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CALCULATION OF THE YANG–MILLS VACUUM WAVE FUNCTIONAL

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Working in the Schrödinger representation and $A_0 = 0$ gauge, an approximate Yang–Mills ground-state wave functional $\Psi[A]$ is constructed in the following way: we begin by constructing the vacuum wave functional $\Psi_0[A]$ of an Abelian gauge-field with global SU(2) symmetry, and then modify and generalize $\Psi_0[A]$ so that it becomes invariant under local SU(2) gauge transformations. This ansatz leads to a solution of the Schrödinger equation $H\Psi[A] = \epsilon_0\Psi[A]$ for the Yang–Mills vacuum, which, although approximate, may correctly describe its confinement properties.

Given $\Psi[A]$, it is argued that the vacuum expectation values of the Wilson loop integral $A(\mathrm{C})$ and of 't Hooft's flux-tube operator $B(\mathrm{C})$ satisfy the Wilson–'t Hooft criteria $\langle A(\mathrm{C})\rangle \sim \mathrm{e}^{-\mathrm{area}\,(\mathrm{C})}$, $\langle B(\mathrm{C})\rangle \sim \mathrm{e}^{-\mathrm{perimeter}\,(\mathrm{C})}$, for the confinement phase of a gauge field. The confinement mechanism is essentially identical to the one discovered by Polyakov in 3-dimensional compact QED. The reason for the similarity is that there is an "analog-gas" approximation to fixed-time vacuum expectation values $\langle\Psi|Q|\Psi\rangle$: the analog gas in this case is a plasma of smoothed Wu–Yang monopoles.

1. Introduction

It is generally recognized that the confinement properties of a non-Abelian gauge theory are intimately related to the structure of the ground state, or "vacuum", of the theory. In fact, it has recently been shown by 't Hooft [1] that the phases of a quantized gauge field can be classified according to the vacuum expectation values (VEVs) of two operators $A(\mathrm{C})$ and $B(\mathrm{C})$, where $A(\mathrm{C})$ is the Wilson loop integral for the closed curve C, and where $B(\mathrm{C})$ acts on states like a singular gauge transformation, creating a line of magnetic flux along the curve C. The confinement phase of a gauge field, with no spontaneous symmetry breaking, is characterized by

$$\langle A(\mathrm{C})\rangle \sim \mathrm{e}^{-\mathrm{area}(\mathrm{C})}, \tag{1.1a}$$

$$\langle B(C)\rangle \sim \mathrm{e}^{-\mathrm{perimeter}(\mathrm{C})}. \tag{1.1b}$$

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If the curve C is chosen to lie entirely in 3-space at fixed time t, then the VEV of A(C), B(C), and in general the VEV of any operator Q evaluated at fixed time, depends only on the ground state of the theory, since

$$\langle Q\rangle_t = \langle\Psi|Q|\Psi\rangle\,, \tag{1.2}$$

where $\langle\ \rangle_t$ denotes the fixed time (or, for N-point functions, equal times) VEV, and where Ψ represents the ground-state wave functional.

The purpose of this paper is to construct an approximate non-perturbative expression for the Yang–Mills ground-state wave functional Ψ, and use this expression to test the confinement properties of the theory. We will work in the Schrödinger representation with a pure Yang–Mills field, SU(2) gauge symmetry, canonically quantized in the $A_0=0$ gauge. The problem, then, is to solve for the ground state of the Schrödinger wave-functional equation

$$\tfrac{1}{2}\int \mathrm{d}^3x\left\{-\frac{\delta^2}{\delta A_k^c(x)^2}+B_k^c(x)^2\right\}\Psi[A]=\mathscr{E}_0\Psi[A]\,, \tag{1.3}$$

subject to the $A_0=0$ gauge subsidiary condition (Gauss's Law)

$$[\delta^{ac}\partial_k + g\epsilon_{abc}A_k^b(x)]\frac{\delta}{\delta A_k^c(x)}\Psi[A]=0\,. \tag{1.4}$$

Our approach to solving (1.3) non-perturbatively is guided by the following considerations: if we were to set the bare coupling g to zero, then (1.3) would reduce to the Schrödinger equation of an Abelian gauge field. Also, for arbitrary g, (1.4) implies that $\Psi[A]$ is invariant under local, infinitesimal, SU(2) gauge transformations. So it is reasonable to construct trial wave functionals based on the idea that the form of the ground-state solution of the Abelian theory should be combined with the requirement of local gauge invariance in the simplest possible way. The strategy adopted in this paper is therefore to begin by solving for the vacuum wave functional $\Psi_0[A]$ of an Abelian gauge field with global SU(2) symmetry, and then modify and generalize this state so that it becomes invariant under local SU(2) gauge transformations. By a careful choice of parameters in the trial wave functional, this approach leads to an approximate solution of the Yang–Mills Schrödinger equation (1.3), which also satisfies the Gauss' law constraint (1.4).

Using this expression for the ground-state $\Psi[A]$, we verify that the confinement criteria of eq. (1.1) are satisfied. The physical mechanism underlying the confinement phenomenon is essentially identical to the one discovered by Polyakov [2] for compact QED in 2+1 dimensions. The reason for this similarity is that fixed-time VEVs in the 3+1 dimensional Yang–Mills theory are found to be closely related to the vacuum-to-vacuum amplitudes of Yang–Mills theory formulated in 3-dimensional Euclidean space. In fact, the basic result of our analysis will be that a first approximation to the VEV $\langle Q\rangle_t$ at fixed time is given by

$$\langle\Psi|Q|\Psi\rangle \approx \frac{\int \mathscr{D}A(\boldsymbol{x})Q \exp\left[-\frac{1}{\mu}\int d^3x[B_k^c(x)]^2\right]}{\int \mathscr{D}A(\boldsymbol{x}) \exp\left[-\frac{1}{\mu}\int d^3x[B_k^c(x)]^2\right]}, \tag{1.5}$$

where the exponent $(1/\mu)\int d^3xB^2$ is the action of the 3-dimensional theory. The residual gauge freedom in (1.5) must be extracted by the standard techniques. As in 3-dimensional compact QED there is an analog-gas approximation to (1.5); in this case the analog gas consists of a plasma of smoothed Wu–Yang monopoles. This picture is not unlike Mandelstam's [3] picture of the vacuum as a coherent state of monopoles, and is also probably compatible with the meron picture of Callen, Dashen and Gross [4].

The organization of this paper is as follows: in sect. 2 we quantize and solve the free Maxwell field theory in wave-functional formalism. Nothing new is contained in this section, but since the wave-functional formalism is rarely used and may be unfamiliar to the reader, the solution of the Abelian theory is carried out in detail. In sect. 3, non-Abelian, locally gauge-invariant wave functionals are constructed, and a solution for the Yang–Mills ground state is obtained. In sect. 4 we set up a crude analog-gas approximation to (1.5), verify the confinement criteria, and discuss the physical picture associated with our vacuum wave functional.

2. The Abelian vacuum

The Hamiltonian of the free Maxwell field $A_k(x)$ is given by

$$H=\tfrac{1}{2}\int d^3x[E_k^2(x)+b_k^2(x)], \tag{2.1}$$

where

$$b_k(x)=\epsilon_{kij}\partial_iA_j(x). \tag{2.2}$$

In quantizing the Maxwell field in $A_0=0$ gauge, we impose the equal-time commutators

$$[E_i(\boldsymbol{x}), A_j(\boldsymbol{x}')]=i\delta_{ij}\delta^3(\boldsymbol{x}-\boldsymbol{x}'), \tag{2.3}$$

and seek solutions to the Schrödinger wave-functional equation

$$H\Psi[A]=i\frac{\partial}{\partial t}\Psi[A], \tag{2.4}$$

subject to the $A_0=0$ gauge subsidiary condition

$$[\partial_kE_k]\Psi[A]=0. \tag{2.5}$$

Working in a basis where $A_k(x)$ is diagonal, the conjugate momentum operator E_k has the form

$$E_k(x) = i\frac{\delta}{\delta A_k(x)}. \tag{2.6}$$

So the problem is to find solutions to the time-independent Schrödinger equation

$$\tfrac{1}{2}\int d^3x\left[-\frac{\delta^2}{\delta A_k(x)^2} + b_k(x)^2\right]\Psi[A] = \mathscr{E}\Psi[A], \tag{2.7}$$

with

$$\partial_i \frac{\delta}{\delta A_i(x)}\Psi[A] = 0. \tag{2.8}$$

Separate $A_k(x)$ into transverse and longitudinal parts

$$\begin{aligned} A_i(x) &= A_i^{\mathrm{T}}(x) + A_i^{\mathrm{L}}(x) \\ &= \frac{1}{(2\pi)^{3/2}}\int d^3k\,[A_i^{\mathrm{T}}(\boldsymbol{k}) + A_i^{\mathrm{L}}(\boldsymbol{k})]\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \end{aligned} \tag{2.9}$$

where

$$\partial_i A_i^{\mathrm{T}} = 0, \qquad A_i^{\mathrm{L}} = \partial_i\varphi. \tag{2.10}$$

Then

$$H = \tfrac{1}{2}\int d^3k\left\{-\frac{\delta^2}{\delta A_i^{\mathrm{T}}(\boldsymbol{k})\delta A_i^{\mathrm{T}}(-\boldsymbol{k})} - \frac{\delta^2}{\delta A_i^{\mathrm{L}}(\boldsymbol{k})\delta A_i^{\mathrm{L}}(-\boldsymbol{k})} + k^2 A_i^{\mathrm{T}}(\boldsymbol{k})A_i^{\mathrm{T}}(-\boldsymbol{k})\right\}. \tag{2.11}$$

The subsidiary condition becomes

$$\partial_k \frac{\delta}{\delta A_k^{\mathrm{L}}(x)}\Psi[A] = 0, \tag{2.12}$$

which implies that $\Psi[A] = \Psi[A^{\mathrm{T}}]$, and therefore

$$\begin{aligned} H\Psi &= \tfrac{1}{2}\int d^3k\left[-\frac{\delta^2}{\delta A_i^{\mathrm{T}}(\boldsymbol{k})\delta A_i^{\mathrm{T}}(-\boldsymbol{k})} + k^2 A_i^{\mathrm{T}}(\boldsymbol{k})A_i^{\mathrm{T}}(-\boldsymbol{k})\right]\Psi \\ &= \mathscr{E}\Psi. \end{aligned} \tag{2.13}$$

Now introduce

$$\begin{aligned} a(\boldsymbol{k},\lambda) &= \frac{1}{\sqrt{2k}}\epsilon_\lambda^i(\boldsymbol{k})\left[\frac{\delta}{\delta A_i^{\mathrm{T}}(-\boldsymbol{k})} + kA_i^{\mathrm{T}}(\boldsymbol{k})\right], \\ a^+(\boldsymbol{k},\lambda) &= \frac{1}{\sqrt{2k}}\epsilon_\lambda^i(\boldsymbol{k})\left[-\frac{\delta}{\delta A_i^{\mathrm{T}}(\boldsymbol{k})} + kA_i^{\mathrm{T}}(-\boldsymbol{k})\right], \end{aligned} \tag{2.14}$$

where ϵ_1, ϵ_2 are polarization vectors orthogonal to each other and to $\boldsymbol{k}$ so that

$$[a(\boldsymbol{k},\lambda), a^+(\boldsymbol{k}';\lambda')] = \delta_{\lambda\lambda'}\delta^3(\boldsymbol{k}-\boldsymbol{k}'),$$

$$A_i^{\mathrm{T}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3k}{\sqrt{2k}}\sum_{\lambda=1}^{2}\epsilon_\lambda^i(\boldsymbol{k})[a(\boldsymbol{k},\lambda)\,\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} + a^+(\boldsymbol{k},\lambda)\,\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}], \tag{2.15}$$

$$H = \int \mathrm{d}^3k\, k \sum_{\lambda=1}^{2}[a^+(\boldsymbol{k},\lambda)a(\boldsymbol{k},\lambda) + \tfrac{1}{2}\delta^3(0)].$$

The creation-annihilation operators in (2.14) supply the correspondence between the usual formalism and the wave-functional formalism.

