ELECTRIC VORTEX LINES FROM THE YANG–MILLS THEORY

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We develop the dynamics of an unstable Yang–Mills field mode previously found by us. We argue that this unstable mode corresponds to the transition to a state where electric vortex lines are created.

Recently we have studied [1] (henceforth ref. [1] is referred to as I) an unstable mode which is produced when a charged vector field responds to an external homogeneous color magnetic field H. The motivation for our work was that if one starts out in the unstable perturbative "ground state", there will probably be vacuum fluctuations characterized by a constant magnetic field (this is supported by simple one-loop calculations of the effective potential as a function of H [2]). Thus, even if the constant magnetic field is likely to disappear in the end, for some time it can influence the dynamics.

In the present note we shall discuss the unstable mode (induced by the magnetic field) further. In I we found the "longitudinal" (i.e., $\mu, \nu = 0, 3$ and $k = (0, k_3, k_4)$ if H points in the 3-direction) vacuum polarization[‡1]

$$\Pi^{\mathrm{L}}_{\mu\nu}(k) = (k^2\delta_{\mu\nu} - k_\mu k_\nu)\, \Pi^{\mathrm{L}}(k^2). \tag{1}$$

The contribution of the unstable mode to Π is given by (see e.g. eq. (4.3) in I)

$$\Pi^{\mathrm{L}}_{\substack{\text{unstable}\\ \text{mode}}}(k^2) = \frac{e^2}{8\pi^2}\, eH \int_0^1 \mathrm{d}\alpha\, \frac{(2\alpha-1)^2}{k^2\alpha(1-\alpha) - eH}. \tag{2}$$

It was noticed in I that the vacuum polarization (2) could be computed from an effective lagrangian

$$\mathcal{L}^{\mathrm{eff}}_{2\mathrm{D}} = -\tfrac{1}{4} f^2_{\mu\nu} - |(\partial_\mu - \mathrm{i}e_{2\mathrm{D}} a_\mu)\phi|^2 + eH|\phi|^2, \tag{3}$$

where a_μ is a $(1+1)$-dimensional vector field ($\mu = 3, 4$), ϕ is a complex $(1+1)$-dimensional scalar field and $f_{\mu\nu} = \partial\mu a_\nu - \partial_\nu a_\mu$. In eq. (3)

$$e_{2\mathrm{D}} = e\sqrt{eH/2\pi}. \tag{4}$$

The two dimensions referred to in eq. (3) are the 3 and 4 directions.

The lagrangian (3) looks very similar to the Higgs lagrangian, and it is natural to ask if a $|\phi|^4$-term does not appear in higher orders than we have included in eq. (2). This becomes more clear if one considers the $(3+1)$-dimensional lagrangian ($A_\mu = A^3_\mu$, $W_\mu = (A^1_\mu + \mathrm{i}A^2_\mu)/\sqrt{2}$ forms an SU(2) Yang–Mills field)

‡1 The metric used here is formally euclidean, with coordinates labelled $1, \ldots, 4$ and $x_4 = \mathrm{i}x_0$, etc.

$$\mathcal{L} = -\tfrac{1}{2}(D^*_\mu W^*_\nu - D^*_\nu W^*_\mu)(D_\mu W_\nu - D_\nu W_\mu) - D^*_\mu W^*_\mu D_\nu W_\nu - \tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$$
$$-\tfrac{1}{2}(\partial_\mu A_\mu)^2 - \mathrm{i}e(\partial_\mu A_\nu - \partial_\nu A_\mu)\, W^*_\mu W_\nu + \tfrac{1}{4} e^2 (W_\mu W^*_\nu - W_\nu W^*_\mu)^2 . \tag{4}$$

