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ON DETERMINATION OF SPIN, DECAY PARAMETERS, AND DENSITY MATRIX OF DECAYING STATES*)

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I. INTRODUCTION

Excited baryons which decay into a baryon (spin one-half) and a meson (spin zero) have the remarkable property that their decays are perfect analyzers of the spin and density matrix of the initial sample. In addition to the spin and elements of the density matrix, the amplitudes for the decay itself may also be obtained from the angular distribution and polarization of the daughter[1]. In this report, we shall discuss in detail decay sequences such as

$$X \rightarrow \Lambda + K, \quad \Lambda \rightarrow p + \pi \tag{1}$$

where X represents a particle with spin J, and K represents particles with spins one-half and zero, respectively, and the decay $\Lambda \rightarrow p + \pi$ serves to measure the polarization of Λ. We shall show how angular correlations in Eq. (1) exhibit the spin, spin orientation (density matrix), and decay amplitudes of X.

We describe the spin orientation of X by spin multipole parameters (analogues of the multipole moments of a charge distribution). These parameters provide a complete specification of the density matrix of X [2]. Owing to their covariance properties (they transform like spherical harmonics), they are convenient tools for expressing consequences of symmetries in the production of X [3]. They are directly measurable because the multipole moments of angular distributions in Eq. (1) are, aside from certain coefficients, the spin multipoles of X.

For parity conserving decays, all even multipole moments may occur in the angular distribution of Λ, and all odd multipole moments may occur in the angular distribution of polarization of Λ. Consequently, the complete density matrix of X may be measured. The transverse and longitudinal polarization of Λ display the <u>same</u> odd multipoles. Their relative magnitudes and sign, however, depend on J and the parity of X. Consequently, both the spin and parity of X may be measured.

For parity violating decays, even and odd multipoles appear in the angular distribution of Λ and of its longitudinal polarization. In addition, two vector functions of the transverse polarization may be non-vanishing: $\vec{P}_\Lambda \times \hat{\Lambda}$ and $\vec{P}_\Lambda - \hat{\Lambda}\hat{\Lambda} \cdot \vec{P}_\Lambda$. All non-vanishing odd multipoles appear also in these functions. Consequently, there are many relations between angular correlations and it is possible to obtain determinations of the spin and decay parameters as well as the spin multipole parameters of X.

In general, the decay $X \to \Lambda + K$ is described by two amplitudes a and b where a is the probability amplitude for orbital angular momentum $\ell = J - \frac{1}{2}$ and b for $\ell = J + \frac{1}{2}$. The relative magnitudes and phase of these amplitudes are given by the parameters[4])

$$\alpha = 2\,\mathrm{Re}\,[a^*b/|a|^2 + |b|^2]$$

$$\beta = 2\,\mathrm{Im}\,[a^*b/|a|^2 + |b|^2] \qquad (2)$$

$$\gamma = [|a|^2 - |b|^2]/|a|^2 + |b|^2 \; .$$

If parity is conserved, a or b is zero; then $\alpha = \beta = 0$ and $\gamma = \pm 1$. (Our notation is such that the probability amplitude $A_{1/2}$ for finding Λ with helicity ½ is a + b. If the decay is weak, time reversal invariance gives, to lowest order in H_{weak}[5])

$$a = |a| e^{i\delta_{J-1/2}} \qquad b = |b| e^{i\delta_{J+1/2}}$$

where δ_ℓ is the scattering phase shift for Λ - K with orbital angular momentum ℓ.) Since parity conserving decays are special cases and their properties are so easily derived from the general case, we shall not treat them separately.

In Section III, we define a Lorentz invariant angular correlation function F for Eq. (1). It is given for X produced in reactions with two- and three-body final states. This definition is easily generalized to other production reactions.

Without a large number of events, accurate measurements of angular correlations are difficult and a maximum likelihood fit of F to the data may be the most reliable means of obtaining information about parameters of interest. In Section IV, the dependence of F on J, α, β, γ, and the multipole parameters is explicitly displayed. (The derivation of F is given in Appendix A.)

The spin multipole parameters are discussed in detail in Sections V and VI where we relate them directly to the density matrix, give their symmetry properties, and express them in terms of the S matrix for production of X. Since they are spherical tensors with respect to spatial rotations in the rest frame of X, their transformation properties under proper Lorentz transformations are easily obtainable[6]. They give a Lorentz covariant specification of the density matrix.

In Section VII, we discuss specific properties of the density matrix of X when produced in a reaction with polarized target. In this case, angular correlations between production and decay allow for determination of the relative parity of X.

Unless otherwise stated, we always assume F describes only (1). In practice, if X is a resonant state, F will contain contaminations from non-resonant background and/or interactions between the decay products of X and other particles in the final state. Such contaminations are briefly discussed in Section VIII.

This report is mainly an account of investigations made in response to discussions with various experimental groups regarding the application of certain published results[1].

In particular, we would like to thank Drs. H.K. Ticho, D.H. Stork, P. Eberhard, and J.B. Shafer for stimulating and profitable discussions. Owing to time limitations, we are not able to provide here a complete bibliography on the literature, and we hope we may have the reader's indulgence in this matter.

II. NOTATION

J = spin of X;

$\vec{\Lambda}$ = momentum of Λ in rest frame of X;

ϑ_1, φ_1 = spherical angles of $\vec{\Lambda}$ referred to any set of axes that are chosen without regard to X decay;

$\vec{p}$ = momentum of p in rest frame of Λ;

ϑ_p, φ_p = spherical angles of $\vec{p}$ referred to space axes obtained from frame in which ϑ_1, φ_1 are measured by Lorentz transformation without rotation;

$\vec{P}_\Lambda$ = polarization of Λ;

α_Λ = asymmetry parameter for $\Lambda \to p + \pi^-$ as given in Ref. 4) (Cronin and Overseth);

Ψ = angle between $\vec{\Lambda}$ and $\vec{p}$, i.e. $\cos\Psi = \hat{\Lambda}\cdot\hat{p}$, measured from $\vec{p}$ to $\vec{\Lambda}$;

$$\Phi = \tan^{-1}\left\{\frac{\sin\vartheta_p \sin(\varphi_1 - \varphi_p)}{\cos\vartheta_p \sin\vartheta_1 - \sin\vartheta_p \cos\vartheta_1 \cos(\varphi_p - \varphi_1)}\right\}$$

= azimuthal angle of $\vec{p}$ when $\vec{\Lambda}$ is chosen as polar axes;

$$\left.\begin{aligned} \alpha_{X\to\Lambda} &= |A_{1/2}|^2 - |A_{-1/2}|^2 \\ \beta_{X\to\Lambda} &= 2\,\mathrm{Im}\, A_{1/2} A_{-1/2}^* \\ \gamma_{X\to\Lambda} &= 2\,\mathrm{Re}\, A_{1/2} A_{-1/2}^* \end{aligned}\right\} \text{ equivalent definitions to Eq.(2);}$$

$A_{\pm 1/2}$ = amplitude for $X \to \Lambda + \pi$ with Λ helicity $\pm\frac{1}{2}$, normalized so that $|A_{1/2}|^2 + |A_{-1/2}|^2 = 1$;

= $(a \pm b)/\sqrt{|a|^2 + |b|^2}$, where a and b are defined in text;

z_L^M = $m_{JL}\, t_L^M$ (spherical tensor referring to same space axes as $\vartheta_1, \varphi_\Lambda$);

$$m_{JL} = (-)^{J-1/2}\sqrt{(2J+1)(2L+1)}\, C(JJL; \tfrac{1}{2}, -\tfrac{1}{2})$$

$C(JJL;\frac{1}{2},-\frac{1}{2})$ = Clebsch-Gordan coefficient as given in Ref. 18);

t_i^M = spin multipole parameter;

$\vec{X}$ = momentum of X in c.m. of production;

$\vec{u}$ = beam momentum in c.m. production;

ϑ, φ = spherical angles of X ($\cos\vartheta = \hat{u}\cdot\hat{X}$);

$\vec{n}$ = $\hat{u}\times\hat{X}$;

$-\sin\varphi$ = $\hat{n}\cdot\hat{p}$ when target is polarized with $\vec{p}\perp\vec{u}$;

$\cos\varphi$ = $\hat{X}\cdot\hat{p}/\sin\vartheta$;

$\mathcal{D}^{(L)}_{MM'}(\varphi,\vartheta,\Phi)$ = $e^{-iM\varphi}\, d^{(L)}_{MM'}(\vartheta)\, e^{-iM'\Phi}$;

$d^{(L)}_{MM'}(\vartheta)$ = $\langle LM'|e^{-iJ_Y\vartheta}|LM\rangle$ = matrix element of rotation operator;

$\mathcal{C}^{(L)}_{M1}(\varphi,\vartheta,\Phi) \equiv -(\frac{1}{2})\left[\mathcal{D}^{(L)}_{M1}(\varphi,\vartheta,\Phi) - \mathcal{D}^{(L)}_{M-1}(\varphi,\vartheta,\Phi)\right]$;

$\mathcal{S}^{(L)}_{M1}(\varphi,\vartheta,\Phi) \equiv (i/2)\left[\mathcal{D}^{(L)}_{M1}(\varphi,\vartheta,\Phi) + \mathcal{D}^{(L)}_{M-1}(\varphi,\vartheta,\Phi)\right]$;

Some properties of these functions:

$\mathcal{C}^{(L)}_{M1} = (-)^M\, \mathcal{C}^{(L)*}_{-M1}$;

$\mathcal{S}^{(L)}_{M1} = (-)^M\, \mathcal{S}^{(L)*}_{-M1}$;

$d_{\mu\lambda}(\vartheta) = (-)^{\mu+\lambda}\, d_{-\mu-\lambda}(\vartheta) = (-)^{\mu-\lambda}\, d_{-\lambda-\mu}(\vartheta)$;

$d^{(L)}_{\lambda\mu}(\vartheta) = (-)^{L-\lambda}\, d^{(L)}_{\lambda-\mu}(\pi-\vartheta)$;

Under spatial inversions: $\vartheta \to \pi-\vartheta$, $\varphi \to \pi+\varphi$, $\Phi \to \pi+\Phi$

and

$$\mathcal{C}^{(L)}_{M1} \to (-)^L\, \mathcal{C}^{(L)}_{M1} ;$$

$$\mathcal{S}^{(L)}_{M1} \to -(-)^L\, \mathcal{S}^{(L)}_{M1} .$$

III. THE CORRELATION FUNCTION F

The function describing the angular correlations in Eq. (1) will be called F. It gives the probability for finding Λ emitted in solid angle $d\Omega_\Lambda$ in the X rest frame, and proton emitted in solid angle $d\Omega_p$ in the Λ rest frame. It is a Lorentz invariant function.