Eq. (2.13) is simply the Schrödinger equation for an infinite-dimensional harmonic oscillator; the ground-state solution is easily seen to be

$$\begin{aligned}\Psi_0[A] &= N\exp\left[-\tfrac{1}{2}\int \mathrm{d}^3k\, kA_i^{\mathrm{T}}(\boldsymbol{k})A_i^{\mathrm{T}}(-\boldsymbol{k})\right]\\ &= N\exp\left[-\tfrac{1}{2}\int \mathrm{d}^3k\,\frac{1}{k}(\boldsymbol{k}\times\boldsymbol{A}(\boldsymbol{k}))\cdot(\boldsymbol{k}\times\boldsymbol{A}(-\boldsymbol{k}))\right]\\ &= N\exp\left[-\frac{1}{4\pi^2}\int \mathrm{d}^3x_2\,\mathrm{d}^3x_1\, b_n(x_2)b_n(x_1)\frac{1}{|x_2-x_1|^2}\right].\end{aligned} \tag{2.16}$$

This solution is given by Wheeler [5]. Excited states are constructed by operating successively on $\Psi_0[A]$ with the creation operator $a^+(\boldsymbol{k},\lambda)$ of eq. (2.14).

Generalization of (2.16) to the case of an Abelian field theory with a *global* SU(2) invariance is trivial. In that case the Hamiltonian is

$$H = \tfrac{1}{2}\int \mathrm{d}^3x\left[-\frac{\delta^2}{\delta A_k^c(x)^2} + b_k^c(x)^2\right], \tag{2.17}$$

where superscript $c = 1, 2, 3$ is the isospin index, and the ground-state wave functional is just

$$\Psi_0[A] = N\exp\left[-\frac{1}{2\pi^2}\int \mathrm{d}^3x_2\,\mathrm{d}^3x_1\,\mathrm{Tr}\,[b_n(x_2)b_n(x_1)]\frac{1}{|x_2-x_1|^2}\right], \tag{2.18}$$

where

$$b_n(x) = \epsilon_{nij}\partial_i A_j(x) = b_n^a L_a = b_n^a \tfrac{1}{2}\tau_a, \tag{2.19}$$

with τ_a the Pauli matrices. Eq. (2.18) is the starting point of our work in the following sections. It can be readily verified that

$$\begin{aligned}H\Psi_0[A] &= \mathscr{E}_0\Psi_0[A]\\ &= \{\tfrac{1}{2}\times 2_{(\mathrm{spin})}\times 3_{(\mathrm{isospin})}\times\delta^3(0)\times\int \mathrm{d}^3k\, k\}\Psi_0[A],\end{aligned} \tag{2.20}$$

which is the appropriate zero-point energy for this free-field theory.

3. The Yang–Mills vacuum

We now confront the problem of solving the Schrödinger equation for the Yang–Mills field:

$$\tfrac{1}{2}\int d^3x\left[-\frac{\delta^2}{\delta A_k^c(x)^2}+B_k^c(x)^2\right]\Psi[A]=\mathscr{E}_0\Psi[A]\,, \tag{3.1}$$

where

$$B_n=B_n^aL_a=\epsilon_{nij}(\partial_iA_j-igA_iA_j)\,, \tag{3.2}$$

subject to the $A_0=0$ gauge subsidiary condition

$$[\delta^{ac}\partial_k+g\epsilon_{abc}A_k^b(x)]\frac{\delta}{\delta A_k^c(x)}\Psi[A]=0\,. \tag{3.3}$$

Our approach is to reconcile the form of the Abelian vacuum (2.18) with the requirement of local SU(2) gauge invariance imposed by (3.3). This leads to the following ansatz for the Yang–Mills ground state:

$$\begin{aligned}\Psi[A]&=N\exp\left[-\int d^3x_2\,d^3x_1\,\mathrm{Tr}\,[B_n(x_2)V_{21}B_n(x_1)V_{12}]\phi(x_2,x_1)\right]\\&=N\,e^{-R[A]}\,,\end{aligned} \tag{3.4}$$

where N is a normalization factor, and where V_{ab} is the path-ordered line integral

$$V_{ab}=V(x_a,x_b,A(x))\equiv \mathrm{P}\exp\left[ig\int_{x_b}^{x_a}A_k(z)\,dz^k\right], \tag{3.5}$$

with the path between x_a and x_b chosen to be a straight line. The function $\phi(x_2,x_1)$ is to be determined.

Substituting the trial wave functional (3.4) in the Schrödinger equation (3.1) gives

$$\tfrac{1}{2}\int d^3x\left[\frac{\delta^2R}{\delta A_k^c(x)^2}-\left\{\frac{\delta R}{\delta A_k^c(x)}\right\}^2+B_k^c(x)^2\right]\Psi=\mathscr{E}_0\Psi\,. \tag{3.6}$$

In the Abelian case, the zero-point energy $\mathscr{E}_0$ comes from the $\delta^2R/\delta A^2$ term, while the quadratic $(\delta R/\delta A)^2$ term cancels the "potential energy" B^2 term. Before evaluating (3.6) in the non-Abelian case, we first introduce some notation:

$$\begin{aligned}\delta B_1&\equiv\frac{\delta B_n(z_1)}{\delta A_k^c(x)}=\epsilon_{nik}\{\partial_i^{(z_1)}L_c-ig[A_i,L_c]\}\delta^3(x-z_1)\,,\\ \delta V_{21}&\equiv\frac{\delta V(z_2,z_1,A)}{\delta A_k^c(x)}=\frac{\delta}{\delta A_k^c(x)}\mathrm{P}\exp\left[ig\int_{z_1}^{z_2}A_k\,dz^k\right].\end{aligned} \tag{3.7}$$

To evaluate δV_{21}, let $(\tau,\boldsymbol{\rho})\equiv(\tau,r,\theta)$ represent coordinates in a cylindrical coordinate system centered on the line passing through points z_1 and z_2. Coordinate τ

runs along this line, with $\tau_1 \equiv \tau(z_1) < \tau_2 = \tau(z_2)$, while $\boldsymbol{\rho} = (\mathrm{r}, \theta)$ labels the position perpendicular to this line. Then

$$\begin{aligned}\delta V_{21} &= ig \int_{z_1}^{z_2} \mathrm{d}z^n\, \delta_{nk} \delta^3(z-x) \exp\left[ig \int_x^{z_2} A \cdot \mathrm{d}z'\right] L_c \exp\left[ig \int_{z_1}^{x} A \cdot \mathrm{d}z'\right] \\ &= ig \int_{\tau_1}^{\tau_2} \mathrm{d}\tau f_k^{z_1 z_2} \delta(\tau - \tau_x) \delta^2(\rho_x) \exp\left[ig \int_x^{z_2} A \cdot \mathrm{d}z'\right] L_c \exp\left[ig \int_{z_1}^{x} A \cdot \mathrm{d}z'\right] \\ &= ig f_k^{z_1 z_2} \delta^2(\rho_x) \exp\left[ig \int_x^{z_2} A \cdot \mathrm{d}z'\right] L_c \exp\left[ig \int_{z_1}^{x} A \cdot \mathrm{d}z'\right], \end{aligned} \tag{3.8}$$

where the line integrals are path-ordered (the P is omitted for simplicity) and

$$f_k^{z_1 z_2}(\tau) = \frac{\mathrm{d}z^k(\tau)}{\mathrm{d}\tau} = -f_k^{z_2 z_1}(\tau). \tag{3.9}$$

Then,

$$\frac{\delta R}{\delta A_k^c(x)} = 2 \int \mathrm{d}^3x_2\, \mathrm{d}^3x_1\, \mathrm{Tr}\,[\delta B_2 V_{21} B_1 V_{12} + B_2 \delta V_{21} B_1 V_{12}] \phi(x_2, x_1), \tag{3.10}$$

where we have used $\phi(z_2, z_1) = \phi(z_1, z_2)$, since $\phi(x_2, x_1)$ is necessarily an even function, so that altogether

$$\begin{aligned} H\Psi[A] = \Big\{ \int & \mathrm{d}^3x_2\, \mathrm{d}^3x_1\, \mathrm{d}^3x\, \phi(x_2, x_1) \\ & \times \{\mathrm{Tr}\,[\delta B_2 V_{21} \delta B_1 V_{12}] && [ZP_0] \\ & + \mathrm{Tr}\,[B_2 \delta(\delta V_{21}) B_1 V_{12}] && [ZP_1] \\ & + \mathrm{Tr}\,[B_2 \delta V_{21} B_1 \delta V_{12}] && [ZP_2] \\ & + 2\mathrm{Tr}[B_2\{V_{21} \delta B_1 \delta V_{12} + \delta V_{21} \delta B_1 V_{12}\}]\Big\} && [ZP_3] \\ -2 \int & \mathrm{d}^3x_2\, \mathrm{d}^3x_1\, \mathrm{d}^3x_2'\, \mathrm{d}^3x_1'\, \mathrm{d}^3x \phi(x_2, x_1) \phi(x_2', x_1') \\ & \times \{\mathrm{Tr}\,[\delta B_2 V_{21} B_1 V_{12}]\, \mathrm{Tr}\,[\delta B_{2'} V_{21}' B_{1'} V_{12}'] && [Q_0] \\ & + \mathrm{Tr}\,[B_2 \delta V_{21} B_1 V_{12}]\, \mathrm{Tr}\,[B_{2'} \delta V_{21}' B_{1'} V_{12}'] && [Q_1] \\ & + 2\,\mathrm{Tr}\,[\delta B_2 V_{21} B_1 V_{12}]\, \mathrm{Tr}\,[B_{2'} \delta V_{21}' B_{1'} V_{12}']\} && [Q_2) \\ + \int & \mathrm{d}^3x\, \mathrm{Tr}\,[B_n(x) B_n(x)]\Big\} \Psi[A]. && [PE] \end{aligned} \tag{3.11}$$

