Magnetic Charge and Quantum Field Theory*

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A quantum field theory of magnetic and electric charge is constructed. It is verified to be relativistically invariant in consequence of the charge quantization condition $eg/\hbar c = n$, an integer. This is more restrictive than Dirac's condition, which would also allow half-integral values.

IT is somewhat disturbing that the symmetry between electric and magnetic charge which is inherent in Maxwell's equations does not seem to be realized in nature. This is accentuated by Dirac's remarkable suggestion[1] that the existence of magnetic charge would imply the quantization of electric charge. Several authors[2] have asserted recently that conjectured properties of relativistic S matrices are violated for a particle that carries magnetic charge. The compatibility of the magnetic-charge concept with the principles of relativistic quantum field theory has not been examined, however.[3] This note is devoted to a general discussion of that problem.

The content of the covariant Maxwell field equations

$$\partial_\nu F^{\mu\nu} = j^\mu, \quad \partial_\nu {}^*F^{\mu\nu} = {}^*j^\mu$$
$${}^*F^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\lambda\kappa}F_{\lambda\kappa}$$

is conveyed by the local conservation laws of electric and magnetic charge,

$$\partial_\mu j^\mu = 0, \quad \partial_\mu {}^*j^\mu = 0,$$

by the three-dimensional transverse components of the equations of motion

$$\partial_0 \mathbf{E} = \nabla\times\mathbf{H} - \mathbf{j}, \quad \partial_0\mathbf{H} = -\nabla\times\mathbf{E} - {}^*\mathbf{j},$$

and by the equations of constraint

$$\nabla\cdot\mathbf{E} = j^0, \quad \nabla\cdot\mathbf{H} = {}^*j^0.$$

Transverse vector potentials are introduced by the definitions

$$\mathbf{H}^T = \nabla\times\mathbf{A}^T, \quad \mathbf{E}^T = -\nabla\times\mathbf{B}^T.$$

The equal-time commutation relation

$$i[A_k{}^T(x), B_l{}^T(x')] = \epsilon_{klm}\partial_m \mathfrak{D}(\mathbf{x}-\mathbf{x}'),$$

where ϵ_{klm} is the completely antisymmetrical tensor, and

$$\mathfrak{D}(\mathbf{x}) = 1/(4\pi|\mathbf{x}|),$$

implies both the known canonical commutator

$$i[A_k{}^T(x), E_l{}^T(x')] = (\delta_{kl}\delta(\mathbf{x}-\mathbf{x}'))^T$$
$$= \delta_{kl}\delta(\mathbf{x}-\mathbf{x}') - \partial_k\partial_l{}'\mathfrak{D}(\mathbf{x}-\mathbf{x}')$$

and its magnetic analog

$$i[B_k{}^T(x), H_l{}^T(x')] = (\delta_{kl}\delta(\mathbf{x}-\mathbf{x}'))^T.$$

A quantum field theory is relativistically invariant if the energy and momentum densities obey the equal-time commutation relation[4]

$$-i[T^{00}(x), T^{00}(x')] = -(T^{0k}(x) + T^{0k}(x'))\partial_k\delta(\mathbf{x}-\mathbf{x}').$$

Of course, the three-dimensional invariance requirements described by the linear- and angular-momentum operators

$$P^k = \int (d\mathbf{x}) T^{0k}, \quad J^{kl} = \int (d\mathbf{x})(x^k T^{0l} - x^l T^{0k})$$

must also be satisfied.

We shall consider a specific model in which electric charge e is carried by the spin-$\frac{1}{2}$ field ψ, and magnetic charge g by the spin-$\frac{1}{2}$ field χ. The energy density for this system is given by

$$T^{00} = \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{H}^2) + \bar\psi\gamma\cdot(-i\nabla - e\mathbf{A}^T - e\mathbf{A}_g)\psi + m_e\bar\psi\psi$$
$$+ \bar\chi\gamma\cdot(-i\nabla - g\mathbf{B}^T - g\mathbf{B}_e)\chi + m_g\bar\chi\chi,$$

where Fermi operator products are antisymmetrized, as is the application of the coordinate derivatives. We have defined

$$\mathbf{E} = \mathbf{E}^T - \nabla\phi,$$
$$\mathbf{H} = \mathbf{H}^T - \nabla\,{}^*\phi,$$

where

$$\phi(x) = \int (d\mathbf{x}')\mathfrak{D}(\mathbf{x}-\mathbf{x}')j^0(x'),$$

$${}^*\phi(x) = \int (d\mathbf{x}')\mathfrak{D}(\mathbf{x}-\mathbf{x}')\,{}^*j^0(x'),$$

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[1] P. A. M. Dirac, Proc. Roy. Soc. (London) **A133**, 60 (1931); Phys. Rev. **74**, 817 (1948).

[2] D. Zwanziger, Phys. Rev. **137**, B647 (1965); S. Weinberg, *ibid.* **138**, B988 (1965); A. S. Goldhaber, *ibid.* **140**, B1407 (1965).

[3] An apparent exception is the contribution of N. Cabibbo and E. Ferrari, Nuovo Cimento **23**, 1147 (1962). However, these authors do not consider the restrictions imposed by the Jacobi identity on the commutators of non-integrable derivatives. I must acknowledge some provocative discussions with S. Coleman, which occurred in the same year as this reference. He had independently devised a similar theory.

