vation laws are also related to the exigencies of expressing diffraction consistently by Regge poles.

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Overlapping Final-State Interactions

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A simple model for a production process with a three-body final state is constructed. Two pairs of particles in the final state are assumed to interact, perhaps strongly, and the structure of the resulting interaction is examined. The dynamics of the model is defined by a dispersion relation, which yields a singular integral equation. This equation is solved in the case of large mass of the particle which is common to the two interacting pairs.

I. INTRODUCTION

THE effects of interactions in the final state on inelastic processes have been appreciated for a long time.[1] For example, such interactions form the basis of much of the current experimental search for resonances. They also are the foundation of various theoretical studies, such as the isobar model[2] and its modifications.[3,4]

If there are three or more particles in the final state, then several pairs can interact. In many of the theoretical analyses of production processes the assumption has been made that the interaction of only one pair is important (e.g., references 2–4)—the remaining final-state interactions may be inherently weak, or else weak in the relevant kinematical regions. (This assumption may be justified in certain other cases as well.[1])

There have also been discussions in which several pairs were assumed to interact. Some of these discussions deal with the regions near the production threshold, where the kinetic energies are small, and the final-state interactions assumed weak.[5–8] The solutions then were based, essentially, on a perturbation expansion. (This approach was also considered[5,8] for the process $K \rightarrow 3\pi$.) Other discussions, in which several interacting pairs were considered, relate to the statistical model[9] and to the isoscalar nucleon structure.[10,11] In these discussions the form of the amplitude was assumed rather than derived. As a final example, in which several interacting pairs are considered, we mention production in the Lee model. For this problem an exact solution has been obtained.[12]

In this paper we examine a very limited problem involving overlapping final-state interactions of particles with finite masses. By *overlapping final-state interactions* we mean the two-body interactions of two pairs of particles, if the two pairs have one particle in common. Thus, for a three-body final state, two interactions are overlapping whenever they operate simultaneously, whereas in the case of four or more particles one can have simultaneous interactions of two independent pairs.

Our approach is based on integral equations which

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9 See e.g., G. Pinski, Nuovo Cimento **24**, 719 (1962).

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are analogous to those constructed for the process $K \rightarrow 3\pi$ by Khuri and Treiman,[8] but our equations are considered as defining a model, rather as approximations related to a physical process. We assume that two of the particles interact with the third but not with each other, and we discuss in detail the case where the mass m of the third particle is sufficiently large, so that a series expansion in powers of m^{-1} may be meaningful. We note that, apart from a perturbation expansion, this solution is the only one that we can construct. We argue (but do not prove) that this expansion should converge asymptotically.

In the limit $m \rightarrow \infty$ our solution has some analogies with the production amplitude for the Lee model.[12] However, our interest lies not so much in the precise form of higher order corrections, but rather, in removing the kinematic restrictions of the static case.

We examine our solution for the case where there are two s wave resonating pairs in the final state. Our results indicate that a Dalitz plot should exhibit the assumed resonances, and in addition, that there should be a strong resonance in the incoming energy when the two final-state resonances intersect in the physical region.

We should emphasize that this paper was motivated not only by the potential experimental interest but we also consider this work as an attempt to apply the S-matrix approach to production processes. However, we were forced to limit our attention to some particular aspect of such processes, because of their complexity. We chose to examine overlapping final state interactions, for the reasons explained in the foregoing.

In Sec. II we define our model and construct the integral equations, for the case of final-state interactions in the s wave. In Sec. III we obtain the solution as a power series in m^{-1}, and we briefly discuss its asymptotic convergence. We consider the physical interpretation of this solution in Sec. IV. Some generalizations of our model, including interactions in higher partial waves and the treatment of anomalous thresholds, are discussed in Sec. V. Finally, Sec. VI contains a few concluding remarks.

II. DESCRIPTION OF THE MODEL

We consider the production process illustrated in Fig. 1. The center-of-mass energy is $s^{1/2}$, where $s=(q+q')^2$, and the outgoing particles have four-momenta q_i and masses m_i. We define for i, $j=1$, 2, 3,

$$s_{ij}=(q_i+q_j)^2, \tag{2.1}$$

and we have the following relation among these invariants:

$$s_{12}+s_{13}+s_{31}=s+m_1^2+m_2^2+m_3^2. \tag{2.2}$$

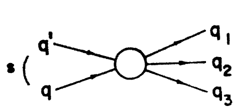

FIG. 1. Production process.

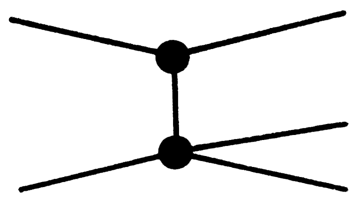

FIG. 2. Pole in a momentum transfer variable.

In order to keep the problem simple, we shall assume that the interaction occurs in a state of total angular momentum $J=0$, and that all particles are spinless. Since we shall be concerned primarily with the behavior at fixed s, the problem is equivalent to that of the decay of a particle into three. It may also be a reasonable description of the production of three particles near the production threshold.

The condition $J=0$ has the following consequence. A production process which involves five particles depends on five kinematical invariants besides the masses. From the invariants introduced above we can select three independent ones, and the remaining two invariants must necessarily involve momentum transfers. Now, the condition $J=0$ requires that the production amplitude be independent of the two momentum transfers. This requires, in turn, that the two incoming particles be distinct from the outgoing ones, and that there be no three-particle vertex interactions of the incoming particles; otherwise we would get poles in the momentum transfer variables, as Fig. 2 shows.

We assume further that there are no antiparticles in our model. The particles are therefore analogous to those of the Lee model[12] but are all relativistic. We shall call the incoming particles $\tilde{V}$, $\tilde{\theta}$ and the outgoing ones N, θ_1, θ_2, with θ_1 and θ_2 identical. We assume that $\tilde{\theta}$ and $\tilde{V}$ are distinct from θ and from the usual V, respectively, for reasons stated in the previous paragraph.