At this stage, it may look doubtful that all the terms multiplying Ψ on the r.h.s. of (3.11) would add up to a constant; moreover, terms ZP_1 and ZP_2 are highly singular. In fact,

$$ZP_1 = \int d^3x_2\, d^3x_1\, d^3x\, \mathrm{Tr}[B_2\delta^2 V_{21} B_1 V_{12}]\phi(x_2, x_1)\,, \tag{3.12}$$

where

$$\begin{aligned}\delta^2 V_{21} &= \frac{\delta}{\delta A_k^c(x)}\delta V_{21}\\ &= (ig)^2 (f_k^{12} f_k^{12})\delta^2(\rho_x)\delta^2(\rho_x)\theta(\tau_x)\\ &\quad\times \exp\left[ig\int_x^{x_2} A\cdot dz\right](\tfrac{1}{2}L_cL_c + \tfrac{1}{2}L_cL_c)\exp\left[ig\int_{x_1}^{x} A\cdot dz\right]\\ &= -\tfrac{3}{4}g^2\delta^2(0)\delta^2(\rho_x)\theta(\tau_x)\exp\left[ig\int_{x_1}^{x_2} A\cdot dz\right],\end{aligned} \tag{3.13}$$

with

$$\theta(\tau_x) = \begin{cases}1, & \tau_1 < \tau_x < \tau_2\\ 0, & \text{otherwise}\end{cases}, \tag{3.14}$$

so that

$$\begin{aligned}ZP_1 &= -\tfrac{3}{4}g^2\delta^2(0)\int d^3x_2\, d^3x_1\, \mathrm{Tr}\,[B_2 V_{21} B_1 V_{12}]\phi(x_2, x_1)\int d^3x\, \delta^2(\rho_x)\theta(\tau_x)\\ &= -\tfrac{3}{4}g^2\delta^2(0)\int d^3x_2\, d^3x_1\, \mathrm{Tr}\,[B_2 V_{21} B_1 V_{12}]\phi(x_2, x_1)|x_2 - x_1|\,.\end{aligned} \tag{3.15}$$

Likewise,

$$\begin{aligned}ZP_2 &= \int d^3x_2\, d^3x_1\, d^3x\, \mathrm{Tr}\,[B_2\delta V_{21} B_1 \delta V_{12}]\phi(x_2, x_1)\\ &= (ig)^2\int d^3x_2\, d^3x_1\, d^3x (f_k^{12} f_k^{21})\delta^2(\rho_x)\delta^2(\rho_x)\theta(\tau_x)\\ &\quad\times \mathrm{Tr}\left[B_2\exp\left[ig\int_x^{x_2} A\cdot dz\right]L_c\exp\left[ig\int_{x_1}^{x} A\cdot dz\right]B_1\right.\\ &\quad\left.\times\exp\left[ig\int_x^{x_1} A\cdot dz\right]L_c\exp\left[ig\int_{x_2}^{x} A\cdot dz\right]\right],\end{aligned} \tag{3.16}$$

and, using

$$\begin{aligned}&\mathrm{Tr}\,[B_2\exp\left[ig\int_x^{x_2} A\cdot dz\right]L_c\exp\left[ig\int_{x_1}^{x} A\cdot dz\right]B_1\exp\left[ig\int_x^{x_1} A\cdot dz\right]L_c\\ &\quad\times\exp\left[ig\int_{x_2}^{x} A\cdot dz\right]]\\ &\quad= -\tfrac{1}{4}\mathrm{Tr}\,[B_2 V_{21} B_1 V_{12}]\,,\\ &f_k^{12} f_k^{21} = -1\,,\end{aligned} \tag{3.17}$$

ZP_2 becomes

$$ZP_2 = -\tfrac{1}{4}g^2\delta^2(0)\int d^3x_2\, d^3x_1\, \mathrm{Tr}[B_2V_{21}B_1V_{12}]\phi(x_2,x_1)|x_2-x_1|\,, \tag{3.18}$$

and therefore

$$ZP_1+ZP_2 = -g^2\delta^2(0)\int d^3x_2\, d^3x_1\, \mathrm{Tr}[B_2V_{21}B_1V_{12}]\phi(x_2,x_1)|x_2-x_1|\,. \tag{3.19}$$

So already the simple form (3.4) has led to an area singularity in (3.11). Apart from this short-distance singularity in (3.11), there is also the possibility of divergences from the integrations over large distances in (3.19). Suppose, for example, that $\phi(x)$ decreases like $|x|^{-2}$ at large distances, as in the Abelian theory. Then $R[A]$ is finite, and likewise $\Psi[A]$ is non-zero, only for configurations $A(x)$ falling faster than $|x|^{-1}$. On the other hand ZP_1+ZP_2 is (infrared) finite only for $A(x)$ falling faster than $|x|^{-3/2}$. So there are configurations for which $\Psi[A]$ is non-zero, while ZP_1+ZP_2 diverges (the fact that the Yang–Mills energy density of the axial-gauge perturbation vacuum is infrared divergent, and the relevance of infrared finite states for confinement, was first pointed out by Mandelstam in ref. [3]). In order to avoid such infrared-divergent contributions to the energy, $\phi(x)$ can be chosen to fall off exponentially at large x. In fact, if $\phi(x)$ is a function sharply peaked at $x=0$, there is the hope of cancelling ZP_1 and ZP_2 against the potential energy $\int B^2$ term.

In order to proceed it will be necessary first to regularize the terms in (3.11), and then to show, as in perturbation theory, that all infinities can be absorbed into infinite rescalings of the bare coupling g and field $A(x)$. We will regularize by using the definition [6] of the functional derivative

$$\frac{\delta}{\delta A_k^c(x)}F[A] = \lim_{\lambda\to 0}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\{F[A_i^a(z)+\epsilon\delta_{ik}\delta^{ac}\delta_\lambda^3(z-x)]-F[A_i^a]\}\,, \tag{3.20}$$

where $\delta_\lambda^{(3)}(z)$ is a δ-sequence which is chosen such that

$$\delta_\lambda^3(z)=0\,,\qquad |z|>\lambda\,,$$

$$\int d^3z\,\delta_\lambda^3(z)=1\,. \tag{3.21}$$

It will be convenient to use

$$\delta_\lambda^3(z)=\begin{cases}\dfrac{1}{\frac{4}{3}\pi\lambda^3}\,, & |z|<\lambda\,,\\ 0\,, & |z|>\lambda\,.\end{cases} \tag{3.22}$$

Now define

$$\frac{\delta_\lambda}{\delta_\lambda A_k^c(x)}F[A] = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\{F[A_i^a(z)+\epsilon\delta_{ik}\delta^{ac}\delta_\lambda^3(z-x)]-F[A_i^a]\}\,, \tag{3.23}$$

and let

$$H = \lim_{\lambda\to 0} H_\lambda$$
$$= \lim_{\lambda\to 0} \tfrac{1}{2}\int d^3x\left[-\frac{\delta_\lambda}{\delta_\lambda A_k^c}\frac{\delta}{\delta A_k^c} + B_k^{c^2}\right]. \quad (3.24)$$

It is not hard to show that H_λ, like H, is a hermitian operator. First note that

$$\left\langle \Psi_1, i\frac{\delta_\lambda}{\delta_\lambda A(x)}\Psi_2\right\rangle = \int \mathscr{D}A\Psi_1^*[A]i\frac{d}{d\epsilon}\Psi_2[A+\epsilon\delta_\lambda^3]\Big|_{\epsilon=0}$$
$$= i\frac{d}{d\epsilon}\int \mathscr{D}A\Psi_1^*[A-\epsilon\delta_\lambda]\Psi_2[A]\Big|_{\epsilon=0}$$
$$= \int \mathscr{D}A\left\{-i\frac{\delta_\lambda}{\delta_\lambda A(x)}\Psi_1^*\right\}\Psi_2[A]$$
$$= \left\langle\left(i\frac{\delta_\lambda}{\delta_\lambda A(x)}\Psi_1\right), \Psi_2\right\rangle, \quad (3.25)$$

so that $i\delta_\lambda/\delta_\lambda A$ is hermitian. Further, by expanding an arbitrary functional $F[A]$ in a functional Taylor series, it is easy to see that

$$\left[i\frac{\delta}{\delta A(x)}, \quad i\frac{\delta_\lambda}{\delta_\lambda A(x')}\right]F[A] = 0. \quad (3.26)$$

Since $i\delta_\lambda/\delta_\lambda A$ is hermitian, and commutes with $i\delta/\delta A$, it follows that H_λ is hermitian. On the other hand, H_λ breaks gauge invariance to order λ. This is an unpleasant feature of our regularization scheme, but it will not introduce any special difficulties in what follows.

Having introduced the regularized Hamiltonian H_λ, the problem is now to find a sequence of wave functionals $\Psi_\lambda[A]$ satisfying

$$\lim_{\lambda\to 0} H_\lambda\Psi_\lambda[A] = \lim_{\lambda\to 0}\mathscr{E}_\lambda\Psi_\lambda[A] = \mathscr{E}_0\Psi[A]. \quad (3.27)$$

That is, for each λ,

$$H_\lambda\Psi_\lambda[A] = (\mathscr{E}_\lambda + \eta_\lambda[A])\Psi_\lambda[A], \quad (3.28)$$

where, for any $A(x)$

$$\lim_{\lambda\to 0}\eta_\lambda[A]\Psi_\lambda[A] = 0. \quad (3.29)$$

The approximation in this procedure is due to the fact that, for each λ, $\Psi_\lambda[A]$ is an eigenstate of H_λ only up to a functional $\eta_\lambda[A]$. And while $\eta_\lambda[A]$ is small for any given $A(x)$, it may nevertheless have an infinite expectation value. In other words, eq. (3.29) does *not* imply that

$$\lim_{\lambda\to 0} \langle \eta_\lambda[A]\rangle = 0\,. \tag{3.30}$$

For this reason, it is not necessarily true that $\mathscr{E}_0$ of eq. (3.27) is the exact vacuum energy, or that $\Psi[A]$ is the exact eigenstate. On the other hand, a solution $\Psi[A]$ of eq. (3.27) *may* be an exact eigenstate of H; and in fact it is quite simple to construct examples of a sequence Ψ_λ converging to a known solution Ψ_0 of a soluble theory, satisfying (3.27) but not (3.30). One such example is given in the appendix. The point is that in systems with infinitely many degrees of freedom, a small deviation of the wave functional Ψ from the true solution can produce infinite corrections to the energy expectation value. But providing that $\eta_\lambda[A]$ is significant only for the very high frequency components of A, a solution $\Psi_\lambda[A]$ of eq. (3.27) should be an excellent approximation, at least in the infrared regime, to the true solution of the Schrödinger equation.

