A NEW CLASSICAL SOLUTION OF THE YANG–MILLS FIELD EQUATIONS

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The classical solutions of conformal invariant field theories have been the object of several interesting recent investigations. Two important examples are the neutral scalar field equation

$$\Box\phi + g\phi^3 = 0 \,, \tag{1}$$

and the Yang–Mills equation

$$\partial_\nu F_{\mu\nu} = g\,[F_{\mu\nu}, A_\nu] \,, \tag{2a}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g\,[A_\mu, A_\nu] \,. \tag{2b}$$

In the first case the classical solution

$$\phi = \sqrt{\frac{2}{g}}\,\frac{2a}{x^2 + a^2} \,, \tag{3}$$

has been suggested recently [1].

On the other hand the "instanton" solution [2] of eqs. (2)

$$A_\mu = -\frac{2\mathrm{i}}{g}\,\frac{\Sigma_{\mu\alpha}x^\alpha}{x^2 + a^2} \,, \tag{4a}$$

$$F_{\mu\nu} = \frac{4\mathrm{i}}{g}\,\frac{a^2\Sigma_{\mu\nu}}{(x^2 + a^2)^2} \,, \tag{4b}$$

has received a great amount of interest ‡1.

The similarity between the expressions (3) and (4) has a deep group theoretical origin. It has indeed been shown [1, 3] that they are both invariant under an O(5) subgroup of the O(5, 1) (Euclidean) conformal group.

As a consequence of their common root, many of their properties are similar: both expressions are non-singular and integrable only in the framework of a Euclidean field theory, and in both cases the energy momentum tensor vanishes.

The particular interest of solution (4) of the Yang–Mills equation follows from the presence (in addition to conformal invariance) of invariance under a group of non-Abelian gauge transformations.

A very important property is that, although the field $F_{\mu\nu}$ has a fast convergence at large distances, the potential A_μ converges much more slowly towards a pure gauge term.

$$A_\mu \to \frac{-2\mathrm{i}}{g}\,\frac{\Sigma_{\mu\alpha}x^\alpha}{x^2} = \frac{1}{g} f^{-1}\,\frac{\partial}{\partial x_\mu} f \,, \tag{5}$$

$$f = (\alpha_\mu x^\mu / r) \,, \tag{6}$$

A remarkable consequence of this fact is seen by considering the pseudoscalar density

$$D(x) = (1/64\pi^2)\,\mathrm{Tr}\,\epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \,, \tag{7}$$

which is the divergence of the operator I_μ:

$$(\partial I^\mu / \partial x^\mu) = D \,, \tag{8}$$

$$I^\mu = (\epsilon^{\mu\nu\alpha\beta}/16\pi^2)\,\mathrm{Tr}\,\{A_\nu(\partial_\alpha A_\beta + \tfrac{2}{3} g A_\alpha A_\beta)\} \,. \tag{9}$$

‡1 The Euclidean metric is $r^2 = \bar{x}^2 + x_4^2$, $\Box = \nabla^2 + (\partial/\partial x_4)^2$. If we set $x_4 = \mathrm{i}x_0$, we get the Minkowski metric $r^2 = \bar{x}^2 - x_0^2$, $\Box = \nabla^2 - (\partial/\partial x_0)^2$.

Because of the slow convergence of the field A_μ, one finds that the Pontryagin number,

$$q = g^2 \int \mathrm{d}^4 x\,, \quad D = g^2 \int I^\mu \mathrm{d}\sigma_\mu\,, \tag{10}$$

is different from zero and can take the values ±1.

The beautiful and exciting features of the classical solutions (4) are suggesting that a systematic investigation of a larger class of classical solutions might lead to a more general understanding of the properties of non-Abelian field theories.

In this direction, we wish to point out the existence of a different kind of solutions of the conformal invariand eqs. (1) and (2), whose properties are in a certain way complementary to solutions (3) and (4).