[4] J. Schwinger, Phys. Rev. **127**, 324 (1962). The form stated in the text characterizes the restricted physical class of local systems. See J. Schwinger, Phys. Rev. **130**, 406 (1963); **130**, 800 (1963); and Nuovo Cimento **30**, 278 (1963). There is a related independent discussion by P. A. M. Dirac, Rev. Mod. Phys. **34**, 592 (1962).

and

$$\mathbf{A}_g(x)=\int (d\mathbf{x}')\mathbf{a}(\mathbf{x}-\mathbf{x}')\,{}^*j^0(x')\,,$$

$$\mathbf{B}_e(x)=\int (d\mathbf{x}')\mathbf{b}(\mathbf{x}-\mathbf{x}')j^0(x')\,.$$

The structure of the vector functions $\mathbf{a}(\mathbf{x})$, $\mathbf{b}(\mathbf{x})$ will be specified later.

The two current vectors are identified as

$$j^\mu=e\bar{\psi}\gamma^\mu\psi\,,\quad {}^*j^\mu=g\bar{\chi}\gamma^\mu\chi.$$

The derived commutation relations

$$-i[j^0(x),T^{00}(x')]=-j^k(x')\partial_k\delta(\mathbf{x}-\mathbf{x}')\,,$$
$$-i[{}^*j^0(x),T^{00}(x')]=-{}^*j^k(x')\partial_k\delta(\mathbf{x}-\mathbf{x}')$$

assure the two local charge conservation laws. These commutators are also contained in

$$-i[\mathbf{E}(x),T^{00}(x')]=-\mathbf{H}(x')\times\nabla\delta(\mathbf{x}-\mathbf{x}')-\mathbf{j}(x')\delta(\mathbf{x}-\mathbf{x}')\,,$$

$$-i[\mathbf{H}(x),T^{00}(x')]=\mathbf{E}(x')\times\nabla\delta(\mathbf{x}-\mathbf{x}')-{}^*\mathbf{j}(x')\delta(\mathbf{x}-\mathbf{x}')\,,$$

and the latter imply the equations of motion for $\mathbf{E}$ and $\mathbf{H}$.

The evaluation of the energy-density commutator is simplified by the delta-function nature of the commutators that connect field strengths with the energy-density contributions of the charged fields. Apart from the commutator between the two charged field terms, one obtains the anticipated form, with the momentum-density identification

$$T_k{}^0=\mathbf{E}\times\mathbf{H}+\bar{\psi}\gamma^0(-i\nabla-e\mathbf{A}^T-e\mathbf{A}_g)\psi+\tfrac{1}{4}\nabla\times(\bar{\chi}\gamma^0\sigma\psi)+\bar{\chi}\gamma^0(-i\nabla-g\mathbf{B}^T-g\mathbf{B}_e)\chi+\tfrac{1}{4}\nabla\times(\bar{\chi}\gamma^0\sigma\chi)|_k.$$

The electromagnetic-field contribution to the momentum density can be rearranged as

$$(\mathbf{E}\times\mathbf{H})_k=\mathbf{E}^T\cdot\partial_k\mathbf{A}^T-\nabla\cdot(\mathbf{E}^TA_k{}^T)+j^0A_k{}^T+{}^*j^0B_k{}^T+\nabla\cdot(A_k{}^T\nabla\phi-\phi\partial_k\mathbf{A}^T)+\nabla\cdot(B_k{}^T\nabla\,{}^*\phi-{}^*\phi\partial_k\mathbf{B}^T)+\nabla\times(\tfrac{1}{2}\phi\nabla\,{}^*\phi-\tfrac{1}{2}{}^*\phi\nabla\phi)|_k.$$

The expression for the total linear momentum obtained in this way is

$$P_k=\int (d\mathbf{x})[\mathbf{E}^T\cdot\partial_k\mathbf{A}^T-\bar{\psi}\gamma^0i\partial_k\psi-\bar{\chi}\gamma^0i\partial_k\chi]-\int (d\mathbf{x})(d\mathbf{x}')j^0(x)\,{}^*j^0(x')[a_k(\mathbf{x}-\mathbf{x}')+b_k(\mathbf{x}'-\mathbf{x})]\,,$$

which is the required translation generator only if

$$\mathbf{b}(\mathbf{x})=-\mathbf{a}(-\mathbf{x})\,.$$

We shall defer the important discussion of angular momentum.

The commutator between the energy-density contributions of the two kinds of charges has several nonlocal contributions. The condition for complete cancellation is

$$\epsilon_{klm}\partial_m\mathfrak{D}(\mathbf{x}-\mathbf{x}')-\partial_la_k(\mathbf{x}-\mathbf{x}')-\partial_kb_l(\mathbf{x}'-\mathbf{x})=0$$

or

$$-\nabla\mathfrak{D}(\mathbf{x})=\nabla\times\mathbf{a}(\mathbf{x})\,.$$

This equation cannot be solved without exception, in view of the contradiction that appears between the left- and the right-hand sides on integrating over a closed surface containing the origin. There are solutions that are valid almost everywhere, however. Thus

$$\mathbf{a}_\mathbf{n}(\mathbf{x})=\mathfrak{D}(\mathbf{x})\frac{1}{2}\left(\frac{\mathbf{n}\times\mathbf{x}}{|\mathbf{x}|+\mathbf{n}\cdot\mathbf{x}}-\frac{\mathbf{n}\times\mathbf{x}}{|\mathbf{x}|-\mathbf{n}\cdot\mathbf{x}}\right)=\mathbf{a}_\mathbf{n}(-\mathbf{x})=\mathbf{a}_{-\mathbf{n}}(\mathbf{x})$$

is a solution at all points that do not lie on the infinite line drawn through the origin in the direction of the unit vector $\mathbf{n}$. The nature of the singularity is indicated by the nonvanishing limit of the integral extended over an arbitrarily small surface pierced by the line $\mathbf{x}=\mathbf{n}|\mathbf{x}|$ or the equivalent one-dimensional integral drawn about the line,

$$\lim\int d\mathbf{S}\cdot\nabla\times\mathbf{a}_\mathbf{n}=\lim\oint d\mathbf{x}\cdot\mathbf{a}_\mathbf{n}=-\tfrac{1}{2}.$$