If, for example, X is produced in the reaction $K^- + p \to X + \pi$, F is related to the differential cross-section $d\sigma$ by

$$d\sigma = R_{X\to\Lambda} R_{\Lambda\to p} F \, d\Omega_\Lambda \, d\Omega_p \, d\Omega \, , \tag{3}$$

where $d\Omega = d \cos \vartheta \, d\varphi$ is the element of solid angle in the $K^- p$ c.m. frame in which X is produced, and

$R_{X\to\Lambda}$ = branching ratio for $X \to \Lambda + K$;

$R_{\Lambda\to p}$ = branching ratio for $\Lambda \to p + \pi^-$;

$d\Omega_\Lambda \equiv d \cos \vartheta_\Lambda \, d\varphi_\Lambda$ = element of solid angle of Λ emission in X rest frame;

$d\Omega_p \equiv d \cos \vartheta_p \, d\varphi_p$ = element of solid angle of proton emission in Λ rest frame.

In this case, F may depend upon the six variables $\vartheta, \varphi, \vartheta_\Lambda, \varphi_\Lambda, \vartheta_p, \varphi_p$, and also the total energy; F is independent of φ if the target is unpolarized.

If X is produced in a reaction with a three-body final state, such as $K^- + p \to X + K^0 + K^+$, the variables ϑ_K and φ_K appear in F and the relation between $d\sigma$ and F is

$$d\sigma = R_{X\to\Lambda} R_{\Lambda\to p} F \, d\Omega_\Lambda \, d\Omega_p \, d\Omega_K \, d\Omega \, , \tag{3'}$$

where $d\Omega_K = d \cos \vartheta_K$ and ϑ_K and φ_K are spherical angles of K^0 emission in the $K^+ K^0$ rest frame.

The structure of F is given in the next section.

IV. THE FORM OF F

Since Λ has spin $\frac{1}{2}$, and $\Lambda \to p + \pi$ is a parity violating decay, the angular distribution of proton emission is linear in the direction cosines of $\vec{p}$. Therefore, F always has the form

$$F = I + \alpha_\Lambda \vec{IP}_\Lambda \cdot \hat{p} \,, \tag{4}$$

where I is proportional to the number of Λ emitted per unit solid angle (in the X rest frame); Eq. (4) gives the complete dependence of F on ϑ_p, φ_p. The dependence of F on ϑ_Λ and φ_Λ can be obtained by generalization of the reasoning which leads to Eq. (4). This is given in Appendix A. The result is, if parity is conserved in X decay,

$$F = \sum_{\substack{L=0 \\ (L\ \text{even})}}^{2J-1} \sum_{M=-L}^{M=L} z_L^M \mathcal{D}_{M0}^{(L)}(\varphi_\Lambda, \vartheta_\Lambda, 0)$$

$$+ \alpha_\Lambda \cos\Psi \sum_{\substack{L=1 \\ (L\ \text{odd})}}^{2J} \sum_{M=-L}^{L} z_L^M \mathcal{D}_{M0}^{(L)}(\varphi_\Lambda, \vartheta_\Lambda, 0) \tag{5}$$

$$+ \gamma_{X\to\Lambda}(2J+1)\,\alpha_\Lambda \sin\Psi \sum_{\substack{L=1 \\ (L\ \text{odd})}}^{2J} \sum_{M=-L}^{L} z_L^M [L(L+1)]^{-\frac{1}{2}} \mathcal{C}_{M1}^{(L)}(\varphi_\Lambda, \vartheta_\Lambda, 0) \,.$$

From comparison with Eq. (4) one sees that the first sum is an expansion of I; the second an expansion of the longitudinal polarization $\alpha_\Lambda \vec{IP}_\Lambda \cdot \hat{\Lambda}\hat{\Lambda} \cdot \hat{p}$; and the last the contribution to F from the transverse polarization of Λ. Note the appearance of the <u>same</u> expansion coefficients in the last two sums. This along with the factor $(2J+1)$ makes <u>unique</u> determinations of J and $\gamma_{X\to\Lambda}$ possible. In this case, $\gamma_{X\to\Lambda} = \pm 1$.

When parity is violated in $X \to \Lambda + K$, F has the form [note that Eq. (6) reduces to Eq. (5) when $\alpha_{X\to\Lambda} = \beta_{X\to\Lambda} = 0$]

$$F = (1 + \alpha_\Lambda \alpha_{X\to\Lambda} \cos \Psi) \sum_{\substack{L=0 \\ (L \text{ even})}}^{2J-1} \sum_M z_L^M \mathcal{D}_{M0}^{(L)}(\varphi_\Lambda, \vartheta_\Lambda, \Phi)$$

$$+ (\alpha_{X\to\Lambda} + \alpha_\Lambda \cos \Psi) \sum_{\substack{L=1 \\ (L \text{ odd})}}^{2J} \sum_M z_L^M \mathcal{D}_{M0}^{(L)}(\varphi_\Lambda, \vartheta_\Lambda, \Phi) \qquad (6)$$

$$+ (2J+1)\, \alpha_\Lambda \sin \Psi \sum_{\substack{L=1 \\ (L \text{ odd})}}^{2J} \sum_M \Big\{ z_L^M [L(L+1)]^{-1/2}$$

$$\times \Big[\beta_{X\to\Lambda} \mathcal{S}_{M1}^{(L)}(\varphi_\Lambda, \vartheta_\Lambda, \Phi) + \gamma_{X\to\Lambda} \mathcal{C}_{M1}^{(L)}(\varphi_\Lambda, \vartheta_\Lambda, \Phi) \Big] \Big\} .$$

Here again the terms proportional to $\alpha_\Lambda \cos \Psi$ represent the expansion of the longitudinal polarization of Λ, the last sum that of the transverse polarization, and the remaining sums are the expansion of I[7].

The functions $\mathcal{D}_{M0}^{(L)}$, $\cos \Psi \mathcal{D}_{M0}^{(L)}$, $\sin \Psi\, \mathcal{C}_{M1}^{(L)}$, and $\sin \Psi\, \mathcal{S}_{M1}^{(L)}$ in Eqs. (5) and (6) are orthogonal over the range of the variables $\vartheta_\Lambda, \varphi_\Lambda, \Psi, \Phi$ (two unit spheres). Their normalization is given by that of the $\mathcal{D}_{MN}^{(L)}$ namely

$$\left[\frac{(2L+1)}{8\pi^2}\right] \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\vartheta \int_0^{2\pi} d\Phi\, \mathcal{D}_{MN}^{(L)}(\varphi, \vartheta, \Phi)\, \mathcal{D}_{M'N'}^{(L')}(\varphi, \vartheta, \Phi)^* = \delta_{LL'}\, \delta_{MM'}\, \delta_{NN'} , \qquad (7)$$

where $\delta_{LL'}$ is a Kronecker delta. Therefore, Eq. (6) is a (generalized) Fourier expansion of F.

For each value of L,M there are four independent moments of F:

$$<\mathcal{D}^{(L)}_{M0}>, <\cos\Psi\,\mathcal{D}^{(L)}_{M0}>, <\sin\Psi\,\mathcal{C}^{(L)}_{M1}>$$

and

$$<\sin\Psi\,\mathcal{S}^{(L)}_{M1}> ;$$

$$<f> = \frac{\int d\Omega_\Lambda\, d\Omega_p\, F\, f}{(\int d\Omega_\Lambda\, d\Omega_p\, F)} \quad . \tag{8}$$

These are multipole moments of I, $I\vec{P}_\Lambda \cdot \hat{\Lambda}$, and the two vector functions $I\vec{P}_\Lambda - \hat{\Lambda}\hat{\Lambda}\cdot I\vec{P}_\Lambda$ and $I\vec{P}_\Lambda \times \hat{\Lambda}$ (see Table I).

Table I

Relation of moments of F to angular distribution I and angular distribution of polarization $I\vec{P}_\Lambda$ of Λ

	I	$I\vec{P}_\Lambda \cdot \hat{\Lambda}$	$(I\vec{P}_\Lambda)_{\text{transverse}}$	
			$I\vec{P}_\Lambda - \hat{\Lambda}\hat{\Lambda}\cdot I\vec{P}_\Lambda$	$I\vec{P}_\Lambda \times \hat{\Lambda}$
L,M moment (multipole moment)	$<\mathcal{D}^{(L)}_{M0}>$	$<\cos\Psi\,\mathcal{D}^{(L)}_{M0}>$	$<\sin\Psi\,\mathcal{C}^{(L)}_{M1}>$	$<\sin\Psi\,\mathcal{S}^{(L)}_{M1}>$

Equation (6) shows that:

i) all moments with $L > 2J$ vanish;

ii) for L even, non-vanishing moments occur only in I and $I\vec{P}_\Lambda \cdot \hat{\Lambda}$;

iii) for L even, moments of I and $I\vec{P}_\Lambda \cdot \hat{\Lambda}$ are related by

$$\alpha_{X\to\Lambda}\, \alpha_\Lambda <\mathcal{D}^{(L)}_{M0}> = 3 <\cos\Psi\,\mathcal{D}^{(L)}_{M0}> ; \tag{9}$$

iv) for L odd, moments of I and $\vec{IP}_\Lambda \cdot \hat{\Lambda}$ are related by

$$\alpha_\Lambda < \mathcal{D}^{(L)}_{M0} > = 3\, \alpha_{X\to\Lambda} < \cos\Psi\, \mathcal{D}^{(L)}_{M0} > \quad ; \tag{10}$$

v) for L odd, moments of $(\vec{IP}_\Lambda)_{transverse}$ are related to those of $\vec{IP}_\Lambda \cdot \hat{\Lambda}$ by

$$\sqrt{L(L+1)}\ < \sin\Psi\, \mathcal{C}^{(L)}_{M1} > \ = \ (2J+1)\ \gamma_{X\to\Lambda}\ < \cos\Psi\, \mathcal{D}^{(L)}_{M0} > \tag{11}$$

$$\sqrt{L(L+1)}\ < \sin\Psi\, \mathcal{S}^{(L)}_{M1} > \ = \ (2J+1)\ \beta_{X\to\Lambda}\ < \cos\Psi\, \mathcal{D}^{(L)}_{M0} > \tag{12}$$

and that

vi)

$$\mathfrak{z}^M_L = < \mathcal{D}^{(L)}_{M0} >^* \quad \text{for L even} \tag{13}$$

$$\alpha_\Lambda\, \mathfrak{z}^M_L = 3 < \cos\Psi\, \mathcal{D}^{(L)}_{M0} >^* \quad \text{for L odd ,} \tag{14}$$

where

$$\mathfrak{z}^M_L = z^M_L/(2L+1)\ z^0_0 \ .$$

Equations (9) to (14) are summarized in Table II.

