where

$$G_E(\vec{q}, \tau) = (i/V)\sum_j e^{i(E_0 - E_j)\tau} |\langle 0|A_{\vec{q}}|j\rangle|^2, \quad \tau > 0$$

$$= (i/V)\sum_j e^{-i(E_0 - E_j)\tau} |\langle 0|A_{\vec{q}}^\dagger|j\rangle|^2, \quad \tau < 0 \tag{21}$$

$$A_{\vec{q}} = \sum_{\substack{|\vec{k}|<k_F \\ |\vec{k}+\vec{q}|>k_F}} a_{\vec{k}}^\dagger a_{\vec{k}+\vec{q}}, \tag{22}$$

$$A_{\vec{q}}^\dagger = \sum_{\substack{|\vec{k}|>k_F \\ |\vec{k}-\vec{q}|<k_F}} a_{\vec{k}}^\dagger a_{\vec{k}-\vec{q}}. \tag{23}$$

Note that the states j in (21) are states of the N-Fermion system. Then, as before,

$$G_E(\vec{q}, \tau) = \lim_{\eta\to 0}(2\pi)^{-1}\int_{-\infty}^{\infty} G_E(\vec{q}, \lambda, \eta) e^{-i\lambda\tau} d\lambda, \tag{24}$$

$$G_E(\vec{q}, \lambda, \eta) = -\frac{1}{V}\sum_j \left[\frac{|\langle 0|A_{\vec{q}}|j\rangle|^2}{\lambda - \epsilon_j + i\eta} + \frac{|\langle 0|A_{\vec{q}}^\dagger|j\rangle|^2}{\lambda + \epsilon_j - i\eta}\right]. \tag{25}$$

Thus except for states which are not created by $A_{\vec{q}}$ or $A_{\vec{q}}^\dagger$ the spectral representation of G_E gives just the excitation energies, ϵ_j, of the system, and it might, therefore, be called the excitation Green's function. Although the numerators of the two terms in (25) look the same, they are not, since the restrictions on the k's in (22) and (23) are different. In fact, it is easy to see that for noninteracting Fermions only the first term contributes. It is interesting to note also that ϵ_p and μ can be obtained by comparing just the lowest poles in the spectral representations of the two Green's functions, G and G_E.

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UNITARITY CONDITION BELOW PHYSICAL THRESHOLDS IN THE NORMAL AND ANOMALOUS CASES*

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It is often the case with dispersion relations that the absorptive part for positive energies extends below the physical threshold. Thus, for example, in the process $2\pi \to N+\bar{N}$ or $\gamma \to N+\bar{N}$, the physical threshold is at $t=4M^2$ (t being the square of the energy), whereas the absorptive part extends to $t=4\mu^2$, the square of the energy of the lowest intermediate state. When calculating the amplitude for these processes,[1] the unitarity condition has to be used in the region $4\mu^2<t<4M^2$, and the question at once arises whether it is really valid there. Indeed, for those processes where there are anomalous thresholds, the imaginary part of the transition amplitude is known to be nonzero in a region where it should be zero according to the unitarity condition. It has therefore not been possible up until now to apply dispersion relations to the calculation of such processes. The object of this Letter is to attempt to justify the use of the unitarity condition below the physical threshold, and to investigate how to modify it in the anomalous case. We shall find that the same treatment is applicable to the normal and anomalous cases—it will confirm that the unitarity condition as used in reference 1 is correct, and will also show how to calculate the imaginary part of the transition amplitude in the anomalous case for the entire region in which it fails to vanish. We shall assume the validity of dispersion relations for partial waves throughout, and shall also approximate the unitarity condition in the physical region by taking only two-particle intermediate states.

First let us quote the results which are obtained in the normal case if the unitarity condition is used below the physical threshold. We consider the process $2\pi \to N+\bar{N}$, and neglect the spin of the nucleon, which merely complicates the algebra. The transition amplitude for the S wave (any other angular-momentum wave could be similarly treated) may then be written as

$$A(t)=A^{(1)}(t)+A^{(2)}(t), \tag{1}$$

where

$$A^{(1)}(t)=\frac{1}{\pi}\int_{-\infty}^{a}dt'\frac{\mathrm{Im}A^{(1)}(t')}{t'-t}, \tag{2a}$$

$$A^{(2)}(t)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'\frac{\mathrm{Im}A^{(2)}(t')}{t'-t}. \tag{2b}$$

The limit a is less than $4\mu^2$. The absorptive part $\mathrm{Im}A^{(1)}$ comes from pion-nucleon scattering and will be assumed known. It is then required to calculate $A^{(2)}$, whose absorptive part comes from the reaction in question, by unitarity.

