expansion of the amplitude in powers of T^2. There are as many Russell-Saunders combinations of (L,S) and (L',S') for each J as there are independent helicity amplitudes. The nonderivative relations occur because not all (L,S) combinations contribute to zeroth order in T^2. For each higher power of T^2 more L values contribute. But if not all possible values of (L,S) occur, there will be as many relations among the successive derivatives of the amplitudes as there are noncontributing (L,S) combinations at each stage. Eventually, of course, all (L,S) values will enter and no further relations can emerge. Equation (40) is an example of a first-derivative relation for $\pi\pi' \to \bar{N}\Delta$ at the $\bar{N}\Delta$ pseudothreshold. Its derivation is given in Appendix C.

V. CROSS SECTIONS AND DECAY DENSITY MATRICES

The t-channel kinematic singularities and the associated threshold relations among amplitudes have an important influence on the structure of the differential cross section and decay density matrices for peripheral processes in the s channel. For the s-channel process $a+b \to c+d$, the differential cross section is

$$\frac{d\sigma}{dt}=\frac{1}{64\pi s p_s{}^2}\frac{1}{(2s_a+1)(2s_b+1)}\sum_\lambda |g_{\lambda_c\lambda_d;\lambda_a\lambda_b}|^2, \quad (54)$$

where $g_{\lambda_c\lambda_d;\lambda_a\lambda_b}$ are the s-channel helicity amplitudes.[24] The orthogonality of the crossing matrix for helicity amplitudes allows the replacement in (54),

$$\sum_\lambda |g_{\lambda_c\lambda_d;\lambda_a\lambda_b}|^2 \to \sum_\lambda |f_{\lambda_c\bar{\lambda}_a;\bar{\lambda}_d\lambda_b}|^2, \quad (55)$$

where $f_{\lambda_c\bar{\lambda}_a;\bar{\lambda}_d\lambda_b}$ are the t-channel amplitudes. Now the s-channel cross section is expressed directly in terms of the sum of the absolute squares of the analytic continuations of the t-channel helicity amplitudes.[7,44] Similarly, the decay density matrix of one of the outgoing particles in the s channel, say c, takes the form

$$\rho_{mm'}{}^{(c)}=\sum_{\bar{\lambda}_a\bar{\lambda}_d\lambda_b} f_{m\bar{\lambda}_a;\bar{\lambda}_d\lambda_b} f_{m'\bar{\lambda}_a;\bar{\lambda}_d\lambda_b}{}^*, \quad (56)$$

provided the quantization axis is chosen as the momentum transfer direction in the rest frame of c.[44] The direct use of t-channel amplitudes has obvious advantages in the treatment of peripheral processes.

[44] K. Gottfried and J. D. Jackson, Nuovo Cimento 33, 309 (1964).

A. Absence of t-Channel Kinematic Singularities in the Cross Section

The first obvious requirement in using t-channel amplitudes is to incorporate the proper kinematic-singularity structure, as given by (26) and (28). This is done automatically in perturbation theory or with the use of invariant amplitudes. But in Regge-pole models with helicity amplitudes the requirements must be consciously imposed. If the kinematic-singularity structure, but not the threshold relations, are imposed, the resulting cross-section expressions contain explicit polelike factors of the form $|t-(m_a \pm m_c)^2|^{-1}$. From (25) it is clear that the phenomenological expression for the differential cross section for $\pi N \to \pi'\Delta$ will then have the form

$$\frac{d\sigma}{dt}=\frac{1}{|(m_\Delta+m_N)^2-t|[(m_\Delta-m_N)^2-t]^2}S(s,t), \quad (57)$$

where $S(s,t)$ is well behaved in t. A collection of Regge-pole formulas for a large class of reactions, showing explicit kinematic singularities of this type, have been given by Wang,[12] and have been used by some authors[13,14] in empirical fits to experimental data. Similarly the standard formulas used to describe high-energy pion-nucleon charge exchange[45] have an over-all t-channel kinematic-singularity factor $(4m^2-t)^{-1}$. A singularity like $(4m^2-t)^{-1}$ is so far away from the s-channel physical region that its presence or absence is of no practical consequence. But for processes like $\pi N \to \pi'\Delta$ the factors exhibited in (57) almost completely determine the t dependence at small t.

The presence of these t-channel kinematic singularities in the physical cross section for the s channel is not consistent with the known singularity structure of the s-channel amplitudes. This point has been stressed by Lin,[15] with emphasis on $t=0$, and by Stack.[16] Consider the expression (54) for the differential cross section. The s-channel amplitudes g_λ possess s-channel kinematic singularities at $s=(m_a \pm m_b)^2$, $(m_c \pm m_d)^2$ and singularities in t on the boundary of the physical region,[9,10] but do not have singularities at the t-channel thresholds $t=(m_a \pm m_c)^2$, $(m_b \pm m_d)^2$. Thus the cross section (54) cannot possess the polelike factors in t shown in (57) and all of the formulas of Ref. 12. The only t-channel singularities allowed in $|g_\lambda|^2$ are dynamic ones (for example, poles corresponding to exchanged particles) whose locations do not depend on the external masses.

The puzzle or inconsistency here can be phrased as follows: The g_λ have no t-channel kinematic singularities; the f_λ do have them. Is the replacement (55) allowed, and if it is, how can we avoid obtaining a cross section possessing the impermissible polelike factors, as

[45] G. Höhler, J. Baacke, and G. Eisenbeiss, Phys. Letters 22, 203 (1966).

in (57)? The first part of the question has a positive answer. The equality of the left-hand and right-hand sides of (55) is assured by the fact that the crossing matrix is a real orthogonal matrix in the physical s channel. Thus, as long as we stay in the physical region of s, the use of (55) is allowed. But, as emphasized by Lin,[15] the equality of the two sides of (55) does not hold outside the physical region where the crossing angles become complex and have singularities. In particular, near the t-channel thresholds the right-hand side of (55) has the singularities discussed in Sec. III, but the left-hand side is well behaved. The second part of the question, how to avoid obtaining expressions like (57), has a subtler answer. We have seen in Sec. IV that when amplitudes have kinematic singularities there are always accompanying threshold relations among the amplitudes of different helicity. The explicit satisfaction of these threshold constraints among the f_λ will always eliminate the kinematic singularities from the right-hand side of (55), *when it is evaluated in the physical region of s*. When the sum of the absolute squares of the f_λ is computed in the unphysical region, it will contain, of course, the threshold kinematic singularities, since each t-channel amplitude possesses them and no cancellation can occur in a sum of absolute squares.

To see how the singularities are cancelled in the physical region in the s channel, but not outside it, we discuss the somewhat academic example of pion-nucleon charge exchange where the amplitudes are singular as $(t-4m^2)^{-1/2}$ at the normal $N\bar{N}$ threshold. This singularity is not important at small t, but the principle involved in its removal from the cross section is the same as for more practical examples such as $\pi N\rightarrow\pi\Delta$, and the details are simpler. We use the Regge-pole model with the exchange of a ρ-meson trajectory as the framework, although the method has wider applicability. The kinematic singularities and threshold relations for $\pi\pi\rightarrow\bar{N}N$ are treated in Appendix B. The two t-channel amplitudes are given by (B2). With the standard Regge pole assumptions (see Appendix D), the amplitudes can be written as

$$f_{++;00}=\frac{\gamma_1(t)}{(t-4m^2)^{1/2}}R(s,t)\,,$$
$$f_{+-;00}=\frac{\alpha(t)\gamma_2(t)}{(t-4m^2)^{1/2}}\left(\frac{(\surd t)\sin\theta_t}{2m\cos\theta_t}\right)R(s,t)\,, \tag{58}$$

where $R(s,t)$ is the usual Regge-pole amplitude for spinless particles [Eq. (D8)], $\alpha(t)$ is the ρ-meson trajectory, and $\gamma_i(t)$ is a reduced residue function, free of all kinematic singularities. In forming the absolute square of $f_{+-;00}$ care must be taken in the interpretation of $(\surd t)\sin\theta_t/\cos\theta_t$. As mentioned below (39), it is given meaning through the expression $(\surd t)\sin\theta_t/\cos\theta_t=2(\surd\varphi)/(s-u)$. According to (54) and (55) the differential cross section is

$$\frac{d\sigma}{dt}=\frac{|R(s,t)|^2}{64\pi sp_s^2}\frac{1}{4m^2-t}\left[\gamma_1^2(t)-\frac{t}{4m^2}\alpha^2(t)\gamma_2^2(t)\right]. \tag{59}$$

In writing (59) it has been assumed that $t<0$, that $\alpha(t)$, $\gamma_1(t)$, and $\gamma_2(t)$ are real, and that $|\cos\theta_t|\simeq|\sin\theta_t|\gg1$. This is the standard Regge-pole formula of Höhler[45] and others. γ_1 and γ_2 are assumed to be arbitrary empirical functions of t, to be determined by fitting the data.

But we know that γ_1 and γ_2 are not completely arbitrary. The amplitudes must satisfy (B5) at $t=4m^2$. In terms of the residue functions this requirement is

$$\left.\frac{\alpha(t)\gamma_2(t)}{\gamma_1(t)}\right|_{t=4m^2}=1\,. \tag{60}$$

To see how this condition removes the polelike factor $(4m^2-t)^{-1}$, in (59), we write, for arbitrary t,

$$\alpha(t)\gamma_2(t)=\gamma_1(t)+\xi(t)(4m^2-t)/4m^2\,, \tag{61}$$

where $\xi(t)$ is not infinite at $t=4m^2$, and is well behaved and real for $t<4m^2$. The threshold condition (60) is now satisfied, but the cross section still involves two arbitrary residue functions, $\gamma_1(t)$ and $\xi(t)$. Substitution of (61) into (59) yields

$$\frac{d\sigma}{dt}=\frac{|R(s,t)|^2}{64\pi sp_s^2}\times\frac{1}{4m^2}\left\{\gamma_1^2-\frac{t}{4m^2}\left[2\gamma_1\xi+\xi^2\left(1-\frac{t}{4m^2}\right)\right]\right\}. \tag{62}$$

We note that the threshold singularity has been cancelled out by the imposition of the threshold constraint (60). The cross section has the proper behavior in t, as required by (54).

