# INSTANTONS AND TRANSVERSE PURE GAUGE FIELDS

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The phenomenon of the ambiguity of the Coulomb gauge in Yang-Mills theories, first discovered by Gribov, is studied for the instanton solutions.

It is shown that in the Coulomb gauge the instanton connects a non-vanishing transverse pure gauge field in the remote (Euclidean) past to an analogous one in the far future.

Quite recently Gribov [1] has found a new "pathological" effect in the Yang-Mills theories. While in Q.E.D. the transversality condition

$$\partial_i A_i = 0 \tag{1}$$

completely fixes the gauge, leading to a unique solution of the Cauchy problem, the same thing does not happen in non-Abelian gauge theories.

In fact, for the latter case, Gribov has shown that in general there exist space dependent gauge transformations that connect different solutions of eq. (1); i.e., the condition (1) is not enough to determine unambiguously the potential $A_i(x)$, once the field $F_{\mu\nu}$ is known.

In particular, for a vanishing field $F_{\mu\nu} = 0$ the transversality condition (1) can be written in the form

$$\partial_i (u^{-1} \partial_i u) = 0 \tag{2}$$

where $U(\boldsymbol{x}, t)$ is a gauge group element that generates a pure gauge potential:

$$A_\mu = (U^{-1} \partial_i U) = 0 \tag{3}$$

In Q.E.D. the analogue of eq. (2) is:

$$\nabla^2 \Lambda = 0 \tag{4}$$

where

$$A_\mu = \partial_\mu \Lambda. \tag{5}$$

Of course, the regular solutions of (4) are space independent and therefore $A_i = 0$ is the only well-behaved potential that in the Coulomb gauge (1) represents the field $F_{\mu\nu} = 0$.

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On the contrary Gribov has shown that for non-Abelian $U$ eq. (2) has non-trivial solutions; in particular he considers the SU(2) case and shows that there exist spherically symmetric solutions of the form:

$$U(\boldsymbol{x}, t) = \exp[\mathrm{i}\alpha(r, t)\, \boldsymbol{\sigma} \cdot \boldsymbol{x}/r], \quad r^2 = \boldsymbol{x}^2 \tag{6}$$

where $\alpha$ satisfies the equation

$$\frac{\partial^2 \alpha}{\partial r^2} + \frac{2}{r}\frac{\partial \alpha}{\partial r} - \frac{\sin 2\alpha}{r^2} = 0; \tag{7}$$

of course for $r \to 0$, $\alpha$ must vanish (mod. $\pi$) at least as $r$ in order to avoid singularities at $r = 0$:

$$\alpha(r) \underset{r \to 0}{\longrightarrow} n\pi + \mathrm{O}(r). \tag{8}$$

The substitution $s = \ln r$ reduces eq. (7) to the equation of a pendulum with friction in a constant gravitational field [‡1]

$$\partial^2 \alpha / \partial s^2 + \partial \alpha / \partial s - \sin 2\alpha = 0. \tag{9}$$

The potential relevant to eq. (9) has the form (fig. 1)

$$u(\alpha) = -2 \sin^2 \alpha. \tag{10}$$

‡1 The equations (7) and (9) were already considered in ref. [2], but they were not treated in a quite correct way.

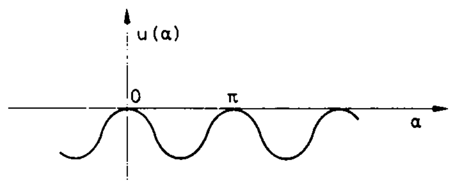


Fig. 1.