It will be simplest to first state the answer to the problem posed above, and then verify it. The solution to (3.27) is provided by the sequence of wave functionals

$$\Psi_\lambda[A] = N_\lambda \exp\left[-\int \mathrm{d}^3x_2\, \mathrm{d}^3x_1\, \mathrm{Tr}\,[B_n(x_2)V_{21}B_n(x_1)V_{12}]\,\frac{\phi_{\lambda_2}(x_2-x_1)}{Z\mu}\right], \tag{3.31}$$

where

$$\phi_{\lambda_2}(|\boldsymbol{x}|) = \frac{1}{2\pi^{3/2}\lambda_2}\,\frac{\mathrm{e}^{-x^2/\lambda_2^2}}{x^2},$$

$$\int \mathrm{d}^3x\, \phi_{\lambda_2}(x) = 1, \tag{3.32}$$

and where Z, a dimensionless constant, and μ, a constant with dimensions of mass, are related to λ_2 and to $\lambda_1 = \lambda$ by

$$\lambda = \lambda_1 = 1/(\mu Z^{4/3})\,,$$

$$\lambda_2 = a/(\mu Z^{2/3}) + \tfrac{1}{12}\sqrt{\pi}\lambda_1\,, \tag{3.33}$$

with a another dimensionless constant whose value will be determined below. Evidently $Z\to\infty$ as $\lambda\to 0$ (μ is fixed). To make sense of $\Psi_\lambda[A]$ in the $\lambda\to 0$ limit, introduce the rescaled coupling g_r and field A^r defined by

$$A = \sqrt{Z}A^\mathrm{r}\,,$$

$$g = \frac{1}{\sqrt{Z}}\,g^\mathrm{r}\,, \tag{3.34}$$

so that, in terms of the rescaled quantities

$$\Psi_\lambda[A^r]=N_\lambda \exp\Big[-\int d^3x_2\, d^3x_1\, \mathrm{Tr}[B_n^r(x_2)V_{21}\times B_n^r(x_1)V_{12}]\frac{1}{\mu}\Big\{\frac{1}{2\pi^{3/2}\lambda_2}\frac{e^{-(x_2-x_1)^2/\lambda_2^2}}{(x_2-x_1)^2}\Big\}\Big],$$

$$H_\lambda=\tfrac{1}{2}\int d^3x\Big[-\frac{1}{Z}\frac{\delta_\lambda}{\delta_\lambda A^r}\frac{\delta}{\delta A^r}+ZB^{r2}\Big]. \qquad (3.35)$$

From here on we will drop the superscript "r", and let $A(x)$, g denote the rescaled, rather than bare, field and coupling. The scaling behavior in (3.33) is different from what one expects in the weak-coupling regime, which again suggests that $\Psi_\lambda[A]$ is correctly describing only the infrared behavior of the theory.

It must first be shown that $\Psi_\lambda[A]$ is normalizable, i.e., that

$$R[A]=\int d^3x_2\, d^3x_1\, \mathrm{Tr}\,[B_2V_{21}B_1V_{12}]\phi_{\lambda_2}(x_2-x_1)/\mu$$

$$=2\int d^3x\, d^3X\, \mathrm{Tr}\,[B_n(\boldsymbol{X}-\boldsymbol{x})V_{21}B_n(\boldsymbol{X}+\boldsymbol{x})V_{12}]\phi_{\lambda_2}(2x)/\mu>0. \qquad (3.36)$$

Make the change of variables

$$\boldsymbol{x}\to(x,\theta,\varphi)=(x,\Omega),$$
$$\boldsymbol{X}\to\boldsymbol{X}^\Omega=[R(\theta,\varphi)]\boldsymbol{X}=(X_1^\Omega,X_2^\Omega,X_3^\Omega), \qquad (3.37)$$

where $R(\theta,\varphi)$ is a rotation matrix chosen such that the X_1^Ω axis points in the (θ,φ) direction, while vectors $\boldsymbol{X}_\perp^\Omega\equiv(0,X_2^\Omega,X_3^\Omega)$ are perpendicular to this direction. Then

$$R[A]=2\int d\Omega\, d\boldsymbol{X}_\perp^\Omega\int_{-\infty}^{\infty}dX_1^\Omega\int_0^\infty x^2\,dx\, \mathrm{Tr}\,[B_n(X_1^\Omega-x,\boldsymbol{X}_\perp^\Omega)V_{21}\times B_n(X_1^\Omega+x,\boldsymbol{X}_\perp^\Omega)V_{12}]\phi_{\lambda_2}(2x)/\mu$$

$$=\int d\Omega\, d\boldsymbol{X}_\perp^\Omega\int_{-\infty}^{\infty}dX_1^\Omega\int_{-\infty}^{\infty}dx\, \mathrm{Tr}\,[B_n(X_1^\Omega-x,X_\perp^\Omega)V_{21}\times B_n(X_1^\Omega+x,\boldsymbol{X}_\perp^\Omega)V_{12}]x^2\phi_{\lambda_2}(2x)/\mu. \qquad (3.38)$$

Now along any given line corresponding to fixed values of $\boldsymbol{X}_\perp$, Ω there exists a gauge transformation such that $V_{21}\to 1$ and $B_n(X_1^\Omega\pm x,\boldsymbol{X}_\perp^\Omega)\to B_n'(X_1^\Omega\pm x,\boldsymbol{X}_\perp^\Omega,\Omega)$ along this line*. Then it is always possible to re-express (3.38) as

$$R[A]=\tfrac{1}{8}\int d\Omega\, d\boldsymbol{X}_\perp^\Omega\, dX_1^\Omega\, dx B_n'^a(X_1^\Omega-x,\boldsymbol{X}^\Omega,\Omega)\times B_n'^a(X_1^\Omega+x,\boldsymbol{X}^\Omega,\Omega)\frac{1}{\mu}\frac{1}{2\pi^{3/2}\lambda_2}e^{-4x^2/\lambda_2^2}. \qquad (3.39)$$

* In other words, suppose L is a line in 3-space. Then it is always possible to find a gauge transformation $\boldsymbol{A}(\boldsymbol{x})\to\boldsymbol{A}'(\boldsymbol{x})=U^{-1}\boldsymbol{A}(\boldsymbol{x})U+iU^{-1}\boldsymbol{\nabla}U$ such that $\boldsymbol{A}'(\boldsymbol{x})\cdot\hat{\boldsymbol{e}}_L=0$ for $x\in L$, where $\hat{\boldsymbol{e}}_L$ is a unit vector in the L direction. But if the component of $\boldsymbol{A}'(\boldsymbol{x})$ parallel to L vanishes for $\boldsymbol{x}$ on L, then $V_{12}\to 1$ for all points $(x_1,x_2)\in L$. There will, of course, be a separate gauge transformation for each line L.

The elimination of the line integrals V_{21} from (3.39) has, of course, a price: $B_n'^a$ is not a function of five variables (position + orientation), rather than three (position only). Now let

$$B_n'^a(X_1^\Omega \pm x, \boldsymbol{X}_\perp^\Omega, \Omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}k B_n'^a(k, \boldsymbol{X}_\perp^\Omega, \Omega)\, \mathrm{e}^{ik(X_1^\Omega \pm x)} . \tag{3.40}$$

Then

$$R[A] = \frac{1}{(32\pi)\mu} \int \mathrm{d}\Omega\, \mathrm{d}X_\perp^\Omega\, \mathrm{d}k \{B_n'^a(k, \boldsymbol{X}_\perp^\Omega, \Omega) B_n'^{a*}(k, \boldsymbol{X}_\perp^\Omega, \Omega)\}\, \mathrm{e}^{-\lambda_2^2 k^2/4} . \tag{3.41}$$

But since the integrand in (3.41) is positive definite, it follows that $R[A] > 0$, Q.E.D. So a sufficient condition for $R[A] > 0$, and therefore for normalizability of $\Psi[A]$ in (3.31), is simply that the one-dimensional Fourier transform of $x^2 \phi_\lambda(x)$ be positive definite; this condition is satisfied by ϕ_λ in (3.32).

To verify that $\Psi_\lambda[A]$ defined by eq. (3.35) is a solution to (3.27), let H_λ operate on Ψ_λ and find

$$\begin{aligned}
H_\lambda \Psi_\lambda[A] = \Big\{ & \frac{1}{Z} \int \mathrm{d}^3x_2\, \mathrm{d}^3x_1\, \mathrm{d}^3x \frac{1}{\mu} \phi_{\lambda_2}(x_2 - x_1) \\
& \times \{\mathrm{Tr}\,[\delta_{\lambda_1} B_2 V_{21} \delta B_1 V_{12}] && [ZP_0] \\
& + \mathrm{Tr}\,[B_2 \delta_{\lambda_1} (\delta V_{21}) B_1 V_{12}] && [ZP_1] \\
& + \mathrm{Tr}\,[B_2 \delta_{\lambda_1} V_{21} B_1 \delta V_{12}] && [ZP_2] \\
& + \mathrm{Tr}\,[B_2 \{V_{21} \delta_{\lambda_1} B_1 \delta V_{12} + \delta V_{21} \delta_{\lambda_1} B_1 V_{12}\} + (\delta_{\lambda_1} \leftrightarrow \delta)]\} && [ZP_3] \\
& - \frac{2}{Z} \int \mathrm{d}^3x_2\, \mathrm{d}^3x_1\, \mathrm{d}^3x_2'\, \mathrm{d}^3x_1'\, \mathrm{d}^3x \frac{1}{\mu^2} \phi_{\lambda_2}(x_2 - x_1) \phi_{\lambda_2}(x_2' - x_1') \\
& \times \{\mathrm{Tr}\,[\delta_{\lambda_1} B_2 V_{21} B_1 V_{12}]\, \mathrm{Tr}\,[\delta B_{2'} V_{21}' B_{1'} V_{12}'] && [Q_0] \\
& + \mathrm{Tr}\,[B_2 \delta_{\lambda_1} V_{21} B_1 V_{12}]\, \mathrm{Tr}[B_{2'} \delta V_{21}' B_{1'} V_{12}'] && [Q_1] \\
& + (\mathrm{Tr}\,[\delta_{\lambda_1} B_2 V_{21} B_1 V_{12}]\, \mathrm{Tr}[B_{2'} \delta V_{21}' B_{1'} V_{12}'] + (\delta_{\lambda_1} \leftrightarrow \delta))\} && [Q_2] \\
& + Z \int \mathrm{d}^3x\, \mathrm{Tr}\,[B_n(x) B_n(x)] \Big\} \Psi_\lambda[A] . && [PE]
\end{aligned} \tag{3.42}$$