There only remains one further remark. In the unphysical region where $t>0$, the sum of the absolute squares of the amplitudes (58) is proportional to the square bracket in (59), but with a plus sign between the two terms. Then substitution of (61) does not result in a common factor of $(4m^2-t)$; the right-hand side of (55) now possesses the known kinematic singularity. The discontinuous behavior as the line $t=0$ is crossed is not surprising. The absolute squares of analytic functions need not be analytic.

B. Cross Section and Decay Correlations for $\pi N\rightarrow\pi'\Delta$

The example of charge-exchange scattering is not very exciting because the normal $\bar{N}N$ threshold is so far away from the region of interest. Empirical fitting with (59) or (62) will lead to substantially the same results, even though (59) is incorrect in principle. But for our favorite reaction, $\pi N\rightarrow\pi\Delta$, the differences

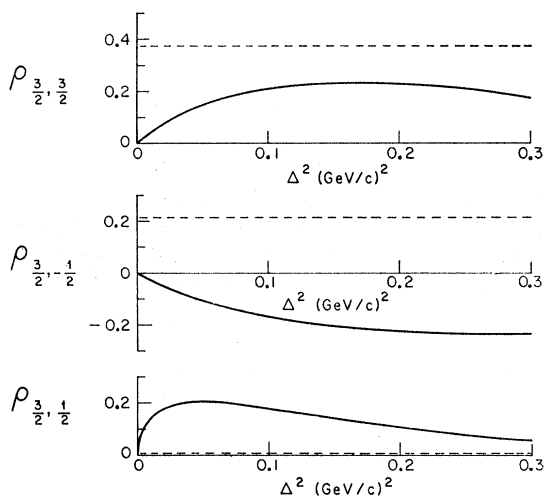

FIG. 4. Density matrix elements $\rho_{mm'}$ for the decay of the Δ in the process $\pi N \to \pi\Delta$, assuming that the threshold relations (39) and (40) hold at the $\bar{N}\Delta$ pseudothreshold, $t=0.09$ (GeV/c)^2, and that the residue functions are not rapidly varying in t [see text below Eqs. (64) and (65)]. The dashed lines are the predictions of the magnetic-dipole coupling model of Stodolsky and Sakurai.

between the formulas of Wang,[12] as used by Krammer and Maor,[13] and the correct expressions are enormous. The $\bar{N}\Delta$ pseudothreshold is at $t=0.09$ (GeV/c)^2. Thus the cross-section formula (57) appears to have a dynamic pole corresponding to the exchange of a particle of mass 300 MeV, far lighter than the ρ meson presumed to be the dominant exchange. This sharply peaked factor governs the small-t behavior and requires a zero in the function $S(s,t)$ between $t=0.09$ and the physical region $t\lesssim 0$ in order to fit the experimental data.

The threshold relations (36), (39), and (40) are required in order to remove the spurious t-channel polelike factors from (57). Within a Regge-pole framework we write the t-channel amplitudes as (see Appendix D)

$$f_{\lambda_3\lambda_4;00}=\frac{1}{T_N'T_P'^2}\frac{\alpha!}{(\alpha-\mu)!}\gamma_{\lambda_3\lambda_4}(t)\times\left(\frac{(\sqrt{t})\sin\theta_t}{M\cos\theta_t}\right)^{\mu}R(s,t)\,, \quad (63)$$

where $\mu=\lambda_3-\lambda_4$, and M is a mass parameter inserted to make all the residues have the same dimensions. It is conveniently chosen to be the pseudothreshold mass $M=m_3-m_4$. The threshold conditions then become relations among the residue functions at $t=(m_3\pm m_4)^2$. To demonstrate the cancellation of the singularities it is necessary to parametrize the residue functions so that the threshold relations are exhibited explicitly. For relations at both thresholds, but no derivative relations, an obvious parametrization is

$$\gamma_\lambda(t)=n_\lambda(t)\left[\frac{t-(m_3-m_4)^2}{4m_3m_4}\right]-p_\lambda(t)\left[\frac{t-(m_3+m_4)^2}{4m_3m_4}\right].$$

Then the residues at the normal threshold are $n_\lambda(t)$ and those at the pseudothreshold are $p_\lambda(t)$. The various relations are then demanded of n_λ and p_λ, respectively. Away from the thresholds, n_λ and p_λ are, of course, arbitrary. But in practice, a smooth functional dependence can be assumed. In verifying the cancellation of the kinematic singularities in the cross section it is sufficient to assume n_λ and p_λ are constant. Functional dependence on t can then be envisioned in terms of Taylor-series expansions around the respective thresholds.

If there are relations among derivatives, as well as the amplitudes, obvious generalizations to the parametrization are necessary, with attendant complications in the algebra. For the example of $\pi N\to\pi\Delta$, we will simplify matters by ignoring the normal $\bar{N}\Delta$ threshold at $t=4.72$ (GeV/c)^2. Then the conditions (39) and (40) at the $\bar{N}\Delta$ pseudothreshold can be satisfied by a parametrization of the form

$$\begin{aligned}\bar{\gamma}_1&\equiv\gamma_{\frac{1}{2}\frac{1}{2}}=\sqrt{3}a+b_1\left(\frac{t-t_p}{t_p}\right),\\ \bar{\gamma}_2&\equiv\alpha\gamma_{\frac{1}{2},-\frac{1}{2}}=-\sqrt{3}a+b_2\left(\frac{t-t_p}{t_p}\right),\\ \bar{\gamma}_3&\equiv\alpha\gamma_{\frac{3}{2}\frac{1}{2}}=a+b_3\left(\frac{t-t_p}{t_p}\right),\\ \bar{\gamma}_4&\equiv\alpha(\alpha-1)\gamma_{\frac{3}{2},-\frac{1}{2}}=-a+b_4\left(\frac{t-t_p}{t_p}\right),\end{aligned} \quad (64)$$

where the derivative relation (40) requires

$$\sqrt{3}(b_1+b_2)-(b_3+b_4)=a \quad (65)$$

at $t=t_p=(m_3-m_4)^2=0.09$ (GeV/c)^2. Apart from the condition (65) at $t=t_p$, the well-behaved functions $a(t)$, $b_1(t)$, $b_2(t)$, $b_3(t)$, and $b_4(t)$ are arbitrary, in the absence of dynamical information. But it is reasonable to hope that they are relatively slowly varying in t, at least for physical t values in the range $|t|\lesssim 3t_p$.

One simple, plausible choice for the residues follows from the presence of a factor of $\alpha(t)$ in $\bar{\gamma}_2$, $\bar{\gamma}_3$, and $\bar{\gamma}_4$ in (64): With the assumption of a linear trajectory, the parameters a and b_i are chosen as constants, and the residues $\bar{\gamma}_2$, $\bar{\gamma}_3$, and $\bar{\gamma}_4$ are made proportional to $\alpha(t)$ This fixes b_2, b_3, and b_4 relative to a. Then the slope parameter b_1 is determined by the derivative relation (65). Assuming that the ρ-meson trajectory vanishes at $t\simeq-0.6$ (GeV/c)^2, the nonflip residue function $\bar{\gamma}_1(t)$ goes from $+\sqrt{3}$ (in some units) at $t=0.09$, to zero at $t\simeq-0.1$, and down to -4.5 at $t\simeq-0.6$, while the other three residues, $\bar{\gamma}_2$, $\bar{\gamma}_3$, and $\bar{\gamma}_4$, change from $-\sqrt{3}$, $+1$, and -1, respectively, to zero in the same interval of t.

The density matrix describing the decay of the Δ can be written, apart from a very small region of t

at $\theta_s \simeq 0°$, as

$$
\begin{aligned}
N\rho_{\frac{1}{2}\frac{1}{2}} &= \bar{\gamma}_1{}^2 + (\Delta^2/t_p)\bar{\gamma}_2{}^2 ,\\
N\rho_{\frac{3}{2}\frac{3}{2}} &= (\Delta^2/t_p)\bar{\gamma}_3{}^2 + (\Delta^2/t_p)^2\bar{\gamma}_4{}^2 ,\\
N\rho_{\frac{3}{2},-\frac{1}{2}} &= (\Delta^2/t_p)(\bar{\gamma}_2\bar{\gamma}_3 - \bar{\gamma}_1\bar{\gamma}_4) ,\\
N\rho_{\frac{3}{2}\frac{1}{2}} &= (\Delta^2/t_p)^{1/2}[\bar{\gamma}_1\bar{\gamma}_3 + (\Delta^2/t_p)\bar{\gamma}_2\bar{\gamma}_4] ,
\end{aligned}
\tag{66}
$$

where

$$N = 2[\bar{\gamma}_1{}^2 + (\Delta^2/t_p)(\bar{\gamma}_2{}^2 + \bar{\gamma}_3{}^2) + (\Delta^2/t_p)^2\bar{\gamma}_4{}^2] ,$$