We now relabel some of the invariants:

$$s(N\theta_1)=\sigma_1, \quad s(N\theta_2)=\sigma_2, \tag{2.3a}$$

also $s(V\theta)=s$, as before. The masses are assumed as follows:

$$m(\tilde{V})=m(N)=m, \quad m(\tilde{\theta})=m(\theta)=\mu. \tag{2.3b}$$

We shall be dealing with the following processes:

$$\tilde{V}+\tilde{\theta} \rightarrow N+\theta+\theta, \tag{2.4a}$$

$$N+\theta \rightarrow N+\theta. \tag{2.4b}$$

The $N-\theta$ interaction is assumed known and pure s wave. (The $\theta-\theta$ interaction is ignored.) The amplitudes corresponding to (2.4a,b) will be denoted by $M(s,\sigma_1,\sigma_2)$ and by $T(\sigma)$, respectively. We can ignore kinematic factors and can put, for $\sigma \geqslant (m+\mu)^2$,

$$T(\sigma)=e^{i\delta(\sigma)}\sin\delta(\sigma). \tag{2.5}$$

The function $\delta(\sigma)$ will be assumed sufficiently smooth, bounded at $\sigma=(m+\mu)^2$, and vanishing sufficiently rapidly at infinity. This last condition is to ensure the convergence of all the integrals defined below, and also is to make the expansion in powers of m^{-1} in Sec. III meaningful. Note that the process (2.4a) is coupled to the processes $\tilde{V}+\tilde{\theta}\to\tilde{V}+\tilde{\theta}$ and $N+\theta+\theta\to N+\theta+\theta$, but these latter processes are not relevant as long as we do not discuss the dependence on s.

We may now state our problem as follows: *to determine, in a suitable dynamical framework, the dependence of M on σ_1 and σ_2 in terms of a given function $\delta(\sigma)$.* We turn, therefore, to a discussion of the dynamical framework.

The dynamics of our model is defined in terms of a dispersion relation. (Such an approach has been adopted previously.[10,13]) We assume the following representation for M:

$$M(s,\sigma_1,\sigma_2)=M(s,\sigma_1{}^0,\sigma_2{}^0)+\frac{\sigma_1-\sigma_1{}^0}{\pi}\int_{(m+\mu)^2}^{\infty}\frac{[M]_{\sigma_1'}d\sigma_1'}{(\sigma_1'-\sigma_1-i\epsilon)(\sigma_1'-\sigma_1{}^0)}+\frac{\sigma_2-\sigma_2{}^0}{\pi}\int_{(m+\mu)^2}^{\infty}\frac{[M]_{\sigma_2'}d\sigma_2'}{(\sigma_2'-\sigma_2-i\epsilon)(\sigma_2'-\sigma_2{}^0)}. \quad (2.6)$$

The subtractions are those required for our solution in Sec. III. The bracket denotes the discontinuity in M across the cuts in σ_1 and in σ_2. The contributions to the cuts are indicated in Fig. 3. In the physical region we take for the discontinuity (see, for example, references 8 and 10):

$$[M(s,\sigma_1;\Omega_1)]_{\sigma_1}=\frac{1}{4\pi}\int T^*(\sigma_1,\Omega_1,\Omega_1')M(s,\sigma_1;\Omega_1')d\Omega_1', \quad (2.7)$$

where Ω_1 is the direction of the outgoing θ_1 and determines σ_2, while Ω_1' is the direction of the intermediate θ_1' in the $N_1'-\theta_1'$ (or $N-\theta_1$) center-of-mass system. In our case T depends only on σ_1, and not on the angle between Ω_1 and Ω_1'; it follows that the only contribution to $[M]_{\sigma_1}$, comes from the s state of N and θ_1, and we may write:

$$[M]_{\sigma_1}=[M(s,\sigma_1;\Omega_1)]_{\sigma_1}=T^*(\sigma_1)M_0(s,\sigma_1), \quad (2.8)$$

$$M_0(s,\sigma_1)=\frac{1}{4\pi}\int M(s,\sigma_1;\Omega_1)d\Omega_1. \quad (2.9)$$

The definition (2.9) is valid only in the physical region, and M_0 is defined outside this region by analytic continuation. The analytic properties of M_0 can be deduced from Eq. (2.16) below, and have the same general properties as partial wave amplitudes for scattering.[14]

We now use the fact that θ_1 and θ_2 are identical. This implies

$$M(s,\sigma_1,\sigma_2)=M(s,\sigma_1{}^0,\sigma_2{}^0)+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}d\sigma'\,T^*(\sigma')M_0(s,\sigma')\left[\frac{\sigma_1-\sigma_1{}^0}{(\sigma_1'-\sigma_1-i\epsilon)(\sigma'-\sigma_1{}^0)}+\frac{\sigma_2-\sigma_2{}^0}{(\sigma'-\sigma_2-i\epsilon)(\sigma'-\sigma_2{}^0)}\right]. \quad (2.10)$$

Equations (2.9) and (2.10) are analogous to Eqs. (21) and (24) of Khuri and Treiman[8] and, in effect, define our model. However, our equations still allow some arbitrariness in the behavior of solutions near the end point (or threshold) $\sigma=(m+\mu)^2$. We remove the arbitrariness by assuming that $M_0(\sigma)$ is bounded near this end point. If we require, moreover, that near this end point

$$T(\sigma)\sim[\sigma-(m+\mu)^2]^{1/2}\equiv K,$$

then our assumption implies $M_0(\sigma)\sim K+(\text{const})$. These considerations can be made precise in terms of the Hölder condition.[15]

(We have suppressed the s dependence. This will usually be done from now on.)

One can interpret the foregoing equations in terms of graphs as follows. If one attempts an iterative solution starting with $M_0{}^{(0)}(\sigma_1)=(\text{const})$, then the successive approximations can be represented by diagrams; a typical example is shown in Fig. 4. Such diagrams are of two types, with the last interaction taking place in the σ_1 channel or in the σ_2 channel. These two channels correspond to the two terms of the integrand, respectively. (The discussion in references 5–8 also is restricted, essentially, to diagrams such as in Fig. 4.) We observe that the triangle diagrams, which ordinarily

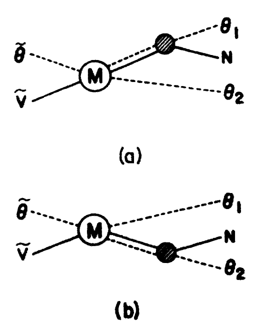

FIG. 3. Contributions to the integral representation (2.6). Shading indicates the interaction T.

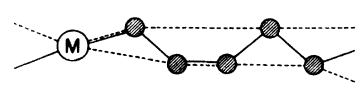

FIG. 4. A typical diagram which occurs in the iterative solution of Eq. (2.10).