The tachyon in eq. (3) is expected to propagate in the loops, thus turning the W^4 interaction into a $|\phi|^4$ interaction. Instead of eq. (3) one then obtains the following effective action for the unstable mode,

$$S^{\mathrm{eff}} = \int \mathrm{d}x_3\, \mathrm{d}x_4\, \mathcal{L}^{\mathrm{eff}}_{2\mathrm{D}}; \qquad \mathcal{L}^{\mathrm{eff}}_{2\mathrm{D}} = -\tfrac{1}{4} f^2_{\mu\nu} - \tfrac{1}{2}(\partial_\mu a_\mu)^2 - |(\partial_\mu - \mathrm{i}e_{2\mathrm{D}} a_\mu)\phi|^2 + eH|\phi|^2 - \lambda|\phi|^4 , \tag{5}$$

$$\lambda = \tfrac{1}{2} e^2 (eH/2\pi). \tag{6}$$

The unstable mode will, of course, interact with other modes, but this interaction is not considered here. The effective action (5) is then expected to reproduce the same "longitudinal" Greens functions as one can compute from the "fundamental" lagrangian (4) as far as the unstable mode is concerned. Notice that there is a formal correspondence between the Feynman diagrams in (4) and (5). The difference is that (4) is (3 + 1)-dimensional and contains (apart from the "photon" field) charged vector fields describing all modes, whereas (5) is (1 + 1)-dimensional and the charged field is a scalar describing only the unstable mode.

It is well known [3] that eq. (5) has non-trivial vortex solutions. The vortex is of an electric type (see ref. [4] where further references can be found), since a_μ only has $\mu = 3, 4$ components and since a_μ depends only on x_3 and x_4.

However, it is important to realize that we can choose other planes – thus, e.g. the (x_2, x_4) plane also contains vortex solutions[‡2]. Since the vortex is electric, one would expect that it can terminate on quarks. The vortex may therefore be relevant for the quark confinement discussed by 't Hooft [4], although this is not clear in detail.

We want to emphasize that although the external field is necessary in order to induce the vortices, the non-trivial topology remains when the field is switched off [1]. This phenomenon is similar to the Prasad–Sommerfield limit [5].

To give a formal proof that the 1 + 1-dimensional effective lagrangian indeed is produced in the way described above, we observe that in order to carry out a perturbative expansion of the path integral

$$Z = \mathrm{const} \int [\mathrm{d}A]\ [\mathrm{d}W]\ [\mathrm{d}W^*]\ \exp(\mathrm{i}S[A_1 W_1 W^*]) \tag{7}$$

(or the corresponding generating functional for Green's functions), where $S = \int\mathcal{L}$, and $\mathcal{L}$ is given by eq. (4), we first have to determine the propagators, which means that a representation of the integration variables must be found for which the part of the Lagrangian which is bilinear in the fields, is diagonal. For the charged vector field W this means that we have to diagonalize the differential operator

$$\mathcal{D}_{\mu\nu} = \delta_{\mu\nu} D^{\mathrm{cl}}_\lambda D^{\mathrm{cl}}_\lambda - 2\mathrm{i}eF_{\mu\nu}, \tag{8}$$

where $F_{12} = -F_{21} = H$ and all other components of F are zero, whereas:

$$D^{\mathrm{cl}}_\lambda = (\partial_{11}\, \partial_2 - \mathrm{i}eHx_{11}\, \partial_{31}\, \partial_0), \tag{9}$$

i.e. it is the covariant derivative with respect to the background field, in contrast to the symbol D_λ occurring in eq. (4) which is the covariant derivative with respect to both the background field and the dynamical variable A_μ.

The last term in (8) acts as a mass matrix. It has two eigenvectors with nontrivial eigenvalues. They are

‡2 Remember that we took $k_\perp = 0$. Thus, in the "transverse" plane we are at large distances.

$$e^{\pm} = (1, \pm\mathrm{i}, 0, 0)/\sqrt{2} \tag{10}$$

with eigenvalues $\pm 2eH$. To get a tachyonic propagator we have to use the positive eigenvalue, so the tachyonic mode is contained in the component of the field W along e^{+}. When acting here, the differential operator $\mathcal{D}$ becomes:

$$\mathcal{D}|_{e^+} = \partial_1^2 - e^2H^2x_1^2 - 2\mathrm{i}eHx_1\partial_2 + \partial_2^2 + \partial_3^2 - \partial_0^2 + 2eH. \tag{11}$$