and $\Delta^2 = -t$. In writing (66) it has been assumed that the residues are real and that all the amplitudes have the same phase. The linear residues of the previous paragraph lead to the density matrix elements shown in Fig. 4. Upon comparison with experiment we find that these predictions are almost as far from the facts as they could be. The data on $\pi^+ p \to \pi^0 \Delta^{++}$ at 3.54 GeV/c,[46] 4.0 GeV/c,[47] and 8.0 GeV/c [48] are all more or less in agreement with the Stodolsky-Sakurai model prediction[42] of $\rho_{\frac{3}{2}\frac{3}{2}} = 0.375$, $\rho_{\frac{3}{2},-\frac{1}{2}} = \sqrt{3}/8 = 0.217$, and $\rho_{\frac{3}{2}\frac{1}{2}} = 0$, shown as dashed lines in Fig. 4. The most disagreeable feature of the results shown in Fig. 4 is the negative value of $\rho_{\frac{3}{2},-\frac{1}{2}}$. This can be blamed in large measure on the ratio $\bar{\gamma}_3/\bar{\gamma}_2 = -1/\sqrt{3}$ at $t = t_p$, in contrast to the magnetic-dipole model's ratio of $+\sqrt{3}$, as given in Eq. (43). Other simple choices for the residue functions, e.g., making $\bar{\gamma}_2$ constant, allowing only linear behavior for $\bar{\gamma}_1$ and $\bar{\gamma}_3$, and imposing the magnetic dipole coupling [Eq. (43)] at $t = m_\rho{}^2$, give results qualitatively similar to those shown in Fig. 4, with $\rho_{\frac{3}{2},-\frac{1}{2}} < 0$ and $\rho_{\frac{3}{2}\frac{1}{2}} > 0$ and of the same order of magnitude. The situation can of course be remedied within the framework of (64) with $a \neq 0$, by choosing sufficiently rapidly varying functions $a(t)$ and $b_i(t)$. But the threshold constraints at $t = t_p$ are a severe hindrance, rather than a help, in obtaining a fit to experiment.

The experimental data on $\pi^+ p \to \pi^0 \Delta^{++}$ imply that the dynamics are such that the threshold constraints are not applicable, as discussed in Sec. IV C. If the amplitudes are finite at $t = t_p$, rather than behaving as $(t - t_p)^{-1}$, i.e., $a = 0$ in (64), then the only relation among the residues is (65) with the right-hand side equal to zero. Clearly there is now a tremendous amount of freedom, even with relatively slowly varying residue functions. The choice $\bar{\gamma}_1 = 0 = \bar{\gamma}_4$ and $\bar{\gamma}_3(t) = \sqrt{3}\bar{\gamma}_2(t)$ of the magnetic dipole coupling model, is one of the possibilities that seems consistent with the decay correlation data. The differential cross sections at 3.54 and 4.0 GeV/c give further evidence of something close to the Stodolsky-Sakurai coupling. They show a dip in the forward direction consistent with a small value of the nonflip amplitude $f_{\frac{1}{2}\frac{1}{2}}$, and also are consistent with a dip in the cross section at $t \simeq -0.6$ GeV/c, as expected from the factors of $\alpha(t)$ in $\bar{\gamma}_2$ and $\bar{\gamma}_3$. The 8-GeV/c data seem to show departures from the $M1$ coupling model, but still imply less than the standard singularity behavior at $t = t_p$. The density matrix elements quoted by Krammer and Maor[13] have a t dependence that indicates the presence of $f_{\frac{1}{2}\frac{1}{2}}$. The differential-cross-section shape[13] is consistent with this, having a definitely nonzero value in the forward direction. In spite of these differences from the lower-energy data (differences that may be hard to explain within the Regge-pole model), the 8-GeV/c results are far from agreeing with the curves shown in Fig. 4. Thus all the experimental data support the idea that the t-channel dynamics are such that the pseudothreshold constraints [Eq. (39)] are circumvented[49] by having less than standard singularity behavior. Within the framework of the Regge-pole model, the only alternative is to have what seem to be unreasonably violent t dependences of the residue functions.

Independently of whether or not the dynamics chooses to make empty the threshold relations, the cross-section formulas used to make empirical fits to the data must be free of the pole-type singularities of (57). The work of Krammer and Maor on $\pi N \to \pi\Delta$ [13] and $KN \to K\Delta$,[50] and of Krammer on $\pi N \to \eta\Delta$ [51] must be reconsidered. Because of the experimental density matrix elements for all these reactions, they were led to empirical residues of roughly the $M1$ variety for both the ρ and A_2 trajectories. But the t dependence of their residues and the fits to the differential cross sections are in error because of the use of Wang's formulas.[12]

C. Other Reactions

The general behavior discussed in Secs. V A and V B holds true for other reactions as well. An example of interest is the process $\pi N \to \rho\Delta$, discussed by Frautschi and Jones[14] with a Regge-pole model of pion exchange. The thresholds of most significance are the $\pi\bar{\rho}$ threshold at $t \simeq 0.38$ (GeV/c)^2 and the $\bar{N}\Delta$ pseudothreshold at $t = 0.09$ (GeV/c)^2. For natural parity exchanges ($\eta = +1$) the threshold kinematic singularities are the same as for $\pi N \to \pi'\Delta$. For pion exchange and others with $\eta = -1$, the threshold singularities are $(T_N T_P T_N{}'^2 T_P{}')^{-1}$, where primes refer to the $\bar{N}\Delta$ channel. Frautschi and Jones keep only the nonflip ($\lambda = 0$, $\mu = 0$) amplitude near $t = 0$, but have the kinematic singularity factors in the cross section. They discuss three models: (1) constant residue, (2) elementary pion exchange, and (3) linear residue. The first model gives an unreasonably peaked cross section in the forward direction because of the polelike factors in t. The elementary pion-exchange model, a dynamical exception in the sense of Sec. IV C, gives amplitudes vanishing at the thresholds

[46] M. Abolins, D. D. Carmony, D.-N. Hoa, R. L. Lander, C. Rindfleisch, and N.-H. Xuong, Phys. Rev. **136**, B195 (1964).

[47] Aachen-Berlin-Birmingham-Bonn-Hamburg-London (I.C.)-München Collaboration, Nuovo Cimento **34**, 495 (1964).

[48] Aachen-Berlin-CERN Collaboration, Phys. Letters **22**, 533 (1966); see also Ref. 13.

[49] Evaded is a very appropriate word here, but "evasion" has been pre-exempted for behavior at $t = 0$.

[50] M. Krammer and U. Maor, Nuovo Cimento **52A**, 308 (1967).

[51] M. Krammer, Nuovo Cimento **52A**, 931 (1967).

as the reciprocal of the standard threshold behavior. This t dependence in the numerator of the amplitude rather than the denominator gives an unacceptably large and broad differential cross section. In their third model Frautschi and Jones argue that for a pionic Regge trajectory the proximity of the pion pole at $t=0.02$ $(\text{GeV}/c)^2$ and the $\bar{N}\Delta$ pseudothreshold at $t=0.09$ $(\text{GeV}/c)^2$ means that the residue function should reflect approximately the exceptional behavior of the elementary pion at this threshold, while at the other thresholds α_π is probably different enough from zero to eliminate the dynamical exceptions. Thus they parametrize the residue as $\gamma(t)\propto(t-b)$, where b is expected to be in the neighborhood of the $\bar{N}\Delta$ pseudothreshold. Comparison with data at 4 and 8 GeV/c shows that $b=0.09$ $(\text{GeV}/c)^2$ gives considerable improvement over model (1), but that $b\simeq 0$ is definitely superior. LeBellac[52] has used this empirical vanishing of the nonflip residue function near $t=0$ as a supporting link in a chain of argument concerning conspiracy and the pion trajectory.

Several remarks can be made. The first is that, as far as the cross section is concerned, a choice of constant or slowly varying residue functions [model (1) of Frautschi and Jones] is possible provided all the amplitudes are kept in the cross section and the various threshold relations are satisfied explicitly. The kinematic-singularity factor in the Frautschi-Jones cross section decreases by a factor of 10 from the pion pole to $t=-0.2$ $(\text{GeV}/c)^2$. Once this is removed by cancellation from the numerator, there is no need for residues which vanish near $t=0$.

Model (3) had its origins in the idea of a dynamical exception, with amplitudes having less than the standard kinematic singularities. The t-channel polelike factors would not then appear in the cross section from the beginning and an acceptable t dependence might result. But the empirical result of a residue vanishing at $t\simeq 0$ is at variance with the original motivation, as is admitted by Frautschi and Jones.[14] From the present viewpoint the vanishing of the residue at $t=0$ is forced by the presence of the improper t-channel polelike factors in the cross section.

The final remark is that the interpretation of the t dependence of the cross section for a process like $\pi N\to\rho\Delta$ at small $(-t)$ values demands considerable care because of the finite widths of the ρ meson and the $\Delta(1236)$ resonance. This has been illustrated by Wolf[53] in his discussion of the energy and t dependence of this reaction. If the events are plotted versus $\cos\theta_s$ instead of t, or equivalently, versus $[-t-(-t)_{\min}]$, where $(-t)_{\min}$ is the minimum kinematically allowed value of $-t$ for each event, there is little or no evidence of a turnover in the cross section at small t values.

The above example illustrates some of the dangers of application of Regge-pole formulas with the correct kinematic singularities included, but without strict attention to the threshold relations among the amplitudes. With pionic Regge exchange it may well be necessary to have a relatively complicated parametrization of the residues, satisfying the elementary pion-exchange requirements at the pion pole, as well as the threshold relations among the amplitudes at the pseudothresholds, at least. In questions of conspiracy and the detailed behavior of cross sections at small t values it is essential to handle *all* aspects of the nearby t-channel thresholds correctly. Otherwise, erroneous inferences may be drawn about presumed dynamics.