First recompute ZP_1 and ZP_2:

$$\begin{aligned}
\delta_{\lambda_1} \delta V_{21} &= \frac{\delta_{\lambda_1}}{\delta_{\lambda_1} A_k^c(x)} \frac{\delta V_{21}}{\delta A_k^c(x)} \\
&= ig \int_{x_1}^{x_2} \mathrm{d}z^k \delta^3(z - x) \frac{\delta_{\lambda_1}}{\delta_{\lambda_1} A_k^c(x)} \left[\exp\left[ig \int_x^{x_2} A \cdot \mathrm{d}x' \right] L_c \exp\left[ig \int_{x_1}^{x} A \cdot \mathrm{d}x' \right] \right]
\end{aligned}$$

$$= ig\int_{x_1}^{x_2} dz^k \delta^3(z-x)\Big\{ ig\int_{x}^{x_2} dz'_k \delta^3_{\lambda_1}(z'-x)\exp\Big[ig\int_{z'}^{x_2} A\cdot dx'\Big]L_c$$

$$\times\exp\Big[ig\int_{x}^{z'} A\cdot dx'\Big]L_c \exp\Big[ig\int_{x_1}^{x} A\cdot dx'\Big]$$

$$+ ig\int_{x_1}^{x} dz''_k \delta^3_{\lambda_1}(z''-x)\exp\Big[ig\int_{x}^{x_2} A\cdot dx'\Big]L_c \exp\Big[ig\int_{z''}^{x} A\cdot dx'\Big]L_c$$

$$\times\exp\Big[ig\int_{x_1}^{z''} A\cdot dx'\Big]\Big\}. \tag{3.43}$$

Now,

$$\exp\Big[ig\int_{x_1}^{x_2} A\cdot dx'\Big]\phi_{\lambda_2}(x_2-x_1) = \{1+O(\lambda_2)\}\phi_{\lambda_2}(x_2-x_1),$$

$$L_c L_c = \tfrac{3}{4}\cdot 1. \tag{3.44}$$

So

$$\delta_{\lambda_1}\delta V_{21} = \tfrac{3}{4}(ig)^2\delta^2(\rho_x)\theta(\tau_x)\int_{x_1}^{x_2} dz'_k f_k^{x_1 x_2}\delta^3_{\lambda_1}(z'-x)[1+O(\lambda_2)]$$

$$= -\tfrac{3}{4}g^2\delta^2(\rho_x)\theta(\tau_x)\frac{1}{\frac{4}{3}\pi\lambda_1^3}2\lambda_1[1+O(\lambda_2)]$$

$$= -\frac{9}{8\pi}g^2\delta^2(\rho_x)\theta(\tau_x)\frac{1}{\lambda_1^2}[1+O(\lambda_2)]. \tag{3.45}$$

Then,

$$ZP_1 = \frac{1}{Z}\int d^3x_2\, d^3x_1\, d^3x\, \mathrm{Tr}[B_2\delta_{\lambda_1}(\delta V_{21})B_1 V_{12}]\frac{1}{\mu}\phi_{\lambda_2}(x_2-x_1)$$

$$= -\frac{1}{Z}\frac{9g^2}{8\pi}\frac{1}{\mu\lambda_1^2}\int d^3x_2\, d^3x_1\, \mathrm{Tr}[B_2B_1]\phi_{\lambda_2}(x_2-x_1)$$

$$\times\int d^3x\,\delta^2(\rho_x)\theta(\tau_x)\{1+O(\lambda_2)\}$$

$$= -\frac{1}{Z}\frac{9g^2}{8\pi}\frac{1}{\mu\lambda_1^2}\int d^3x_2\, d^3x_1\, \mathrm{Tr}[B_2^2]\phi_{\lambda_2}(x_2-x_1)|x_2-x_1|\{|1+O(\lambda_2)\}$$

$$= -\frac{1}{Z}\frac{9g^2}{8\pi^{3/2}}\frac{\lambda_2}{\mu\lambda_1^2}\Big\{\int d^3x\, \mathrm{Tr}[B^2]+O(\lambda_2^2)\Big\}, \tag{3.46}$$

where, since terms of order λ_2 inside the integral are odd under $\boldsymbol{x}_2\to\boldsymbol{x}_2$, $(\boldsymbol{x}_2-\boldsymbol{x}_1)\to$

$-(\boldsymbol{x}_2-\boldsymbol{x}_1)$, surviving higher-order terms are $O(\lambda_2^2)$. Next,

$$ZP_2=\frac{1}{Z\mu}\int d^3x_2\,d^3x_1\,d^3x\{igf_k^{x_1x_2}\delta^2(\rho_x)\theta(\tau_x)\}$$
$$\times\mathrm{Tr}[B_2L_cB_1\delta_{\lambda_1}V_{12}]\phi_{\lambda_2}(x_2-x_1)(1+O(\lambda_2))$$
$$=\frac{(ig)^2}{Z\mu}\int d^3x_2\,d^3x_1\,d^3x\left\{f_k^{12}\delta^2(\rho_x)\theta(\tau_x)\int_{x_2}^{x_1}dz'_k\delta^3_{\lambda_1}(z'-x)\right\}$$
$$\times\mathrm{Tr}[B_2L_cB_1L_c]\phi_{\lambda_2}(x_2-x_1)(1+O(\lambda_2))$$
$$=\frac{(ig)^2}{Z\mu}\frac{-1}{4}\int d^3x_2\,d^3x_1\,\mathrm{Tr}[B_2B_1]\phi_{\lambda_2}(x_2-x_1)$$
$$\times\int d^3xf_k^{12}\delta^2(\rho_x)\int_{x_2}^{x_1}dz^k\delta^3_{\lambda_1}(z-x)(1+O(\lambda_2))$$
$$=-\frac{1}{Z}\frac{3}{8}\frac{g^2}{\pi^{3/2}}\frac{\lambda_2}{\mu\lambda_1^2}\left\{\int d^3x\mathrm{Tr}[B^2]+O(\lambda_2^2)\right\}. \tag{3.47}$$

The leading contribution of the ZP_3 term is basically the same as ZP_1 and ZP_2, differing only by a numerical factor. ZP_3 is given by

$$ZP_3=\frac{1}{Z\mu}\int d^3x_2\,d^3x_1\,d^3x\,\mathrm{Tr}\left[B_n(x_2)\frac{\delta_{\lambda_1}}{\delta_{\lambda_1}A_k^c(x)}\{V_{21}\delta B_1V_{12}\}\right]\phi_{\lambda_2}(x_2-x_1)+(\delta_{\lambda_1}\leftrightarrow\delta)$$
$$=\frac{1}{Z\mu}\int d^3x_2\,d^3x_1\,d^3x\left\{ig\int_{x_1}^{x_2}dz^m\delta_{mk}\delta^3_{\lambda_1}(z-x)\right\}$$
$$\times\epsilon_{nik}\,\mathrm{Tr}[B_n(x_2)V_{2x}L_cV_{x1}(\partial_iL_c-ig[A_i,L_c])_{x_1}\delta^3(x-x_1)V_{12}$$
$$-B_n(x_2)V_{21}(\partial_iL_c-ig[A_i,L_c])_{x_1}\delta^3(x-x_1)V_{1x}L_cV_{x_2}]\phi_{\lambda_2}(x_2-x_1)+(\delta_{\lambda_1}\leftrightarrow\delta), \tag{3.48}$$

which after some manipulations is found to be

$$ZP_3=-\frac{1}{Z}\frac{3g^2}{4\pi^{3/2}}\frac{(\lambda_2-\frac{1}{4}\sqrt{\pi}\lambda_1)}{\mu\lambda_1^2}\left\{\int d^3x\,\mathrm{Tr}\,[B_n(x)^2]+O(\lambda_2^2)\right\}. \tag{3.49}$$

So now we have

$$ZP_1+ZP_2+ZP_3+PE$$
$$=\left[Z-\frac{1}{Z}\frac{9g_r^2}{4\pi^{3/2}}\frac{(\lambda_2-\frac{1}{12}\sqrt{\pi}\lambda_1)}{\mu\lambda_1^2}\right]\int d^3x\,\mathrm{Tr}[B^2]-\frac{1}{Z}\frac{\lambda_2}{\mu\lambda_1^2}O(\lambda_2^2), \tag{3.50}$$

where we have again added the subscript "r" to emphasize that it is the finite, rescaled coupling that appears in (3.50). In order to have the r.h.s. of (3.50) vanish in the $\lambda \to 0$, $Z \to \infty$ limit, let

$$a = \frac{4\pi^{3/2}}{9g_r^2} \tag{3.51}$$

in eq. (3.33), so that

$$Z^2 = \frac{9g_r^2}{4\pi^{3/2}} \frac{(\lambda_2 - \frac{1}{12}\sqrt{\pi}\lambda_1)}{\mu\lambda_1^2}, \tag{3.52}$$

and therefore

$$ZP_1 + ZP_2 + ZP_3 + PE = 0 + O(Z^{-1/3}). \tag{3.53}$$

The ZP_0 term contains, in the Abelian theory, the zero-point energy of eq. (2.20). In the present case it is given by

$$\begin{aligned} ZP_0 &= \frac{1}{Z\mu}\int d^3x_2\, d^3x_1\, d^3x\, \mathrm{Tr}[\delta_{\lambda_1}B_2 V_{21}\delta B_1 V_{12}]\phi_{\lambda_2}(x_2 - x_1) \\ &= \frac{1}{Z\mu}\int d^3x_2\, d^3x_1\, d^3x\, \mathrm{Tr}[(\partial_i L_c - ig[A_i, L_c])_{x_2}\delta^3_{\lambda_1}(x_2 - x) V_{21} \\ &\quad \times (\partial_i L_c - ig[A_i, L_c])_{x_1}\delta^3(x_1 - x) V_{12}] \\ &\quad \times (\epsilon_{nik})^2 \phi_{\lambda_2}(x_2 - x_1), \end{aligned} \tag{3.54}$$

which becomes, after some integration by parts,

$$\begin{aligned} ZP_0 = \frac{1}{Z\mu}\int d^3x_2\, d^3x_1 (\epsilon_{nik})^2 \{ &\mathrm{Tr}(L_c V_{21} L_c V_{12}]\delta^3_{\lambda_1}(x_2 - x_1)(-\partial_i^2)\phi_{\lambda_2}(x_2 - x_1) \\ &+ \mathrm{Tr}[L_c(\partial_i - igA_i)_{x_2} V_{21} L_c V_{12}]\delta^3_{\lambda_1}(x_2 - x_1)\partial_i^{x_1}\phi_{\lambda_2}(x_2 - x_1)(+\text{like terms}) \\ &+ \mathrm{Tr}[L_c(\partial_i - igA_i)_{x_2} V_{21}(\partial_i - igA_i)_{x_1} V_{12}]\delta^3_{\lambda_1}(x_2 - x_1)\phi_{\lambda_2}(x_2 - x_1)(+\text{like terms})\}. \end{aligned} \tag{3.55}$$

But notice that

$$\{(\partial_i - igA_i)_{x_2} V_{21}\}\partial_i \phi_{\lambda_2}(x_2 - x_1) = 0. \tag{3.56}$$

This is because, if we let $\partial_\parallel$ denote differentiation in the $\boldsymbol{x}_2 - \boldsymbol{x}_1$ direction, and let $\vec{\partial}_\perp$ denote differentiation in the orthogonal directions, then $(\partial_\parallel - igA_\parallel)V_{21} = 0$ and $\vec{\partial}_\perp \phi_{\lambda_2}(|\boldsymbol{x}_2 - \boldsymbol{x}_1|) = 0$. Furthermore

$$\begin{aligned} &\{(\partial_i - igA_i)_{x_2} V_{21}\}\phi_{\lambda_2}(x_2 - x_1) \\ &\quad = [0 + (\tfrac{1}{2}ig\epsilon_{nik}B_n(x_2)f_k^{12})|\boldsymbol{x}_2 - \boldsymbol{x}_1| + O(|\boldsymbol{x}_2 - \boldsymbol{x}_1|^2)]\phi_{\lambda_2}(x_2 - x_1), \end{aligned} \tag{3.57}$$

so that

$$ZP_0 = \mathscr{E}_\lambda + O(Z^{-1/3}), \tag{3.58}$$

where $\mathscr{E}_\lambda$, the zero-point energy, is given by

$$\mathscr{E}_\lambda = \frac{3V}{Z\mu}\int d^3x\,\delta^3_{\lambda_1}(x)(-\nabla^2)\phi_{\lambda_2}(x)$$

$$= Z^5\left(\frac{9}{a\pi^{3/2}}\right)V\mu^4\,, \tag{3.59}$$

which diverges as $\lambda\to 0$ (V is the volume of 3-space), as expected.