Let us start from the scalar case and look for an elementary solution which, at the same time, is Lorentz invariant and possesses simple dilatation properties. It is easily seen that

$$\phi = \frac{1}{\sqrt{g}}\frac{1}{\gamma}, \quad r = \sqrt{x^2}\,, \tag{11}$$

is one such solution. Even in a Euclidean space, solution (11) is not acceptable, since it is singular both for $r = 0$ and $r = \infty$ [which are the characteristic points of the O(1, 1) dilatation group]. We can, however, improve this solution by using a conformal transformation in order to shift the singularities to two arbitrary positions u and v. We shall thus get

$$\phi = \frac{1}{\sqrt{g}} \sqrt{\frac{(u-v)^2}{(x-u)^2(x-v)^2}}\,. \tag{12}$$

One can choose without loss of generality $u = -v = a$

$$\phi = \frac{2}{\sqrt{g}} \frac{a}{\sqrt{(x-a)^2(x+a)^2}}\,. \tag{13}$$

We can now go to the physical Minkowski space by writing $x_4 = \mathrm{i}x_0$, and orient the vector a_μ along the time direction, $a_\mu = (1,0,0,0)$. Our solution becomes

$$\phi = \frac{2}{\sqrt{g}} \frac{1}{\sqrt{(1+t_+^2)(1+t_-^2)}}\,, \tag{14}$$

where $t_\pm = x_0 \pm |\bar{x}|$ are the advanced, retarded times, respectively. Solution (14) is regular everywhere and gives rise to a finite action in the Minkowski space.

The form of solution (14), which is due to Castell [4], has a simple interpretation in the framework of the conformal group. If we introduce the six-dimensional co-ordinates

$$\xi_i = x_i \quad (i = 1,2,3); \qquad \xi_0 = x_0\,;$$
$$\xi_5 = \tfrac{1}{2}(1 + x_0^2 - \boldsymbol{x}^2); \qquad \xi_6 = \tfrac{1}{2}(1 - x_0^2 + \boldsymbol{x}^2)\,,$$
$$\xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_0^2 + \xi_5^2 - \xi_6^2 = 0\,,$$

we can write solution (14) in the form

$$\phi = \frac{1}{\sqrt{g}} \frac{1}{\sqrt{(\xi_6 + \mathrm{i}\xi_0)(\xi_6 - \mathrm{i}\xi_0)}}\,, \tag{15}$$

exhibiting invariance under the compact O(4) × O(2) subgroup made out of the (1, 2, 3, 5) × (0, 6) rotations‡2 This is the largest compact subgroup contained in the Minkowski conformal group.

We wish now to derive the analogue of the scalar solution (13) for the case of the Yang–Mills equation. Let us start from the solution of eqs. (2) which has simple properties under the Lorentz and dilatation group. It is indeed quite easy to find the following solutions‡3

$$A_\mu = -\frac{\mathrm{i}}{g} \frac{\Sigma_{\mu\alpha} x^\alpha}{x^2}\,, \tag{16a}$$

$$F_{\mu\nu} = \frac{1}{g} [\Sigma_{\mu\alpha} x^\alpha, \Sigma_{\nu\beta} x^\beta] \left[\frac{1}{x^2}\right]^2\,. \tag{16b}$$

It is important to notice that our solution can be written in the form

$$A_\mu = \frac{1}{2g} f^{-1} \partial_\mu f\,. \tag{17}$$

The appearance of the factor $\frac{1}{2}$ as compared to the

‡2 The general six-dimensional form of solution (12) is

$$\phi = \frac{1}{\sqrt{2g}} \left\{ \frac{(\alpha_\mu \beta_\mu)}{(\alpha_\mu \xi_\mu)(\beta_\mu \xi_\mu)} \right\}^{1/2}\,,$$

where α_μ and β_μ are arbitrary six-dimensional vectors of zero length. Correspondingly the general form of solution (3) is

$$\phi = \sqrt{\frac{2}{g}} \frac{1}{(\xi_\mu \lambda_\mu)}\,; \quad \lambda_\mu \lambda_\mu = 1\,.$$

‡3 Solutions (4) and (16) can be derived simultaneously by inserting in eqs. (2) the "ansatz" $A_\mu = -2\mathrm{i}c((\Sigma_{\mu\alpha} x^\alpha)/(a^2 + x^2))$. One is led to the two equations $\frac{3}{2}a^2(1 - cg) = 0$; $(1 - cg)(\frac{1}{2} - cg) = 0$, which shows that for $a \neq 0$ we must have $cg = 1$ whereas in the special case $a = 0$ we have the supplementary solution $cg = \frac{1}{2}$.

asymptotic form (5) is essential and it accounts for the presence of a non-vanishing $F_{\mu\nu}$.