The complete equation obeyed by $\mathbf{a}_\mathbf{n}(\mathbf{x})$ is, therefore,

$$\nabla\times\mathbf{a}_\mathbf{n}(\mathbf{x})=-\nabla\mathfrak{D}(\mathbf{x})+\mathbf{h}_\mathbf{n}(\mathbf{x})$$

where

$$\mathbf{h}_\mathbf{n}(\mathbf{x})=-\tfrac{1}{2}\mathbf{n}(\mathbf{n}\cdot\mathbf{x}/|\mathbf{x}|)\delta_\mathbf{n}(\mathbf{x})$$

and $\delta_\mathbf{n}(\mathbf{x})$ is the two-dimensional delta function in the plane orthogonal to $\mathbf{n}$. The vector field $\mathbf{h}_\mathbf{n}(\mathbf{x})$ obeys

$$\nabla\cdot\mathbf{h}_\mathbf{n}(\mathbf{x})=-\delta(\mathbf{x})\,.$$

Note also that

$$\nabla\cdot\mathbf{a}_\mathbf{n}(\mathbf{x})=0$$

since

$$4\pi\mathbf{a}_\mathbf{n}(\mathbf{x})=\nabla\times\left\{\mathbf{n}\tfrac{1}{2}\ln\frac{|\mathbf{x}|-\mathbf{n}\cdot\mathbf{x}}{|\mathbf{x}|+\mathbf{n}\cdot\mathbf{x}}\right\}\,,$$

and that

$$\mathbf{b}_\mathbf{n}(\mathbf{x})=-\mathbf{a}_\mathbf{n}(\mathbf{x})$$

is an axial vector, in conformity with the space-reflection characteristics of $\mathbf{B}^T$.

The change from one singularity line to another is a gauge transformation, almost everywhere, on the vector potentials

$$\mathbf{A}_g(x)=\int(d\mathbf{x}')\mathbf{a}_\mathbf{n}(\mathbf{x}-\mathbf{x}')\,{}^*j^0(x')\,,$$

$$\mathbf{B}_e(x)=-\int(d\mathbf{x}')\mathbf{a}_\mathbf{n}(\mathbf{x}-\mathbf{x}')j^0(x')\,.$$

The corresponding charged field transformations are

$$\psi'(x)=\exp\left[ie\int_\infty^x d\mathbf{x}'\cdot(\mathbf{A}_g'-\mathbf{A}_g)(x')\right]\psi(x)\,,$$

$$\chi'(x)=\exp\left[ig\int_\infty^x d\mathbf{x}'\cdot(\mathbf{B}_e'-\mathbf{B}_e)(x')\right]\chi(x)\,,$$

where

$$\int_\infty^x d\mathbf{x}'\cdot(\mathbf{A}_g'-\mathbf{A}_g)=\int(d\mathbf{x}')f_{\mathbf{n}'\mathbf{n}}(\mathbf{x}-\mathbf{x}')\,{}^*j^0(x')\,,$$

$$\int_\infty^x d\mathbf{x}'\cdot(\mathbf{B}_e'-\mathbf{B}_e)=-\int(d\mathbf{x}')f_{\mathbf{n}'\mathbf{n}}(\mathbf{x}-\mathbf{x}')j^0(x')\,,$$

and

$$f_{\mathbf{n}'\mathbf{n}}(\mathbf{x})=\int_\infty^0 d\boldsymbol{\xi}\cdot(\mathbf{a}_{\mathbf{n}'}-\mathbf{a}_\mathbf{n})(\mathbf{x}+\boldsymbol{\xi})=-f_{\mathbf{n}'\mathbf{n}}(-\mathbf{x})\,.$$

This transformation preserves the formal structure of T^{00} and T^{0k}. It also maintains the commutation properties of ψ and χ since

$$\psi'(x)=U^{-1}\psi(x)U\,,\quad \chi'(x)=U^{-1}\chi(x)U\,,$$

where

$$U=\exp\left[i\int(d\mathbf{x})(d\mathbf{x}')j^0(x)f_{\mathbf{n}'\mathbf{n}}(\mathbf{x}-\mathbf{x}')\,{}^*j^0(x')\right]$$

is a unitary operator. There is nothing in this discussion, incidentally, that depends upon the particular spins of the charged fields.