[13] F. Zachariasen, Phys. Rev. **121**, 1851 (1961).
[14] S. W. McDowell, Phys. Rev. **116**, 774 (1959).
[15] N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff N. V., Groningen, Holland, 1953).

lead to anomalous thresholds in production amplitudes, in this case have an internally unstable mass. The contribution of such diagrams is, therefore, superimposed on the physical cut and need not be considered separately.[16,17]

One can also obtain Eq. (2.10) from a more formal argument. Assume, on the basis of perturbation theory,[16,17] analyticity in the product of cut σ_1 and σ_2 planes. The most general function having this property and an appropriate behavior at infinity is[18]

$$M(\sigma_1,\sigma_2)=M(\sigma_1{}^0,\sigma_2{}^0)+(\sigma_1-\sigma_1{}^0)(\sigma_2-\sigma_2{}^0)\int\int_{(m+\mu)^2}^{\infty}\frac{dx_1dx_2\rho(x_1,x_2)}{(x_1-\sigma_1)(x_1-\sigma_1{}^0)(x_2-\sigma_2)(x_2-\sigma_2{}^0)}$$
$$+(\sigma_1-\sigma_1{}^0)\int_{(m+\mu)^2}^{\infty}\frac{dx_1\varphi(x_1)}{(x_1-\sigma_1)(x_1-\sigma_1{}^0)}+(\sigma_2-\sigma_2{}^0)\int_{(m+\mu)^2}^{\infty}\frac{dx_2\varphi(x_2)}{(x_2-\sigma_2)(x_2-\sigma_2{}^0)}. \quad (2.11)$$

Since for $\sigma_2<(m+\mu)^2<\sigma_1$ the discontinuity (2.8) depends only on σ_1, we must have $\rho=0$, and the expression for the discontinuity leads to the integral equation (2.10). The condition $\rho=0$ expresses the fact that there are no diagrams which imply a simultaneous dependence on σ_1 and σ_2, such as the four-point single loop diagram.

Let us return to Eqs. (2.9)–(2.10). We notice that the s dependence of the solution of this system comes from two sources. Firstly, Eq. (2.9) is explicitly s dependent due to the fact that the boundary of the physical region, limiting the Ω_1 integration, depends on s if σ_1 and σ_2 are taken as kinematic variables instead of σ_1 and Ω_1. Secondly, there is the s dependence of the term $M(s,\sigma_1{}^0,\sigma_2{}^0)$. This contributes to the s dependence of $M(s,\sigma_1,\sigma_2)$ in a rather trivial way, since the term $M(s,\sigma_1{}^0,\sigma_2{}^0)$ is merely a factor of the complete solution. Note that we have a homogeneous linear equation, and therefore $\chi(s)M(s,\sigma_1,\sigma_2)$ is a solution if M is one. This second source of s dependence of M is not relevant to the problem at hand, and can be examined only if our dynamical framework is extended.

In order to discuss Eq. (2.9) in more detail, we have to specify the direction Ω_1 more precisely. With reference to the process of Fig. 3, we can choose a coordinate system[19,20] in which the center of mass of N and θ_1 is at rest and in which the polar axis is along the momentum of θ_2. We specify the direction Ω_1 by polar angles (α_1,ϕ_1); the choice of origin for ϕ_1 is immaterial. By a straightforward application of the conservation laws and Lorentz transformations, one can show that

$$\sigma_2=\xi(\sigma_1)+\eta(\sigma_1)\cos\alpha_1\equiv\xi(\sigma_1)+\eta(\sigma_1)x_1, \quad (2.12a)$$

where

$$\xi(\sigma_1)=s+\mu^2-(2\sigma_1)^{-1}(s+\sigma_1-\mu^2)(s+\mu^2-m^2), \quad (2.12b)$$

$$\eta(\sigma_1)=(2\sigma_1)^{-1}\{[(s-\sigma_1-\mu^2)^2-4\sigma_1\mu^2]\times[(\sigma_1-m^2-\mu^2)^2-4m^2\mu^2]\}^{1/2}. \quad (2.12c)$$

We now write

$$M_0(\sigma_1)=\frac{1}{4\pi}\int d\Omega_1M(\sigma_1;\Omega_1)=\frac{1}{4\pi}\int_{-1}^{+1}dx_1\int_0^{2\pi}d\phi_1M(\sigma_1,\sigma_2). \quad (2.13)$$

The integration over ϕ_1 at fixed x_1 represents an integral over one of the momentum transfers, of which M has been assumed independent, and hence gives just 2π. The x_1 integration can be transformed to give

$$M_0(\sigma_1)=[2\eta(\sigma_1)]^{-1}\int_{L-}^{L+}M(\sigma_1,\sigma_2)d\sigma_2, \quad (2.14)$$

where

$$L_\pm=\xi(\sigma_1)\pm\eta(\sigma_1). \quad (2.15)$$

The Jacobian $(2\eta)^{-1}$ is independent of σ_2. Note that the boundary conditions on σ_2 are the conditions for the three final-state momenta to be collinear ($\cos\alpha_1=\pm1$).

We now insert (2.10) into (2.14) and obtain a linear integral equation for M_0:

$$M_0(\sigma_1)=M(\sigma_1{}^0,\sigma_2{}^0)+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}d\sigma'\,T^*(\sigma')M_0(\sigma')\left[\frac{\sigma_1-\sigma_1{}^0}{(\sigma'-\sigma_1-i\epsilon)(\sigma'-\sigma_1{}^0)}+\frac{1}{2\eta}\int_{L-}^{L+}\frac{(\sigma_2-\sigma_2{}^0)d\sigma_2}{(\sigma'-\sigma_2-i\epsilon)(\sigma'-\sigma_2{}^0)}\right]. \quad (2.16)$$

where we have interchanged orders of integration. This can be justified easily if σ_1 is considered a complex variable. We recall also that $T(\sigma')$ is assumed to vanish at infinity sufficiently rapidly. The second term in the brackets gives

$$\frac{1}{2\eta(\sigma_1)}\ln\left(\frac{\sigma'-L_+-i\epsilon}{\sigma'-L_--i\epsilon}\right)-\frac{1}{\sigma'-\sigma_2{}^0}. \quad (2.17)$$

Equation (2.16) is a singular integral equation. How-

[16] L. F. Cook, Jr., and J. Tarski, J. Math. Phys. 3, 1 (1962)' especially Appendix A.
[17] G. Barton and C. Kacser, Nuovo Cimento 21, 593 (1961).
[18] See, for instance, A. S. Wightman, in *Relations de dispersion et particules élémentaires*, edited by C. De Witt and R. Omnès (Hermann et Cie, Paris, 1961), pp. 281 ff.
[19] G. C. Wick, Ann. Phys. (N. Y.) 18, 65 (1962).
[20] L. F. Cook, Jr., and B. W. Lee, Phys. Rev. 127, 283 (1962).

ever, the theory as expounded by Muskhelishvili[15] cannot be applied immediately, since the kernel (2.17) has singularities but is not a Cauchy kernel. We shall comment briefly on Eq. (2.16) at the end of the next section.