Of the remaining terms, Q_1 and Q_2 go to zero faster than $1/Z$, since both terms contain a factor

$$F = \frac{1}{Z\mu}\int d^3x_2\,d^3x_1\,\mathrm{Tr}[B_2\delta V_{21}B_1V_{12}]\phi_{\lambda_2}(x_2-x_1)$$

$$= \frac{ig}{Z\mu}\int d^3x_2\,d^3x_1 f_k^{12}\delta^2(\rho_x)\,\mathrm{Tr}[B_2L_cB_2]\phi_{\lambda_2}(x_2-x_1) + \mathrm{O}(\lambda_2/Z)\,. \tag{3.60}$$

But note that $\mathrm{Tr}\,[B_n(x_2)L_cB_n(x_2)] = 0$; therefore $F = 0 + \mathrm{O}(\lambda_2/Z)$, and likewise

$$Q_1 = 0 + \mathrm{O}(\lambda_2/Z)\,,$$

$$Q_2 = 0 + \mathrm{O}(\lambda_2/Z)\,. \tag{3.61}$$

Finally, there is the Q_0 term (which, in the Abelian theory, with $\phi(x_1-x_2)\sim 1/(x_1-x_2)^2$ would cancel the potential energy $b_n^2(x)$ term)

$$Q_0 = -\frac{2}{Z\mu}\int d^3x_2\,d^3x_1\,d^3x_1'\,d^3x_2'\,d^3x\,\phi_{\lambda_2}(x_2-x_1)\phi_{\lambda_2}(x_2'-x_1')$$

$$\times\mathrm{Tr}\,[\delta_{\lambda_1}B_2V_{21}B_1V_{12}]\,\mathrm{Tr}[\delta B_{2'}V'_{21}B_{1'}V'_{12}]$$

$$= \left\{-\frac{1}{Z\mu^2}\int d^3x\,\mathrm{Tr}[\{\partial_iB_n - ig[A_i, B_n]\}\{\partial_jB_m - ig[A_j, B_m]\}]\right.$$

$$\left.\times\,\epsilon_{nik}\epsilon_{mjk}\right\} + \mathrm{O}(\lambda_2/Z)\,. \tag{3.62}$$

So Q_0 is $\mathrm{O}(1/Z)$.

Putting everything together, we have

$$H_\lambda\Psi_\lambda[A] = \{\mathscr{E}_\lambda + \mathrm{O}(Z^{-1/3}) + \mathrm{O}(Z^{-1}) + \ldots\}\Psi_\lambda[A]\,, \tag{3.63}$$

where $\mathscr{E}_\lambda$ is given by eq. (3.59). The other terms in brackets in eq. (3.63), which are all on the order of various inverse powers of Z, correspond to $\eta_\lambda[A]$ in eq. (3.28). Inspection of the actual terms in $\eta_\lambda[A]$ shows that the product $\eta_\lambda\Psi_\lambda$ is infrared finite in the sense that the functional $\eta_\lambda[A]\Psi_\lambda[A]$ is finite regardless of the long-distance behavior of $A(x)$. If $\eta_\lambda[A]$ diverges (which occurs for $A(x)$ falling as $x^{-1/2}$ or slower) like $(\mathrm{Volume})^P$, then $\Psi[A]\to 0$ like $\exp\{-(\mathrm{Volume})^P\}$. For any sufficiently smooth configuration $A(x)$, the product $\eta_\lambda[A]\Psi_\lambda[A]\to 0$ in the $\lambda\to 0$, $Z\to\infty$ limit, so that eq. (3.29) is satisfied (although the expectation value $\langle\eta_\lambda\rangle$ is infinite). The restriction to "sufficiently smooth" (in this case twice differentiable) configurations is necessary for this reason: if the derivative of the non-Abelian magnetic field $B_n(x)$ is not everywhere finite, then it is possible for $Q_0[A]$ to be infinite while $\Psi[A]$ remains finite, violating eq. (3.29). But with this one restriction to differentiable field-strengths, it has been shown that the sequence of gauge-invariant wave functionals $\Psi_\lambda[A]$ in (3.35) does indeed satisfy (3.27), and in the sense of eq. (3.27) represents an approximate eigenstate of the Yang–Mills Hamiltonian.

To summarize: we have found that satisfying Gauss' law exactly as a constraint on states introduces new, infinite contributions (ZP_1 and ZP_2) to the energy density, so that a gauge-invariant version of the perturbative ground state (i.e., $\phi(x_2, x_1) \sim 1/(x_2 - x_1)^2$ in eq. (3.4)) is a very poor choice for the Yang–Mills vacuum. However, with the ansatz (3.4) and a scheme for regularizing infinities and rescaling the bare field and coupling, it is possible not only to eliminate the unpleasant infinite terms, but even to construct a wave functional which is an approximate eigenstate of the Yang–Mills Hamiltonian. As in renormalizable perturbation theory, the regularization is to be removed only at the end of a calculation; it can be seen, for example, that

$$\lim_{\lambda\to 0} H_\lambda \Psi_\lambda \neq \Big[\lim_{\lambda\to 0} H_\lambda\Big]\Big[\lim_{\lambda\to 0} \Psi_\lambda\Big]. \tag{3.64}$$

In fact, since the bare coupling g is 0, taking the $\lambda \to 0$ limit prematurely would reduce the non-Abelian theory to an Abelian one.

4. Confinement properties of the Yang–Mills vacuum

The non-Abelian ground state Ψ_λ of eq. (3.35) is, in the usual terminology, a periodic $\theta = 0$ vacuum. This fact was guaranteed at the outset, since Ψ_λ, or any wave functional of the form (3.4), is invariant under any finite, non-singular, gauge transformation*. This means in particular that if $g_n(\boldsymbol{x}) \in SU(2)$ is a time-independent topologically non-trivial gauge transformation in the nth homotopy class ($g_n : S^3 \to S^3$), then

$$\Psi_\lambda[g_n \circ A] = \Psi_\lambda[A], \tag{4.1}$$

which is just the definition of $\theta = 0$ vacuum periodicity.

The ansatz (3.4) does not specify the function $\phi(x_1, x_2)$, which can only be found by explicit calculation. In sect. 2 it was found for the Abelian theory that $\phi = 1/2\pi^2(\boldsymbol{x}_2 - \boldsymbol{x}_1)^2$, while for the non-Abelian theory we have seen that $\phi = \phi_\lambda(\boldsymbol{x}_2 - \boldsymbol{x}_1)/\mu$. This has an immediate consequence for the long-distance behavior

* The Gauss' law constraint requires only that Ψ be invariant under infinitesimal gauge transformations.

of vacuum fluctuations. A comparison of Ψ_{Abelian} of eq. (2.18) and $\Psi_\lambda[A]$ of eq. (3.35) shows that, for the Abelian theory, only configurations falling faster than $1/r$ at spatial infinity have non-zero amplitude, while for the non-Abelian theory it is only necessary to fall faster than $1/\sqrt{r}$. Now it is believed that configurations with $1/r$ (non-pure-gauge) long-distance behavior are necessary to satisfy the Wilson confinement criterion. Such configurations are suppressed in the Abelian vacuum, but are present with finite amplitude in the Yang–Mills vacuum. This fact is suggestive concerning confinement, but not yet conclusive. The test of whether a non-Abelian gauge field is in the unbroken confinement phase is whether or not the Wilson–'t Hooft criteria of eq. (1.1a) (confinement) and (1.1b) (no symmetry breaking) are actually satisfied.

Let $Q[A]$ be an operator which depends of $A(\boldsymbol{x})$ only at a fixed time t. Then, using (3.35), the fixed-time VEV $\langle Q[A]\rangle$ is given by

$$\begin{aligned}\langle Q[A]\rangle &= \lim_{\lambda\to 0}\langle\Psi_\lambda|Q|\Psi_\lambda\rangle \\ &= \lim_{\lambda\to 0} N_\lambda^2\int \mathscr{D}A(\boldsymbol{x})Q[A]\exp\left[-\frac{2}{\mu}\int d^3x_2\, d^3x_1\, \mathrm{Tr}[B_2V_{21}B_1V_{12}]\phi_{\lambda_2}(x_2-x_1)\right].\end{aligned} \tag{4.2}$$

Of course, eq. (4.2) is only as good as the sequence Ψ_λ, which may itself only approximate the true ground state. Let us now make a further approximation to (4.2) by taking the limit $\lambda\to 0$ inside the functional integral

$$\begin{aligned}\langle Q[A]\rangle &\approx N^2\int \mathscr{D}A(\boldsymbol{x})Q[A]\exp\left[-\frac{2}{\mu}\int d^3x\,\mathrm{Tr}[B^2]\right] \\ &= N^2\int \mathscr{D}A(\boldsymbol{x})Q[A]\exp\left[-\frac{1}{\mu}\int d^3x\,\mathrm{Tr}[F_{ij}^2]\right].\end{aligned} \tag{4.3}$$

The validity of this last step is hard to estimate; in sect. 3 it was seen that taking the $\lambda\to 0$ limit prematurely can be dangerous. But hopefully for those operators $Q[A]$ which are mainly sensitive to the large wavelength-long distance structure of the theory, eq. (4.3) is a reasonable approximation. In other words, one expects (4.3) to be valid when the scale of $Q[A]$ (e.g., the dimensions of a Wilson loop) is in a regime where the effective coupling is large. The residual gauge freedom in (4.3) must be extracted by the standard techniques.

Eq. (4.3) is identical to an expression for the VEV of $Q[A(\boldsymbol{x})]$ in a *three* Euclidean dimensional Yang–Mills theory*. A rather similar problem has been studied by Polyakov, in the context of three-dimensional compact QED (QED embedded in the three-dimensional Georgi–Glashow model). The important feature of the Georgi–Glashow model is the existence of 't Hooft–Polyakov monopoles, which are pseudoparticles in the 3-dimensional theory. The functional integral corresponding

* The coupling is $g^r\mu^{1/2}$. There is no loss in generality in setting $g^r = 1$.

to (4.3) can be evaluated for this theory by saddlepoint methods around multi-monopole configurations, and the result

$$\langle A(\mathrm{C})\rangle \sim \mathrm{e}^{-\gamma S(\mathrm{C})} \tag{4.4}$$

is obtained, where γ is constant and $S(\mathrm{C})$ is the surface area enclosed by C. The behavior (4.4) implies the confinement of electric charge in the three-dimensional Georgi–Glashow model.