The phase factor that multiplies $\psi(x)$, say, in the above gauge transformation is still a unitary operator. The transformation that it generates is

$$\exp\left[-ie\int(d\mathbf{x}'')f_{\mathbf{n}'\mathbf{n}}(\mathbf{x}-\mathbf{x}'')\,{}^*j^0(x'')\right]\chi(x')$$
$$\times\exp\left[ie\int(d\mathbf{x}'')f_{\mathbf{n}'\mathbf{n}}(\mathbf{x}-\mathbf{x}'')\,{}^*j^0(x'')\right]$$
$$=\exp[iegf_{\mathbf{n}'\mathbf{n}}(\mathbf{x}-\mathbf{x}')]\chi(x')\,.$$

Relative to the point $\mathbf{x}'$ as origin, $f_{\mathbf{n}'\mathbf{n}}(\mathbf{x}-\mathbf{x}')$ is the difference of the line integrals of $\mathbf{a}_{\mathbf{n}'}$ and $\mathbf{a}_\mathbf{n}$ extended up to the point $\mathbf{x}$. This difference is unaltered by all deformations of the integration path that maintain the topology associated with the two singularity lines defined by $\mathbf{n}$ and $\mathbf{n}'$. But should the continuously deformed path cut through one of the lines, $f_{\mathbf{n}'\mathbf{n}}(\mathbf{x}-\mathbf{x}')$ would change discontinuously by $\pm\frac{1}{2}$. Only if

$$eg/4\pi=n\,,$$

where n is an integer, can the gauge transformation be unique, almost everywhere. The integer-valued quantity is $eg/\hbar c$, in conventional Gaussian units. This charge-quantization condition differs from that of Dirac; the smallest nonzero charge product is unity rather than one-half. The discrepancy has arisen from our use of an infinite discontinuity line, in accordance with space-reflection considerations, rather than the semi-infinite line employed by Dirac. Another deviation between the two approaches will emerge shortly. We shall then identify the slight subtlety that has been overlooked in using a semi-infinite discontinuity line.

The line integral involved in this gauge transformation discussion can be evaluated easily, if $\mathbf{n}$ and $\mathbf{n}'$ differ infinitesimally

$$\mathbf{n}'-\mathbf{n}=\delta\mathbf{n}$$

through the effect of an infinitesimal rotation

$$\delta\boldsymbol{\omega}=\mathbf{n}\times\delta\mathbf{n}\,.$$

Indeed,

$$\delta_\mathbf{n}4\pi\mathbf{a}_\mathbf{n}(\mathbf{x})=-\nabla\tfrac{1}{2}\delta\boldsymbol{\omega}\cdot\left[\frac{\mathbf{x}}{|\mathbf{x}|+\mathbf{n}\cdot\mathbf{x}}+\frac{\mathbf{x}}{|\mathbf{x}|-\mathbf{n}\cdot\mathbf{x}}\right]$$

and

$$f_{\mathbf{n}'\mathbf{n}}(\mathbf{x})=-\delta\boldsymbol{\omega}\cdot\left[\mathbf{x}\times\mathbf{a}_\mathbf{n}(\mathbf{x})+\left(\frac{1}{4\pi}\right)\frac{\mathbf{x}}{|\mathbf{x}|}\right]$$
$$=\delta\boldsymbol{\omega}\cdot\mathbf{f}_\mathbf{n}(\mathbf{x})\,.$$

Thus, the infinitesimal gauge transformation is generated by the unitary operator

$$U(\delta\boldsymbol{\omega})=1+i\delta\boldsymbol{\omega}\cdot\mathbf{j}_\mathbf{n}\,,$$

where

$$\mathbf{j}_\mathbf{n}=\int(d\mathbf{x})(d\mathbf{x}')j^0(x)\mathbf{f}_\mathbf{n}(\mathbf{x}-\mathbf{x}')\,{}^*j^0(x')\,.$$

Only the components of $\mathbf{j}_\mathbf{n}$ perpendicular to $\mathbf{n}$ are relevant to this transformation. But, in fact, the parallel component vanishes,

$$\mathbf{n}\cdot\mathbf{j}_\mathbf{n}=0\,,$$

since

$$\mathbf{n}\times\mathbf{x}\cdot4\pi\mathbf{a}_\mathbf{n}(\mathbf{x})=-(\mathbf{n}\cdot\mathbf{x})/|\mathbf{x}|\,.$$

Now we consider the total angular momentum of the system. It is given by

$$\mathbf{J}=\mathbf{J}^{(0)}+\int(d\mathbf{x})(\phi\nabla{}^*\phi-{}^*\phi\nabla\phi)$$
$$-\int(d\mathbf{x})(d\mathbf{x}')j^0(x)(\mathbf{x}-\mathbf{x}')\times\mathbf{a}_\mathbf{n}(\mathbf{x}-\mathbf{x}')\,{}^*j^0(x')\,,$$

where

$$\mathbf{J}^{(0)}=\int(d\mathbf{x})[\bar{\psi}\gamma^0(-i\mathbf{x}\times\nabla+\tfrac{1}{2}\boldsymbol{\sigma})\psi+\bar{\chi}\gamma^0(-i\mathbf{x}\times\nabla+\tfrac{1}{2}\boldsymbol{\sigma})\chi+\mathbf{E}^T\cdot(\mathbf{x}\times\nabla)\mathbf{A}^T+\mathbf{E}^T\times\mathbf{A}^T]$$

is the conventional total-angular-momentum operator. The additional static-field contribution

$$\int(d\mathbf{x})(\phi\nabla{}^*\phi-{}^*\phi\nabla\phi)=-\int(d\mathbf{x})(d\mathbf{x}')j^0(x)\left(\frac{1}{4\pi}\right)\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}{}^*j^0(x')$$

has long been recognized.[5] But it is only part of the extra angular momentum, which is given completely by

$$\mathbf{J}=\mathbf{J}^{(0)}+\mathbf{j}_{\mathbf{n}}.$$

The meaning of the supplementary angular momentum is evident. It generates the gauge transformation needed to maintain the relation between the singularity line and the coordinate system. The vector $\mathbf{j}_{\mathbf{n}}$ is not an independent angular momentum, however. Its components are commutative, and the combination $\mathbf{J}^{(0)}+\mathbf{j}_{\mathbf{n}}$ obeys angular-momentum commutation relations in consequence of the differential property

$$\delta\omega_1\cdot\mathbf{x}\times\nabla\delta\omega_2\cdot\mathbf{f}_{\mathbf{n}}(\mathbf{x})-\delta\omega_2\cdot\mathbf{x}\times\nabla\delta\omega_1\cdot\mathbf{f}_{\mathbf{n}}(\mathbf{x})=-\delta\omega_1\times\delta\omega_2\cdot\mathbf{f}_{\mathbf{n}}(\mathbf{x}).$$