Table II

Value of moments with $L \le 2J$ according to Eq. (6) $\left(\mathfrak{z}^M_L \equiv z^M_L/(2L+1)\ z^0_0\right)$

	L even	L odd
$<\mathcal{D}^{(L)}_{M0}>^*$	$\mathfrak{z}^M_L$	$\alpha_{X\to\Lambda}\mathfrak{z}^M_L$
$3<\cos\Psi\,\mathcal{D}^{(L)}_{M0}>^*$	$\alpha_{X\to\Lambda}\ \alpha_\Lambda \mathfrak{z}^M_L$	$\alpha_\Lambda \mathfrak{z}^M_L$
$3<\sin\Psi\,\mathcal{C}^{(L)}_{M1}>^*$	0	$(2J+1)\ \gamma_{X\to\Lambda}\ \alpha_\Lambda \mathfrak{z}^M_L/\sqrt{L(L+1)}$
$3<\sin\Psi\,\mathcal{S}^{(L)}_{M1}>^*$	0	$(2J+1)\ \beta_{X\to\Lambda}\ \alpha_\Lambda \mathfrak{z}^M_L/\sqrt{L(L+1)}$

Note that Table II contains and is a generalization of well-known relations; for example, for L = 0 the second row gives the well-known relation[8]

$$\alpha_{X\to\Lambda}\,\alpha_{\Lambda} = 3 < \cos\Psi > .$$

Since the α, β, γ parameters satisfy

$$\alpha_{X\to\Lambda}^2 + \beta_{X\to\Lambda}^2 + \gamma_{X\to\Lambda}^2 = 1 ,$$

one has an additional constraint on moments with L odd,

$$(2J+1)^2 = \frac{L(L+1)\left(< \sin\Psi\, \mathcal{C}_{M1}^{(L)} >^2 + < \sin\Psi\, \mathcal{S}_{M1}^{(L)} >^2\right)}{(1-\alpha_{X\to\Lambda}^2) < \cos\Psi\, \mathcal{D}_{M0}^{(L)} >^2} . \tag{15}$$

Therefore, if there is a sufficient number N of events to obtain averages[8] directly from the data, and at least one odd multipole ($\mathfrak{z}_L^M$ with L odd) is appreciable, the spin of X may be determined using Eq. (15). Since Eq. (15) is a ratio, the error associated with such an experimental determination of J is not Gaussian distributed. We discuss this in detail in Appendix B. Here we wish to point out that, in principle, J, $\alpha_{X\to\Lambda}$, $\beta_{X\to\Lambda}$, $\gamma_{X\to\Lambda}$, and all the z_L^M may be obtained from F. Table II along with Eq. (15) shows this.

The z_L^M describe angular correlations between production and decay of X. They contain the dependence of F on the production angles ϑ and φ. (They will also depend on the total energy and, in general, other variables associated with production.) From the well-known transformation properties of the $\mathcal{D}_{MN}^{(L)}$ functions[9] and the invariance of F (a scalar function), it follows that the z_L^M transform like spherical harmonics under spatial rotations in the X rest frame, namely[10]:

$$R_{(\alpha,\beta,\gamma)}\, z_L^M = \sum_{M'} z_L^{M'} \; {}^{(L)}_{M'M} (\alpha,\beta,\gamma) , \tag{16}$$

where α, β, γ are the Euler angles of the rotation. The z_L^M are proportional to the spin multipole parameters of X and are discussed in detail in the next sections.

V. THE MULTIPOLE PARAMETERS z_L^M

The parameters z_L^M describe the spin orientation of X. Owing to their role in F, however, they are all proportional to the Clebsch-Gordan coefficient C(JJL; ½, - ½), i.e.

$$z_L^M = n_{JL}\, t_L^M \ , \tag{17}$$

where

$$n_{JL} = (-)^{J-\frac{1}{2}} \sqrt{(2J+1)(2L+1)} \quad C(JJL; \tfrac{1}{2}, -\tfrac{1}{2}) \tag{18}$$

and t_L^M are the spin multipole parameters of X. In terms of the spin density matrix ρ of X, t_L^M are defined by the expansion

$$\rho = (2J+1)^{-1} \sum_{L,M} (2L+1)\, t_L^{M*}\, T_L^M \ , \tag{19}$$

where T_L^M are polynomials of degree L in the spin matrices of X, formed from components of spin as spherical harmonics are formed from components of a unit vector. The T_L^M are trace-orthogonal, and normalized such that

$$\mathrm{Trace}\left(T_L^M\, T_{L'}^{M'\dagger}\right) = \delta_{LL'}\ \delta_{MM'}\,(2J+1)/(2L+1) \ ; \tag{20}$$

[with this normalization, $T_1^0 = S_z/\sqrt{J(J+1)}$ [11].] In a representation where S_z is diagonal (diagonal element m), the matrix elements of T_L^M are Clebsch-Gordan coefficients, namely

$$\left(T_L^M\right)_{mm'} = C(JLJ;\ m'M): \quad m = m' + M \ . \tag{21}$$

From Eqs. (19) and (20), one sees that

$$t_L^M = \mathrm{Trace}\left(\rho\ T_L^M\right) \ . \tag{22}$$

Table III

Density matrix for $J = \frac{1}{2}$ in terms of multipole parameters z_L^M

$$J = \tfrac{1}{2}$$

$$S_z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

$$\rho = \tfrac{1}{2} \begin{pmatrix} z_0^0 + z_1^0 & + z_1^{-1} \sqrt{2} \\ - z_1^{+1} \sqrt{2} & z_0^0 - z_1^0 \end{pmatrix}$$

Table IV

Density matrix for $J = \frac{3}{2}$ in terms of multipole parameters z_L^M

$$J = \tfrac{3}{2}$$

$$\rho = \rho^{\dagger} \text{ and when } S_z = \begin{pmatrix} \frac{3}{2} & & & \\ & \frac{1}{2} & & \\ & & -\frac{1}{2} & \\ & & & -\frac{3}{2} \end{pmatrix}$$

$$4\rho = \begin{pmatrix} z_0^0 + 3z_1^0 - z_2^0 - \frac{1}{3} z_3^0 & z_1^{-1}\sqrt{6} - z_2^{-1}\sqrt{2} - z_3^{-1}\frac{2}{3} & - z_2^{-2}\sqrt{2} - z_3^{-2}\frac{\sqrt{10}}{3} & - \frac{\sqrt{20}}{3} z_3^{-3} \\ & z_0^0 + z_1^0 + z_2^0 + z_3^0 & 2\sqrt{2}\, z_1^{-1} + \frac{2}{\sqrt{3}} z_3^{-1} & -\sqrt{2}\, z^{-2} + \frac{\sqrt{10}}{3} z_3^{-2} \\ & & z_0^0 - z_1^0 + z_2^0 - z_3^0 & +\sqrt{6}\, z_1^{-1} + 2\sqrt{3}\, z_2^{-1} - \frac{2}{3} z_3^{-1} \\ & & & z_0^0 - 3z_1^0 - z_2^0 + \frac{1}{3} z_3^0 \end{pmatrix}$$

Since ρ is Hermitian, and

$$T_L^{M\dagger} = (-)^M T_L^{-M} \quad , \tag{23}$$

the t_L^M satisfy

$$t_L^{M*} = (-)^M t_L^{-M} \quad . \tag{24}$$

Therefore, the t_L^M yield a set of $(2J+1)^2$ real numbers which completely specify ρ.

Under spatial rotations (in the rest frame of X), the matrices T_L^M transform like the components of spherical tensors, namely

$$\sum_{m,m'} \left[T_L^M\right]_{mm'} \mathcal{D}_{mp}^{(J)}(\alpha,\beta,\gamma) \mathcal{D}_{qm'}^{(J)}(\alpha,\beta,\gamma)^* = \sum_{M'} \left[T_L^{M'}\right]_{pq} \mathcal{D}_{M'M}^{(L)}(\alpha,\beta,\gamma) \; . \tag{25}$$

where α, β, γ are the Euler angles of the rotation. [Note that since ρ is scalar, Eq. (16) follows from Eqs. (22) and (25).]

The t_L^M obtain their directional properties from the production of X; for example, if X is produced in the reaction

$$\pi + p \rightarrow X + K \tag{26}$$

with unpolarized target, the t_L^M are spherical tensors composed from components of the ingoing and outgoing momenta (in the c.m. frame).

If parity is conserved in the production, all t_L^M are invariant against space inversions, since they are averages of spin operators.

Independently of the production mechanism, the z_L^M obey certain restrictions owing to general properties of ρ. We shall give some of these here. For this purpose, it is convenient to normalize ρ such that Trace $\rho = 1$; then we have $t_0^0 = 1$ and the diagonal elements of ρ are probabilities. Since these must lie between zero and one, we have

$$-1 \leq \sum_{L=1}^{2J} (2L+1)\; C(JLJ;\, m'0)\; t_L^0 \leq 2J \quad \text{for all } m' \; . \tag{27}$$

In addition, the condition that Trace $\rho^2 \leq 1$ yields

$$\sum_{L=1}^{2J} \sum_{M} (2L+1) \left| t_L^M \right|^2 \leq 2J \quad . \tag{28}$$

As was pointed out some years ago by Lee and Yang[8], these restrictions allow for a determination of a lower bound for J. When ρ represents an incoherent superposition of Q pure states (e.g., if X is produced in $\pi + p \to X + K$ with unpolarized target), and $Q < 2J+1$, not all t_L^M may vanish. Capps' generalization[1] of the Eberhard-Good theorem Q Trace $\rho^2 \geq 1$ (the proof of this theorem is given in Appendix C) yields

$$Q \sum_{L=1}^{2J} \sum_{M} (2L+1) \left| t_L^M \right|^2 \geq 2J+1-Q \quad . \tag{29}$$

The range of each t_L^M is bounded by Eq. (22) and the fact that the eigenvalues of ρ are less than or equal to one. So, for the diagonal matrices T_L^o, one has

$$\text{minimum eigenvalue of } T_L^o \leq t_L^o \leq \text{maximum eigenvalue of } T_L^o \quad . \tag{30}$$

For $M \neq 0$, one may form the Hermitian matrices

$$R_L^M = \left(-\frac{1}{2}\right)\left(T_L^M + T_L^{M+}\right) \text{ and } I_L^M = \left(\frac{1}{2i}\right)\left(T_L^M - T_L^{M+}\right) , \tag{31}$$

and using the notation r_L^M and j_L^M for the eigenvalues of R and I, one has

$$\begin{aligned} \left(r_L^M\right)_{min} &\leq \text{Re } t_L^M \leq \left(r_L^M\right)_{max} \\ \left(j_L^M\right)_{min} &\leq \text{Im } t_L^M \leq \left(j_L^M\right)_{max} \quad . \end{aligned} \tag{32}$$

Owing to restrictions such as Eqs. (27) and (28), the t_L^M do not vary independently over these ranges; they cannot simultaneously attain the maximal magnitudes quoted above.