Obviously, the operator is diagonal when it acts upon a function which is a plane wave in the x_2, x_3 and x_0 variables and a Hermite function in the x_1 variable. The eigenfunctions which produce the unstable mode are:

$$F^{(\mathrm{tach})}_{k_2,k_3,k_0}(x) = \sqrt[4]{eH/\pi}\,\exp[-\tfrac{1}{2}eH(x_1 - k_2/eH)^2]\,\exp[\mathrm{i}(k_2x_2 + k_3x_3 - k_0x_0)], \tag{12}$$

for which

$$\mathcal{D}|_{e^+}F^{(\mathrm{tach})}_{k_2,k_3,k_0}(x) = (k_0^2 - k_3^2 + eH)\,F^{(\mathrm{tach})}_{k_2,k_3,k_0}(x). \tag{13}$$

The propagator of the unstable mode is thus in the momentum-space representation:

$$(k_3^2 - k_0^2 - eH - \mathrm{i}\epsilon)^{-1}, \tag{14}$$

which clearly corresponds to the propagation in the 3–0 plane of a tachyon with mass-squared eH.

In coordinate space the unstable mode can be described formally by a field $U(x)$, the propagator of which according to (12) and (14) is given by:

$$\mathrm{i}\langle TU(x)U^*(x')\rangle^{(0)} = \int\frac{\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}k_0}{(2\pi)^3}\,\frac{F^{(\mathrm{tach})}_{k_2,k_3,k_0}(x)\,F^{(\mathrm{tach})*}_{k_2,k_3,k_0}(x')}{k_3^2 - k_0^2 - eH - \mathrm{i}\epsilon}$$

$$= \int\frac{\mathrm{d}k_3\,\mathrm{d}k_0}{(2\pi)^2}\,\frac{\exp\{\mathrm{i}[k_3(x_3 - x_3') - k_0(x_0 - x_0')]\}}{k_3^2 - k_0^2 - eH - \mathrm{i}\epsilon}\,K(x_1, x_2; x_1', x_2'), \tag{15}$$

$$K(x_1, x_2; x_1', x_2') = \frac{eH}{2\pi}\exp[\tfrac{1}{2}\mathrm{i}eH(x_1 + x_1')(x_2 - x_2')]\,\exp\{-\tfrac{1}{4}eH[(x_1 - x_1')^2 + (x_2 - x_2')^2]\}, \tag{16}$$

while its interaction with itself and with the "photon" field A_μ, if only the $\mu = 3, 0$ components of A_μ are present, according to (4) just is the interaction part of a Higgs lagrangian:

$$\mathcal{L}_{\mathrm{int}}[A, U, U^*] = -\mathrm{i}eA_\mu U^*\overleftrightarrow{\partial}_\mu U - e^2A_\mu^2U^*U - \tfrac{1}{2}e^2(U^*U)^2. \tag{17}$$

Furthermore the kernel K has properties which essentially make the dynamics of the unstable mode 1 + 1-dimensional. It firstly fulfills:

$$\int\mathrm{d}x_1'\,\mathrm{d}x_2'\,K(x_1, x_2; x_1', x_2')\,K(x_1', x_2'; x_1''x_2'') = K(x_1, x_2; x_1'', x_2''), \tag{18}$$

i.e. it reproduces itself if two propagators are joined by a vertex at which no transfer of transverse momentum takes place. Thus, a one-loop Feynman integral into which no transverse momentum flows is given by a Feynman integral where the propagators are free 1 + 1-dimensional scalar tachyons propagators, and the interaction is described by eq. (5). This follows if one carries out all the integrations over the transverse coordinates, using the chain rule (18). One is left with:

$$\int \mathrm{d}x_1\, \mathrm{d}x_2\, K(x_1, x_2; x_1, x_2) = \frac{eH}{2\pi}(2\pi)^2\, \delta^{(2)}(0), \tag{19}$$

where the δ-function expresses conservation of the transverse momenta (which were all set equal to zero). The remaining (x_3, x_0)-integrations are then just (1 + 1)-dimensional. A special case of this result was found in I.