In most of the calculations performed up till now, the perturbation result for $A^{(1)}(t)$ is taken. The contribution of the nucleon poles to the scattering amplitude (not just the S waves) is

$$\frac{g^2}{s-M^2}-\frac{g^2}{u-M^2}, \tag{3}$$

where s and u are the squares of the energies of the pion-nucleon and the crossed pion-nucleon systems. Expressed in terms of the energy and angle for the reaction $2\pi \to N+\bar{N}$,

$$s=2pq\cos\theta-p^2-q^2, \tag{4a}$$

$$u=-2pq\cos\theta-p^2-q^2, \tag{4b}$$

p and q being the momenta of the nucleon and meson, given by

$$p^2=\tfrac{1}{4}t-M^2, \tag{5a}$$

$$q^2=\tfrac{1}{4}t-\mu^2. \tag{5b}$$

On inserting these expressions into (3) and isolating the S wave, we find that

$$A^{(1)}(t)=-\frac{g^2}{pq}\ln\frac{-p^2-q^2-M^2+2pq}{-p^2-q^2-M^2-2pq}. \tag{6}$$

As long as the signs of the square roots are defined to be the same inside and outside the logarithm, it is immaterial which sign is chosen. We take that branch of the logarithm which is real for $t>4\mu^2$. As q is imaginary and p real for $4\mu^2<t<4M^2$, (6) is often written as an inverse tangent.

The unitarity condition for $\mathrm{Im}A^{(2)}$ in the two-pion approximation is

$$\mathrm{Im}A^{(2)}(M',t)=(q/8\pi w)\{A^{(1)}(M',t)+A^{(2)}(M',t)\}B(t), \tag{7}$$

where $B(t)$ is the S-wave amplitude for pion-pion scattering. The arguments M' are inserted for future reference and should be ignored now. w is the center-of-mass energy $\sqrt{t}$. Equations (7) and (2b) can now be solved for $A^{(2)}$. This has been done by Frazer and Fulco[1] and by Omnès.[2] Either form is equally suitable for us at the moment, but the Omnès solution will be slightly more convenient in the subsequent discussion and we shall adopt it here. The result is then

$$A^{(2)}(M',t)=\frac{1}{8\pi^2D(t)}\int_{4\mu^2}^{\infty}dt'\frac{q'N(t')A^{(1)}(M',t')}{w'(t'-t)}. \tag{8}$$

D and N are defined by

$$D(t)=\exp\left\{-\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'\frac{\delta(t')}{t'-t}\right\}, \tag{9a}$$

$$N(t)=D(t)B(t)=\frac{8\pi w}{q}\sin\delta\exp\left\{P\frac{1}{\pi}\int_{4\mu'}^{\infty}dt'\frac{\delta(t')}{t'-t}\right\}, \tag{9b}$$

δ being the pion-pion phase shift. For t real, $D(t)$ is defined in the usual way by letting t approach the real axis from above. The phase factor of $D(t)$ when t is real and greater than $4\mu^2$ is $e^{-i\delta}$, so that $N(t)$ is real and has no cut over this range of values of t.

It is now necessary to justify Eq. (8) without using the unitarity condition below the physical threshold. To do this we consider the Green's function (in momentum space) rather than the scattering amplitude for our process. The Green's function is defined for arbitrary values of the squares of the momenta of the particles; if we multiply it by i and by a factor (p^2-m^2) for each particle, where m is the relevant mass, we obtain a quantity which is equal to the scattering amplitude when the squares of the momenta are equal to the squares of the corresponding masses. We shall, however, allow the squares of the nucleon momenta to have an arbitrary value M'^2 (the same for both nucleons); we can then isolate the S wave and obtain a quantity $A(M',t)$, which is equal to $A(t)$ when $M'=M$.

By the usual technique of evaluating the imaginary part of the Green's function, it is then easy to see that $A(M',t)$ satisfies the unitarity equation (7) if we are in the physical region, i.e., if $t > 4M'^2$. Instead of (6), $A^{(1)}(M', t)$ will be given by

$$A^{(1)}(M', t) = -\frac{g^2(M')}{p'q} \ln \frac{-p'^2 - q^2 - M^2 + 2p'q}{-p'^2 - q^2 - M^2 - 2p'q}, \qquad (10)$$

where

$$p'^2 = \tfrac{1}{4}t - M^2. \qquad (11)$$

Note that M^2 in (6) is not replaced by M'^2, as it refers to the mass of a real intermediate state. $g^2(M')$ is the square of the vertex function when the square of the momentum of one of the nucleons is M'^2.