The case of 3-dimensional Yang–Mills theory is very similar to that of the 3-dimensional Georgi–Glashow model. As in the Georgi–Glashow model, the functional integrals are to be performed around multi-monopole configurations. However, the pseudoparticles of 3-dimensional pure Yang–Mills theory are Wu–Yang, rather than 't Hooft–Polyakov, monopoles, and this leads to a few complications. Wu–Yang monopoles, of the form

$$A_k^a(\boldsymbol{x}) = \epsilon_{kja}\frac{x_j}{r^2}, \tag{4.5}$$

are infinite action solutions of the Euclidean field equations, due to the singularity at $r=|\boldsymbol{x}|=0$. These solutions correspond to maxima, rather than minima, of the Euclidean action. For the purpose of constructing analog–gas approximations, it is necessary to consider instead a smoothed Wu–Yang monopole

$$A_k^a(\boldsymbol{x}) = \epsilon_{kja}\frac{x_j}{r^2}f(r), \tag{4.6}$$

where

$$f(0)=0, \qquad f(r)=1, \quad r>R. \tag{4.7}$$

Banks, Myerson and Kogut [7] have shown that a function $f(r)$ exists such that (4.6) is a solution of the Euclidean field equations everywhere except on a shell $r=R$. The Euclidean action of the smoothed monopole is finite and depends on the smoothing radius R; it will be denoted S_R. By an appropriate gauge transformation the non-Abelian magnetic field due to (4.6) can be made to point along any arbitrary direction $\hat{w}$ in isospace, so that the transformed field at $r>R$ is

$$A_k^a = q w^a \epsilon_{kij}\frac{x_i n_j}{4\pi r(r-\boldsymbol{x}\cdot\hat{\boldsymbol{n}})},$$

$$B_k^a = q w^a \frac{x_k}{r^3} + B^{\text{string}}, \tag{4.8}$$

where $\hat{\boldsymbol{n}}$ is a unit vector pointing along the string direction, and $q=\pm 1$. Now consider a superposition of N monopoles ($q_i=+1$) and antimonopoles ($q_i=-1$) which are all aligned in the same direction $\hat{w}$ of isospace, and centered at points

$\{x_i\}$. Because of the alignment in isospace, the superposition of monopole–anti-monopole configurations is itself a solution of the Euclidean field equations, in the regions where $|x - x_i| > R_i$. Providing the monopole separations $|x_i - x_j|$ are much greater than the monopole smoothing radii, the action of this configuration is

$$S_N = \sum_{i=1}^{N} S_{R_i} + \beta \sum_{i>j} \frac{q_i q_j}{|\boldsymbol{x}_i - \boldsymbol{x}_j|}, \tag{4.9}$$

where $\beta = 8\pi/\mu$.

Because the monopole smoothing radii R must be extracted by collective coordinate methods, and because allowance should be made for variation in monopole isospace alignment over large (relative to monopole separation) distances, the quantitative evaluation of eq. (4.3) by analog-gas–saddlepoint methods is arduous (although not impossible). But qualitative information may still be obtained from (4.3) with the help of the following "semi-Abelian" simplifications. We consider only those multimonopole configurations in which (i) all monopoles of a given configuration are exactly aligned in isospace; and (ii) the smoothing radius of each monopole is smaller than a fixed cutoff radius R_c. These restrictions may not be too unrealistic. If a multimonopole configuration is to be an approximate solution of the field equations, it is necessary that the variation of isospace alignment be very gradual, and the monopole smoothing radius small, on the scale of average monopole separation. In that case the monopole interaction has the Coulomb form of eq. (4.9), and depends only on positions $|\boldsymbol{x}_i - \boldsymbol{x}_j|$, and charges q_i.

Let A_m represent a single monopole configuration of the form (4.8), centered at point $\boldsymbol{x}_1$, with smoothing radius R_1. The contribution of quantum fluctuations about A_m to the 3-dimensional vacuum-to-vacuum amplitude is

$$\begin{aligned} Z_1 &= \int \mathscr{D}a(\boldsymbol{x})\, \mathrm{e}^{-S[A_m + a]} \\ &\approx \int \mathscr{D}a(\boldsymbol{x}) \exp\left[-S_{R_1} - \tfrac{1}{2} \int \mathrm{d}^3y\, \mathrm{d}^3x\, a(y) \frac{\delta^2 S}{\delta A^2}\bigg|_{A_m} a(x) \right], \end{aligned} \tag{4.10}$$

where

$$S[A] = \frac{2}{\mu} \int \mathrm{d}^3z\, \mathrm{Tr}[B^2]. \tag{4.11}$$

The operator $\int \delta^2 S/\delta A^2|_{A_m}$ has zero modes corresponding to translation invariance and global gauge rotations. There is also a negative eigenvalue corresponding to variations of the smoothing radius. Integration over these modes can be replaced, *via* the collective coordinate technique of introducing appropriate constraints [8], by integration over $\boldsymbol{x}_1$, $\boldsymbol{R}_1$, and $\hat{\boldsymbol{w}}$. Gauge-fixing by the Faddeev–Popov method is also required. The result is simply

$$Z_1 = \int \mathrm{d}\hat{\boldsymbol{w}} \int \mathrm{d}\boldsymbol{x}_1 K, \tag{4.12}$$

where K, a constant depending on μ and R_c, can be calculated in principle. For N monopoles, we have approximately

$$Z_N \approx \int d\hat{w} \int dx_1 \dots dx_N \frac{K^N}{N!} \sum_{\{q_n\}} \exp\left[-\beta \sum_{i>j} \frac{q_i q_j}{|x_i - x_j|} \right], \tag{4.13}$$

and finally the contribution from all multimonopole configurations is

$$Z = \int d\hat{w} \sum_{\substack{N \\ \{q_n\}}} \frac{K^N}{N!} \int \prod_{k=1}^{N} dx_k \exp\left[-\beta \sum_{i>j} \frac{q_i q_j}{|x_i - x_j|} \right]. \tag{4.14}$$

Eq. (4.14) is just the grand partition function for a Coulomb gas, which in this case is a plasma of magnetic monopoles [2]. So in this rather crude analog-gas approximation, eq. (4.3) becomes

$$\begin{aligned} \langle Q[A] \rangle &\approx Z[Q]/Z \\ &= \frac{1}{Z} \int d\hat{w} \sum_{\substack{N \\ \{q_n\}}} \frac{K^N}{N!} \int \prod_{k=1}^{N} dx_k Q\left[\sum_{i=1}^{N} A_{m_i}(x) \right] \exp\left[-\beta \sum_{i>j} \frac{q_i q_j}{|x_i - x_j|} \right], \end{aligned} \tag{4.15}$$

where $A_{m_i}(x)$ is the potential, of the form (4.8), due to the ith monopole.

Eq. (4.15) can now be used to evaluate the Wilson loop $\langle A(C) \rangle$ and 't Hooft's operator $\langle B(C) \rangle$. We have

$$\begin{aligned} \langle A(C) \rangle &= N \int \mathscr{D}A \, \mathrm{Tr}[\mathrm{P}\, e^{i\oint A \cdot dl}]\, e^{-S[A]} \\ &\approx Z[A(C)]/Z. \end{aligned} \tag{4.16}$$

Now,

$$\mathrm{Tr}\left[\mathrm{P} \exp\left[i \sum_i \oint A_{m_i} \cdot dl \right] \right] = \mathrm{Tr}\left[\exp\left[i \sum_i q_i \int dS \cdot \frac{(x - x_i)}{|x - x_i|^3} w^a L_a \right] \right], \tag{4.17}$$

and therefore

$$\begin{aligned} \langle A(C) \rangle = \frac{2}{Z} \left\{ \int d\hat{w} \right\} \sum_{\substack{N \\ \{q_n\}}} \frac{K^N}{N!} \int \prod_k dx_k \\ \times \exp\left[\tfrac{1}{2} i \sum_i q_i \int dS \cdot \frac{(x - x_i)}{|x - x_i|^3} \right] \exp\left[-\beta \sum_{i>j} \frac{q_i q_j}{|x_i - x_j|} \right], \end{aligned} \tag{4.18}$$

where we have used the fact that

$$\mathrm{Tr}\,[(w^a L_a)^n] = \begin{cases} 2^{1-n}, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \tag{4.19}$$

and the fact that, because of the sum over $\{q_i\}$, products of an odd number of q_i in (4.18) must vanish. The integral in eq. (4.18) has been evaluated by Polyakov; the details are presented in ref. [2]. The final result is that

$$\langle A(\mathrm{C})\rangle \sim \mathrm{e}^{-\gamma S(\mathrm{C})}\,, \tag{4.4}$$

and so (1.1a) is satisfied.

The 't Hooft operator acts on states like a singular gauge transformation, so that

$$\begin{aligned}\langle B(\mathrm{C})\rangle &= \langle\Psi|B(\mathrm{C})|\Psi\rangle = \langle\Psi[A]|\Psi[A']\rangle\\ &= N\int \mathscr{D}A\,\exp\left[-\frac{1}{\mu}\int \mathrm{d}^3x\,\mathrm{Tr}[B_nB_n + B'_nB'_n]\right],\end{aligned} \tag{4.20}$$

where

$$A'_k(\boldsymbol{x}) = \Omega^{-1}(\boldsymbol{x})A_k\Omega(\boldsymbol{x}) + \frac{i}{g}\Omega^{-1}\partial_k\Omega\,, \tag{4.21}$$

and

$$\Omega(\theta = 2\pi) = \mathrm{e}^{i\pi}\Omega(\theta = 0)\,, \tag{4.22}$$

where $0 \leqslant \theta \leqslant 2\pi$ parametrizes a curve winding once around C. Operator $B(\mathrm{C})$ acting on physical states creates a tube of non-Abelian magnetic flux along the curve C. The excited state $B(\mathrm{C})\Psi$ must also be gauge-invariant (i.e., physical). In the monopole-gas approximation

$$\begin{aligned}S[A'] &= \frac{1}{\mu}\int \mathrm{d}^3x\,\mathrm{Tr}[B'_n(x)^2] = S_{\text{flux loop}} + S\Big[\sum_i A_{\mathrm{m}_i}\Big]\,,\\ &= \alpha L(\mathrm{C}) + S\Big[\sum_i A_{\mathrm{m}_i}\Big]\,.\end{aligned} \tag{4.23}$$

There is no interaction between the flux loop and the monopoles since, if B_k^{mon} is the magnetic field due to a set of isospin-aligned monopoles, the interaction would be proportional to the loop integral

$$\oint_{\mathrm{C}} B_k^{\text{mon}}\,\mathrm{d}x^k = 0\,. \tag{4.24}$$

$S_{\text{flux loop}}$ is just the 3-dimensional Euclidean action of the flux loop, which is proportional to the length $L(\mathrm{C})$ of the loop. So, in the analog-gas approximation to eq. (4.20)

$$\langle B(\mathrm{C})\rangle = \mathrm{e}^{-\alpha L(\mathrm{C})}\,. \tag{4.25}$$

This means that $\langle B(\mathrm{C})\rangle$ has a perimeter-law falloff, and so eq. (1.1b) is satisfied.