Additional restrictions have been given by Henry and de Rafaele[6] and Ademollo and Gatto[1].

VI. CONSEQUENCES OF SYMMETRIES IN THE PRODUCTION PROCESS AND THE RELATION OF THE MULTIPOLE PARAMETERS TO PRODUCTION MATRIX ELEMENTS

Consequences of space-time symmetries in the production of X may be deduced with ease from Eq. (16). We shall now give some examples.

If X is produced in a two-body reaction like Eq. (26) with unpolarized target, many z_L^M vanish for forward and backward productions, since for $\vartheta = 0$ or π, the system has axial symmetry about the beam direction $\hat{u}$. Therefore, the z_L^M must be symmetric with respect to rotations about $\hat{u}$. Taking the beam direction as polar axis in the rest frame of X, one has

$$z_L^M = 0 \text{ for } M \neq 0 \text{ when } \vartheta = 0 \text{ or } \pi; \quad (\text{polar axis } \hat{u}) \; . \qquad (33)$$

In addition, parity conservation in Eq. (26) gives t_L^M which are invariant against spatial inversions and then it follows that

$$z_L^M = 0 \text{ for } L \text{ odd when } \vartheta = 0 \text{ or } \pi; \qquad (34)$$

this follows because the z_L^M must be symmetric with respect to rotations of 180° about any axis perpendicular to $\vec{u}$, and $\mathcal{D}^{(L)}_{MM'}(0, \pi, 0) = (-)^{L-M}\, \delta_{-MM'}$. The relation (34) is the generalization of the well-known fact that, owing to parity conservation, polarizations [L = 1] vanish in the forward and backward directions.

Another example of the use of Eq. (16) to deduce restrictions on the z_L^M is the derivation of the "checkerboard" theorem of Capps[1]. If parity is conserved in Eq. (26), the collection of X (at any angle) has the symmetry of spatial inversion followed by rotation of 180° about the normal $\hat{n}$ to the production plane[12]. Consequently, the z_L^M must be invariant against rotations of 180° about $\hat{n}$, and with $\hat{n}$ as polar axis, Eq. (16) yields $e^{iM\pi}\, z_L^M = z_L^M$. Thus, one has

$$z_L^M = 0 \text{ for } M \text{ odd (polar axis } \hat{n}) \ . \tag{35}$$

Referring to Eq. (19), one sees that, in the representation Eq. (21), ρ must be of the form of a checkerboard with zero and (possibly) non-zero entries when Eq. (35) holds.

When X is produced in weak interactions, one may deduce restrictions on the z_L^M owing to time reversal invariance of H_{weak}. If interactions between the particles in the final state of the production reaction are negligible, to lowest order in H_{weak}, one has (see Section VI)

$$z_L^M(\vec{u}, \vec{X}) = (-)^L\ z_L^M(-\vec{u}, -\vec{X}) \tag{36}$$

when one sums over the spins, etc., of the other particles in the final state (in this case, there are only two vectors upon which the z_L^M can depend - the incident beam direction $\vec{u}$, and $\vec{X}$). If $\vec{n} = \vec{u} \times \vec{X}$ is taken as polar axis and Eq. (16) is evaluated for a rotation of 180° about $\vec{n}$ (since then $\vec{u} \rightarrow -\vec{u}$ and $\vec{X} \rightarrow -\vec{X}$), one has

$$z_L^M(-\vec{u}, -\vec{X}) = e^{iM\pi}\ z_L^M(\vec{u}, \vec{X}) \tag{37}$$

and thus, the result that

$$z_L^M = 0 \text{ when } L + M \text{ is odd (polar axis } \vec{n}) \ . \tag{38}$$

In particular, for L = 1 and a two-body production reaction, this result states that the component of the polarization vector of X along the normal to the production plane must vanish[13].

The above restrictions on multipole parameters owing to the production symmetries may be generalized to three-body final states. For reactions like

$$K^- + p \rightarrow X + K^0 + K^+ \qquad (3')$$

the z_L^M depend upon ϑ_K, φ_K as well as X production angles. We define ϑ_K, φ_K as the spherical angles of K^0 production measured in the rest frame of K^0K^+, which is obtained from the X rest frame by a Lorentz transformation without rotation along $\vec{X}$ [the momentum of X in the c.m. of Eq. (3')]. The generalizations of Eqs. (33) to (35) may be expressed in terms of $z_L^M(L', M')$ defined by ($d\Omega_K = d\cos\vartheta_K \, d\varphi_K$)

$$z_L^M(L', M') = \int d\Omega_K \, z_L^M \mathcal{D}^{(L')}_{M'0}(\varphi_K, \vartheta_K, 0) \quad . \qquad (39)$$

Since $\mathcal{D}^{(\ell)*}_{m0}(\varphi, \vartheta, 0) = \sqrt{4\pi/2\ell+1)}\, \mathcal{Y}_\ell^m(\vartheta, \varphi)$, corresponding to Eq. (33) is the relation

$$z_L^M = (4\pi)^{-1} \sum_{L', M'} (2\ell + 1) \; z_L^M(L', M') \mathcal{D}^{(L')}_{M'0}(\varphi_K, \vartheta_K, 0)^* \quad . \qquad (40)$$

The generalization of Eq. (35) is

$$z_L^M(\ell, m) = 0 \text{ when } \ell + m + M \text{ is odd (polar axis along } \hat{n}) \text{ .} \qquad (41)$$

Similarly for Eq. (33), one has

$$z_L^M(\ell, m) \rightarrow 0 \text{ as } \vartheta \rightarrow 0 \text{ or } \pi, \text{ for } M + m \neq 0 \text{ (polar axis along } \hat{u}) \qquad (42)$$

and when parity is conserved,

$$z_L^M(\ell, m) \rightarrow 0 \text{ as } \vartheta \rightarrow 0 \text{ or } \pi \text{ for } L + \ell \text{ odd .} \qquad (43)$$

To show explicitly the relation of the z_L^M to production S-matrix elements, for simplicity, we first give the relations when X is produced in a two-body reaction where in the initial and in the final state only one particle has spin; j and J, respectively. In this case, there are $(2j+1)\,(2J+1)$ helicity amplitudes[14] $f_{\lambda_X;\lambda}(\vartheta,\varphi)$. For discussing the spin orientation of X, however, the canonical spin representation[15] is more convenient. In this representation, the relativistic S matrix elements are given in terms of the projection of spin of the particles along, for example, the normal $\hat{n}$ to the production plane[22]. Since cross-sections, etc., depend on the azimuthal production angle φ only when the target is polarized, we shall use amplitudes which are not explicit functions of φ; they are related to helicity amplitudes by[16]

$$S_{m;\alpha}(\vartheta) = \sum_{\lambda_X,\lambda} \mathcal{D}^{(J)*}_{\lambda_X m} \mathcal{D}^{(j)}_{\lambda\alpha}\, e^{-i\lambda_X\varphi} f_{\lambda_X;\lambda}(\vartheta,\varphi) \quad , \tag{44}$$

where m and α are the projections of the spins along $\hat{n}$ and the arguments of the $\mathcal{D}$ functions are such that they rotate space axes (in the rest frame of each particle) from the helicity frame (z axis along c.m. momentum of particle, y along $\hat{n}$) to z axis along $\hat{n}$ and x axis along the c.m. momentum of the particle. The normalization is such that the total cross-section for production with unpolarized target is

$$\sigma_T = 2\pi\int_{-1}^{1} d\cos\vartheta \sum_{m,\alpha} |S_{m;\alpha}|^2 \Big/ (2j+1) \quad . \tag{45}$$

The density matrix (ρ) for productions with unpolarized targets (see next section for discussion of productions from polarized targets), is given by

$$\rho_{mm'} = \sum_{\alpha} S_{m;\alpha}\, S^*_{m';\alpha} \Big/ (2j+1) \; . \tag{46}$$

From the transformation properties of $\mathcal{D}^{(\ell)}_{mo}$ one sees that the analogue of Eq. (48) is

$$S^{(\ell)}_{m\mu;\alpha}(\vartheta) = \eta(-)^{J-\mu+\ell-m-j+\alpha} \, S^{(\ell)}_{m\mu;\alpha}(\vartheta) \; . \qquad (55)$$

We shall use this relation in the next section where we shall show how for Eq. (50) η may be measured.

VII. POLARIZED TARGETS

In this section, we shall assume the incident particles A have spin 0, the target particles p have spin ½, and the polarization of the target $\vec{P}$ is normal to the beam direction $\hat{u}$.

Corresponding to the canonical spin representation used in the previous section, the target polarization is described by the density matrix $\rho^{(i)}$ where

$$\rho^{(i)} = \tfrac{1}{2}(1 - P \sin\varphi\sigma_3 + P \cos\varphi\sigma_2) \tag{56}$$

where σ_i are usual Pauli spin matrices and φ is the azimuthal angle of X measured in the c.m. from $\vec{P}$ in a right-hand sense about $\hat{u}$; i.e.,

$$\begin{aligned} -P \sin\varphi &= \hat{n}\cdot\vec{P} \\ P\cos\varphi &= \hat{X}\cdot\vec{P}/\sin\vartheta \quad . \end{aligned} \tag{57}$$

In place of Eq. (46), one has now

$$\rho = S\rho^{(i)} S^{\dagger} \tag{58}$$

and the matrix elements have the form

$$\rho_{mm'} = a_{mm'} - P\sin\varphi\, b_{mm'} + P\cos\varphi\, c_{mm'} \quad , \tag{59}$$

where

$$\begin{aligned} a_{mm'} &= \tfrac{1}{2}\left(S_{m;+} S^*_{m';+} + S_{m;-} S^*_{m';-}\right) \\ b_{mm'} &= \tfrac{1}{2}\left(S_{m;+} S^*_{m';+} - S_{m;-} S^*_{m';-}\right) \\ c_{mm'} &= \frac{1}{2i}\left(S_{m;+} S^*_{m';-} - S_{m;-} S^*_{m';+}\right) \end{aligned} \tag{60}$$

$$S_{m;+} = S_{m;\frac{1}{2}},\ S_{m-} = S_{m;-\frac{1}{2}} \ .$$

(Note that one may solve for all the production amplitudes given the matrices a and b and one matrix element of c.) The a matrix gives the angular correlations $[z^M_L(P = 0)]$ one measures with unpolarized target;

the b matrix the right-left asymmetry of these correlations, and c the up-down asymmetry; i.e., if $(z_L^M)_{right}$ is the average of z_L^M for all events produced to the right looking down stream along the incident beam with $\vec{P}$ up

$$\left(t_L^M\right)_{right} - \left(t_L^M\right)_{left} = \left(\frac{4}{\pi}\right) P\, b_L^M \tag{61}$$

and, similarly,

$$\left(t_L^M\right)_{up} - \left(t_L^M\right)_{down} = \left(\frac{4}{\pi}\right) P\, c_L^M \tag{62}$$

where [see Eq. (21)]

$$b_L^M = \mathrm{Trace}\left(b\, T_L^M\right) \quad \text{and} \quad c_L^M = \mathrm{Trace}\left(c\, T_L^M\right) \quad . \tag{63}$$

If parity is conserved, either S_{m+} or S_{m-} vanishes [see Eq. (48)]. In this case, the only new information the b matrix contains is the relative parity η.