If now $M' < \mu$, the whole range of values $t > 4\mu^2$ where $\mathrm{Im}A^{(2)}(M', t)$ is nonzero is in the physical region, so that Eq. (7) is always applicable. We can therefore solve it and obtain Eq. (8) for $A^{(2)}(M', t)$. This solution is thus justified if $M' < \mu$.

To obtain the physically interesting case where $M' = M$, we now make an analytic continuation of this solution as a function of M'. This is permissible, as the Green's function in the physical region is known to be the boundary value of an analytic function.[3] We therefore have to examine the analytic properties of $A^{(1)}(M', t)$, given by (10). The quantity $g^2(M')$ is analytic if $M' < (M + \mu)^2$, since it satisfies a dispersion relation in M'^2, which can in fact be proved.[4] The remainder of the expression will have a branch point where the numerator in the logarithm vanishes, i.e., at

$$t = t_0 \equiv -M^2 + 2(M'^2 + \mu^2) - (M'^2 - \mu^2)^2/M^2. \qquad (12)$$

This value of t is always less than $4\mu^2$ unless

$$M'^2 = M^2 + \mu^2, \qquad (13)$$

when $t_0 = 4\mu^2$. If $M'^2 < M^2 + \mu^2$, the right-hand side of Eq. (10) has no branch point at $p' = 0$ or $q = 0$.

Equation (8) can now be analytically continued from values of M' less than μ to $M' = M$. The equality (13) is then never satisfied, so that the singularity at the point given by (12) is always at a value of t less than $4\mu^2$, i.e., below the range of integration in (8). There is also never a singularity at $q = 0$. The right-hand side of (8) is accordingly an analytic function of M'^2 in the relevant range, so that it can be continued onto the mass shell. Equation (8) is thus correct if $M' = M$, and we get the same result as if we had applied unitarity directly, with $M' = M$, below the physical threshold.

We next examine the calculation when there is an anomalous threshold. Suppose the theory contained, in addition to the meson and nucleon, a baryon of mass B, where

$$M^2 > B^2 + \mu^2. \qquad (14)$$

Equation (3) would then contain terms with M replaced by B, corresponding to an intermediate state of this baryon in pion-nucleon scattering. Let us now calculate the amplitude for the process $2\pi \to N + \bar{N}$, and use these new terms in the calculation of $A^{(1)}$. Equation (10) is thus replaced by

$$A^{(1)}(M', t) = -\frac{g^2(M')}{p'q} \ln \frac{-p'^2 - q^2 - B^2 + 2p'q}{-p'^2 - q^2 - B^2 - 2p'q}. \qquad (15)$$

The solution (8) can be derived as before if $M' < \mu$ and, for these values of M', there is no anomalous threshold. Equation (12) for the singularity becomes

$$t = t_0 \equiv -B^2 + 2(M'^2 + \mu^2) - (M'^2 - \mu^2)^2/B^2. \qquad (16)$$

t_0 is less than $4\mu^2$ except when

$$M'^2 = B^2 + \mu^2. \qquad (17)$$

If $M'^2 > B^2 + \mu^2$, (15) will behave like $-2\pi i/p'q$ at $q = 0$.

The analytic continuation in M' is now not straightforward, as the value (17) is reached before $M' = M$ and, when M'^2 has this value, $t_0 = 4\mu^2$, so that the singularity in $A^{(1)}(M', t)$ is at the end of the range of integration in (8). In order to avoid this, let us give M'^2 a small positive imaginary part. As M' increases, the singularity at $t = t_0$ moves as indicated by the dotted line in Fig. 1, ultimately reaching the point t_1, given by

$$t_1 = -B^2 + 2(M^2 + \mu^2) - (M^2 - \mu^2)^2/B^2. \qquad (18)$$

When M'^2 was less than $B^2 + \mu^2$, i.e., before the singularity went round the point $t = 4\mu^2$, the integral in (8) was along the real axis from $4\mu^2$

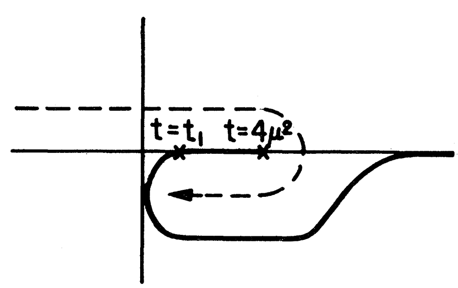

FIG. 1. The singularities in the integrand, and the path of integration, for the anomalous case.

to ∞. When $M'^2 > B^2+\mu^2$, therefore, we have to integrate along the solid curve shown. Note that the integrand in (8) is an analytic function of t in the relevant region, so that we are permitted to deform the contour—we have already pointed out that $N(t)$ has no cut along the positive real axis.