VI. SUMMARY AND CONCLUSIONS

The analytic structure of helicity amplitudes for two-body processes near kinematic thresholds has been discussed without recourse to the crossing matrix. The tools are those of nonrelativistic quantum mechanics, e.g., channel spin $\mathbf{S}$ and Russell-Saunders coupling of $\mathbf{L}+\mathbf{S}=\mathbf{J}$, as befits a situation where $p\to 0$, with the standard partial-wave threshold behavior [Eq. (20)]. The kinematic singularities of the helicity amplitudes are shown to follow from a mismatch between J and L for each term in the partial-wave series. There can be no question about the applicability of these methods, including use of (20), at the normal thresholds in each channel. The behavior of the amplitudes at pseudothresholds can also be discussed within this framework, provided changes are made in the formal assignments of parities and phase factors, as described in Sec. II E. Implicit here are the assumptions of Lorentz invariance and analyticity, in common with the approaches using the crossing matrix. The general result for the kinematic-singularity structure is contained in Eqs. (26) and (28).

Going along with the singularities of the helicity amplitudes at the normal thresholds and pseudothresholds are relations among the amplitudes and perhaps their derivatives with respect to the channel energy. These relations can be understood as occurring because only the lowest L value for each J survives at threshold; the Russell-Saunders amplitudes corresponding to higher L values vanish as higher powers of the momentum. If the number of Russell-Saunders amplitudes contributing at threshold for each J is less than the number of independent helicity amplitudes, there will be relations among the helicity amplitudes. Similarly, if, to first order in the energy above threshold (i.e., to next order in p^2), there are still more helicity amplitudes than there are different Russell-Saunders amplitudes, there will be relations among the first derivatives, and so on. The explicit construction of the relations among the amplitudes for $\pi N\to\pi'\Delta$ in the t and s channels is presented in Secs. IV A and IV D, respectively. The simpler process, $\pi N\to KY$, is treated in Appendix B.

For comparison, the helicity amplitudes are expressed in terms of invariant amplitudes and it is shown that their usea utomatically incorporates both the kinematic

[52] M. LeBellac, Phys. Letters **25B**, 524 (1967).
[53] G. Wolf, Phys. Rev. Letters **19**, 925 (1967).

singularity structure and the accompanying relations among the amplitudes at the thresholds, only provided we assume that the invariant amplitudes have nothing but dynamic singularities. The use of invariant amplitudes for both s- and t-channel processes is illustrated for $\pi N \to KY$ and $\pi N \to \pi'\Delta$.

An important aspect of the threshold relations is the possibility of dynamical exceptions. For dynamical reasons the amplitudes may be less singular at one or more thresholds than expected from the standard formulas (26) and (28). Absence of one or more values of channel spin is one reason for such behavior. In perturbation theory the limitation of J values to less than the maximum channel spin is another (see Sec. IV C). In such circumstances, the threshold relations among the amplitudes reduce to zero equals zero, while relations among the derivatives may or may not exist in modified form, depending on the degree of departure from the standard singularity behavior and other details. The reaction $\pi N \to \pi\Delta$ is an example of a process which very likely avoids the imposition of threshold relations on its amplitudes at the $\bar{N}\Delta$ pseudothreshold, $t=0.09$ $(\mathrm{GeV}/c)^2$. This is discussed in detail in Secs. IV C and V B.

The use of t-channel helicity amplitudes in the expression for the s-channel cross section is standard in all models of peripheral process at high energies for obvious reasons. The replacement, (55), is justified in the physical s channel by the orthogonality of the crossing matrix.[6,7] Since the t-channel helicity amplitudes in general possess kinematic singularities of the inverse-square-root type at t-channel thresholds which may lie close to the physical s channel, one is led to explicit exhibition of such kinematic factors in the s-channel differential cross section.[12–14] On general grounds it is known that such t-channel kinematic factors cannot occur in the s-channel cross section.[15,16] Explicit satisfaction of the various threshold relations among the amplitudes is sufficient to cancel all the t-channel polelike factors, provided the variables (s,t) lie in the physical region of the s channel. The academic example of $\pi^- p \to \pi^0 n$ with Regge-pole exchange is discussed in Sec. V A to illustrate this point. The more significant reaction, $\pi N \to \pi\Delta$, with its highly singular behavior at the $\bar{N}\Delta$ pseudothreshold, is treated in detail in Sec. V B with special emphasis on the density matrix of the Δ. It is shown that the requirements on the amplitudes at the pseudothreshold, while removing the spurious second-order pole at $t=0.09$ $(\mathrm{GeV}/c)^2$ in the cross section, tend to give decay correlations of the Δ in sharp disagreement with experiment. Apart from the possibility of unpalatably rapid variations in t for the residue functions, experiment thus indicates that the amplitudes for this process are less singular than expected at the thresholds. This, of course, is another way of eliminating the t-channel polelike factors in the s-channel cross section. It leaves only one relation among the four helicity amplitudes at the $\bar{N}\Delta$ pseudothreshold and encompasses a large class of models, including the magnetic-dipole coupling which gives decay correlations more or less in agreement with existing data.

The results obtained concerning the kinematic singularities of two-body helicity amplitudes are not new. But it is believed that the derivation of the singularity structure at threshold by means of nonrelativistic quantum-mechanical principles appropriate to that threshold, and without recourse to the crossing matrix, is simpler and more transparent than the other methods,[8–11] as well as being an aid to the physical understanding of these singularities. Similarly, the existence of relations among the different helicity amplitudes at thresholds has been discussed by others.[10,17,18] But again, the same framework of nonrelativistic quantum theory yields in a straightforward way the threshold relations without resort to elaborate relativistic formalism. The two most important points, for applications at least, are (1) the nonexistence of kinematic-singularity factors in the cross sections, a result that can be assured provided the threshold relations are imposed in the parametrization of any model, and (2) the possibility of avoidance of the requirements at a threshold by means of some dynamical mechanism which lowers the degree of the singularity.

One apparent lesson from this work is that helicity amplitudes are a bad representation, with many peculiarities and subtleties which must be looked after with great care. The use of invariant amplitudes or M functions, with the kinematic structure all exhibited explicitly, offers a more painless approach. But for high spins, invariant amplitudes and their attendant kinematics are not easy to construct.[4] Furthermore, they do not have simple angular momentum and parity properties. Helicity amplitudes will, in all probability, continue to be used because of their elegant angular momentum properties and their general applicability to arbitrary spins.

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APPENDIX A

The Wigner d functions can be written in the following form:

$$d_{\lambda\mu}{}^{J}(\theta)=\frac{(\sin\frac{1}{2}\theta)^{\mu-\lambda}(\cos\frac{1}{2}\theta)^{\mu+\lambda}}{[(J+\mu)!(J-\mu)!(J+\lambda)!(J-\lambda)!]^{1/2}}\sum_{\alpha=0}^{J-\mu}\binom{J-\mu}{\alpha}\times\frac{(J+\mu+\alpha)!(J-\lambda)!}{2^{\alpha}\,(\mu-\lambda+\alpha)!}(z-1)^{\alpha},\quad\text{(A1)}$$

where $z=\cos\theta$ and we have assumed $\lambda,\ \mu\geqslant 0$ and $\lambda\leqslant\mu$. Other possibilities for λ and μ can be obtained from the symmetries of the d functions (see Ref. 5, for example). The leading term (highest power of z) is

$$d_{\lambda\mu}{}^{J}(\theta)\sim\frac{(\sin\theta)^{\mu}(\cot\frac{1}{2}\theta)^{\lambda}(2J)!\,z^{J-\mu}}{2^{J}[(J+\mu)!(J-\mu)!(J+\lambda)!(J-\lambda)!]^{1/2}}. \quad \text{(A2)}$$

In the Russell-Saunders expansion (32) and (37) there occur the angular momentum Clebsch-Gordan coefficients $\langle LS0\mu|J\mu\rangle$, with special values $L=J-1$, $S=1$, 2 and $L=J-2$, $S=2$. The coefficients are particular examples of $\langle ab\alpha\beta|(a+b)\gamma\rangle$ and $\langle ab\alpha\beta|(a+b-1)\gamma\rangle$, given explicitly by Brink and Satchler.[54] The three values needed in Sec. IV A are

$$\langle(J-1)10\mu|J\mu\rangle=\frac{1}{(J-1)![J(2J-1)]^{1/2}}\times\left[\frac{(J+\mu)!(J-\mu)!}{(1+\mu)!(1-\mu)!}\right]^{1/2}, \quad \text{(A3)}$$

$$\langle(J-1)20\mu|J\mu\rangle=\frac{-2\mu}{(J-2)!}\left[\frac{(2J-3)!3!}{(2J+2)(2J)!}\right]^{1/2}\times\left[\frac{(J+\mu)!(J-\mu)!}{(2+\mu)!(2-\mu)!}\right]^{1/2}, \quad \text{(A4)}$$

$$\langle(J-2)20\mu|J\mu\rangle=\frac{1}{(J-2)!}\left[\frac{(2J-4)!4!}{(2J)!}\right]^{1/2}\times\left[\frac{(J+\mu)!(J-\mu)!}{(2+\mu)!(2-\mu)!}\right]^{1/2}. \quad \text{(A5)}$$

The combinations of the leading term (A2) of $d_{\lambda\mu}{}^{J}(\theta_t)$ and the various Clebsch-Gordan coefficients are conveniently written in the form

$$\langle(J-1)10\mu|J\mu\rangle d_{0\mu}{}^{J}(\theta_t)=a_1(J)z^{J-1}d_{0\mu}{}^{1}(\theta_t), \quad \text{(A6)}$$

$$\langle(J-1)20\mu|J\mu\rangle d_{0\mu}{}^{J}(\theta_t)=\mu a_2(J)z^{J-2}d_{0\mu}{}^{2}(\theta_t), \quad \text{(A7)}$$

$$\langle(J-2)20\mu|J\mu\rangle d_{0\mu}{}^{J}(\theta_t)=b_2(J)z^{J-2}d_{0\mu}{}^{2}(\theta_t). \quad \text{(A8)}$$

Note that Eqs. (A6)–(A8) hold only for the highest power of z, namely $z^{J-\mu}$, on boths ides of each equation. The exhibition of $d_0{}^{\mu S}(\theta_t)$, rather than powers of z, serves two purposes. One is to remind the reader that a particular channel spin S is involved and the other is to show that all the nontrivial dependence on the helicity index μ is isolated in this d function that is independent of J. The coefficients in (A6)–(A8) are

$$a_1(J)=\frac{(2J)!}{2^{J}J!(J-1)![J(2J-1)]^{1/2}}, \quad \text{(A9)}$$

$$a_2(J)=\frac{-2[(2J-3)!(2J)!]^{1/2}}{2^{J}J!(J-2)![3(J+1)]^{1/2}}, \quad \text{(A10)}$$

$$b_2(J)=2(\sqrt{\tfrac{2}{3}})\frac{[(2J-4)!(2J)!]^{1/2}}{2^{J}J!(J-2)!}. \quad \text{(A11)}$$

The specific form of $a_1(J)$, $a_2(J)$, and $b_2(J)$ are of no real concern, but it is perhaps worthwhile to note that their asymptotic forms for large J are $2^{J}/(2\pi J)^{1/2}$ times 1, $-1/\sqrt{3}$, and $1/\sqrt{3}$, respectively.