The relative parity η may be determined with polarized target because a polarized target always induces a right-left asymmetric spin orientation of X whose sign yields η. To see this, note that one has, from Eqs. (60) and (48)

$$b_L^M = \eta\ \mathrm{Trace}\left(a\, Y T_L^M\right) \quad , \tag{64}$$

where the diagonal matrix Y has elements

$$Y_{mm'} = (-)^{J-m}\, \delta_{mm'} \quad . \tag{65}$$

Since any $(2J+1) \times (2J+1)$ matrix can be expanded as a sum of T_L^M matrices, one has, remembering that $(T_L^M)_{mm'} = 0$ for $m \neq m' + M$,

$$Y T_L^M = \sum_K (2K+1)\ y_{LM;K}\ T_K^M \quad , \tag{66}$$

with

$$y_{LM;K} = (2J+1)^{-1}\ \mathrm{Trace}\left(YT_L^M\ T_K^{M\dagger}\right) . \qquad (67)$$

Now since $Y^{-1} = Y$, Eq. (66) gives

$$T_L^M = \sum_K (2K+1)\ y_{LM;K}\ YT_K^M$$

and one obtains

$$\eta\ a_L^M = \sum_K (2K+1)\ y_{LM;K}\ b_K^M . \qquad (68)$$

This shows that, for each non-vanishing moment obtained with unpolarized target, there is a corresponding right-left asymmetric spin orientation when polarized target is used: in every case there will be some right-left asymmetric spin orientation, since for L = M = 0 Eq. (68) gives

$$\eta \left(\frac{d\sigma}{d\Omega}\right)_{P=0} = \sum_K (2K+1)\ y_K\ b_K^o \qquad (69)$$

or, using Eq. (61),

$$\eta P \left(\frac{d\sigma}{d\Omega}\right)_{P=0} = \frac{\pi}{4} \sum_K (2K+1)\ y_K \left[\left(t_K^o\right)_{right} - \left(t_K^o\right)_{left}\right] \qquad (70)$$

where

$$y_K = (2J+1)^{-1}\ \mathrm{Trace}\left(YT_K^o\right) = (2J+1)^{-1} \sum_m (-)^{J-m}\ C(JKJ;m o) . \qquad (71)$$

(Note that $y_K = 0$ for K+ 2J odd.) When $X \rightarrow \Lambda + K$, the t_K^o with K odd are moments of Λ polarization. Therefore, one sees that the Λ will be right-left asymmetrically polarized when X is produced with a polarized target.

In terms of the angular correlation function F, one has, from Eqs. (70) and (14)

$$\eta P = \frac{\pi}{4}\left(<f>_{\text{right}} - <f>_{\text{left}}\right) \tag{72}$$

where[19])

$$f = \frac{3}{\alpha_\Lambda} \sum_K (2K+1)^2 \; (y_K/m_{JK}) \; \cos\Psi \mathcal{D}^{(K)}_{00}(\varphi_\Lambda, \vartheta_\Lambda, 0) \; . \tag{73}$$

Equation (70) is the generalization for $J > \frac{1}{2}$ of the well-known relation[20]) for $J = \frac{1}{2}$ which states that the right-left asymmetry of polarization of X produced with polarized target yields η.

If some spin orientation is found when X is produced with unpolarized target, then Eq. (68) with $L \neq 0$, may also be used to obtain η. In particular, one may use the inverted form

$$b^0_0 = \eta \sum_K (2K+1)\; y_K\, a^0_K \tag{74}$$

which can be obtained directly from Eqs. (64) and (66). This gives the right-left asymmetry in number of X produced with polarized target, namely

$$\frac{N_R - N_L}{N_R + N_L} = \frac{2}{\pi}\,\eta P\left(<f>_{\text{right}} + <f>_{\text{left}}\right) = \frac{4}{\pi}\,\eta P <f>_{P=0} \tag{75}$$

and is the generalization for $J \geq \frac{1}{2}$ of the relation between the polarization of X when produced with unpolarized target and the right-left asymmetry in the number of X produced with polarized target.

When parity is conserved in the production reaction, the same considerations as lead to Eq. (35) applied to Eq. (59) yield the restrictions

$$\begin{aligned} a^M_L = b^M_L &= 0 \text{ when M is odd (polar axis along } \hat{n}) \\ c^M_L &= 0 \text{ when M is even (polar axis along } \hat{n}). \end{aligned} \tag{76}$$

For neutrino reactions with polarized target, Eq. (59) and similar considerations as led to Eq. (37) yield

$$\begin{aligned} a_L^M &= 0 \text{ when } L+M \text{ is odd (polar axis along } \hat{n}) \\ b_L^M = c_L^M &= 0 \text{ when } L+M \text{ is even (polar axis along } \hat{n}) \end{aligned} \tag{77}$$

when H_W is time reversal invariant.

For forward or backward productions with a polarized target, one may choose as polar axis $\hat{n}$ any normal to $\hat{u}$; then the z_L^M also have the form (59), namely

$$z_L^M = a_L^M + \vec{P}\cdot\hat{n}\, b_L^M + \vec{P}\cdot\hat{n}\times\hat{u}\, c_L^M \quad . \tag{78}$$

A rotation of axes 180° about $\hat{u}$ yields $z_L^M(-\vec{P})$ so since

$$R_{\hat{u}}(\pi)\; z_L^M = (-)^{L+M}\; z_L^M \tag{79}$$

one has

$$(-)^{L+M}\; z_L^M(\vec{P}) = z_L^M(-\vec{P})$$

and

$$\begin{aligned} a_L^M &= 0 \text{ for } L+M \text{ odd (polar axis } \perp \hat{u}) \\ b_L^M = c_L^M &= 0 \text{ for } L+M \text{ even (polar axis } \perp \hat{u}) \end{aligned} \tag{80}$$

when

$$\vartheta = 0 \text{ or } \pi \quad .$$

Similarly from a 90° rotation about $\hat{u}$, one may obtain equations relating b_L^M to c_L^M. (Arbitrary rotations about $\hat{u}$ yield restrictions on M for non-vanishing a, b, c.)

Parity conservation requires, taking $\hat{n}$ along $\vec{P}$ and considering a rotation of 180° about $\vec{P}$, $z_L^M = 0$ for M odd when $\vartheta = 0$ or π.

The matrix c depends upon the relative phases of $S_{m;+}$ and $S_{m';-}$. If a and b are known, c yields only one new parameter. To see this explicitly, first consider the case of parity conservation. Then, for fixed m, either $S_{m;+}$ or $S_{m;-}$ vanishes. For example, if $\eta = +1$, $S_{J;-}$, $S_{J-1;+}$, etc. vanish and the matrix elements of a give $|S_{J;+}|$, $|S_{J-1;-}|$, etc., the relative phases of all amplitudes with $\alpha = +\frac{1}{2}$, and the relative phases of all amplitudes with $\alpha = -\frac{1}{2}$. Thus, aside from an over-all phase, all $S_{m;\alpha}$ are determined if in addition one relative phase such as $\arg\{S_{J;+}\, S^{*}_{J-1;-}\} = X$ is known. The multipole parameters of c depend on X and a in a very simple way. If, for example, the two diagonal elements of a, a_{JJ} and $a_{J-1,J-1}$ are different from zero, from Eq. (60) [and Eq. (48)] one sees that

$$c_L^M = -i\eta\, e^{iX}\left(aT_L^M a\right)_{J-1} \Big/ \sqrt{a_{JJ}\, a_{J-1,\,J-1}} \tag{81}$$

where

$$X = \arg S_{J;+}\, S^{*}_{J-1;-} \qquad \text{if } \eta = +1$$

$$= \arg S_{J;-}\, S^{*}_{J-1;+} \qquad \text{if } \eta = -1 \quad .$$

Consequently, if a is known, any non-vanishing up-down asymmetry of F suffices to determine X. [Owing to Eq. (76), angular correlations which exhibit an up-down asymmetry depend upon φ_Λ.]

For parity violation in production of X, both a and b are needed to determine the magnitudes of the production amplitudes, and the relative phases of amplitudes with $\alpha = +\frac{1}{2}$ and those with $\alpha = -\frac{1}{2}$. Assuming a and b are known, one has

$$\begin{aligned} S_{m;+}\, S^{*}_{m';+} &= a_{mm'} + b_{mm'} \\ S_{m;-}\, S^{*}_{m';-} &= a_{mm'} - b_{mm'} \end{aligned} \tag{82}$$

and c is needed to obtain the relative phase of $S_{m;-}$ and $S_{m;+}$. If $S_{J;+}$ and $S_{J;-}$ are both different from zero, let $X_J = \arg\{S_{J;+}\, S^{*}_{J;-}\}$. If X_J is known, all relative phases are determined. Using Eq. (60) one sees that

$$c_L^M = \frac{1}{2i} \frac{e^{iX_J}[(a-b)\ T_L^M(a+b)]_{JJ} - e^{iX_J}[(a+b)\ T_L^M(a-b)]_{JJ}}{\sqrt{(a-b)_{JJ}\ (a+b)_{JJ}}} \quad . \quad (83)$$

[If $S_{J;+}$ or $S_{J;-}$ is zero, of course, one would choose for X the relative phase of some other pair and obtain Eq. (83) with appropriately modified indices. For example, using J and J - 1 one obtains the generalization of Eq. (81).] In this case also, only one non-vanishing up-down angular correlation asymmetry suffices to complete the determination of the $S_{m;\alpha}$ matrix.

Note that, owing to Eq. (64), there are many relations [like Eq. (72) and Eq. (75)] that relate angular correlations with $P = 0$ to asymmetric angular correlations when $P \neq 0$. In general, these are J dependent and may be used to determine J [21).]