III. SOLUTION IN THE CASE OF LARGE m

We now study our equations for the case of large m, by constructing the solution as a power series in m^{-1}. The lowest approximation, which corresponds to the static limit ($m \to \infty$), is particularly simple; this is, of course, expected from the Lee model.[12] The next approximation, of order m^{-1}, can also be written down quite simply, while that of order m^{-2} is already much more involved. We shall conclude this section with a brief discussion of a few mathematical details, which include the question of convergence.

To study the equations for large m, we introduce new variables in place of σ_1, etc., since the latter approach infinity with m. We therefore define α, β, and ω as the positive quantities which satisfy

$$\sigma_1=(m+\alpha)^2,\quad \sigma_2=(m+\beta)^2, \tag{3.1a,b}$$

$$s=(m+\omega)^2. \tag{3.1c}$$

In the static limit, α, β, ω are the total energies of the particles θ_1, θ_2, $\tilde{\theta}$, respectively. The new variable of integration will be γ', where

$$\sigma'=(m+\gamma')^2. \tag{3.1d}$$

In this section we shall use the quantities α, etc., as arguments of the various functions previously introduced, with the natural notation $T(\alpha)\equiv T[(m+\alpha)^2]$, etc. We shall require the expansion of ξ and η for large m:

$$\xi=(m+\omega-\alpha)^2+O(m^{-1}), \tag{3.2a}$$

$$\eta=2K(\alpha)[1+(2m)^{-1}(\omega-2\alpha)]+O(m^{-2}), \tag{3.2b}$$

where

$$K(\alpha)=\{(\alpha^2-\mu^2)[(\omega-\alpha)^2-\mu^2]\}^{1/2}. \tag{3.2c}$$

Elementary calculations show that, for large m, Eq. (2.10) reduces to

$$M(\alpha,\beta)=M(\alpha^0,\beta^0)+\frac{1}{\pi}\int_\mu^\infty d\gamma' T^*(\gamma')M_0(\gamma')\left[\frac{\alpha-\alpha^0}{(\gamma'-\alpha-i\epsilon)(\gamma'-\alpha^0)}+\frac{\beta-\beta^0}{(\gamma'-\beta-i\epsilon)(\gamma'-\beta^0)}\right]+O(m^{-2}). \tag{3.3}$$

We next want to project the s wave out of this equation. However, in the static limit the physical region lies on the line

$$\alpha+\beta=\omega,$$

and the transformation $\int d\Omega_1 \to \int d\beta$, which we used previously, becomes singular. We therefore proceed by calculating M as a function of α and x_1 [cf. Eqs. (2.12)]. We expand in powers of $\eta x_1/\xi$, which is of order m^{-1}, and obtain

$$\begin{aligned}M(\alpha;x_1)=M(\alpha^0;x_1^0)&-\frac{K(\alpha^0)x_1^0}{m\pi}\int_\mu^\infty \frac{d\gamma'\,T^*(\gamma')M_0(\gamma')}{(\gamma'-\omega+\alpha^0)^2}\\&+\frac{\alpha-\alpha^0}{\pi}\int_\mu^\infty d\gamma'\,T^*(\gamma')M_0(\gamma')\left[\frac{1}{(\gamma'-\alpha-i\epsilon)(\gamma'-\alpha^0)}-\frac{1}{(\gamma'-\omega+\alpha-i\epsilon)(\gamma'-\omega+\alpha^0)}\right]\\&+\frac{K(\alpha)x_1}{m\pi}\int_\mu^\infty\frac{d\gamma'\,T^*(\gamma')M_0(\gamma')}{(\gamma'-\omega+\alpha-i\epsilon)^2}+O(m^{-2}).\end{aligned} \tag{3.4}$$

Let us now construct the solution of this equation. The first two terms combine to give $M(\alpha^0;0)$ (e.g., set $\alpha=\alpha^0$ and $x_1=0$). The s wave is given by

$$\begin{aligned}M_0(\alpha)&=\frac{1}{2}\int_{-1}^{+1}M(\alpha;x_1)dx_1\\&=M_0(\alpha^0)+\frac{\alpha-\alpha^0}{\pi}\int_\mu^\infty d\gamma'\,T^*(\gamma')M_0(\gamma')\left[\frac{1}{(\gamma'-\alpha-i\epsilon)(\gamma'-\alpha^0)}-\frac{1}{(\gamma'-\omega+\alpha-i\epsilon)(\gamma'+\omega-\alpha^0)}\right]+O(m^{-2}),\end{aligned} \tag{3.5}$$

where we have set

$$M(\alpha^0;0)=M_0(\alpha^0).$$

This equality is, in fact, implied by our procedure, and has a natural interpretation. The amplitude $M_0(\alpha^0)$ has a well-defined meaning in the static limit, and therefore should be the limit of $M(\alpha^0,\beta^0)$ for $\beta^0=\omega-\alpha^0$. This, in turn, corresponds to $x_1^0=0$.