For the general problem with nonvanishing helicities in the initial and final states the coefficients of highest powers of z in $e_{\lambda\mu}{}^{J\pm}(z)$ are needed. From the definitions in Ref. 26 they are found to be, with the same restrictions on λ, μ as in (A2),

$$e_{\lambda\mu}{}^{J+}(z)=\frac{(2J)!\,z^{J-\mu}}{2^{J}[(J+\mu)!(J-\mu)!(J+\lambda)!(J-\lambda)!]^{1/2}}\times\left[1+O\left(\frac{1}{z^2}\right)\right], \quad \text{(A12)}$$

$$e_{\lambda\mu}{}^{J-}(z)=-\frac{\lambda(J-\mu)}{Jz}e_{\lambda\mu}{}^{J+}(z)\left[1+O\left(\frac{1}{z^2}\right)\right]. \quad \text{(A13)}$$

The m dependence of the Clebsch-Gordan coefficients $\langle LS0m|Jm\rangle$ needed in the general case is[54]

$$\langle(J-S)S0m|Jm\rangle=\left[\frac{(J+m)!(J-m)!}{(S+m)!(S-m)!}\right]^{1/2}a(J,S), \quad \text{(A14)}$$

$$\langle(J-S+1)S0m|Jm\rangle=m\left[\frac{(J+m)!(J-m)!}{(S+m)!(S-m)!}\right]^{1/2}b(J,S), \quad \text{(A15)}$$

$$\langle(J-S+2)S0m|Jm\rangle=[S(J+1)-m^2(2J-2S+3)]\times\left[\frac{(J+m)!(J-m)!}{(S+m)!(S-m)!}\right]^{1/2}c(J,S). \quad \text{(A16)}$$

Evidently, the combination of (A14) or (A15) with (A12) allows the J dependence to be factored from the λ or μ dependence, as required in the development from (32) to (35), for example.

APPENDIX B

In this Appendix we consider the relatively simple process of $0^-+\frac{1}{2}^+\to 0^-+\frac{1}{2}^+$. In order to maintain the most general kinematics, the s-channel reaction will be called $\pi N\to KY$. But πN or KN elastic scattering

[54] D. M. Brink and G. R. Satchler, *Angular Momentum* (Oxford University Press, London, 1962), p. 114.

can be obtained by considering the appropriate limits. For convenience in writing formulas, the simplified notation $m_1=\mu'$, $m_2=\mu$, $m_3=m'$, $m_4=m$ is used where the ordering corresponds to the t-channel process $\bar{K}\pi\to Y\bar{N}$.

1. t-Channel Kinematic Singularities

The channel spins are $S=0$ and $S'=0, 1$ for the initial and final states. There are thus no kinematic singularities at the $\bar{K}\pi$ thresholds. Table II shows the allowed orbital angular momenta at the $Y\bar{N}$ normal threshold and pseudothreshold for successive J values. For $J>0$, the minimum L' value is seen to be $L'=J-1$ ($S'=1$ only) at the normal threshold, and $L'=J$ (both $S'=0, 1$) at the pseudothreshold. The kinematic singularity at the normal threshold is [see Eq. (20), (21), or (26)] $(T_N')^{-1}$. There is no kinematic singularity at the pseudothreshold. The helicity amplitudes can therefore be written as

$$f_{\lambda'\lambda;00}=\frac{(\sqrt{\varphi})^{|\lambda'-\lambda|}}{[t-(m+m')^2]^{1/2}}A_{\lambda'\lambda}{}^t(s,t)\,, \qquad \text{(B1)}$$

in analogy to (25), with $A_{\lambda'\lambda}(s,t)$ free of all kinematic singularities. Explicitly, we have

$$\begin{aligned} f_{++;00}&=A_{++}{}^t(s,t)[t-(m+m')^2]^{-1/2}\,,\\ f_{+-;00}&=2(\sqrt{t})pp'\sin\theta_t A_{+-}{}^t(s,t)[t-(m+m')^2]^{-1/2}\,. \end{aligned} \qquad \text{(B2)}$$

Note that the first (second) helicity index is for $Y(\bar{N})$.

2. t-Channel Threshold Constraints

At threshold the two amplitudes $f_{++;00}$ and $f_{+-;00}$ are related. First consider the normal threshold. From Table II we see that only channel spin $S'=1$ is present and that $L'=J\pm1$. The Russell-Saunders decomposition [Eq. (30)] of the partial-wave amplitude thus has two terms. But at threshold only the lowest L' value gives a nonvanishing contribution to $A_{\lambda'\lambda}$. Furthermore, the analog of (31) is

$$F_N{}^J(L'=J-1, S'=1)=(T_N')^{J-1}(T_P'T_NT_P)^J\tilde{F}_N{}^J\,.$$

This means that at threshold the helicity amplitude (B1) has the partial-wave expansion

$$T_N'f_{\lambda'\lambda;00}=\langle\tfrac{1}{2}\tfrac{1}{2}\,\lambda'-\lambda|1\mu\rangle\sum_J(J+\tfrac{1}{2})\tilde{F}_N{}^J(4tpp')^J \times\langle(J-1)10\mu|J\mu\rangle d_{0\mu}{}^J(\theta_t)\,. \qquad \text{(B3)}$$

Use of Eq. (A6) gives an equation similar to (34). Within the region of convergence of the partial-wave sum (a finite segment of the line AB in Fig. 3), the result is

$$T_N'f_{\lambda'\lambda;00}=y_N(s)\langle\tfrac{1}{2}\tfrac{1}{2}\,\lambda'-\lambda|1\,\mu\rangle(4tpp')d_{0\mu}{}^1(\theta_t)\,, \qquad \text{(B4)}$$

where $y_N(s)$ is the sum over J. In terms of the $A_{\lambda'\lambda}{}^t(s,t)$

TABLE II. Orbital angular momentum values L' at the normal thresholds and pseudothresholds for the final state in $\bar{K}\pi\to Y\bar{N}$.

J^P	Normal $S'=0^-$	Normal $S'=1^-$	Pseudo $S'=0^+$	Pseudo $S'=1^+$
0^+	...	1	0	...
1^-	...	0, 2	1	1
2^+	...	1, 3	2	2
3^-	...	2, 4	3	3

in (B2) this threshold condition becomes

$$\left.\frac{A_{++}{}^t(s,t)}{A_{+-}{}^t(s,t)}\right|_{t=(m+m')^2}=2(m+m')pp'z$$
$$=(m+m')(s+mm')-(m\mu'^2+m'\mu^2)\,. \qquad \text{(B5)}$$

At the pseudothreshold we see from Table II that both $S'=0$ and $S'=1$ contribute. This means that instead of (B3) there will be an expansion involving two terms for each J, one for each S' value. Use of counterparts of (A6) and (A7) and summation of the resulting J series leads to an expression like (35), involving two independent functions of s. Since there are only two distinct helicity amplitudes to begin with, there are no relations at the pseudothreshold.

The structure exhibited in (B2), plus the threshold relation (B5), is thus a complete specification of the restrictions imposed on the t-channel helicity amplitudes by kinematics alone.

3. Description in Terms of Invariant Amplitudes

The familiar description in terms of invariant amplitudes A, B automatically displays the kinematic singularities and threshold constraints. For the s-channel process $\pi N\to KY$ the Feynman amplitude is

$$M=\bar{u}_{\lambda'}(p')[-A+i\gamma\cdot\tfrac{1}{2}(q+q')B]u_\lambda(p)\,, \qquad \text{(B6)}$$

where $q_\mu(q_\mu')$ is the 4-momentum of $\pi(K)$ and $u(p)$ $(u(p'))$ is the Dirac spinor for N (Y). The t-channel amplitude is

$$M=\bar{u}_{\lambda'}(p')[-A+i\gamma\cdot\tfrac{1}{2}(q-\bar{q})B]v_\lambda(\bar{p})\,, \qquad \text{(B7)}$$

where now $\bar{q}_\mu(\bar{p}_\mu)$ is the 4-momentum of $\bar{K}(\bar{N})$. The negative-energy spinor is conveniently written as $v_\lambda(p)=(-1)^{\lambda-1/2}\gamma_5u_{-\lambda}(p)$, where λ is the helicity of $\bar{N}$ and

$$\gamma_5=\begin{pmatrix}0 & -I\\ -I & 0\end{pmatrix},$$

with I being the 2×2 unit matrix.