Thus, according to the arguments above, the expression (3.35) for the Yang–Mills vacuum satisfies both the criteria (1.1) for the unbroken, confinement phase of a quantized gauge field. Apart from the simplicity of this expression, it also provides an attractive picture of the confinement mechanism in terms of the color magnetic properties of the vacuum. A basic property of a plasma of magnetic monopoles is the tendency to screen magnetic fields, and it can be seen that in the analog-gas approximation

$$\langle A(\mathrm{C})\rangle = \langle \mathrm{Tr}\,[\mathrm{P}\,\mathrm{e}^{i\oint \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{x}}]\rangle = \mathrm{e}^{-\mathscr{E}_{\mathrm{loop}}}\,, \tag{4.26}$$

where $\mathscr{E}_{\mathrm{loop}}$ is the energy of an electric loop inserted into a monopole plasma (note that, since all monopoles in a given configuration are aligned in isospace, the situation is essentially Abelian). By Ampere's law, a current loop sets up a magnetic field, which is screened by the plasma by formation of magnetic dipoles on a surface enclosed by loop. The energy $\mathscr{E}_{\mathrm{loop}}$ required for screening is therefore proportional to $S(\mathrm{C})$, and this is the physical explanation of the result (4.4).

It is likely that the wave functional Ψ_λ of (3.35) has much in common with the trial ground state proposed by Mandelstam [3], which represents a coherent state of smoothed Wu–Yang monopoles. Callan, Dashen and Gross have also speculated, in their meron picture, that the vacuum wave functional is dominated by monopole-like configurations [4]. In fact, the general idea of non-Abelian charge confinement *via* the mechanism outlined in the last paragraph has been around for some time; the wave functional (3.35) simply provides a specific realization of this idea. The confining properties of wave functionals Gaussian in $B(x)$ have also been discussed, in the context of $2+1$ dimensional QCD, by Halpern [9], who has noted that wave functionals of the form

$$\Psi \sim \exp\left[-\int \mathrm{d}^2x\ \mathrm{Tr}[B^2]\right] \tag{4.27}$$

will satisfy the Wilson criterion in the $2+1$ dimensional theory.

The results achieved so far are encouraging. Starting from first principles (i.e., the Schrödinger equation + Gauss' law), we have been able to construct an approximate ground state of the theory and to verify the confinement criteria. However, these results were arrived at by a series of approximations whose reliability is as yet unknown. In particular:

(i) The analog-gas manipulations leading to (4.15) were not an actual calculation, but only an argument regarding how the real calculation should go. This argument needs to be checked quantitatively, to insure that the various assumptions implicit in the analog-gas approximation are realistic.

(ii) The functional integral (4.3) is not identical to (4.2). Taking the $\lambda \to 0$ limit inside the integral sets $V_{12} \to 1$, and this can have a drastic effect. We have seen, e.g., that $(\delta^2/\delta A^2)\Psi_\lambda$ differs from $(\delta^2/\delta A^2)\lim_{\lambda\to 0}\Psi_\lambda$ by an infinite amount. So the extent to which (4.3) is a good approximation to (4.2) is unknown, and will no doubt depend on the form of $Q[A]$.

(iii) Suppose (4.3) were an *exact* expression for $\langle Q\rangle$. This would lead to the following contradiction [10]: let $D_3(\boldsymbol{x}_1-\boldsymbol{x}_2)$ and $D_4(x_1^\mu - x_2^\mu)$ be gauge-invariant 2-point functions corresponding to the VEV $\langle Q_1 Q_2\rangle$ in the 3- and 4-dimensional

Yang–Mills theories, respectively. Then according to (4.3),

$$D_3(\boldsymbol{x}_2-\boldsymbol{x}_1)=D_4(x_1^\mu-x_2^\mu)|_{x_1^0=x_2^0}\,. \tag{4.28}$$

But, on general principles, D_3 and D_4 should both have spectral decompositions of the form

$$D_3(\boldsymbol{x})=\int \mathrm{d}m^2\rho_3(m^2)\Delta_3(\boldsymbol{x},m^2)\,,$$

$$D_4(x^\mu)=\int \mathrm{d}m^2\rho_4(m^2)\Delta_4(x^\mu,m^2)\,, \tag{4.29}$$

where

$$\Delta_n(x,m^2)=\int \mathrm{d}^nK\frac{\mathrm{e}^{iK\cdot x}}{K^2+m^2}\,. \tag{4.30}$$

But if D_3 and D_4 have the standard analytic structure in terms of cuts and poles, then (4.29) is incompatible with (4.28). Now we have seen that (4.2) and (4.3) are not identical, so hopefully this problem does not occur for N-point functions calculated *via* (4.2). Until this resolution can be demonstrated, however, analyticity remains a potential problem.

(iv) The extent to which $\langle Q\rangle$ in (4.2) depends on the choice of δ-sequence, and on how closely Ψ_λ approximates the true ground state, is also unclear. Again, this dependence will probably vary for different operators.

(v) It has been shown that Ψ_λ approximates an eigenstate of the Yang–Mills Hamiltonian, but this eigenstate is not necessarily the ground state. Nevertheless, the extreme simplicity of Ψ_λ, and its similarity in form to the Abelian vacuum, favors the idea that Ψ_λ is in fact the ground state.

(vi) The consistency of the limiting procedure used in sect. 3 needs to be investigated further. It would be interesting, for example, to see if eq. (4.3) could be recovered in a lattice formulation, where gauge invariance is preserved in the regulated Hamiltonian.

Dangers exist; but assuming that our approximations are justifiable, gains in understanding the confinement phase have been made. It is hoped that the approximate ground state found here will lead to further insight into the confinement phenomenon, and other properties of quantized Yang–Mills fields.

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Appendix

In this appendix we do two things: first, we show by a simple example why the condition (3.30) is too strong to impose on a sequence of wave functionals Ψ_λ when dealing with a system of infinitely many degrees of freedom. Secondly, we construct another *exact* solution of the Yang–Mills Schrödinger equation (1.3) with finite bare coupling g. This solution is non-normalizable and therefore does not correspond to a physical state; nevertheless, it is remarkable that an equation of such complexity has such a mathematically straightforward solution.

Consider a denumerably infinite system of uncoupled harmonic oscillators $\{x_i\}$, where x_i is the displacement of the ith oscillator from equilibrium, and where

$$H=\sum_i \left\{ -\frac{1}{2m}\frac{\partial^2}{\partial x_i^2}+\tfrac{1}{2}\alpha x_i^2 \right\} \tag{A.1}$$

is the Hamiltonian of the system. The ground-state wave function of this system is just

$$\Psi_0[\{x_i\}]=N\exp\left[-\tfrac{1}{2}m\omega\sum_i x_i^2\right], \tag{A.2}$$

with $\omega=\sqrt{\alpha/m}$, and zero-point energy

$$\mathscr{E}_0=\sum_{i=1}^{\infty}\tfrac{1}{2}\omega. \tag{A.3}$$

Now consider the sequence of wave functions

$$\Psi_\lambda[\{x_i\}]=N\exp\left[-\sum_i(\tfrac{1}{2}m\omega x_i^2+\lambda x_i^4)\right]. \tag{A.4}$$

It is clear that $\lim_{\lambda\to 0}\Psi_\lambda=\Psi_0$. Let $H_\lambda=H_0$, and substitute into (3.27). The result is

$$H_\lambda\Psi_\lambda[\{x_i\}]=(\mathscr{E}_0+\eta_\lambda[\{x_i\}])\Psi_\lambda[\{x_i\}], \tag{A.5}$$

where $\mathscr{E}_0$ is given by (A.3), and where, to lowest order in λ,

$$\eta_\lambda[\{x_i\}]=\lambda\sum_i\left(\frac{6}{m}x_i^2-4\omega x_i^4\right)+\text{higher orders}. \tag{A.6}$$

Although it is true that

$$\lim_{\lambda\to 0}\eta_\lambda[\{x_i\}]\Psi_\lambda[\{x_i\}]=0, \tag{A.7}$$

it is also true that, for any λ arbitrarily small but non-zero,

$$\langle\eta_\lambda[\{x_i\}]\rangle=\infty, \tag{A.8}$$

since $\langle\eta_\lambda\rangle$ involves an infinite sum of terms of order λ. Thus, while Ψ_λ smoothly approaches the exact solution Ψ_0, the expectation value of the deviation $\langle\eta_\lambda\rangle$ is unbounded. For this reason, the condition (3.30) is too strong to require of a sequence Ψ_λ converging to an exact solution Ψ_0 of a system with infinitely many degrees of freedom.

The second task of this appendix is to display another solution of the Yang–Mills Schrödinger equation (1.3). The solution is ($\mathscr{E}_0=0$)

$$\Psi[A]=\exp\left[-\int \mathrm{d}^3x\,\mathrm{Tr}\left[\epsilon_{ijk}\left(A_i\partial_jA_k-\frac{2g}{3}A_iA_jA_k\right)\right]\right], \tag{A.9}$$

which may be readily verified by plugging (A.9) into (1.3) and doing a little algebra. This functional also satisfies the Gauss' law condition (1.4), since the exponent is invariant under infinitesimal gauge transformations. In fact, the exponent in (A.9) can be recognized as the expression for the winding number $\times 8\pi^2$ in the $A_0=0$ gauge. As in sect. 3, the wave functional (A.9) is arrived at by starting with a solution of the Schrödinger equation for the Maxwell field (2.1):

$$\Psi_0[A]=\exp\left[-\int \mathrm{d}^3x\,\mathrm{Tr}[\epsilon_{ijk}A_i\partial_jA_k]\right], \tag{A.10}$$

and then modifying Ψ_0 so that it becomes invariant under local, infinitesimal SU(2) gauge transformations.

Neither of the functionals in (A.9) and (A.10) are normalizable, since the exponents are not negative definite. There are analogous solutions in quantum mechanics. For example, the two-dimensional harmonic oscillator

$$\left[-\frac{1}{2m^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\tfrac{1}{2}K^2(x^2+y^2)\right]\psi(x,y)=\mathscr{E}\psi(x,y) \tag{A.11}$$

has a solution

$$\psi(x,y)=\mathrm{e}^{-m\omega(x^2-y^2)/2}, \tag{A.12}$$

which is not normalizable, and therefore does not correspond to a physical state. The solution (A.10) is very analogous to (A.12) , with the x and y directions replaced by field components of definite polarization.

So, because $\Psi[A]$ of eq. (A.9) is non-normalizable, it is probably of no physical importance whatever. Still, it is quite remarkable that eq. (1.3), which is a partial differential equation in infinitely many variables, and for which there is no obvious way of separating variables, can have such a simple solution for finite, non-infinitesimal coupling g. We regard this as further evidence of the value of using exact gauge invariance plus the solutions to the Abelian theory in finding solutions of the non-Abelian theory.

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