APPENDIX A

DERIVATION OF GENERAL FORM OF
COMPLETE CORRELATION FUNCTION F

Derivation of Eqs. (5) and (6)[24]:

We shall use helicity states[14] to describe the $\Lambda - \pi$ system. The density matrix for Λ in the usual notation is

$$\rho^{f} = \frac{I}{2}\,(1 + \vec{P}\cdot\vec{\sigma}) \tag{A.1}$$

where $\vec{P} = \vec{P}_{\Lambda}$. To define symbols, we evaluate ρ^{f} using for $\vec{\sigma}$ the usual Pauli spinors

$$\rho^{f} = \frac{1}{2}\begin{pmatrix} I + I\,P_0 & \sqrt{2}\,I\,P_{-1} \\ -\sqrt{2}\,I\,P_{+1} & I - I\,P_0 \end{pmatrix} \quad ; \tag{A.2}$$

where $P_0 = P_Z$, $\sqrt{2}\,P_{\pm 1} = \mp(P_X \pm iP_Y)$ and the components of $\vec{P}$ refer to any co-ordinate system S, which is specified independently of the decay vector. However, if Eq. (A.1) is evaluated in the helicity state representation, the components of $\vec{P}$ in Eq. (A.2) refer to a co-ordinate system S′. It is obtained by rotating S through the Euler angles $(\varphi_\Lambda, \vartheta_\Lambda, 0)$ where ϑ_Λ and φ_Λ are the spherical angles of $\vec{\Lambda}$ measured in S. Here we shall use helicity states which differ from those defined in Ref. 14) by the phase factor $e^{i\lambda\varphi}$ [see G.C. Wick, Ann.Phys. 18, 65 (1962)]. In this helicity representation

$$(\rho^{f})_{\lambda\lambda'} = \frac{1}{2}\begin{pmatrix} I + I\,\vec{P}\cdot\hat{\Lambda} & \sqrt{2}\,P_{-} \\ -\sqrt{2}\,P_{+} & I - I\,\vec{P}\cdot\hat{\Lambda} \end{pmatrix} \tag{A.3}$$

where

$$P_{-} = -\,P_{+}^{*} \tag{A.4}$$

and

$$P_{+} = \sum_{m} \mathcal{D}^{(1)}_{m1}(\varphi_\Lambda, \vartheta_\Lambda, 0)\; P_m \tag{A.5}$$

$\mathcal{D}^{(1)}_{mm'}(\varphi_\Lambda, \vartheta_\Lambda, 0)$ is the rotation matrix element for the rotation of axes $S \rightarrow S'$.

The final state's density ρ^f is related to the density matrix of X by

$$\rho^f = S \rho S^\dagger \tag{A.6}$$

where S is the transition matrix for $X \to \Lambda + \pi$. In the X rest frame, the transition from a state with quantum numbers JM to a $\Lambda\pi$ state with helicity λ is given by[14)]

$$S_{\lambda M} = \sqrt{\frac{2J+1}{4\pi}}\, \mathcal{D}^{(J)}_{M\lambda}(\varphi, \vartheta, 0)^* A_\lambda \tag{A.7}$$

where A_λ is defined in Section III. Substituting Eq. (A.7) in Eq. (A.6) and using Eqs. (19) and (21), one obtains

$$\rho^f_{\lambda\lambda'} = \frac{A_\lambda A^*_{\lambda'}}{4\pi} \sum_{\substack{L,M'' \\ M,M'}} (2L+1)\, t^{M''^*}_L\, C\,(JLJ;MM'')\, \mathcal{D}^{(J)^*}_{M\lambda}\, \mathcal{D}^{(J)}_{M'\lambda'} \quad . \tag{A.8}$$

Using the relation[14)]

$$\mathcal{D}^{(J)}_{M\lambda}(\varphi, \vartheta, 0)^*\, \mathcal{D}^{(J)}_{M'\lambda'}(\varphi, \vartheta, 0)$$

$$= \sum_{\ell} C\,(JJ\ell;M-M')\, C\,(JJ\ell;\lambda-\lambda')(-)^{M'-\lambda'}\, \mathcal{D}^{(\ell)}_{M-M',\lambda-\lambda'}(\varphi, \vartheta, 0)^* \tag{A.9}$$

and the relation $(M' + M'' = M)$

$$C\,(JLJ;M',M'') = (-)^{J+M'+1} \sqrt{\frac{2J+1}{2L+1}}\, C\,(JJL;M,-M')$$

and the orthogonality of the Clebsch-Gordan coefficients, one obtains

$$\rho^{(f)}_{\lambda\lambda'} = (-)^{J-\lambda} \left(\frac{A_\lambda A^*_{\lambda'}}{4\pi}\right) \sqrt{2J+1} \sum_{L,M} \sqrt{2L+1}\, C\,(JJL',\lambda,-\lambda')\, t^{M^*}_L\, \mathcal{D}^{(L)^*}_{M,\lambda-\lambda'} \quad . \tag{A.10}$$

Note that in Eqs. (A.10), (A.11) and (A.13), $\mathcal{D}^{(L)}_{MN} \equiv \mathcal{D}^{(L)}_{MN}(\varphi_\Lambda, \vartheta_\Lambda, 0)$. Using Eqs. (A.3), (A.10) and (17) and the relations

$$C(JJL;\tfrac{1}{2},-\tfrac{1}{2}) = (-)^{2J+L}\, C(JJL;-\tfrac{1}{2},\tfrac{1}{2})$$

$$\alpha_{X\to\lambda} = |A_{1/2}|^2 - |A_{-1/2}|^2$$

one obtains

$$I\vec{P}_\Lambda \cdot \hat{\Lambda} = \rho^{(f)}_{\frac{1}{2}\frac{1}{2}} - \rho^{(f)}_{-\frac{1}{2}-\frac{1}{2}} = \alpha_{X\to\Lambda} \sum_{\substack{L(\text{even})\\ M}} z_L^M\, \mathcal{D}^{(L)}_{M0} + \sum_{\substack{L(\text{odd})\\ M}} z_L^M\, \mathcal{D}^{(L)}_{M0} \qquad (A.11)$$

since

$$z_L^{M^*} = (-)^M z_L^{-M} \text{ and } \mathcal{D}^{(L)^*}_{M0} = (-)^M \mathcal{D}^{(L)}_{-M0} \quad .$$

For the off diagonal element $\rho^f_{\frac{1}{2},-\frac{1}{2}}$ using

$$\beta_{X\to\Lambda} = 2 \text{ Im } A_{1/2} A^*_{-1/2}$$

$$\gamma_{X\to\Lambda} = 2 \text{ Re } A_{1/2} A^*_{-1/2}$$

and

$$\sqrt{L(L+1)}\; C(JJL;\tfrac{1}{2}\tfrac{1}{2}) = (2J+1)\; C(JJL;\tfrac{1}{2},-\tfrac{1}{2}) \text{ when L is odd} \qquad (A.12)$$

one has

$$\rho^{(f)}_{\frac{1}{2},-\frac{1}{2}} = -\frac{2J+1}{2}\,(\gamma_{X\to\Lambda} + i\beta_{X\to\Lambda}) \sum_{L,M} [L(L+1)]^{-\frac{1}{2}}\, \mathcal{D}^{(L)^*}_{M1}\, z_L^{M^*} \quad . \qquad (A.13)$$

Thus $\rho^{(f)}$ is given in terms of z_L^M. With Eq. (A.2), one may evaluate $I\vec{P}_\Lambda \cdot \hat{p}$ from Eqs. (A.13) and (A.11) as using

$$I\vec{P}_\Lambda \cdot \hat{p} = \cos\Psi\left(\rho^{(f)}_{\frac{1}{2}\frac{1}{2}} - \rho^{(f)}_{-\frac{1}{2}-\frac{1}{2}}\right) + \sin\Psi\left(\rho^{(f)}_{\frac{1}{2},-\frac{1}{2}}\, e^{i\Phi} + \rho^{(f)}_{-\frac{1}{2},\frac{1}{2}}\, e^{-i\Phi}\right) \quad (A.14)$$

where Ψ and Φ, defined in Section III, are spherical angles of $\hat{p}$ in the co-ordinate system of Eq. (A.2). Incorporating the factors $e^{i\Phi}$ in Eq. (A.14) into the $\mathcal{D}_{M1}^{(L)^*}$ in Eq. (A.13) and using the relation

$$\mathcal{D}_{M1}^{(L)}(\varphi_\Lambda, \vartheta_\Lambda, \Phi)^* = -(-)^M \mathcal{D}_{-M,-1}^{(L)}(\varphi_\Lambda, \vartheta_\Lambda, \Phi) \qquad (A.15)$$

and $z_L^{M^*} = (-)^M z_L^{-M}$, one obtains for $F = I + \alpha_\Lambda I \vec{P}_\Lambda \cdot \hat{p}$ the form (6) and (5) when $\alpha_{X\to\Lambda} = \beta_{X\to\Lambda} = 0$.

APPENDIX B

STATISTICAL ERROR ANALYSIS FOR EXPERIMENTAL DETERMINATIONS OF SPIN, DENSITY MATRIX ELEMENTS, ETC.

The structure of F shows that J, z_L^M and α, β, γ may be completely determined. Generally maximum likelihood techniques can be used to obtain "best" values. However, if the number of events N is sufficiently large, the right-hand side of Eq. (8) may be evaluated by taking the indicated average in the data[8,25]; i.e., for any (real) function $f_\alpha(\vartheta_\Lambda, \varphi_\Lambda, \Psi, \Phi)$, let

$$f_{\alpha i} = f_\alpha(\vartheta_{\Lambda i}, \varphi_{\Lambda i}, \Psi_i, \Phi_i) \tag{B.1}$$

where $\vartheta_{\Lambda i}, \varphi_{\Lambda i}, \Psi_i, \Phi_i$ are the observed angles in the i^{th} event. If there is no systematic error, one may take $z_0^o = N$ and

$$< f_\alpha > = \left(\frac{1}{N}\right) \sum_{i=1}^{N} f_{\alpha i} \pm \sqrt{\left(\frac{1}{N}\right) \sum_{i=1}^{N} f_{\alpha i}^2 - \left[\left(\frac{1}{N}\right) \sum_{i=1}^{N} f_{\alpha i}\right]^2} \quad . \tag{B.2}$$

However, if one needs to evaluate a set of moments from one set of data, the associated errors are correlated. Taking

$$< f_\alpha > = \left(\frac{1}{N}\right) \sum_{i=1}^{N} f_{\alpha i} \quad ,$$

one may estimate errors by evaluating the error matrix ϵ whose elements are

$$\epsilon_{\alpha\beta} = \left(\frac{1}{N}\right) \sum_{i=1}^{N} f_{\alpha i} f_{\beta i} - \left(\frac{1}{N}\right)^2 \sum_{i=1}^{N} \sum_{j=1}^{N} f_{\alpha i} f_{\beta j} \tag{B.3}$$

[$\sqrt{\epsilon_{\alpha\alpha}}$ is the estimated error given in Eq. (B.2)].