The Jacob-Wick helicity amplitudes are obtained by evaluating (B6) or (B7) in the center-of-momentum frame. For the t channel, straightforward manipulation leads to the expression (B1) with $A_{++}{}^t$ and $A_{+-}{}^t$

given by

$$A_{++}{}^t=[(m+m')^2-t]A$$
$$+\tfrac{1}{2}(m+m')\left[s-u+(\mu'^2-\mu^2)\left(\frac{m'-m}{m'+m}\right)\right]B, \quad \text{(B8)}$$
$$A_{+-}{}^t=B.$$

We see by inspection that $A_{++}{}^t$ and $A_{+-}{}^t$ are free of all but dynamical singularities if A and B are. Hence the kinematic singularity of the helicity amplitudes at the normal threshold is automatically incorporated in the forms (B6) and (B7). Similarly, we note that at the normal threshold the coefficient of A in $A_{++}{}^t$ vanishes. Both helicity amplitudes become proportional to B, with their ratio given by (B5). Thus the threshold relation is also satisfied automatically.

4. Dynamical Exceptions

The singularities and relations at thresholds hold in general merely because of kinematics. But dynamics may give rise to exceptions. As an illustration, suppose that in the t channel only the $J=1^-$ state occurs, or more correctly, that a vector-meson (V) exchange is evaluated in perturbation theory. The invariant amplitudes in this case are

$$A=\frac{g}{m_V{}^2-t}\left[G_V\frac{(m'-m)(\mu'^2-\mu^2)}{m_V{}^2}+G_T\frac{(u-s)}{m'+m}\right],$$
$$B=\frac{2g(G_V+G_T)}{m_V{}^2-t}, \quad \text{(B9)}$$

where G_V and G_T are the Dirac and Pauli coupling constants at the $VY\bar{N}$ vertex, g is the $\bar{K}\pi V$ coupling constant, and m_V is the mass of the vector meson. For arbitrary G_V and G_T the helicity amplitudes have the standard kinematic singularity and relation at threshold. But if $G_T=-G_V$ the B amplitude is zero. Then $f_{+-;00}$ vanishes identically and $f_{++;00}$ vanishes as $T_N{}'$ at threshold. This is an example of an exception to the restrictions of (B1) and (B5). In general, if the amplitudes are less singular than the requirements of kinematics imply, constraint equations such as (B5) do not apply. The threshold constraints hold for the most singular parts of the amplitudes, i.e., for the nonvanishing parts of the $A_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)$ in (26). Another example, where the helicity amplitudes themselves do not vanish at threshold, is afforded by vector-meson exchange in $\pi N\to\pi'\Delta$ (see Sec. IV C).

5. *s*-Channel Kinematic Singularities

The s-channel reaction $\pi N\to KY$ can be treated analogously. Equation (15) takes the form

$$F_{+;+}{}^\eta=\frac{1}{\sqrt{2}}\left(\frac{1}{\cos\frac{1}{2}\theta_s}g_{+;+}-\eta\frac{1}{\sin\frac{1}{2}\theta_s}g_{+;-}\right), \quad \text{(B10)}$$

where the s-channel helicity amplitudes are denoted by $g_{\lambda';\lambda}$. It is easy to show that $F_{+;-}{}^\eta=-\eta F_{+;+}{}^\eta$. Thus we only need $F_{+;+}{}^\eta$ and for simplicity of notation we write $F^\eta=F_{+;+}{}^\eta/\sqrt{2}$ in what follows. The two equations in (B10) give[55]

$$g_{+;+}=\cos\tfrac{1}{2}\theta_s\,(F^-+F^+),$$
$$g_{+;-}=\sin\tfrac{1}{2}\theta_s\,(F^--F^+). \quad \text{(B11)}$$

Either by considering L and L' values, as in Sec. III A and Sec. 1 of this Appendix, or directly from the general results of (26) and (28), we obtain the kinematic-singularity structure

$$F^\eta=(4sp_sp_s{}')^{\frac{1}{2}(1+\eta)}A^\eta/\sqrt{s}, \quad \text{(B12)}$$

provided the mesons are assumed lighter than the baryons. It should be recalled that the $\sqrt{s}$ singularity in the denominator comes from the factor $(\sin\frac{1}{2}\theta_s)^{-1}$ in (B10) [see above Eq. (24)]. Hence the sum (F^-+F^+) will not contain it, while the difference (F^--F^+) will. The end result is

$$g_{+;+}=\cos\tfrac{1}{2}\theta_s\,[A^-(s,t)+4sp_sp_s{}'A^+(s,t)]/\sqrt{s},$$
$$g_{+;-}=\sin\tfrac{1}{2}\theta_s\,[A^-(s,t)-4sp_sp_s{}'A^+(s,t)]/\sqrt{s}, \quad \text{(B13)}$$

where the functions A^- and A^+ are related at $\sqrt{s}=0$ according to

$$A^-=-(m^2-\mu^2)(m'^2-\mu'^2)A^++O(\sqrt{s}) \quad \text{(B14)}$$

in order that $g_{+;+}$ be well behaved at $\sqrt{s}=0$.

6. *s*-Channel Threshold Relations

Inspection of (B13) shows that in the limit $4sp_sp_s{}'\to 0$ the amplitudes depend only on A^-. Consequently, at the four normal and pseudothresholds,

$$g_{+;+}/\cos\tfrac{1}{2}\theta_s=g_{+;-}/\sin\tfrac{1}{2}\theta_s. \quad \text{(B15)}$$

Another way of establishing these threshold relations is by using (B10) and (B12). Jones[17] and Trueman[43] obtained these constraints for πN scattering and utilized them to determine linear combinations of amplitudes having more rapidly converging asymptotic behavior for $\sqrt{s}\to\infty$.

7. Invariant-Amplitude Description in the *s*-Channel

The expressions for F^+ and F^- in terms of the invariant amplitudes A and B of (B6) are obtained by reduction of the Dirac spinors to Pauli form. The results are

$$F^+=[(E-m)(E'-m')]^{1/2}\times[A-((\sqrt{s})+\tfrac{1}{2}(m+m'))B],$$
$$F^-=-[(E+m)(E'+m')]^{1/2}\times[A+((\sqrt{s})-\tfrac{1}{2}(m+m'))B], \quad \text{(B16)}$$

[55] The amplitudes $F^\pm$ are seen to be related to the Pauli amplitudes f_1 and f_2 by $F^-=-8\pi Wf_1$, $F^+=-8\pi Wf_2$, where $M=-8\pi W\langle\chi'|f_1+f_2\sigma\cdot\hat{p}'\sigma\cdot\hat{p}|\chi\rangle$.

where E and E' are the energies of the baryons in the center-of-momentum frame. Using $4s(E\pm m)(E'\pm m') = [((\sqrt{s})\pm m)^2-\mu^2][((\sqrt{s})\pm m')^2-\mu'^2]$, we can verify that $F^\pm$ have the proper threshold behavior, (B12), provided A and B possess no kinematic singularities. The vanishing of F^+ at all four thresholds automatically implies the threshold relation (B15). Similarly, it can be checked that (F^-+F^+) is finite and regular at $\sqrt{s}=0$, while (F^--F^+) goes as $1/\sqrt{s}$. The $A^\pm$ amplitudes of (B13) can be expressed in terms of A and B:

$$A^\pm=\tfrac{1}{2}\{[((\sqrt{s})+m)^2-\mu^2][((\sqrt{s})+m')^2-\mu'^2]\}^{\mp 1/2} \times\left[\pm A-\left((\sqrt{s})\pm\frac{m+m'}{2}\right)B\right], \quad \text{(B17)}$$

verifying the analytic behavior of $A^\pm$ as functions of $\sqrt{s}$ for $\mathrm{Re}(\sqrt{s})>0$.

APPENDIX C

The threshold relations (39) for $\pi\pi'\to\bar{N}\Delta$ at $t=(m_3-m_4)^2$ do not exhaust the relations among the amplitudes. There is a further connection among their derivatives with respect to t at the pseudothreshold. To establish this relation it is necessary to go beyond the expansion (37) and keep the next-order terms in $T_P'^2$. The Russell-Saunders decomposition will now contain contributions with $S'=1$, $L'=J$ and $S'=2$, $L'=J$, in addition to higher terms with $S'=2$ and $L'=J-2$. It is necessary to know the corrections to (A2) to order z^{-2}. For our present purposes we need the result only for $\lambda=0$:

$$d_{0\mu}{}^J(\theta)=\frac{(\sin\theta)^\mu(2J)!\,z^{J-\mu}}{2^J J![(J+\mu)!(J-\mu)!]^{1/2}} \times\left[1+\frac{\mu}{2z^2}-\frac{J(J-1)+\mu^2}{2(2J-1)z^2}+\cdots\right]. \quad \text{(C1)}$$

From (39) it is evident that it is useful to define amplitudes with the boundary function and some other factors removed. Thus we introduce

$$\tilde{f}_{\lambda_3\lambda_4;00}=T_N'T_P'^2(\sqrt{\varphi})^{-\mu}(2(m_3-m_4)pp'z)^{\mu-2} \times f_{\lambda_3\lambda_4;00}. \quad \text{(C2)}$$

The connections (39) now read

$$\begin{aligned}
\tilde{f}_{\frac12\frac12;00}{}^{(0)}&=3\sqrt{2}(m_3+m_4)^2 y_3(s)\,,\\
\tilde{f}_{\frac12,-\frac12;00}{}^{(0)}&=-\tilde{f}_{\frac12\frac12;00}{}^{(0)}\,,\\
\tilde{f}_{\frac32,\frac12;00}{}^{(0)}&=+\frac{1}{\sqrt{3}}\tilde{f}_{\frac12\frac12;00}{}^{(0)}\,,\\
\tilde{f}_{\frac32,-\frac12;00}{}^{(0)}&=-\frac{1}{\sqrt{3}}\tilde{f}_{\frac12\frac12;00}{}^{(0)}\,,
\end{aligned} \quad \text{(C3)}$$

where the superscript zero indicates evaluation at the pseudothreshold.