The eigenvalue spectrum of $\mathbf{J}$ is of the same integral or half-integral nature as that of $\mathbf{J}^{(0)}$, since

$$\mathbf{n}\cdot\mathbf{J}=\mathbf{n}\cdot\mathbf{J}^{(0)}.$$

Angular-momentum considerations shed some light on the charge-quantization condition, in relation to the use of a semi-infinite discontinuity line. Consider the alternative function

$$\mathbf{a}_{\mathbf{n}}'(\mathbf{x})=-\mathbf{b}_{\mathbf{n}}'(-\mathbf{x})=-\mathfrak{D}(\mathbf{x})\frac{\mathbf{n}\times\mathbf{x}}{|\mathbf{x}|-\mathbf{n}\cdot\mathbf{x}},$$

which obeys

$$\nabla\times\mathbf{a}_{\mathbf{n}}'(\mathbf{x})=-\nabla\mathfrak{D}(\mathbf{x})+\mathbf{h}_{\mathbf{n}}'(\mathbf{x}).$$

The function $\mathbf{h}_{\mathbf{n}}'(\mathbf{x})$ satisfies

$$\nabla\cdot\mathbf{h}_{\mathbf{n}}'(\mathbf{x})=-\delta(\mathbf{x}),$$

and is given by

$$\mathbf{h}_{\mathbf{n}}'(\mathbf{x})=-\mathbf{n}\tfrac{1}{2}(1+(\mathbf{n}\cdot\mathbf{x})/|\mathbf{x}|)\delta_{\mathbf{n}}(\mathbf{x}).$$

Now, if one encircles the line $\mathbf{x}=\mathbf{n}|\mathbf{x}|$,

$$\lim\oint d\mathbf{x}\cdot\mathbf{a}_{\mathbf{n}}'=-1$$

and the gauge-transformation discussion supplies the uniqueness condition

$$eg/4\pi=\tfrac{1}{2}n,\quad\text{(Dirac)}$$

where n is an integer. The angular-momentum construction is changed in only one detail, which derives from

$$\mathbf{n}\cdot\mathbf{f}_{\mathbf{n}}'(\mathbf{x})=-\mathbf{n}\cdot[\mathbf{x}\times\mathbf{a}_{\mathbf{n}}'(\mathbf{x})+(1/4\pi)\mathbf{x}/|\mathbf{x}|]=1/4\pi.$$

Accordingly,

$$\mathbf{n}\cdot\mathbf{J}=\mathbf{n}\cdot\mathbf{J}^{(0)}+(1/4\pi)Q\,{}^*Q,$$

which introduces the total charge operators

$$Q=\int(d\mathbf{x})j^0(x),\quad {}^*Q=\int(d\mathbf{x})\,{}^*j^0(x).$$

The eigenvalues of $(1/4\pi)Q\,{}^*Q$ are integer multiples of the unit $eg/4\pi$. If this term is considered simply as an additive angular-momentum component, we are again led to the Dirac quantization condition. But that does violence to the geometrical equivalence of a rotation about an axis through the angle 2π with the identity transformation, apart from the characteristic sign reversal of half-integral spin fields. Thus, let χ be a field of spin s. Then we have

$$\exp(-2\pi i\mathbf{n}\cdot\mathbf{J})\chi\exp(2\pi i\mathbf{n}\cdot\mathbf{J})=(-1)^{2s}\exp(\tfrac{1}{2}igQ)\chi$$

which is not merely $(-1)^{2s}\chi$, if the magnitude of $eg/4\pi$ assumes one of the half-integral values $\frac{1}{2},\frac{3}{2},\cdots$. This is another indication that the charge quantization condition must be the stronger integer requirement for $eg/4\pi$.

The apparent contradiction between the two procedures is resolved by a more careful consideration of the discontinuous-line integrals. Let $\mathbf{n}$ define the positive z axis and consider the limiting values of the line integrals, drawn positively about the z axis, for $\mathbf{a}_{\mathbf{n}}'$ and $\mathbf{a}_{\mathbf{n}}$. We distinguish three domains: $z>0$, $z=0$, $z<0$. The values are

$$\lim\oint d\mathbf{x}\cdot\mathbf{a}_{\mathbf{n}}'=-1,\ z>0$$
$$=-\tfrac{1}{2},\ z=0$$
$$=0,\ z<0$$

and

$$\lim\oint d\mathbf{x}\cdot\mathbf{a}_{\mathbf{n}}=-\tfrac{1}{2},\ z>0$$
$$=0,\ z=0$$
$$=+\tfrac{1}{2},\ z<0,$$

where the assignment at $z=0$ is taken as the average of the two limits from opposite sides. The smallest nonzero magnitude of the line integrals is the significant one, and this is now recognized to be $\frac{1}{2}$, in both procedures. Hence the correct charge quantization condi-