Hence, remembering the initial condition (8), one gets that, beyond the trivial solution $\alpha(s) = n\pi$, other solutions are possible: at $s = -\infty$ ($r = 0$), the pendulum starts from a point of unstable equilibrium $\alpha = n\pi$ and then for $s \to +\infty$ ($r \to \infty$) it asymptotically reaches the stable equilibrium at $\alpha = (n \pm \frac{1}{2})\pi$; therefore the solutions of eq. (7) are characterized by

$$\alpha(r\to\infty) \quad \alpha(r=0) = \begin{cases} 0 & \text{trivial solution} \\ \pm\dfrac{\pi}{2} & \text{non-trivial solutions}\,. \end{cases} \tag{11}$$

The non-trivial solutions of eq. (7) inserted in (6) and (3) give nonvanishing transverse pure gauge fields.

In this letter we are going to show that configurations of this kind are present in the instanton solution [3] of the Euclidean Yang-Mills equations. The instanton can be written in the form:

$$A_\mu = \frac{x^2}{x^2+\lambda^2}\, g^{-1}\partial_\mu g \tag{12}$$

where the SU(2) element $g(x)$ is

$$g(x) = \frac{x_4 - \mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{x}}{\sqrt{x^2}} \equiv \exp\left[\mathrm{i}\beta(r,x_4)\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{r}\right], \tag{13}$$

with

$$\beta(r,x_4) = -\operatorname{arctg}(r/x_4). \tag{14}$$

The instanton solution (12) can be written in the Coulomb gauge (1) performing a spherically symmetric gauge transformation:

$$B_\mu = h^{-1}A_\mu h + h^{-1}\partial_\mu h, \tag{15}$$

where

$$h(x) = \exp[\mathrm{i}\gamma(r,x_4)\,\boldsymbol{\sigma}\cdot\boldsymbol{x}/r]. \tag{16}$$

The transversality condition

$$\partial_i B_i = 0 \tag{17}$$

expressed in terms of $\gamma$ takes the form [2]:

$$\frac{\partial^2\gamma}{\partial r^2} + \frac{2}{r}\frac{\partial\gamma}{\partial r} = \frac{x_4^2 - r^2 + \lambda^2}{r^2(x^2+\lambda^2)}\sin 2\gamma - \frac{2x_4}{r(x^2+\lambda^2)}\cos 2\gamma + \frac{2x_4(x_4^2+\lambda^2)}{r(x^2+\lambda^2)^2}. \tag{18}$$

At large Euclidean four-dimensional distances ($x^2 \gg \lambda^2$), eq. (18) becomes much simpler if one uses the variable:

$$\alpha(r,x_4) = \beta(r,x_4) + \gamma(r,x_4), \tag{19}$$

where $\beta$ is given by (14).

In fact, for $x^2 \gg \lambda^2$ one gets that such an $\alpha$ just satisfies the Gribov equation (7).

This fact is not surprising: in fact for $x^2 \gg \lambda^2$ the instanton becomes a pure gauge field:

$$B_\mu = U^{-1}\partial_\mu U(1 + \mathrm{O}(\lambda^2/x^2)), \tag{20}$$

where

$$U(x) = g(x)h(x) = \exp[\mathrm{i}\alpha(r,x_4)\,\boldsymbol{\sigma}\cdot\boldsymbol{x}/r] \tag{21}$$

and hence the transversality condition (17) takes the form (7).

A priori, for $|x_4| \gg \lambda$ the instanton field could trivially vanish, once the Coulomb gauge is chosen; on the contrary, we will show that it corresponds, both for positive and negative large $x_4$, to non-trivial solutions of the Gribov equation (7).

To this aim we discuss the behaviour of $\alpha(r,x_4)$ as a function of $r$ at large fixed $x_4$; in particular, we look at the limit in the left-hand-side of (11), for $|x_4| \gg \lambda$. The definition (14) says that at $r = 0$ the phase $\beta$ is an integer multiple of $\pi$; we choose a determination such that (fig. 2)

$$\beta(r=0,x_4) = 0, \qquad x_4 < 0. \tag{22}$$

Moreover, one gets from (14) that keeping $x_4$ fixed (and negative) and by increasing $r$, also $\beta$ increases, until $\pi/2$ [‡2] is reached for $r \to \infty$:

$$\beta(r\to\infty,x_4) = \pi/2, \qquad x_4 < 0. \tag{23}$$

Actually, in the limit $r \to \infty$, arctg $\beta$ passes from $+\infty$ to $-\infty$ when $x_4$ changes sign, but $\beta$ does not change [it passes from $\pi/2 - \epsilon$ to $\pi/2 + \epsilon$ with $\epsilon = \mathrm{O}(x_4/r)$]; therefore, one also has:

$$\beta(r\to\infty,x_4) = \pi/2, \qquad x_4 > 0. \tag{24}$$

Finally, when $r$ decreases, at fixed (and positive) $x_4$, $\beta$ increases and reach the value $\pi$ at $r = 0$.

$$\beta(r=0,x_4) = \pi, \qquad x_4 > 0. \tag{25}$$

‡2 Let us note that eq. (23) does not depend on the actual value of $x_4$; of course a completely different result would have been obtained if the $\lim_{x_4\to\infty}$ were performed before the $\lim_{r\to\infty}$; however, as we are interested in the behaviour of the field all over the space at a given time, the $\lim_{x_4\to\pm\infty}$ must be performed at the end.

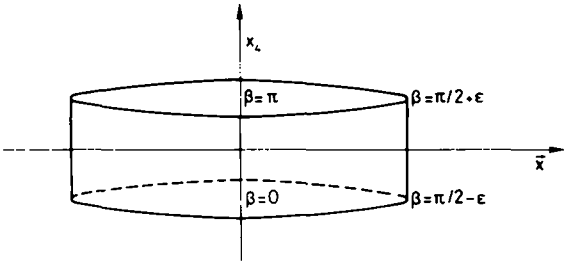


Fig. 2.

The behaviour of $\gamma(r, x_4)$ can be inferred by eq. (18), by imposing that $h(x)$ [defined in eq. (16)] is regular all over the space-time (we do not want to change the topological number of the instanton).

At $r = 0$ the phase $\gamma$ must be a multiple of $\pi$; the continuity of $\gamma$ prevents jumps and then we can choose

$$\gamma(r=0, x_4) = 0 \tag{26}$$

for any $x_4$; therefore by (19), (22) and (25) we get:

$$\alpha(r=0, x_4) = \beta(r=0, x_4) = \begin{cases} 0 & x_4 < 0 \\ \pi & x_4 > 0 \end{cases}. \tag{27}$$

By deriving eq. (18) with respect to $x_4$, one gets [2] that for any value of $x_4$

$$\partial\gamma/\partial x_4 \xrightarrow[r\to\infty]{} O(1/r); \tag{28}$$

then for large $r$ the phase $\gamma$ does not depend on $x_4$.

At $x_4 = 0$ eq. (18) becomes (using the variable $s = \ln r$)

$$\frac{\partial^2\gamma}{\partial s^2} + \frac{\partial\gamma}{\partial s} + \left(\frac{1-\lambda^2 e^{-2s}}{1+\lambda^2 e^{-2s}}\right)\sin 2\gamma = 0. \tag{29}$$

Thus, for $s \to +\infty$, it behaves as if the potential $u(\alpha)$ were reversed with respect to that one depicted in fig. 1; hence, one expects that $\gamma(r=\infty, x_4)$ is an integer multiple of $\pi$.

By combining the information from eq. (9), summarized in (11), with those contained in (27), one realizes that the only possibility is

$$\gamma(r\to\infty, x_4) = 0 \tag{30}$$

and therefore

$$\alpha(r\to\infty, x_4) = \beta(r\to\infty, x_4) = \pi/2 \tag{31}$$

for any value of $x_4$.

Finally, the insertion of (27) and (31) into the left-hand side of (11) shows that the instanton solution performs a transition from a non-trivial transverse pure gauge field at $x_4 = -\infty$ to another one at $x_4 = +\infty$.