A treatment similar to that of Sec. IV A in obtaining (36) from (32) and (39) from (37), but using (A16) as well as (A14) and (A15) and (C1) instead of (A2), leads to an expansion around $T_P'=0$ of the form

$$\tilde{f}_{\lambda_3\lambda_4;00}=\tilde{f}_{\lambda_3\lambda_4;00}{}^{(0)}\left\{1+\frac{\mu}{2z^2}+T_P'^2\left[X(s)-\frac{\mu}{2(m_3-m_4)^2}+\mu^2 Y(s)\right]\right\} +T_P'^2(-1)^{1/2-\lambda_4}\mu\frac{\langle\frac32\,\frac12\,\lambda_3-\lambda_4|1\,\mu\rangle}{[(1+\mu)!(1-\mu)!]^{1/2}}Z(s)\,, \quad \text{(C4)}$$

where the zeroth-order terms are given by (38) or (C3). The various contributions to (C4) arise as follows: (i) The $(\mu/2z^2)$ comes from the second term in the expansion (C1) and is of order $T_P'^2$. (ii) $X(s)$ has contributions from the expansion in t of the $S'=2$, $L'=J-2$ partial waves, as well as the leading contribution for $S'=2$, $L'=J$ and derivatives of μ-independent factors connecting $\tilde{f}$ and f. (iii) The $[\mu/2(m_3-m_4)^2]$ term comes from differentiation of a factor $(\sqrt{t})^{-\mu}$ in $(\sqrt{\varphi})^{-\mu}$ of (C2). (iv) $\mu^2 Y(s)$ arises from the third term in (C1) for the $S'=2$, $L'=J-2$ amplitudes and part of the $S'=2$, $L'=J$ amplitudes [see (A16)]. (v) The last term is the leading contribution of the $S'=1$, $L'=J$ partial waves.

Since the first-order terms in (C4) involve three unknown functions of s, there exists one relation among the derivatives of the amplitudes at the pseudothreshold. It is easy to show from (C4) that this relation is

$$\left[\frac{\tilde{f}_{\frac12\frac12;00}+\tilde{f}_{\frac12,-\frac12;00}-(1/\sqrt{3})(\tilde{f}_{\frac32\frac12;00}+\tilde{f}_{\frac32,-\frac12;00})}{t-(m_3-m_4)^2}\right]^{(0)} =\frac{\tilde{f}_{\frac12\frac12;00}{}^{(0)}}{3(m_3-m_4)^2}\left[1-\frac{(m_3-m_4)^2}{(T_P'z)^2}\right]. \quad \text{(C5)}$$

The square bracket on the right-hand side can be expressed in terms of masses and s. For $m_1=m_2=\mu$, this bracket becomes

$$\left[1-\frac{(m_3-m_4)^2}{(T_P'z)^2}\right]=1+\frac{4m_3m_4[(m_3-m_4)^2-4\mu^2]}{(s-u)^2}, \quad \text{(C6)}$$

showing that in the limit of large s it approaches unity.

The three unknown functions, X, Y, Z, in (C4) represent $T_P'^2$ contributions from $(S'=2, L'=J-2)$, $(S'=2, L'=J)$, and $(S'=1, L'=J)$. The one remaining Russell-Saunders combination, $(S'=2, L'=J+2)$, only enters at $T_P'^4$. This is the reason for the existence of the relation (C5). The second and higher derivatives with respect to t will receive contributions from all four Russell-Saunders combinations and so will have no relations among them. At the normal threshold the zeroth order in $T_N'^2$ involves $(S'=1, L'=J-1)$ and $(S'=2, L'=J-1)$, while the first order in $T_N'^2$ has contributions from $(S'=1, L'=J+1)$ and $(S'=2, L'=J+1)$ as well. Thus there are only the relationships contained in (36), and none among the derivatives.

The derivative relation (C5) can be cast into various forms by using amplitudes other than the $\tilde{f}$ of (C2). In particular, the derivation of threshold relations by means of transversity amplitudes[10] leads to derivative relations for the normal helicity amplitudes akin to (C5), but with zero on the right-hand side and a left-hand side differing by certain factors of i^k that correspond to powers of $(\sin\theta_t/\cos\theta_t)^k$ at threshold.

APPENDIX D

In this Appendix we give a brief discussion of Regge-pole amplitudes with emphasis on exhibiting the helicity dependence in a reasonable, factorable way. We consider a single Regge pole of definite signature η_s and parity factor η in the t channel; more elaborate exchanges can be built up by linear superposition. From (17) and (18) it is evident, in the limit of large $z=\cos\theta_t$, that only amplitudes with the same value of η as the Regge pole will survive. Thus the helicity amplitude in the limit of large z is

$$f_{\lambda_3\lambda_4;\lambda_1\lambda_2}(t,\theta_t)=(\sqrt{2}\cos\tfrac{1}{2}\theta_t)^{|\lambda+\mu|}(\sqrt{2}\sin\tfrac{1}{2}\theta_t)^{|\lambda-\mu|}\times\tfrac{1}{2}F_{\lambda_3\lambda_4;\lambda_1\lambda_2}{}^{\eta}(t,z). \quad \text{(D1)}$$

It is sufficient to choose $\lambda=\lambda_1-\lambda_2$, $\mu=\lambda_3-\lambda_4$ both non-negative. Amplitudes for other values of λ, μ can be found from the Jacob-Wick parity relation

$$f_{-\lambda_3-\lambda_4;-\lambda_1-\lambda_2}=(\eta_3\eta_4/\eta_1\eta_2)(-1)^{s_3+s_4-s_1-s_2+\lambda-\mu}\times f_{\lambda_3\lambda_4;\lambda_1\lambda_2} \quad \text{(D2)}$$

and

$$f_{\lambda_3\lambda_4;-\lambda_1-\lambda_2}=\eta(-1)^{m-\mu}\eta_1\eta_2(-1)^{s_1+s_2}\left(\frac{1-\cos\theta_t}{1+\cos\theta_t}\right)^n f_{\lambda_3\lambda_4;\lambda_1\lambda_2}, \quad \text{(D3)}$$

where m and n are the larger and smaller of (λ,μ), respectively. Equation (D3) follows from (15), to the leading power of z. If lower powers of z are retained, (D1) and (D3) are more complicated, but it is still sufficient to take λ, μ non-negative.

1. Singularity Structure and Residue Behavior

The kinematic-singularity structure can be exhibited explicitly by means of (26). The amplitude (D1) can be conveniently written in the form

$$f_{\lambda_3\lambda_4;\lambda_1\lambda_2}=\left(\frac{(\sqrt{t})\sin\theta_t}{\cos\theta_t}\right)^{\lambda+\mu}\left(\frac{-\cos\theta_t}{1-\cos\theta_t}\right)^n(4pp'\cos\theta_t)^m\times\tfrac{1}{2}K(t)A_{\lambda_3\lambda_4;\lambda_1\lambda_2}{}^{\eta}(s,t), \quad \text{(D4)}$$

where $K(t)$ is the kinematic-singularity factor [the square bracket in (26)] for $\lambda=\mu=0$. A factor of $(-1)^n$ has been inserted into (D4) for convenience since we assume the masses to be such that $\cos\theta_t=-1$ on the boundary of the s-channel physical region. The behavior in $(\sqrt{t})\sin\theta_t$ is appropriate for the dependence (23) on $\sqrt{\varphi}$ at this boundary.[56] The analytic amplitude A^{η} is assumed to be dominated by a single Regge pole. From (17) and the analytic continuation in J of (A12) we see that for large z, A^{η} will be given by

$$A_{\lambda_3\lambda_4;\lambda_1\lambda_2}{}^{\eta}=(\alpha+\tfrac{1}{2})\beta_{\lambda_3\lambda_4;\lambda_1\lambda_2}(t)\times\frac{(2\alpha)!\,z^{\alpha-m}}{2^{\alpha}[(\alpha-\mu)!(\alpha+\mu)!(\alpha-\lambda)!(\alpha+\lambda)!]^{1/2}}\times\left(\frac{\eta_s+e^{-i\pi\alpha}}{\sin\pi\alpha}\right), \quad \text{(D5)}$$

where $\alpha(t)$ is the trajectory of the pole and $\beta(t)$ is the residue function. $\beta(t)$ must have appropriate singularities in t so that A^{η} is well behaved.

The specific dependence of $\beta(t)$ on α, λ, and μ depends on dynamical assumptions, such as whether the trajectory chooses "sense" or "nonsense" at integer values of α less than λ or μ. These are discussed in footnotes 9 and 10 of Ref. 12. For a trajectory that chooses "sense" the residue function has a factor $[(\alpha-J)(\alpha+J+1)]^{1/2}$ for each "sense-nonsense" value of J, i.e., $n\leqslant J<m$, and a factor $(\alpha-J)(\alpha+J+1)$ for "nonsense-nonsense" values of J, i.e., $J<n$. Thus the the residue is proportional to

$$\alpha(\alpha+1)(\alpha-1)(\alpha+2)\cdots(\alpha-n+1)(\alpha+n)\times[(\alpha-n)(\alpha+n+1)\cdots(\alpha-m+1)(\alpha+m)]^{1/2}.$$

This can be written as

$$[(\alpha+\lambda)!(\alpha+\mu)!/(\alpha-\lambda)!(\alpha-\mu)!]^{1/2}.$$

[56] The powers of $\sqrt{t}$ used in obtaining (D4) are not quite those of (26). They are the powers necessary to compensate for the behavior of $\xi(\theta)$ at $t=0$. See the discussion below Eq. (23). Another point worthy of note: (D4) is appropriate for all masses unequal. For processes like $\bar{N}N\to\pi\rho$, with two masses equal, the kinematic-singularity structure is different (see Ref. 10). With $K(t)$ defined by (26) in the limit $m_1=m_2$, the right-hand side of Eq. (D4) must be multiplied by $(\sqrt{t})^N$, where $N=-n+\frac{1}{2}\eta\eta_1\eta_2[1-(-1)^n]$.