Equation (3.5) can be easily solved[12,16,21] in the static

[21] See also R. Omnès, Nuovo Cimento 18, 316 (1958).

limit $m\to\infty$. (See also below.) We have

$$M_0(\alpha)=Ce^{\Delta(\alpha)+\Delta(\omega-\alpha)}, \tag{3.6}$$

where

$$C=M_0(\alpha^0)e^{-\Delta(\omega-\alpha_0)}, \tag{3.7a}$$

$$\Delta(\alpha)=\frac{\alpha-\alpha^0}{\pi}\int_\mu^\infty \frac{\delta(\gamma')d\gamma'}{(\gamma'-\alpha-i\epsilon)(\gamma'-\alpha^0)}$$

$$=\rho(\alpha)+i\delta(\alpha); \tag{3.7b}$$

$$\rho(\alpha)=\frac{\alpha-\alpha^0}{\pi}P\int_\mu^\infty \frac{\delta(\gamma')d\gamma'}{(\gamma'-\alpha)(\gamma'-\alpha^0)}. \tag{3.7c}$$

Note also that in the static limit

$$M(\alpha,\beta)=M_0(\alpha)=M_0(\beta),$$

and that Eq. (3.6) can be written in the equivalent forms

$$M(\alpha,\beta)=Ce^{\Delta(\alpha)+\Delta(\omega-\alpha)}=Ce^{\Delta(\beta)+\Delta(\omega-\beta)}$$
$$=Ce^{\Delta(\alpha)+\Delta(\beta)}.$$

Let us now consider the complete amplitude to order m^{-1}. Equations (3.3) and (3.4) yield expressions of the form

$$M(\alpha,\beta)=M(\alpha^0,\beta^0)+F(\alpha,\alpha^0)+F(\beta,\beta^0) \tag{3.8}$$

$$\begin{aligned}M(\alpha;x_1)&=M_0(\alpha^0)+F(\alpha,\alpha^0)+F(\omega-\alpha,\,\omega-\alpha^0)\\&\quad+m^{-1}K(\alpha)x_1F'(\omega-\alpha)+O(m^{-2})\\&=Ce^{\Delta(\alpha)+\Delta(\omega-\alpha)}+m^{-1}K(\alpha)x_1F'(\omega-\alpha)\\&\quad+O(m^{-2}),\end{aligned} \tag{3.9}$$

where $F'(\alpha)$ is the derivative of $F(\alpha,\alpha^0)$ with respect to α, which is independent of α^0. We see that to order m^{-1}, only the s and the p wave contribute to M.

To obtain the next approximation we first have to calculate $M(\alpha;x_1)$ to order m^{-2}. Then we can project the s wave, and the rest of the problem is as before. We indicate a few more steps. We obtain

$$M_0(\alpha)=M_0(\alpha^0)+\frac{\alpha-\alpha^0}{\pi}\int_\mu^\infty d\gamma'\,T^*(\gamma')M_0(\gamma')\times\left[\frac{1}{(\gamma'-\alpha-i\epsilon)(\gamma'-\alpha^0)}-\frac{1}{(\gamma'-\omega+\alpha-i\epsilon)(\gamma'-\omega+\alpha^0)}+\frac{R}{m^2}\right]+O(m^{-3}), \tag{3.10}$$

The function R can be calculated by straightforward kinematics, but the arithmetic becomes tedious. We next write

$$M_0(\alpha)=M_0^{(0)}(\alpha)+m^{-2}M_0^{(2)}(\alpha)+O(m^{-3}),$$

where

$$M_0^{(0)}(\alpha)=Ce^{\Delta(\alpha)+\Delta(\omega-\alpha)}.$$

We substitute into Eq. (3.10) and neglect the term $m^{-4}RM_0^{(2)}(\alpha)$. The result is an inhomogeneous equation for $M_0^{(2)}$, which can be solved by standard methods[15,21] (provided the function R is sufficiently smooth). We can select that solution which is proportional to C, since any remaining part of the solution could be absorbed by redefining C. We may add that, even if the function R should have a simple form in some limiting cases, the functional form of $M_0^{(2)}$ would still be far from simple.

Let us now consider a few mathematical details. First, the expansion in powers of m^{-1} leading to Eq. (3.4) is clearly valid for complex α, but the limit $\mathrm{Im}\alpha\to 0$ requires some care. One simple way to study this limit is to differentiate in the equation $((u-i\epsilon)^{-1}=\pi i\delta(u)+P/u$. We obtain the following relation:

$$\int_a^b \frac{f(x)dx}{(x-y-i\epsilon)^2}=\pi i f'(y)+\lim_{\eta\to 0+}\left[\left(\int_a^{y-\eta}+\int_{y+\eta}^b\right)\frac{f(x)dx}{(x-y)^2}-\frac{2f(y)}{\eta}\right]. \tag{3.11}$$

Hence as $\mathrm{Im}y\to 0$, the expression $(x-y-i\epsilon)^{-2}$ becomes symbolic for a distribution. This type of argument clearly applies as well to higher powers, i.e., to $(x-y-i\epsilon)^{-n}$.

Second, we note some symmetry properties:

$$M_0(\alpha^*)=M_0(\alpha)^*, \tag{3.12a}$$

$$M_0^{(0)}(\alpha)=M_0^{(0)}(\omega-\alpha^*). \tag{3.12b}$$

The first of these follows from Eqs. (2.9)–(2.10). It relates M_0 in the two half-planes when the cuts overlap [cf. Eq. (2.16)]. This symmetry also applies to $M_0^{(0)}$, as defined by Eq. (3.5). In this latter equation we may separate the cuts slightly, or else we may assume that $M_0^{(0)}(\omega,\alpha^0)$ is analytic in ω, and continue from the region[12] $\omega<2\mu$. Next, Eq. (3.12b) tells how the $i\epsilon$'s for α^0 and for $\omega-\alpha^0$ are related. This equation is obviously valid if $\omega<2\mu$ and if α^0 is away from the cut.

Our third remark has to do with the behavior near the end points of integration. We consider only the solution $M_0^{(0)}(\alpha)$. This clearly has the asymptotic behavior which is implicit in the dispersion relation. It is also not difficult to verify that our condition for $\sigma_1=(m+\mu)^2$, or for $\alpha=\mu$, is fulfilled. We note that if we were to abandon our condition for $\alpha=\mu$, then we could write[15,21]

$$M_0(\alpha)=\left(\frac{a\alpha+b}{\alpha-\mu}+\frac{a(\omega-\alpha^*)+b}{\omega-\alpha^*-\mu}\right)e^{\Delta(\alpha)+\Delta(\omega-\alpha)},$$

with two arbitrary parameters instead of one. (Stronger singularities at $\alpha=\mu$ would lead to divergence.) Without our condition at $\alpha=\mu$ we could also have nonsubtracted dispersion relations.