If $\alpha_{X\to\Lambda}$ and J are known and $\alpha_{X\to\Lambda} \neq 0$, one need only evaluate the $< \mathcal{D}_{M0}^{(L)} >$ to obtain the X density matrix. In this case, the angular distribution I of Λ yields the entire density matrix of X, and an over-all measure of the error can be obtained from the function I′ where

$$I' \equiv \sum_{L,M} (2L+1) \langle \mathcal{D}^{(L)}_{M0} \rangle^{*} \mathcal{D}^{(L)}_{M0}(\varphi_\Lambda, \vartheta_\Lambda, 0) \quad . \qquad (B.4)$$

To do this, one may compute $\sqrt{N}$ times the ratio of the deviation of the logarithm of the likelihood function $\mathcal{L}$ to its variance; i.e.,

$$\sqrt{\frac{(\langle \ln \mathcal{L} \rangle - \overline{\ln \mathcal{L}})^2}{N\sigma^2}} \qquad (B.5)$$

where

$$\langle \ln \mathcal{L} \rangle = \left(\frac{1}{N}\right) \sum_{i=1}^{N} \ln I'_i$$

$$\overline{\ln \mathcal{L}} = \left(\frac{1}{4\pi}\right) \int d\Omega_\Lambda \; I' \ln I'$$

$$\sigma^2 = \left(\frac{1}{N}\right) \sum_{i=1}^{N} (\ln I'_i - \overline{\ln \mathcal{L}})^2 \quad .$$

If the moments of I are accurately given by taking

$$\langle \mathcal{D}^{(L)}_{M0} \rangle = \left(\frac{1}{N}\right) \sum_{i} \mathcal{D}^{(L)}_{M0}(\varphi_{\Lambda i}, \vartheta_{\Lambda i}, 0) \quad , \qquad (B.6)$$

Eq. (B.5) will be small compared to one.

In the following, we shall assume N is sufficiently large and averages like Eq. (B.6) are good measures of the moments of F.

If J and the decay parameters are to be determined, one needs to measure the Λ polarization. We discuss the case $\alpha = \beta = 0$ (parity conservation in X decay) first. In this case, $\gamma_{X\to\Lambda} = \pm 1$ and one has, for all odd values of L and $-L \leq M \leq L$,

$$\gamma_{X\to\Lambda}(2J+1) \langle \cos\Psi \, \mathcal{D}^{(L)}_{M0} \rangle = \sqrt{L(L+1)} \langle \sin\Psi \, \mathcal{C}^{(L)}_{M1} \rangle \quad ; \qquad (B.7)$$

Eq.(B.7) is non-trivial only when the corresponding z_L^M is appreciable. Since $\gamma_{X\to\Lambda} = \pm 1$, it can be determined from Eq. (B.7) using usual methods [see Ref. 25)]. However, the experimental value of J obtained using relations (B.7) needs some further discussion.

The quantity 2J+1 derived from data will not be an integer, in general. A determination of J from this data may be obtained from consideration of the function $P_J(A)$ where $P_J(A)\, dA$ is the probability of finding the value A in the interval dA for 2J+1 given that the spin is J. It depends also on the values found for $\langle f_\alpha \rangle$ in the experiment and their associated errors.

We now outline a method for calculating $P_J(A)$. For definiteness, first consider the relation (B.7) with M = 0 [all quantities in Eq. (B.7) are real] and assume $\gamma_{X\to\Lambda}$ is known. Let Eq. (B.7) be represented by the equation

$$y = Ax \qquad \text{(B.8)}$$

where y and x are the experimental values of $\sqrt{L(L+1)} \times \langle \sin\Psi\, \mathcal{C}_{00}^{(L)} \rangle$ and $\langle \cos\Psi\, \mathcal{D}_{00}^{(L)} \rangle$, respectively. Assume the error in y and x is Gaussian distributed with standard deviations σ_y and σ_x as given in Eq. (B.2). [For simplicity, we assume x and y may take on any values between $-\infty$ and $+\infty$. Actually they are restricted by relations like (27), (28) and (32).] We shall ignore error correlations and then show how the result may be generalized to take correlated errors into account.

The error distributions in x and y are described by

$$\begin{aligned} \rho_L(x) &= [2\pi\sigma_x^2]^{-1/2}\, e^{-(x-x_0)^2/2\sigma_x^2} \\ \rho_T(x) &= [2\pi\sigma_y^2]^{-1/2}\, e^{-(y-y_0)^2/2\sigma_y^2} \,. \end{aligned} \qquad \text{(B.9)}$$

Owing to Eq. (B.7), one expects that $\sigma_y \approx (2J+1)\,\sigma_x$. Therefore, we calculate $P_J(A)$ in terms of J, x_0, σ_x and σ_y with:

$$y_0 = (2J+1)\, x_0$$

$$x_0 = \left(\frac{1}{N}\right) \sum_i \cos \Psi_i \left(\mathcal{D}\, {}^{(L)}_{00}\right)_i \qquad \text{(B.10)}$$

$$\sigma_x^2 = \left(\frac{1}{N}\right) \left\{ \left(\frac{1}{N}\right) \sum_i \left[\cos \Psi_i \left(\mathcal{D}\, {}^{(L)}_{00}\right)_i \right]^2 - \left[\left(\frac{1}{N}\right) \sum_i \cos \Psi_i \left(\mathcal{D}\, {}^{(L)}_{00}\right)_i \right]^2 \right\}$$

$$\sigma_y^2 = \left[\frac{L(L+1)}{N}\right] \left\{ \left(\frac{1}{N}\right) \sum_i \left[\sin \Psi_i \left(\mathcal{C}\, {}^{(L)}_{01}\right)_i \right]^2 - \left(\frac{1}{N}\right)^2 \left[\sum_i \sin \Psi_i \left(\mathcal{C}\, {}^{(L)}_{01}\right)_i \right]^2 \right\}$$

The joint probability of finding x and A in the intervals dx and dA is $P_J(A,x)\, dA\, dx$ where $P_J(A,x) = \mathcal{P}_L(x)\, \mathcal{P}_T(A,x)$. It then follows from Eq. (B.9) that

$$P_J(A) = (2\pi\, \sigma_x\, \sigma_y)^{-1} \int_{-\infty}^{\infty} dx \times e^{-\frac{1}{2}\left[(x-x_0)^2/\sigma_x^2 + (Ax-(2J+1)x_0)^2/\sigma_y^2\right]} . \qquad \text{(B.11)}$$

Carrying out the integration, one finds

$$P_J(A) = |x_0| \left[\frac{\sigma_y^2 + (2J+1)\, A\, \sigma_x^2)}{\sqrt{\pi}\,(\sigma_y^2 + A^2\, \sigma_x^2)^{3/2}}\right] \times e^{-(2J+1-A)^2 x_0^2 / 2(\sigma_y^2 + A^2 \sigma_x^2)} \qquad \text{(B.12)}$$

Note how A appears in the argument of the exponential, as a result $P_J(A)$ is skewed about its maximum. The average value of A in $P_J(A)$ tends to be larger than 2J + 1 while, as shown by Eq. (B.16), the maximum tends to occur at $A < 2J + 1$. Clearly, if σ_x^2 is small $P_J(A)$ is Gaussian with standard deviation σ_y/x_0. However, if σ_x^2 is appreciable compared with x_0^2, the skewed nature of $P_J(A)$ may be important. In this case, the various values of $P_J(A)$ for A = value given by Eq. (B.8) and $J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, etc. should be calculated.

To take error correlations into account, one may easily generalize Eq. (B.11). Let ϵ be the 2 × 2 error matrix (B.3) and, using matrix notations,

$$\mathcal{a} = \begin{pmatrix} 1 \\ A \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 1 \\ 2J+1 \end{pmatrix}, \quad \nu = x\,\mathcal{a} - x_0\,\mathcal{J} \qquad \text{(B.13)}$$

one may write Eq. (B.11) as

$$P_J(A) = \left(\frac{n_J}{\pi}\right) \int_{-\infty}^{\infty} dx \times e^{-\frac{1}{2}(V\epsilon^{-1}V)} \tag{B.14}$$

with ϵ^{-1} = inverse of ϵ, and

$$P_J(A) = \left(|x_0| \frac{n_J}{\sqrt{\pi}}\right) (a\epsilon^{-1}J)(a\epsilon^{-1}a)^{-3/2} e^{-(2J+1-A)^2 x_0^2/2(a\epsilon^{-1}a)} \det\epsilon \tag{B.15}$$

with det ϵ = determinant of ϵ.

When terms of order $(\epsilon_{\alpha\beta}/x_0^2)^2$ are neglected, the maximum of Eq. (B.15) occurs at

$$A \cong (2J+1)\left[1 - \frac{2}{x_0^2}\left\{\epsilon_{xx} - (2J+1)^{-1} \epsilon_{xy}\right\}\right] . \tag{B.16}$$

The skewness of $P_J(A)$ is considerable when $-\ln P_J(A) \approx (x_0^2/\epsilon_{xx})$.

If parity is violated in X decay, one must use relations like Eq. (15) to determine J[26]. In this case one has a relation of the form

$$y^2 = A^2 x^2 \tag{B.17}$$

where (taking M = 0)

$$x^2 = L(L+1) \left[< \sin\Psi\, S_{01}^{(L)} >^2 + < \sin\Psi\, C_{01}^{(L)} >^2\right] \tag{B.18}$$

and if $\alpha_{X\to\Lambda}$ is not already known,

$$y^2 = (1 - < f_0 >^2) < \cos\Psi\, \mathcal{D}_{00}^{(L)} >^2 \tag{B.19}$$

where $< f_0 > = (3/\alpha_\Lambda) < \cos\Psi >$. Thus from Eq. (B.17) a measured value A of 2J+1 may be obtained. To estimate the associated error, an error matrix ϵ for x and y can be calculated from Eq. (B.3). Letting $y^2 = (1 - < f_0 >^2) < f_3 >^2$, $x^2 = < f_1 >^2 + < f_2 >^2$, one has:

$$\epsilon_{yy} = y^{-2} \sum_{\substack{\alpha=1,2\\ \beta=1,2}} < f_\alpha > \epsilon_{\alpha\beta} < f_\beta >$$

$$\epsilon_{xy} = (xy)^{-1} \sum_{\alpha=1,2} \{(1 - < f_0 >^2) < f_3 > \epsilon_{3\alpha} < f_\alpha > - < f_3 >^2 < f_0 > \epsilon_{0\alpha} < f_\alpha >\} \qquad (B.20)$$

$$\epsilon_{xx} = x^{-2} < f_3 >^2 \{(1 - < f_0 >^2) \epsilon_{33} - 2(1 - < f_0 >^2) < f_0 > \epsilon_{03} < f_3 >$$

$$+ < f_3 >^2 < f_0 >^2 \epsilon_{00}\}$$

assuming normal distributions for the errors in $< f_\alpha >$. Then Eq. (B.15) is a reasonable estimate of $P_J(A)$ when $\sigma_x^2/x_0^2 < 1$ and

$$\frac{(A(2J+1)\ \sigma_x^2}{\sigma_y \sqrt{\sigma_y^2 + A_0^2\ x^2})^2} > \frac{\sigma_x^2}{x_0^2} \ .$$

Actually, the $P_J(A)$ for Eq. (B.17) differs from Eq. (B.15) owing to the restrictions $x^2 \geq 0$, $y^2 \geq 0$, $A^2 > 0$. These give rise to a different behaviour for $P_J(A)$ near $A = 0$, but do not change $P_J(A)$ appreciably when A is of order $2J+1$.