[5] H. A. Wilson, Phys. Rev. **75**, 309 (1949).

tion is the integral restriction

$$eg/4\pi = n\,.$$

These two vector potentials are members of a class of such functions for which the semi-infinite discontinuity line $z>0$ is weighted by a factor α, and the line $z<0$ with a factor β, where

$$\alpha+\beta=1\,.$$

The three limiting values of the line integral are: $-\alpha$, $\frac{1}{2}(-\alpha+\beta)$, β, in the same order as above. The integral charge condition is maintained since the difference of successive values is just $\frac{1}{2}$. The possible values of α and β are

$$\alpha=k/2n\,,\quad \beta=1-(k/2n)$$

where k is an integer. When one of these functions is used, $\mathbf{n}\cdot(\mathbf{J}-\mathbf{J}^{(0)})$ becomes $(1/4\pi)Q^*Q(\alpha-\beta)$. The eigenvalues of the latter operator are integral multiples of $k-n$, an integer.

What conclusions concerning relativistic invariance should be drawn from all these considerations? Given a pair of equal-time points, the arbitrary singularity line can be chosen so that the fundamental energy-density commutator condition is satisfied, verifying relativistic invariance. It is also true, however, that the commutator condition does not appear to be obeyed everywhere, when the discontinuity line is fixed. One suspects that the breakdown which occurs when the two points are connected by the discontinuity line is a failure of the formal apparatus, rather than a violation of relativistic invariance. We shall confirm this by exhibiting a limiting definition[6] of the charged-field contribution to the energy density, which removes this deficiency.

Let us observe that

$$\begin{aligned}-\bar\psi(x)\boldsymbol{\gamma}\cdot(\boldsymbol{\nabla}-ie\mathbf{A}(x))\psi(x)\\ =\lim\Bigg[\bar\psi(x+\tfrac{1}{2}\epsilon)\frac{3\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}}{\boldsymbol{\epsilon}^2}\psi(x-\tfrac{1}{2}\epsilon)\\ \times\exp\bigg(ie\int_{x-\frac{1}{2}\epsilon}^{x+\frac{1}{2}\epsilon}d\mathbf{x}_1\cdot\mathbf{A}(x_1)\bigg)\Bigg],\end{aligned}$$

where the integration path is a straight line connecting the two equal-time points, and an average is to be performed over all directions of $\boldsymbol{\epsilon}$ before letting $|\boldsymbol{\epsilon}|\to 0$. A similar limiting construction applies with ψ and $e\mathbf{A}$ replaced by χ and $g\mathbf{B}$. It is sufficient to consider the operator combinations

$$\begin{aligned}F=\bar\psi_a(x+\tfrac{1}{2}\epsilon)\psi_b(x-\tfrac{1}{2}\epsilon)\\ \times\exp\bigg(ie\int_{x-\frac{1}{2}\epsilon}^{x+\frac{1}{2}\epsilon}d\mathbf{x}_1\cdot(\mathbf{A}^T+\mathbf{A}_g)(x_1)\bigg)\end{aligned}$$

$$\begin{aligned}G=\bar\chi_c(x'+\tfrac{1}{2}\epsilon')\chi_d(x'-\tfrac{1}{2}\epsilon')\\ \times\exp\bigg(ig\int_{x'-\frac{1}{2}\epsilon'}^{x'+\frac{1}{2}\epsilon'}d\mathbf{x}_2\cdot(\mathbf{B}^T+\mathbf{B}_e)(x_2)\bigg)\,.\end{aligned}$$

They have the commutation property

$$FG=GF\exp(iegC)\,,$$

where

$$\begin{aligned}C&=\int_{x-\frac{1}{2}\epsilon}^{x+\frac{1}{2}\epsilon}d\mathbf{x}_1\times\int_{x'-\frac{1}{2}\epsilon'}^{x'+\frac{1}{2}\epsilon'}d\mathbf{x}_2\\ &\qquad\cdot[\boldsymbol{\nabla}\times\mathbf{a}_{\mathbf{n}}(\mathbf{x}_1-\mathbf{x}_2)+\boldsymbol{\nabla}\mathfrak{D}(\mathbf{x}_1-\mathbf{x}_2)]\\ &=\int_{-\frac{1}{2}}^{\frac{1}{2}}d\lambda\int_{-\frac{1}{2}}^{\frac{1}{2}}d\lambda'\boldsymbol{\epsilon}\times\boldsymbol{\epsilon}'\cdot\mathbf{h}_{\mathbf{n}}(\mathbf{x}-\mathbf{x}'+\lambda\boldsymbol{\epsilon}-\lambda'\boldsymbol{\epsilon}')\,.\end{aligned}$$

In general, $\mathbf{x}-\mathbf{x}'$ has a nonzero projection on the plane perpendicular to $\mathbf{n}$. Choose the vectors $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ to be of smaller length than this projection. Then $\mathbf{h}_{\mathbf{n}}$ vanishes for all λ and λ', $C=0$, and F commutes with G. It is different if $\mathbf{x}-\mathbf{x}'$ is parallel to $\mathbf{n}$. If we use the vector function symmetrically associated with the discontinuity line, for example, and note that

$$\boldsymbol{\epsilon}\times\boldsymbol{\epsilon}'\cdot\mathbf{n}\delta_{\mathbf{n}}(\lambda\boldsymbol{\epsilon}-\lambda'\boldsymbol{\epsilon}')=\frac{\boldsymbol{\epsilon}\times\boldsymbol{\epsilon}'\cdot\mathbf{n}}{|\boldsymbol{\epsilon}\times\boldsymbol{\epsilon}'\cdot\mathbf{n}|}\delta(\lambda)\delta(\lambda')\,,$$

we get

$$C=-\frac{1}{2}\frac{\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')|}\frac{\boldsymbol{\epsilon}\times\boldsymbol{\epsilon}'\cdot\mathbf{n}}{|\boldsymbol{\epsilon}\times\boldsymbol{\epsilon}'\cdot\mathbf{n}|}\,.$$