This result can be intuitively understood by looking at the topological charge $q$ of the instanton $B_\mu$, calculated as a flux across the surface depicted in fig. 2. One has

$$q = \varphi_+ - \varphi_- + \varphi_L, \tag{32}$$

where $\varphi_+$ and $\varphi_-$ are the fluxes across the top and bottom surfaces of the cylinder of fig. 2:

$$\varphi_\pm = -\frac{1}{24\pi^2}\int d^3x\, \epsilon_{ijk}\, \mathrm{Tr}(B_i B_j B_k)\Big|_{x_4=\pm\infty} \tag{33}$$

and $\varphi_L$ is the flux across the lateral surface of the same cylinder [‡3]

$$\varphi_L = \frac{1}{8\pi^2}\int_{-\infty}^{+\infty} dx_4 \int d^2S_i\, \epsilon_{ijk}\, \mathrm{Tr}(B_j B_k B_4). \tag{34}$$

By using $B_\mu$ defined in (15) one gets by direct calculation

$$\varphi_L = \frac{1}{\pi}\left\{\gamma(r\to\infty, x_4) + \frac{1}{2}\sin 2\gamma(r\to\infty, x_4)\right\}\Bigg|_{x_4=-\infty}^{x_4=+\infty} \tag{35}$$

The transversality condition (18) implies that $\gamma(r=\infty, x_4)$ does not depend on $x_4$ (eq. (28)); therefore in the Coulomb gauge $\varphi_L = 0$ and the whole topological charge $q = 1$ must be shared between $\varphi_+$ and $-\varphi_-$.

If in the remote past the potential $B_\mu$ were rapidly vanishing, at $x_4 \ll -\lambda$ the phase $\alpha(r, x_4)$ would vanish for any value of $r$ and one would have $\varphi_- = 0$; hence, one should have $\varphi_+ = 1$ and then, at $x_4 \gg \lambda$, $\alpha(r=\infty, x_4) = 0$ and $\alpha(r=0, x_4) = \pi$. But in the Coulomb gauge such a possibility is forbidden by eq. (7), that implies (11); what actually happens is that in the Coulomb gauge one gets (in disagreement with ref. [2]):

$$\varphi_+ = -\varphi_- = \tfrac{1}{2}. \tag{36}$$

The appearance of half-integer "topological" charges is eqsily understood remembering that under the condition (31) the value of the group element $\lim_{x\to\infty} h(x)$, at large fixed $x_4$, does depend on the direction $x/r$. Therefore the Euclidean space $R_3$ does *not* be-

[‡3] Both in (33) and in (34) the limit $x_4 \to \pm\infty$ must be taken after having performed the space integral.

come $S_3$; in such a case there is no topological reason to compel the fluxes $\varphi_\pm$ to be integer; it is for that reason that we have used quotation marks to the expression: half integer "topological" charges.

In conclusion, one can say that, in the Coulomb gauge there are three degenerate vacuum states ($F_{\mu\nu} = 0$) (at the classical level):

i) the usual perturbative vacuum $A_i = 0$, with vanishing topological charge,

ii) a state with "topological" charge $\varphi = -\frac{1}{2}$, corresponding to $\alpha(r \to \infty) - \alpha(0) = \pi/2$,

iii) a state with "topological" charge $\varphi = +\frac{1}{2}$, corresponding to $\alpha(r \to \infty) - \alpha(0) = -\pi/2$, and that the instanton represents a tunnelling effect between the vacua (ii) and (iii), without affecting the usual perturbative vacuum (i).

We remark, however, that our discussion (and Gribov's one) deals only with spherically symmetric gauge transformations [of the form (6) and (16)]. It is possible that under more general gauge transformations the structure of the classical vacuum is much richer; in such a case the instanton would connect the vacuum i) with a new kind of vacuum, different from those described in ii) and iii). Works are in progress to fully clarify these points and to understand the role of many-instanton solutions.

## References

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