We may now let the curves in Fig. 1 approach the real axis. The difference between the value of (15) on the two sides of the cut for $t_0 < t < 4\mu^2$ is $-2\pi i/p'q$, so that (8) is replaced by

$$A^{(2)}(M',t) = -\frac{1}{4\pi D(t)}\int_{t_0}^{4\mu^2} dt' \frac{N(t')}{|p'|w'(t'-t)} + \frac{1}{8\pi^2 D(t)}\int_{4\mu^2}^{\infty} dt' \frac{q'N(t')A^{(1)}(M',t')}{w'(t'-t)}. \quad (19)$$

$A^{(2)}(M',t)$, given by Eq. (8) if $M'^2 < B^2+\mu^2$ and by Eq. (19) if $M'^2 > B^2+\mu^2$, is thus an analytic function of M'^2 if $t > 4\mu^2$. It might be thought that there is a branch point when $M'^2 = B^2+\mu^2$, but this is not the case, as we would have obtained exactly the same result had we given M'^2 a negative instead of a positive imaginary part. The curves in Fig. 1 would then be reflected across the real axis, but the sign change in the difference of the logarithm on the two sides of the cut would be cancelled by the change of sign of q' in the integrand of Eq. (8).

When $M'^2 > B^2+\mu^2$, we have seen that $A^{(1)}(M',t)$ behaves like $-2\pi i/p'q$ at $q=0$, so that $\mathrm{Im}A^{(2)}(M',t')$ is equal to $-g^2(M')B(4\mu^2)/16\pi\mu(M'^2-\mu^2)^{1/2}$ instead of zero at $t=4\mu^2$. The cut in $A^{(2)}$ will extend to the anomalous threshold at $t=t_0$. By putting $M'=M$ in (19) we obtain the result of physical interest, with an anomalous threshold at $t=t_1$. The method of applying the unitarity condition only above the physical threshold, but for varying masses, and then continuing analytically in the masses, can thus handle both the normal and anomalous cases. The extra cut in the anomalous case is a mathematical consequence of the analytic continuation, and does not appear to have any precise physical significance.

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NEW TEST FOR $\Delta I = 1/2$ IN K^+ DECAY*

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We wish to suggest a practicable test of the $\Delta I = 1/2$ rule, based on a comparison of the pion-energy distributions in τ^+ and $\tau^{+\prime}$ decay. At present, the only check of $\Delta I = 1/2$ in these processes is the successful prediction of the τ'/τ branching ratio.[1] However, it is well known that the branching ratio tells us that $\Delta I = 5/2$ and $\Delta I = 7/2$ are absent but tells us almost nothing about the possible presence of $\Delta I = 3/2$ terms. The only symmetric three-pion states have $I=1$ or $I=3$, and the other, nonsymmetric and hence inhibited states with $I=1$ or $I=2$ (which could be produced by a $\Delta I = 3/2$ term) cannot interfere with the symmetric states in a measurement of decay rates. It is of course very important to learn whether the nonleptonic weak interactions involve a mixture of $\Delta I = 1/2$ and $\Delta I = 3/2$. In particular, it has been noted that such a mixture would result if these interactions arose from a folding of a $\Delta I = 1/2$ strangeness-nonconserving current with the usual $\Delta I = 1$ β-decay current.[2] From experience with the τ'/τ ratio we see that a test for $\Delta I = 3/2$ terms must depend on measurements of pion asymmetries of some sort.[3]

Suppose we let $A_\tau(T_1T_2T_3)$ and $A_{\tau'}(T_1T_2T_3)$ be the Lorentz-invariant amplitudes for K^+ decay into $\pi^+\pi^+\pi^-$ or $\pi^0\pi^0\pi^+$ with kinetic energies T_1, T_2, and T_3, respectively. The Bose statistics of pions implies that

$$A_j(T_1T_2T_3) = A_j(T_2T_1T_3) \quad (1)$$

for $j=\tau$ or τ'. We shall break up A_j into symmetric and nonsymmetric parts:

$$A_j(T_1T_2T_3) = A_j^S(T_1T_2T_3) + A_j^N(T_1T_2T_3), \quad (2)$$