The number C does not vanish when the two points are connected by the discontinuity line. Yet, as a consequence of the charge quantization condition,

$$\exp(iegC)=1$$

and F commutes with G. This discussion shows clearly how relativistic invariance will appear to be violated in any treatment based on a perturbation expansion. Field theory is more than a set of "Feynman's rules."

The relativistic quantum field theory of magnetic and electric charge is of such beauty that we must repeat after Dirac: "One would be surprised if Nature had made no use of it."

Added note: While the manuscript of this paper was being typed, the 8 November 1965 issue of *The Physical Review* appeared. It contains a paper by C. R. Hagen on the same subject [Phys. Rev. **140**, B804 (1965)], in which the opposite conclusion is reached concerning relativistic invariance. This author insists on manifest rotational invariance of the formalism, whereas we permit the apparent asymmetry of the singularity line, while maintaining the rotational and Lorentz invariance of the content of the theory. Incidentally, the linear model of Boulware and Gilbert [Phys. Rev. **126**, 1563 (1962)], which is discussed by Hagen, does not give an acceptable idealization of magnetic and electric charge.

[6] ". . . localized field operator products must be understood as the limit of products defined for noncoincident points." J. Schwinger, Phys. Rev. Letters 3, 296 (1959).

Since no exponential functions occur, the fact that a change of the singularity line is not everywhere a gauge transformation cannot be overcome by a charge quantization condition.

[*Note added in proof*. The field theory of electric and magnetic charge requires the use of potentials with arbitrary singularity lines. All orientations of the singularity line must be equivalent. Yet, given any particular choice, pairs of points that are connected by the singularity line appear to have special characteristics. It is the function of the charge quantization condition to remove their anomalous position and restore the equivalence of all space-time points, without exception. This is the decisive feature that distinguishes the present field theory from the earlier particle theory of Dirac, who expressly forbids an electrically charged particle to lie on the singularity line (string") associated with a magnetically charged particle. The discussion of relativistic invariance in the text illustrates this field theoretic viewpoint. The implications of the demand for equivalence of all points were not completely understood, however. Somewhat surprisingly, the potentials with an infinite singularity line, which originally suggested the integer charge quantization condition, can be used only when the integer is even.

Let $\mathbf{a}_{\mathbf{n}}(\mathbf{x})$ here refer to the function that is singular on the semi-infinite line $\mathbf{x}=\mathbf{n}|\mathbf{x}|$. There is a unique unitary transformation associated with the substitution of $\mathbf{n}'$ for $\mathbf{n}$ if

$$\exp[iegf_{\mathbf{n}'\mathbf{n}}(\mathbf{x})],$$

$$f_{\mathbf{n}'\mathbf{n}}(\mathbf{x})=\int_{\infty}^{\mathbf{x}} d\mathbf{x}'\cdot(\mathbf{a}_{\mathbf{n}'}-\mathbf{a}_{\mathbf{n}})(\mathbf{x}')$$

is independent of the integration path, for all points $\mathbf{x}$. On comparing two such paths, the difference of line integrals can be expressed by an integration extended over the surface that is defined by the specified boundaries. The uniqueness condition reads

$$\exp\left[ieg\int d\mathbf{S}'\cdot(\mathbf{h}_{\mathbf{n}'}-\mathbf{h}_{\mathbf{n}})(\mathbf{x}')\right]=1$$

which must hold for all surfaces, without exception. When constructed from limits of products (the F and G of the text), operator structures such as T^{00} are then unambiguously gauge invariant. The effect on F, for example, of replacing fields and potentials with those associated with a new singularity line is to multiply it by the operator factor

$$\exp\left[ie\int_C d\mathbf{x}'\cdot(\mathbf{A}_{g'}-\mathbf{A}_g)(x')\right]$$
$$=\exp\left[ie\int(d\mathbf{x}'')\int d\mathbf{S}'\cdot(\mathbf{h}_{\mathbf{n}'}-\mathbf{h}_{\mathbf{n}})(\mathbf{x}'-\mathbf{x}'')*i^0(x'')\right],$$

where the contour C begins at infinity, move successively to $\mathbf{x}-\frac{1}{2}\boldsymbol{\varepsilon}$, $\mathbf{x}+\frac{1}{2}\boldsymbol{\varepsilon}$, and then returns to infinity. The complete identity of this operator with the unit operator follows from the nature of magnetic charge density eigenvalues-sums of three-dimensional delta functions multiplied by $\pm g$.