The complete residue function is therefore[57]

$$\beta_{\lambda_3\lambda_4;\lambda_1\lambda_2}(t)=\left(\frac{1}{M}\right)^{\lambda+\mu}\left(\frac{1}{4s_0}\right)^{m}\left[\frac{(\alpha+\mu)!(\alpha+\lambda)!}{(\alpha-\mu)!(\alpha-\lambda)!}\right]^{1/2}\left(\frac{pp'}{s_0}\right)^{\alpha-m}\times\frac{\gamma_{\lambda_3\lambda_4;\lambda_1\lambda_2}(t)}{(\alpha+\frac{1}{2})!}. \quad \text{(D6)}$$

Here M is a mass parameter inserted so that all the reduced residues $\gamma(t)$ will have the same dimensions, and s_0 is the usual scale parameter. When (D6) is combined with (D5) and inserted into (D4) the result is

$$f_{\lambda_3\lambda_4;\lambda_1\lambda_2}=X_{mn}(s,t)\left(\frac{\sqrt{-t}}{M}\right)^{\lambda+\mu}\frac{\alpha!\alpha!}{(\alpha-\mu)!(\alpha-\lambda)!}\times K(t)\gamma_{\lambda_3\lambda_4;\lambda_1\lambda_2}(t)R(s,t), \quad \text{(D7)}$$

where the factor $X_{mn}(s,t)$ is discussed in the next section and

$$R(s,t)=\frac{1}{\sqrt{\pi}}\frac{1}{\alpha!}\left(\frac{s-u}{2s_0}\right)^{\alpha}\left(\frac{\eta_s+e^{-i\pi\alpha}}{2\sin\pi\alpha}\right). \quad \text{(D8)}$$

$R(s,t)$ is the standard Regge amplitude for spinless particles. In writing (D8) we have appealed to the work of Freedman and Wang[58] and others in order to make the replacement $(4pp'z)\to(s-u)$, even for unequal masses. Notice that in (D7) we have written $\sqrt{-t}$ in order to have real quantities in the physical region of the s channel (assumed to have $t<0$). The reduced residue $\gamma(t)$ is real and analytic in t; it may contain a "ghost-killing" factor of α for even-signature trajectories.

2. The Very-Small-t Region

The factor X_{mn} is equal to unity over most of the physical range. It has t dependence only in a very small interval near the forward direction in the s channel. Explicitly, we have

$$(\sqrt{-t})^{m+n}X_{mn}=\left(\frac{(\sqrt{t})\sin\theta_t}{\cos\theta_t}\right)^{m+n}\left(\frac{-\cos\theta_t}{1-\cos\theta_t}\right)^{n}\times\left(\frac{4pp'\cos\theta_t}{s-u}\right)^{m}$$
$$=\left(\frac{\sqrt{\varphi}}{2pp'\cos\theta_t}\right)^{m+n}\left(\frac{-\cos\theta_t}{1-\cos\theta_t}\right)^{n}\times\left(\frac{4pp'\cos\theta_t}{s-u}\right)^{m}. \quad \text{(D9)}$$

With $\sqrt{\varphi}\geqslant 0$ and $-\cos\theta_t\geqslant 1$ in the physical s channel, it can be seen that X_{mn} is positive and real there. Furthermore, for large s, $-\cos\theta_t$ increases rapidly away from the exact forward direction. This means that X_{mn} also rapidly approaches unity. The region of t over which variation occurs is measured in units of the minimum value of momentum transfer. Since the minimum transfer falls off as s^{-1} or s^{-2} for large s, this region of t where the mass differences are important is very small. Accurate approximations can be given for X_{mn}. We distinguish two cases: (a) equal masses in either the initial or final state of the t channel, and (b) unequal masses in both initial and final t-channel states. We define a dimensionless variable x such that

$$x^2=-t/(-t)_{\min}. \quad \text{(D10)}$$

$x=1$ for $\theta_s=0°$, and $x>1$ away from the forward direction.

(a) *Equal masses in one t-channel state* (*e.g.*, $\pi\pi\to\bar{N}\Delta$, $\pi\bar{p}\to\bar{N}N$).

For this case, $(-t)_{\min}\propto 1/s^2$ asymptotically,[59] and $-\cos\theta_t\simeq x$ is valid for x values from zero to where X_{mn} is close to unity. The approximation to X_{mn} is thus

$$X_{mn}(x)=\frac{(x-1)^{(m+n)/2}(x+1)^{(m-n)/2}}{x^m}. \quad \text{(D11)}$$

For the reaction $\pi N\to\pi\Delta$, for which

$$(-t)_{\min}\simeq 0.1/P_{\text{lab}}^2(\text{GeV}/c)$$

in units of $(\text{GeV}/c)^2$, the transition region in which X_{mn} rises from zero (if $m\neq 0$) to unity is confined to such small values of t that present experiments cannot possibly explore it.

(b) *Masses unequal in both t-channel states* (*e.g.*, $\pi\bar{p}\to\bar{N}\Delta$, $N\bar{\Delta}\to N\bar{\Delta}$).

Here $(-t)_{\min}\propto 1/s$ asymptotically, and $-\cos\theta_t\simeq 2x^2-1$. Then we obtain

$$X_{mn}(x)=(1-1/x^2)^{(m+n)/2}. \quad \text{(D12)}$$

It is of interest to note that, independently of whether or not some of the masses are equal, at some fixed small value of $-t$, for example, $t=-m_\pi^2$, $X_{mn}=1-O(1/s)$. This can be seen by expanding (D11) and (D12) in powers of x^{-1} and x^{-2}, respectively, and noting the dependence of $(-t)_{\min}$ on s in each case.

[57] We are here ignoring the presence of fixed poles in the partial-wave amplitudes at nonsense, wrong-signature points, discussed by S. Mandelstam and L. L. Wang, Phys. Rev. **160**, 1490 (1967) and A. H. Mueller and T. L. Trueman, *ibid.* **160**, 1296 (1967). If the effects of the third double-spectral function are small, the simple behavior presented here is expected to be approximately valid.

[58] D. Z. Freedman and J.-M. Wang, Phys. Rev. Letters **17**, 569 (1966); Phys. Rev. **153**, 1596 (1967).

[59] A rough approximation in the few-GeV/c region is $(-t)_{\min}=(m_c^2-m_a^2)^2/4P_{\text{lab}}^2$ if the s-channel process is $ab\to cd$, and $m_b=m_d$.

For all inelastic processes at very high energies, then, the transition region in t becomes unimportant as s^{-1} and all the relevant t dependence is contained in (D7) with $X_{mn}=1$. The explicit helicity dependence in t is given by the factor $(\sqrt{-t})^{\lambda+\mu}$ or its equivalent.[56]

3. Trajectory Chooses "Nonsense"

The choice of factorials in the square root in (D6) is such as to cause the residue to behave "sensibly," that is, to vanish when α becomes equal to an unphysical integer J value, $J<m$. Another possibility is to have the Regge trajectory choose "nonsense," that is, to have a residue which behaves the same for "sense-nonsense" helicity values, but with the roles of "sense-sense" and "nonsense-nonsense" helicities interchanged. This means that the residue is proportional to

$$\begin{aligned}(\alpha-m)(\alpha+m+1)\cdots(\alpha-S+1)(\alpha-S)\\ \times[(\alpha-n)(\alpha+n+1)(\alpha-n-1)(\alpha+n+2)\cdots\\ \times(\alpha-m+1)(\alpha+m)]^{1/2},\end{aligned}$$

where for integer $J<S$ the trajectory chooses "nonsense" ($S>m$). This can be written as

$$\beta\propto\frac{(\alpha+S)!}{(\alpha-S)!}\left(\frac{(\alpha-\lambda)!(\alpha-\mu)!}{(\alpha+\lambda)!(\alpha+\mu)!}\right)^{1/2}. \qquad \text{(D13)}$$

We note that the square root in (D13) is just the reciprocal of that occurring in (D6), an acceptable alternative for combination with (D5) to give an analytic amplitude. The choosing of "nonsense" (sometimes called the Gell-Mann mechanism) has as its consequence the replacement in (D7)

$$\frac{\alpha!\alpha!}{(\alpha-\mu)!(\alpha-\lambda)!}\rightarrow\frac{(\alpha+S)!}{(\alpha-S)!}\frac{\alpha!}{(\alpha+\mu)!}\frac{\alpha!}{(\alpha+\lambda)!}. \qquad \text{(D14)}$$

Depending on the values of S, λ, and μ and signature, it may be necessary to multiply (D14) by additional factors in order to prevent "ghost" poles at negative integral values of α. In practice, only the point $\alpha=0$ is important. Slowly varying factors from (D14) can then be incorporated into the reduced residue $\gamma(t)$ in (D7). As a final comment we note that when lower-order powers of z are kept in (D5), compensating trajectories are needed to prevent singularities in the amplitude at "nonsense" values of J ($0\leqslant J<n$) (see Appendix B of Ref. 26).

Note added in proof. Between submission of this paper and receipt of galley proofs, unpublished reports concerned with various aspects of kinematic singularities and threshold constraints and using a variety of techniques have been received from J. P. Ader, M. Capdeville and H. Navelet, J. Franklin, D. Z. Freedman, E. Gotsman and U. Maor, F. S. Henyey, A. Kotanski, A. McKerrell, and H. P. Stapp. The work of Franklin closely parallels our own in the use of L-S coupling techniques. Mention should also be made of a somewhat related work by M. Barmawi [Phys. Rev. **166**, 1846 (1968)] who considers L-S coupling for the partial-wave series and then makes a Watson-Sommerfeld transformation to obtain Regge poles with L-S coupling.