An over-all check on the reliability of the values used for $< f_\alpha >$ can be made by computing Eq. (B.5) for the complete function F.

APPENDIX C

PROOF OF GENERALIZED EBERHARD-GOOD THEOREM

When ρ describes an incoherent mixture of Q pure states,

$$\rho = \sum_{i=1}^{Q} \rho_i \quad \text{with} \quad \text{Trace } \rho_i^2 = (\text{Trace } \rho_i)^2 \tag{C.1}$$

and

$$\text{Trace } \rho^2 = \sum_i \text{Trace } \rho_i^2 + 2 \sum_{i<j} \text{Trace } \rho_i \rho_j \quad . \tag{C.2}$$

Since the diagonal elements of ρ_i are positive (or zero), Trace $\rho_i \rho_j \geq 0$ and

$$\text{Trace } \rho^2 \geq \sum_i \text{Trace } \rho_i^2 \quad . \tag{C.3}$$

The minimum the right-hand side of Eq.(C.3) may attain, consistent with Eq. (C.1), is when Trace $\rho_i = (1/Q)$ Trace ρ for all i. Therefore, one has

$$Q \text{ Trace } \rho^2 \geq (\text{Trace } \rho)^2 \quad . \tag{C.4}$$

APPENDIX D

DETERMINATION OF PRODUCTION AMPLITUDES FOR INTEGER SPIN

Consider a boson of spin ℓ which decays into K^+ and K^o-two spin zero particles. Let the production matrix elements be $S_m^{(\ell)}$; then the angular distribution of K^+ is given by

$$f(\vartheta_K, \varphi_K) = \left| \sum_m S_m^{(\ell)} \sqrt{2\ell+1}\, \mathcal{D}_{mo}^{(\ell)}(\varphi_K, \vartheta_K)^* \right|^2 \tag{D.1}$$

and, using Eq. (52) with

$$\begin{pmatrix} \ell\ell' L \\ mm' -M \end{pmatrix} = (-)^{\ell+\ell'+L} \begin{pmatrix} \ell L \ell' \\ m -Mm' \end{pmatrix} = (-)^{\ell-\ell'+M} [2\ell+1]^{-1/2} \; C(\ell\ell' L; mm') \;, \tag{D.2}$$

one has

$$f(\vartheta_K, \varphi_K) = \sum_{L,M} z_L^M \mathcal{D}_{MO}^{(L)}(\varphi_K, \vartheta_K, 0) \tag{D.3}$$

where

$$z_L^M = (-)^\ell \sqrt{(2L+1)(2\ell+1)}\; C(\ell\ell L;00)\; t_L^M \equiv m_{\ell L}\, t_L^M \tag{D.4}$$

[note the coefficient of t_L^M may be obtained from Eq. (18) replacing ½ by zero] and t_L^M is given by Eq. (22)

$$\rho_{mm'} = S_m^{(\ell)} S_{m'}^{(\ell)*} \quad . \tag{D.5}$$

Since $C(\ell\ell L;00)$ vanishes for L odd, only those t_L^M with L even are observable in this decay. Knowledge of the even multipole is, in general, sufficient to determine the production amplitudes $S_m^{(\ell)}$. To see this, note that

$C(\ell L\ell;m0) = C(\ell L\ell;-m0)$ for even L. Therefore, one has, using Eq. (19),

$$|S_m|^2 + |S_{-m}|^2 = \left(\frac{2}{2\ell+1}\right) \sum_{\substack{L=0 \\ (L\ \text{even})}}^{2\ell} (2L+1)\ C(\ell L\ell;m0)\ t_L^0 \quad . \tag{D.6}$$

The t_L^M with $M \neq 0$ will yield relative magnitudes and phases. [Note consequences of theorems such as Eqs. (35) and (48).]

APPENDIX E

ADDITIONAL DECAY CORRELATIONS WHEN TWO UNSTABLE PARTICLES ARE PRODUCED AND DETERMINATIONS OF PARITY AND SPIN OF SECOND PARTICLE

We now consider the reaction

$$\pi + p \to X + B \qquad \text{(E.1)}$$
$$\quad\quad\quad\quad \downarrow_{\to} K^+ + K^0$$

where B has spin ℓ. In this case, the density matrix for X depends upon ϑ_K, φ_K (X-B spin-spin correlations). It has the form (53) with $L' = 0,2,\ldots,2\ell$. The matrices $\rho^{(L',M')}$

$$\rho^{(L',M')} = a^{(L',M')} - P \sin\varphi\, b^{(L',M')} + P \cos\varphi\, c^{(L',M')} \qquad \text{(E.2)}$$

when the target is polarized [see Section VII; particularly, Eqs. (59) and (60)]. The corresponding multipole parameters of X are given by Eq. (39).

The production matrix elements $S^{(\ell)}_{m\mu;\alpha}$ $(-\ell \le m \le \ell,\ -J \le \mu \le J,\ \alpha = \pm \frac{1}{2})$ obey[12]

$$S^{(\ell)}_{m\mu;\alpha}(\vartheta) = \eta(-)^{\ell-m+J-\mu-\frac{1}{2}+\alpha}\, S^{(\ell)}_{m\mu;\alpha}(\vartheta) \qquad \text{(E.3)}$$

when parity is conserved (η = product of intrinsic parities). To see how η may be determined, one follows the same procedure as in Section VII. For each μ and $|m|$, from Eq. (E.3), $S^{(\ell)}_{m\mu;\alpha} = 0$ if $\alpha = \frac{1}{2}$ or $-\frac{1}{2}$. Using Eq. (D.6), let (note $c_L = d_L = 0$ for L odd)

$$c_L = \left[\frac{(2L+1)}{(2\ell+1)}\right] \sum_m C(\ell L \ell; m0) = \left[\frac{(2L+1)}{(2\ell+1)}\right] \text{Trace } T_2^0 = \delta_{L0}$$

$$d_L = \left[\frac{(2L+1)}{(2\ell+1)}\right] \sum_m (-)^m C(\ell L \ell; m0) \qquad \text{(E.4)}$$

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and from the matrices

$$A = \alpha^{(o,o)}; \; A_{\mu\mu'} = \sum_{m} S^{(\ell)}_{m\mu;\alpha} \, S^{(\ell)}_{m\mu';\alpha}$$

$$B = \sum_{L=0}^{2\ell} d_L \, b^{(L,0)}; \; B_{\mu\mu'} = \sum_{m} (-)^m S^{(\ell)}_{m\mu;\alpha} \, S^{(\ell)*}_{m\mu';\alpha} (-)^{\frac{1}{2}-\alpha} \quad . \tag{E.5}$$

Then the analogue of Eq. (64) is

$$\text{Trace}\left(BT^M_L\right) = (-)^{\ell} \eta \, \text{Trace}\left(AYT^M_L\right) . \tag{E.6}$$

The same reasoning as yields Eq. (72) gives

$$(-)^{\ell} \; \eta P = \frac{\pi}{4} \left(<gf>_{right} - <gf>_{left}\right) \tag{E.7}$$

where f is given by Eq. (73) and

$$g = \sum_{L=0}^{2\ell} d_L \left[\frac{(2L+1)}{m_{\ell L}} \right] \mathcal{D}^{(L)}_{00} (\varphi_K, \vartheta_K, 0) . \tag{E.8}$$

Many additional relations, like Eq. (75) for example, may also be obtained for this case.

The dependence on ℓ of the r.h.s. of Eq. (E.7) [and other such relations obtainable from Eq. (E.6)] shows that the value of ℓ, if it is not known, may also be obtained from studies of right-left asymmetries in Eq. (E.1) with polarized target

$$<g>_{right} - <g>_{left} = \frac{4}{\pi} \eta P <hf>$$

$$h = \sum_{L=0}^{2\ell} c_L \left[\frac{(2L+1)}{m_{\ell L}} \right] \mathcal{D}^{(L)}_{00} (\varphi_K, \vartheta_K, 0) = 1 \quad .$$

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7) Equations (5) and (6) differ in notation only from Eqs. (11) - (16) of Byers and Fenster [Ref. 1)]. Note that $\beta_{X\to\Lambda} = -\beta$.

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9) See, e.g., M.E. Rose, "Elementary theory of angular momentum". J. Wiley and Sons, New York (1957).

10) The notation here is as on page 52 of Ref. 9).

11) With this normalization

$$T_L^0 = \frac{\sqrt{2J+1}\; P_{T_1}\left(S_2\sqrt{J(J+1)}\right)}{\sqrt{(2L+1)\sum_{m'=-J}^{J}\left[P_L\left(\frac{m'}{\sqrt{J(J+1)}}\right)\right]^2}}$$

where P_L is the L^{th} Legendre polynomial.

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16) Explicitly, in Eq. (44)

$$\mathcal{D}^{(j)}_{\lambda\alpha} \equiv \mathcal{D}^{(j)}_{\lambda\alpha}(+\pi/2, +\pi/2, \pi)$$

$$\mathcal{D}^{(j)}_{\lambda_x m} \equiv \mathcal{D}^{(j)}_{\lambda_x m}(\pi/2, \pi/2, \pi) \ .$$

17) Using Eq. (44), one easily obtains Eq. (48) from corresponding relations for $f_{\lambda_x;\lambda}$ given in Ref. 14) [Eq. (44)].

18) See, e.g., J.M. Blatt and V.F. Weisskopf, "Theoretical nuclear physics", J. Wiley and Sons, New York.

19) Here we have used the longitudinal polarization of Λ to obtain the t^0_K. Since, t^0_K with K odd give moments of both longitudinal and transverse Λ polarization (see Table II) alternative forms for f may be used. Indeed when $\alpha_{X\to\Lambda} \neq 0$ and $\beta_{X\to\Lambda} \neq 0$, there are four such forms.

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23) Restrictions on ρ owing to Eqs. (46) and (47) have been given by M. Ademollo and R. Gatto [Ref. 1)].

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