Also, even if some transverse momentum flow through the loop, the structure of K induces a gaussian cut-off:

$$k_\perp^2 \lesssim eH, \tag{20}$$

so the dynamics of the unstable mode remains almost (1 + 1)-dimensional.

These arguments one can give a more precise form by going back to the path integral Z. The unstable mode is, according to the discussion in the preceding paragraphs, isolated in the first term on the right-hand side, when the field W is split into orthogonal components according to:

$$W_\mu(x) = e_\mu^+ \int \frac{\mathrm{d}k_2}{2\pi}\, {}^4\!\sqrt{eH/\pi}\, \exp[-\tfrac{1}{2}eH(x_1 - k_2/eH)^2]\, \exp(\mathrm{i}k_2x_2)\, \phi_{k_2}(x_3, x_0) + W_\mu(x), \tag{21}$$

where $\phi_{k_2}(x_3, x_0)$ is the dynamical variable (the integration variable in the path integral) of the unstable mode. The perturbation series characterized by the propagator (15), and the interaction (17) is generated by taking in eq. (7) only the self-interaction of the variable $\phi_{k_2}(x_3, x_0)$ as well as its interaction with the field A_μ (μ = 3, 0) into account. Eq. (7) then reduces to:

$$Z^{(\text{unst. mode})} = \int [\mathrm{d}A]\ [\mathrm{d}\phi]\ [\mathrm{d}\phi^*]\ \exp\{\mathrm{i}S^{(\text{unst. mode})}\ [A, \phi, \phi^*]\}\,, \tag{22}$$

where $S^{(\text{unst. mode})}$, apart from the restriction on A_μ mentioned above, emerges from S through the replacement of W_μ by the first term on the right-hand side of (21). In the new action the integrals over x_1 and x_2 can be carried out explicitly. The part involving only ϕ and ϕ^* becomes:

$$\begin{aligned} S^{(\text{unst. mode})}\ [\phi, \phi^*] &= \int \frac{\mathrm{d}k_2}{2\pi} \int \mathrm{d}x_3\, \mathrm{d}x_0\, \{-|\partial_\mu \phi_{k_2}|^2 + eH|\phi_{k_2}|^2\}\, (x_3, x_0) \\ &\quad - \frac{e^2}{2} \int \frac{\mathrm{d}k_2}{2\pi}\, \frac{\mathrm{d}k_2'}{2\pi}\, \frac{\mathrm{d}k_2''}{2\pi}\, \frac{\mathrm{d}k_2'''}{2\pi}\, 2\pi\delta(k_2 + k_2' - k_2'' - k_2''')\sqrt{eH/2\pi}\, \exp\left\{-\frac{k_2^2 + k_2'^2 + k_2''^2 + k_2'''^2 - (k_2 + k_2')^2}{2eH}\right\} \\ &\quad \times \int \mathrm{d}x_3\, \mathrm{d}x_0\, \phi^*_{k_2''}(x_3, x_0)\, \phi^*_{k_2'''}(x_3, x_0)\, \phi_{k_2}(x_3, x_0)\, \phi_{k_2'}(x_3, x_0) \end{aligned} \tag{23}$$

while the part describing the interaction between ϕ, ϕ^* and A becomes:

$$\begin{aligned} S^{(\text{unst. mode})}[A, \phi, \phi^*] &= \int \mathrm{d}x_3\, \mathrm{d}x_0 \int \frac{\mathrm{d}k_1\, \mathrm{d}k_2}{(2\pi)^2}\, \exp\{-(k_1^2 + k_2^2)/4eH\} \int \frac{\mathrm{d}k_2'}{2\pi}\, \frac{\mathrm{d}k_2''}{2\pi}\, 2\pi\delta(k_2 + k_2' - k_2'') \\ &\quad \times \exp\{\mathrm{i}k_1(k_2' + k_2'')/2eH\}\, \{-\mathrm{i}eA_\mu(k_1, k_2; x_3, x_0)\, (\phi^*_{k_2''}\, \overset{\leftrightarrow}{\partial}_\mu \phi_{k_2'})\, (x_3, x_0) \\ &\quad - e^2A_\mu^2(k_1, k_2; x_3, x_0)\, (\phi^*_{k_2''}\, \phi_{k_2'})\, (x_3, x_0)\}. \end{aligned} \tag{24}$$

Here the gaussian factor in the integrand obviously gives rise to a cut-off in the transverse momentum transferred by the "photon"-field, in agreement with (20). Eqs. (22) and (23) correspond to a Higgs-type interaction if ϕ_{k_2} is only allowed to have a narrow distribution in k_2.

If the perturbative vacuum does not produce fluctuations corresponding to a constant magnetic field, our work is clearly without any interest. In view of the importance of this point we shall now indicate that there exist a *definition* of the renormalized coupling which gives vacuum fluctuations of the constant magnetic field type (provided the number of flavors is less than or equal to 8 for color SU(3)). The effective Lagrangian $\mathrm{Re}\,\mathcal{L}$ as a function of the magnetic field H has been computed [2]. The result is

$$\mathrm{Re}\ \partial\mathcal{L}/\partial(\tfrac{1}{2}H^2) = -e^2/\bar{e}(t)^2; \quad t = \ln eH/\mu^2, \tag{25}$$

$$\text{where} \quad \bar{e}(0) = e \quad \text{and} \quad \mathrm{d}\bar{e}/\mathrm{d}t = \bar{\beta}(\bar{e}), \bar{\beta} = \beta/(1+\beta/e). \tag{26}$$

Following the procedure of 't Hooft [6], we can now *define* the renormalized coupling $\bar{e}$ such that

$$\bar{\beta}(\bar{e}) = \beta_1\bar{e}^3 + \beta_2\bar{e}^5. \tag{27}$$

With this definition it is easily seen that in general $\mathrm{Re}\,\mathcal{L}$ has a minimum away from the origin [1,2], corresponding to $\bar{e}(t) = \infty$ for some finite value of t (i.e., corresponding to the Landau tachyon which, interestingly enough, disappears from the usual Green's functions when the definition (27) is accepted, as one can see from eq. (26)). From I, we do not expect this to be a stable minimum, but for our purpose it is enough that this state exists for some time.

The condition for the Landau tachyon to occur is that $\beta_2 < 0$. For color SU(3) this implies that the number of flavors satisfies

$$N_\mathrm{f} < 153/19 \approx 8.05 \tag{28}$$

so there should be at most 8 flavors in order that the mechanism discussed above works.

Thus, in conclusion, if $N_\mathrm{f} \leqslant 8$ then no matter how short the constant field fluctuations live, they produce a soup of electric vortex lines (and anti-vortex lines), which in the usual way could be responsible for confinement.

Finally we shall make some comments on the problem of Lorentz invariance. The calculations in I as well as our derivation of the Higgs lagrangian in the present paper are all done by means of a constant homogeneous field pointing in some direction. This set up of course breaks rotational invariance. On the other hand, as a function of $\mathrm{tr}F_{\mu\nu}^2$ the minimum arising from eq. (25) can be realized in an infinity of ways, one of which is our realization. A rotational invariant realization is $H^a{}_b = H\delta^a{}_b$ where a is an isospin index, and b is a spatial index. It is plausible that no matter how one realizes the minimum in $\mathrm{Re}\,\mathcal{L}$ in eq. (25), all physical results should be the same, including the effective Higgs lagrangian describing the dynamics of the unstable mode ‡3.

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‡3 If the rotational invariant realization is chosen, it is to be expected that the (x_4, x_3)-plane is replaced by the (x_4, r)-plane, where $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. The vortices then form an electric bag.

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