Finally, we come to the question of the convergence of the series solution for $M_0(\alpha)$. We do not expect pointwise convergence, for the following reason: Given s, or ω, the exact solution is singular at those values of σ_1 for which $L_\pm(s,\sigma_1)=(m+\mu)^2$, while the corresponding singularity in the approximate solutions occurs at $\alpha=\omega-\mu$. However, the position of the former singularities approaches the position of the latter as $m\to\infty$, and one may hope that the series converges asymptotically:

$$\lim_{m\to\infty} m^n[M_0^{(0)}(\alpha)+\cdots+M_0^{(n)}(\alpha)-M_0(\alpha)]=0. \quad (3.13)$$

We have already mentioned that the general theory[15] does not apply to Eq. (2.16), and we are not certain about properties of the exact solution such as existence, uniqueness, continuity, or dependence on parameters. However, let us suppose that $T(\alpha)$ is analytic in a neighborhood of the arc $\mu<\alpha<\infty$. Then the assumption of an analytic solution $M_0(\alpha)$ is consistent with the elementary facts about analytic properties of integrals. [For example, reference 16, Sec. 3(A).] It might be unnecessary to assume analytic $T(\alpha)$, but the higher order approximations require higher derivatives of T, as Eq. (3.11) shows.

If we make suitable assumptions on the functions $T(\alpha)$ and $M_0(\alpha)$, etc., then it can be shown that the neglected terms, which are formally of order m^{-n-1}, are indeed small. The relation (3.13) follows. Note that the complex singularities[14] of M_0 can be ignored, since they are displaced toward the real axis as m increases.

IV. DISCUSSION OF THE SOLUTION

We now wish to discuss the physical meaning of the results of the previous section. We recall that our investigation was aimed at the following question: What is the structure of the amplitude when two final-state interactions are simultaneously present? Let us suppose that, if only one of these two interactions should be present, then the amplitude would be characterized by $e^{\Delta(\sigma)}\equiv 1+a(\sigma)$. In the case of two simultaneous interactions, the simplest functions which might describe the effect are[5–11]

$$1+a(\sigma_1)+a(\sigma_2) \quad\text{and}\quad [1+a(\sigma_1)][1+a(\sigma_2)]. \quad (4.1\text{a,b})$$

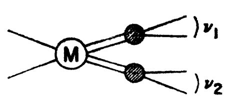

FIG. 5. Example of independent final-state interactions.

If the interaction is sufficiently weak, then the two expressions do not differ greatly, and the first is more convenient. The second expression clearly applies if the two interactions are independent. For example, consider the production process shown in Fig. 5, with the final-state interactions related to the energies ν_1 and ν_2 as indicated. Then we may write

$$M_{l_1l_2}(\nu_1,\nu_2)=M_{l_1l_2}(\nu_1^0,\nu_2)+\frac{\nu_1-\nu_1^0}{\pi}\int_{\nu_0}^{\infty}\frac{T_{l_1}^*(\nu_1')M_{l_1l_2}(\nu_1',\nu_2)d\nu_1'}{(\nu_1'-\nu_1-i\epsilon)(\nu_1'-\nu_1^0)}$$
$$=(1\leftrightarrow 2)$$

and the solution is $Ce^{\Delta_1(\nu_1)+\Delta_2(\nu_2)}$.

Let us now return to our model. We will examine our solution from the following point of view. Let us suppose that the scattering amplitude $T(\sigma)$ exhibits a strong and narrow resonance centered at $\sigma=\bar{\sigma}$; how does this affect M? If we consider, e.g., the region where $\sigma_1\approx\bar{\sigma}$ but σ_2 is away from $\bar{\sigma}$, then the two expressions (4.1a,b) are both approximated by $1+a(\sigma_1)$. This is also expected for our model, since then only one of the two terms in Eq. (2.10) is large. But the interesting case $\sigma_1\approx\sigma_2\approx\bar{\sigma}$ has to be examined more carefully.

Our representation for M is in the form [see (3.8)] of a sum

$$M(s,\sigma_1,\sigma_2)=M(s,\sigma_1^0,\sigma_2^0)+F(s,\sigma_1,\sigma_1^0)+F(s,\sigma_2,\sigma_2^0), \quad (4.2)$$

where by construction $F(s,\sigma_1,\sigma_1^0)$ is large near $\sigma_1=\bar{\sigma}$. However, near the static limit, Eq. (3.9) shows that

$$M(\alpha,\beta)=\tfrac{1}{2}[M(\alpha;x_1)+M(\beta;x_2)]$$
$$=G(\alpha,\alpha_0)+G(\beta,\beta_0)+O(m^{-1}); \quad (4.3\text{a})$$

$$G(\alpha,\alpha_0)=\tfrac{1}{2}C(\omega,\alpha_0)e^{\Delta(\alpha)+\Delta(\omega-\alpha)}. \quad (4.3\text{b})$$

The factor $e^{\Delta(\alpha)}$ is large near $\alpha=\bar{\alpha}$ (corresponding to $\bar{\sigma}$). The factor $e^{\Delta(\omega-\alpha)}$ is large near $\beta=\bar{\alpha}$, since, for large m, we have $\alpha+\beta\approx\omega$ in the physical region. Now, this latter factor can be interpreted as enhancement of the interaction for those values of σ_1, or of α, which kinematically allow the resonance in the σ_2-channel. We may think of this enhancement as a pseudoresonance in σ_1, or, as a reflection of the resonance in σ_2.

When the point $\alpha=\beta=\bar{\alpha}$ lies outside the physical region, e.g., in the case $\omega=\omega_1$ in Fig. 6, the resonance

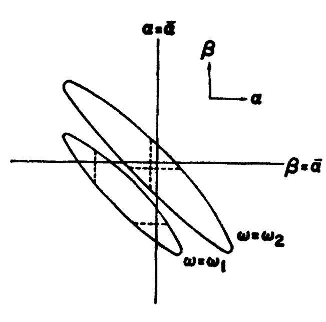

FIG. 6. Regions of enhancement of M for large m. The dashed lines represent the pseudoresonances.

and the pseudoresonance cannot be large simultaneously. In such a case each of the terms $G(\alpha,\alpha^0)$ and $G(\beta,\beta^0)$ contributes to the enhancement at $\alpha=\bar{\alpha}$ and also at $\beta=\bar{\alpha}$, and these two contributions are added. However, as ω increases to ω_2, then the resonance and the pseudoresonance overlap, and we have two contributions to each G which are *multiplied*. We emphasize that this effect is brought about by a *variation in* ω, and is not of the form (4.1b). [One has to assume here that neither the subtraction constant α^0 nor $\omega-\alpha^0$ is near $\bar{\alpha}$, and that $M_0(\omega,\alpha^0)$ varies slowly with ω in the region in question. See Eq. (3.7a).]