Our solutions (16) are singular both for $x_\mu \to 0$ and $x_\mu \to \infty$. As in the scalar case, we take advantage of conformal invariance and displace the singular points to u and v. After a somewhat cumbersome calculation, which also involves a gauge transformation, one gets

$$A_\mu = (-2\mathrm{i}/g)\Sigma_{\mu\alpha} s^\alpha \,, \tag{18a}$$

$$F_{\mu\nu} = (4/g)\,[\Sigma_{\mu\alpha} w^\alpha, \Sigma_{\nu\beta} w^\beta]\,, \tag{18b}$$

where

$$s_\alpha = \frac{1}{2}\left\{\frac{(x-u)_\alpha}{(x-u)^2} + \frac{(x-v)_\alpha}{(x-v)^2}\right\},$$

$$w_\alpha = \frac{1}{2}\left\{\frac{(x-u)_\alpha}{(x-u)^2} - \frac{(x-v)_\alpha}{(x-v)^2}\right\}. \tag{19}$$

It is easy to verify, independently of any heuristic derivation, that the expressions (18) are indeed solutions of the Yang–Mills equations (2). Of course, setting $u \to 0, v \to \infty$, one is led back to solution (16).

Although the natural framework in which our solution should be applied is the physical Minkowski space, it is instructive [also for the sake of comparison with solution (4)] to start by looking at its properties in a Euclidean space.

The asymptotic form of our solution (18):

$$A_\mu \to \frac{1}{g} f^{-1} \frac{\partial}{\partial x_\mu} f\,, \tag{20}$$

does coincide with the asymptotic form, eqs. (5), (6), of the "instanton" solution and gives rise to the same value of the Pontryagin surface integral (10). On the other hand, it is easy to see [from the commutator form of eq. (18b)] that the pseudoscalar density (18b)] that the pseudoscalar density $D(x)$ vanishes. The solution of this apparent paradox comes from the fact that the expressions for $F_{\mu\nu}$ are singular at $x \to u$ and $x \to v$. If one computes the surface integral $\int I_\mu \, \mathrm{d}\sigma_\mu$ on two small spheres centered at the points $x = u$, $x = v$, respectively, one is led to the result

$$D(x) = \tfrac{1}{2} q \{\delta^4(x-u) + \delta^4(x-v)\}\,. \tag{21}$$

In this respect, our Euclidean result differs from solution (4) by the fact that in our case the pseudoscalar density $D(x)$ is concentrated in two points instead of being spread over all space (uniformly on the Adler hypersphere). In both cases the same values of the topological number emerge.

The fundamental feature of our expressions (18) is that they can be simply continued to the Minkowski space, leading to a solution which is non-singular and normalizable.

As in the scalar case, we take $-u_\mu = v_\mu = a_\mu$ and orient a_μ along the time direction. We can now continue to the Minkowski space time by setting $x_4 = \mathrm{i}x_0$ in eqs. (18), (19).

The Minkowski vectors s_α and w_α will now take the form:

$$s_\mu = \frac{1}{2}\left(\frac{t_+}{1+t_+^2} Y_\mu^+ + \frac{t_-}{1+t_-^2} Y_\mu^-\right),$$

$$\hat{w}_\mu = -\mathrm{i} w_\mu = \frac{1}{2}\left(\frac{1}{1+t_+^2} Y_\mu^+ + \frac{1}{1+t_-^2} Y_\mu^-\right), \tag{22}$$

$$Y_\mu^\pm = \frac{\partial}{\partial x_\mu} t_\pm = \left(1, \pm \frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right).$$

From eqs. (18), (22), one clearly sees that our Minkowski solution is well-behaved asymptotically and is regular everywhere.