Suppose a surface cuts the line $\mathbf{x}=\mathbf{n}|\mathbf{x}|$ at a point with $|\mathbf{x}|>0$ where the directed normal to the surface is $\boldsymbol{\nu}$. Then, the possible integral contributions are

$$\int d\mathbf{S}\cdot\mathbf{h}_{\mathbf{n}}(x)=\begin{cases}-1, & \boldsymbol{\nu}\cdot\mathbf{n}>0\\ 0, & \boldsymbol{\nu}\cdot\mathbf{n}=0\\ 1, & \boldsymbol{\nu}\cdot\mathbf{n}<0.\end{cases}$$

If the intersection occurs at $\mathbf{x}=0$, however, the values of the integral must be multiplied by $\frac{1}{2}$. A factor of $\frac{1}{2}$ also appears should the singularity line not pierce the surface, but be tangential to it. Now consider the possible values of $\int d\mathbf{S}\cdot(\mathbf{h}_{\mathbf{n}'}-\mathbf{h}_{\mathbf{n}})$. If neither singularity line is tangential to the surface, this integral equals some integer. Surfaces that cut both lines at the origin are no exception, since the possible magnitudes of such contributions are $\frac{1}{2}\pm\frac{1}{2}$. It is different if either line is tangential to the surface. In general, the integral will differ from an integer by $\frac{1}{2}$. Consider, for example, a semi-infinite plane surface that cuts the line defined by $\mathbf{n}$ and touches the line $\mathbf{n}'$ at a point. The value of the integral is, say, $(-\frac{1}{2})-(-1)=\frac{1}{2}$. Now let the surface rotate about the point until the surface intersects the origin. The integral becomes $(0)-(-\frac{1}{2})=\frac{1}{2}$. Further rotation by an angle less than π gives $(\frac{1}{2})-(0)=\frac{1}{2}$. The other values achieved by continued rotation are 0 and $-\frac{1}{2}$, before $\frac{1}{2}$ is regained. When all possible surfaces are considered, the smallest nonvanishing magnitude assumed by the surface integral is $\frac{1}{2}$. The condition for the uniqueness of the singularity line transformation is therefore the integer quantization rule

$$eg/4\pi=n.$$

The criterion for relativistic invariance can be presented as

$$\exp\left[ieg\int d\mathbf{S}''\cdot\mathbf{h}_{\mathbf{n}}(\mathbf{x}'')\right]=1,$$

where $\mathbf{x}''$ is integrated over an arbitrarily small surface that passes through the point $\mathbf{x}-\mathbf{x}'\neq0$. Since the possible magnitudes of the surface integral are 0, $\frac{1}{2}$, 1, this criterion is satisfied without exception.

Now consider the class of infinite singularity line potentials. Let α be the weight associated with the semi-infinite line defined by $\mathbf{n}$, and $\beta=1-\alpha$ that of its image. The uniqueness of singularity line transformations requires that

$$\tfrac{1}{2}eg\alpha=2\pi n_1\,,\quad \tfrac{1}{2}eg\beta=2\pi n_2\,,$$
$$eg/4\pi=n=n_1+n_2$$

and

$$n\alpha=n_1\,,\quad n\beta=n_2\,.$$

Notice, however, that if α, or β, is specified as an irreducible fraction, the integer n must be divisible by the denominator of that fraction. Thus, in order to use symmetrical potentials $\alpha=\beta=\frac{1}{2}$, the integer n must be even.

Conversations with Bruno Zumino were helpful in stimulating this closer examination of the theory.]

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Positive-Pion Production Asymmetry with Polarized Bremsstrahlung Near Second Resonance*

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The azimuthal asymmetry $\Sigma=(\sigma_\perp-\sigma_\parallel)/(\sigma_\perp+\sigma_\parallel)$ in π^+ photoproduction by linearly polarized bremsstrahlung was measured at photon energies from 475 to 750 MeV at 90° and 135° in the center-of-mass system. The experimental results show that even in this energy region, π^+ are produced predominantly in the plane of the magnetic vector.

I. INTRODUCTION

PHOTOPRODUCTION of pions from nucleons has been studied for over a decade and much information concerning the pion-nucleon system has been gathered in the first resonance region.[1–3] In the second resonance region, however, there is still much uncertainty. The usual experimental information consisted of an angular distribution expressed by a series expansion in $\cos\theta$, θ being the angle between the outgoing-pion and the incident-photon directions in the center-of-mass system. By examining the coefficients in the expansion, one can try to infer the total angular momentum J of the final state. For a given value of J, there are, however, two values of l with different parity. A method of identifying the parity of the state would be to measure the asymmetry in the photoproduction with linearly polarized photons, thus discriminating between electric and magnetic transitions.[4–7]

In the region of the second resonance, many partial-wave amplitudes contribute simultaneously to the process and the analysis is correspondingly involved and less reliable, making experiments sensitive to the interference terms of great interest. The differential-cross-section measurements allow no separation of the real and the imaginary parts of interference terms. However, using unpolarized photons, a measurement of the recoiling-nucleon polarization gives information on the imaginary part of the interference terms and, using linearly polarized photons, a measurement of production asymmetry is sensitive to the behavior of the real part of the interference terms. The recoiling-proton polarization in π^0 photoproduction has, of course, already been extensively studied,[8] but no corresponding study has been carried out in π^+ case. In the case of π^+ photoproduction, where the resonance behavior is more evident ($I=\frac{1}{2}$), and the angular distribution is already very complex, a different experimental approach may be of interest. Asymmetry measurements with polarized photons can, it is hoped, give an additional constraint and help in evaluating the relative importance of the different multipole contributions.

The present experiment was to study this region by measuring the asymmetry in the production of positive pions with respect to the polarization plane of the incident photons. It was an extension to a higher energy region of similar work done in this laboratory[9,10] at the first resonance region. At these energies, contamination due to pion pair production is unavoidable, if reasonable polarization of the photon beam is to be achieved. However, this background cannot alter any inference that we may draw from the results of this experiment, as explained later.

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