It is instructive to look more closely at the structure of this pseudoresonance. Let us first consider the case when the width of the resonance is not small compared with m^{-1}, but m is large. The width of the phase space in the (α,β) plane is of order m^{-1}. Then those parts of the physical region where $|e^{\Delta(\beta)}|\gg 1$, and $|e^{\Delta(\omega-\alpha)}|\gg 1$ are largely overlapping, and the two functions $e^{\Delta(\beta)}$ and $e^{\Delta(\omega-\alpha)}$ describe roughly the same effect, even though one depends only on β, and the other, only on α.

The connection between the two terms can be seen more clearly by examining the correction of order m^{-1} to the amplitude $M(\alpha;x_1)$ [Eq. (3.9)]:

$$M(\alpha;x_1)=Ce^{\Delta(\alpha)+\Delta(\omega-\alpha)}+m^{-1}K(\alpha)x_1F'(\omega-\alpha)+O(m^{-2}).$$

The second term, when Δ has a resonance at $\omega-\alpha$, has the effect of shifting the position of the peak, the shift being in opposite directions for $x_1>0$ and for $x_1<0$ (i.e., for $\beta>\omega-\alpha$ and for $\beta<\omega-\alpha$, respectively). In other words, it tends to rotate the pseudoresonance towards a line of constant β. This must obviously be the case, since the correction considered is the second term in the expansion

$$F(\beta,\beta^0)=F(\omega-\alpha,\,\omega-\alpha^0)+m^{-1}Kx_1F'(\omega-\alpha)+O(m^{-2}).$$

For a narrow resonance, as, e.g., in Fig. 6, one must take enough terms in the expansion to effectively rotate the pseudoresonance. One should keep in mind, however, that our solution is expected to converge asymptotically only, and, therefore, may not be reliable with respect to fine details.

The resonance and pseudoresonance are completely identical in the static limit. This is a reflection of the fact that in this limit both pairs can be in pure s states simultaneously. The infinite mass of the common particle of the two pairs serves to decouple their motion completely, so that the interactions behave as if they were nonoverlapping.

Let us summarize. In the case of large m, the pseudoresonance does not cause enhancement of any part of the phase-space region (for fixed ω) which cannot already be enhanced by the direct resonance in α or β. However, the pseudoresonance does correlate the dependence on α and β with that on ω. Therefore, it describes the formation of isobars in three-body final states, and it indicates a possible strong energy dependence of their excitation probability. This qualitative result may be valid outside the rather restrictive model assumed here.

We remark that models for such higher isobars have also been constructed in the framework of strong coupling physics.[22]

V. SOME GENERALIZATIONS

In constructing our model we have made a number of simplifying assumptions. However, this model can be generalized in some respects without altering its basic structure, and we now give a few examples.

A. Interactions in Other Partial Waves

Let us consider the case where the final-state interactions are in a partial wave with angular momentum $l\neq 0$. Much of the discussion of Sec. II can be easily adapted to this problem. In place of Eqs. (2.8)–(2.9) we now have

$$[M]_{\sigma_1}=T_l^*(\sigma_1)M_l(\sigma_1)P_l(x_1), \quad (5.1a)$$

$$M_l(\sigma_1)=\frac{2l+1}{2}\int_{-1}^{+1}P_l(x_1)M(\sigma_1;x_1)dx_1, \quad (5.1b)$$

where $x_1(\sigma_1,\sigma_2)$ is given by Eq. (2.12a), and in place of Eq. (2.10) we have

$$M(\sigma_1,\sigma_2)=M(\sigma_1^0,\sigma_2^0)+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}d\sigma' T_l^*(\sigma')M_l(\sigma')\left\{\frac{P_l[x_1(\sigma',\sigma_2)](\sigma_1-\sigma_1^0)}{(\sigma'-\sigma_1-i\epsilon)(\sigma'-\sigma_1^0)}+\frac{P_l[x_2(\sigma_1,\sigma')](\sigma_2-\sigma_2^0)}{(\sigma'-\sigma_2-i\epsilon)(\sigma'-\sigma_2^0)}\right\}. \quad (5.2)$$

We next exchange the independent variables (σ_1,σ_2) to (σ_1,x_1), and project out the partial wave $M_l(\sigma_1)$, in order to obtain an equation for this function. The treatment of the first term in the braces requires some care, as follows:

$$P_l[x_1(\sigma',\sigma_2)]=P_l\left(\frac{\sigma_2-\xi'}{\eta'}\right)=P_l\left(\frac{\xi_1+\eta_1x_1-\xi'}{\eta'}\right)$$
$$\equiv P_l(a_1x_1+b_1). \quad (5.3)$$

We used the notation $\xi(\sigma_1)=\xi_1$ and $\xi(\sigma')=\xi'$, and did the same for the function η. The projection of $M_l(\sigma_1)$ can now be carried out. This first term yields the integral

$$\frac{2l+1}{2}\int_{-1}^{+1}P_l(x_1)P_l(a_1x_1+b_1)dx_1=a_1^{\,l}.$$

We hope now, as in Sec. II, that the equation for $M_l(\sigma_1)$

[22] E.g., C. J. Goebel, Phys. Rev. **109**, 1846 (1958).

has a well-behaved solution. The other partial waves, and the complete amplitude, are determined by M_l.

The expansion in powers of m^{-1}, however, cannot be obtained as easily as in Sec. III. This was to be expected, since in the static limit the cosine x_1 is no longer determined by the invariant energies. An attempt to construct such a series solution leads to the following complication, among others. For $l=1$, e.g., and for large m we have the order-of-magnitude relation $M_0(\alpha) \sim m M_1(\alpha)$ (cf. Sec. III). Therefore, we cannot compute $M_1(\alpha)$ in the static limit as the first step of the solution.

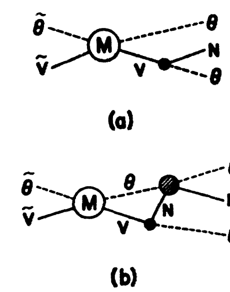

FIG. 7. Diagrams involving the V particle.

B. Anomalous Thresholds

Vertex singularities frequently arise in production amplitudes[16] and give rise to what are known as anomalous thresholds. We shall therefore describe one modification of our model, one which gives rise to vertex singularities, but in which these can be easily taken into account. Let us admit into our model V particles[12] in addition to $\tilde{V}$, $\tilde{\theta}$, N, and θ, and let us consider again the process $\tilde{V}+\tilde{\theta} \rightarrow N+\theta+\theta$. Then the representation (2.10) has to be modified by including the contribution of one-particle poles and of the cuts associated with the anomalous thresholds. These contributions are illustrated in Fig. 7. We assume that the diagram of Fig. 7(b) is the only diagram which yields anomalous thresholds.