We see that our expression for $F_{\mu\nu}$, and therefore for all observable quantities, depends solely on the vector $\hat{w}_\mu$. As shown in eq. (22) the vector $\hat{w}_\mu$ is always timelike (contained in the future light cone) and its modulus is given by

$$\hat{w}^2 = \frac{-1}{(1+t_+^2)(1+t_-^2)}\,. \tag{23}$$

It is also amusing to note that

$$\hat{w}_\mu = -\frac{\mathrm{i}}{4}\frac{\partial}{\partial x_\mu} \lg \frac{(1+\mathrm{i}t_+)(1+\mathrm{i}t_-)}{(1-\mathrm{i}t_+)(1-\mathrm{i}t_-)}\,, \tag{24}$$

The group theoretical meaning of eq. (24) is clearly understood if one remarks that the surfaces

$$-\frac{\mathrm{i}}{4}\lg \frac{(1+\mathrm{i}t_+)(1+\mathrm{i}t_-)}{(1-\mathrm{i}t_+)(1-\mathrm{i}t_-)} = \delta\,, \qquad -\frac{\pi}{2} < \delta < \frac{\pi}{2}\,, \tag{25}$$

are invariant under the (1, 2, 3, 5) rotations and are transformed into each other by the (0, 6) rotations. The physical meaning of our solution is simple exhibited by using a *local* Lorentz frame of reference in which $\hat{w}_\mu$ is oriented along the time direction. In this frame, eq. (18b) becomes:

$$\boldsymbol{E}=0, \quad \boldsymbol{H}=\frac{2}{g}\frac{\mathrm{i}\boldsymbol{\sigma}}{(1+t_+^2)(1+t_-^2)} . \tag{26}$$

Equation (26) clearly shows that the pseudoscalar density $D(x)$, which is proportional to the invariant product $\boldsymbol{E}\cdot\boldsymbol{H}$, vanishes everywhere.

Our solution leads for the different quantities of physical interest the following expressions.

Action. The Lagrangian density $\mathcal{L}$ is given by

$$\mathcal{L}=-\tfrac{1}{8}\operatorname{Tr}F_{\mu\nu}F^{\mu\nu}=(12/g^2)(\hat{w}^2)^2 = \frac{12}{g^2}\left\{\frac{1}{(1+t_+^2)(1+t_-^2)}\right\}^2 , \tag{27}$$

leading to a total action:

$$A=\int \mathcal{L}\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}x_0 = 3\pi^3/2g^2 . \tag{28}$$

Energy momentum tensor. The tensor $\theta_{\mu\nu}$ is given by

$$\theta_{\mu\nu}=-\tfrac{1}{2}\operatorname{Tr}\{F_{\mu\rho}F^{\rho}_{\nu}-\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu}\} = -\frac{4}{g^2}\{4\hat{w}_\mu\hat{w}_\nu-\hat{w}^2g_{\mu\nu}\}\hat{w}^2 , \tag{29}$$

leading to a total energy:

$$E=\int\theta_{00}\,\mathrm{d}^3x=\frac{12}{g^2}\int\frac{\mathrm{d}^3x}{(1+x^2)^4}=\frac{3}{2}\frac{\pi^2}{g^2} . \tag{30}$$

We finally recall that, in the Minkowski framework, the Pontryagin number vanishes.

The previous discussion clearly shows the complementarity between the Euclidean and Minkowski solutions of the Yang-Mills equations. In the latter case, we no longer have the beauty and excitement of a non-vanishing topological number, this has been given up in exchange for the property (essential in a Minkowski description) of having finite non-vanishing energy and action.

Our results indicate that much progress can still be expected by the combined study of the gauge and conformal properties of Yang–Mills theory ‡4.

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‡4 General relations between solutions of eqs. (1) and (2) have been recently discussed in a Princeton preprint by F. Wilczek.

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