To construct the representation for M we have to introduce the amplitude $f(s)$ for the interaction $\tilde{V}+\tilde{\theta} \rightarrow V+\theta$ (assumed s wave), and the coupling constant g for the vertex $V \leftrightarrow N+\theta$. We take the mass of the V particle to be m also. The following representation results:

$$M(\sigma_1,\sigma_2)=M(\sigma_1^0,\sigma_2^0)+\left\{gf(s)\left(\frac{1}{\sigma_1-m^2}-\frac{1}{\sigma_1^0-m^2}\right)+\frac{gf(s)(\sigma_1-\sigma_1^0)}{\pi}\int_{\Gamma(s)}\frac{T^*(\sigma')\mathcal{L}(\sigma')d\sigma'}{(\sigma'-\sigma_1-i\epsilon)(\sigma'-\sigma_1^0)}\right.$$
$$\left.+\frac{\sigma_1-\sigma_1^0}{\pi}\int_{(m+\mu)^2}^{\infty}\frac{T^*(\sigma')M_0(\sigma')d\sigma'}{(\sigma'-\sigma_1-i\epsilon)(\sigma'-\sigma_1^0)}\right\}+\{1\leftrightarrow 2\}. \quad (5.4)$$

The contour[16] $\Gamma(s)$ has as its end points the vertex singularity and $(m+\mu)^2$, and $\mathcal{L}(\sigma')$ is one-half of the usual discontinuity function[23] (since we have contributions from positive energies only). Note that $T(\sigma')$ is now needed in an unphysical region.

The anomalous threshold terms and the pole (Born) terms together form the inhomogeneous part of the equation. This equation can be investigated by the same methods as we used before.

We again assume a well-behaved solution $M_0(\sigma_1)$. The general solution is then the sum of a particular solution of the inhomogeneous equation and of the general solution of the homogeneous equation. Let us select for the particular solution the one which is proportional to $gf(s)$. The two parts of the solution can now be interpreted as the sums of those diagrams which contain an intermediate V particle, and of those which do not contain one, respectively.

C. Distinct Interacting Pairs

Our solution can be applied to the case where the two particles θ_1 and θ_2 are not identical. Let their (s wave) interactions with N be described by the respective functions

$$T_{[1]}(\sigma_1)=e^{i\delta_1(\sigma_1)}\sin\delta_1(\sigma_1),$$
$$T_{[2]}(\sigma_2)=e^{i\delta_2(\sigma_2)}\sin\delta_2(\sigma_2).$$

The construction of the dynamical equations proceeds as before, but we have to deal with two s waves $M_{[1]0}(\sigma_1)$ and $M_{[2]0}(\sigma_2)$. (We have, in fact, two distinct partial-wave expansions.) For the static limit we obtain

$$M_{[1]0}(\alpha)=M(\alpha^0,\omega-\alpha^0)+\frac{\alpha-\alpha^0}{\pi}\int_{\mu}^{\infty}d\gamma'\left[\frac{T_{[1]}^*(\gamma')M_{[1]0}(\gamma')}{(\gamma'-\alpha-i\epsilon)(\gamma'-\alpha^0)}-\frac{T_{[2]}^*(\gamma')M_{[2]0}(\gamma')}{(\gamma'-\omega+\alpha-i\epsilon)(\gamma'-\omega+\alpha^0)}\right], \quad (5.5)$$

and a similar equation for $M_{[2]0}$. We have assumed equal masses, μ, for θ_1 and for θ_2.

The two functions $M_{[1]0}$ and $M_{[2]0}$ satisfy the symmetry relation

$$M_{[1]0}(\alpha)=M_{[2]0}(\omega-\alpha^*). \quad (5.6)$$

We can now eliminate $M_{[2]0}$ from Eq. (5.5), and the resulting equation yields

$$M_{[1]0}(\alpha)=Ce^{\Delta_1(\alpha)+\Delta_2(\omega-\alpha)}. \quad (5.7)$$

This solution can be used as the lowest order term in an

[23] R. E. Cutkosky, J. Math. Phys. 1, 429 (1960).

expansion in powers of m^{-1}. The remainder of the solution can be carried out as in Sec. III.

It is obvious that the generalizations A–C can be combined in various ways. One can easily construct further examples, where, e.g., final-state interactions take place in more than one partial wave, or where the particles have spin, or where the total angular momentum J is different from zero (but has a definite value). In this latter case the amplitude depends on the momentum transfers, but this dependence is known. See, for instance, reference 11 for $J=1$.

VI. CONCLUSION

Let us try to consider what is the value of our model. This model can be looked at from three points of view: as a model field theory, as a qualitative description of overlapping resonances, and as a basis for approximate calculations of realistic production amplitudes. We shall consider these aspects in turn.

With regard to model theories, our model and its solution are very similar to the Lee model, if production is considered. However, our model admits three independent kinematical variables, and this is a basic extension from the two variables in the Lee model. We have compared the static and the nonstatic cases elsewhere in the paper.

With regard to overlapping resonances we may say this: Our model enables us to make here some qualitative conclusions, which may be valid independently of the exact details of the interaction. The conclusions have been described fully in Sec. IV. These conclusions could also be conjectured on the basis of a static model, but it is not altogether satisfactory to generalize when a larger number of variables enters into the problem. Of course, our model also can be applied to nonoverlapping resonances, but then we merely reproduce the isobar model.

With regard to realistic production amplitudes, we have seen how our model can be adapted to various kinds of particles in the final state. However, two of the simplifications in this model seem particularly difficult to improve upon: the neglect of diagrams such as the single-loop diagram with four external lines, and the restriction to a definite total angular momentum J. The diagrams just described are known to have singularities near the physical region,[24,25] and may therefore be difficult to incorporate into a model such as ours. In connection with the total angular momentum, we have already noted that the approximation $J=0$ may be valid near the production threshold. But for large kinetic energies the momentum transfer dependence of the process becomes crucial, and our model is clearly inadequate.

ACKNOWLEDGMENTS

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[25] G. Bonnevay, Proc. Roy. Soc. (London) **A266**, 68 (1962).