PHYSICAL REVIEW D VOLUME 20, NUMBER 12 15 DECEMBER 1979

Instantons as a bridge between weak and strong coupling in quantum chromodynamics

Curtis G. Callan, Jr.
Princeton University, Princeton, New Jersey 08540

Roger F. Dashen
The Institute for Advanced Study, Princeton, New Jersey 08540

David J. Gross
Princeton University, Princeton, New Jersey 08540
(Received 19 July 1979)

The transition from weak to strong coupling in quantum chromodynamics is discussed using semiclassical methods to calculate the effective coupling $g(a)$ of an effective lattice theory as a function of a. We demonstrate that the renormalization effects due to instantons cause a sharp transition from weak to strong coupling in the range $g(a) \approx 1.5$–3. This is verified by showing that the "string tension" as well as the β function calculated by semiclassical methods match their strong-coupling behavior for $g \approx 1.5$–3. An independent test of this notion is provided by a bag-model calculation of the string tension.

I. INTRODUCTION

In the past few years a reasonably simple picture of the workings of quantum chromodynamics (QCD) has emerged. It assigns different physics to different distance scales by introducing the notion of an effective coupling, $g(d)$, governing the quantum fluctuations on distance scale d. At small distances, $g(d)$ is small [in fact $1/g^2(d)$ varies as $\ln(1/\Lambda d)$ as required by asymptotic freedom] and the fluctuations are those of weak-coupling perturbation theory. The effective coupling grows with increasing d and beyond some critical value, d_c (presumably roughly equal to the size of a typical hadron), it is assumed to become large enough that the fluctuations are essentially those of a strong-coupling lattice gauge theory. This strong-coupling limit is automatically confining and completely characterized by the string constant, σ, or the energy density per unit length of the flux tube which joins any two separated charges.

This picture of the limiting behaviors of QCD, even if correct, in turn poses some difficult physics problems. First, there is the issue of dimensional transmutation: The string constant of the strong-coupling limit is dimensional; the only dimensional quantity in the theory is the renormalization scale parameter, Λ, describing the asymptotic-freedom variation of the coupling in the weak-coupling limit; evidently it must be possible to establish a relation of the form $\sigma = \phi\Lambda^2$, where ϕ is a pure number, and to make a precise numerical connection between the physics of weak and strong coupling. Second, there is the problem of computing the properties of hadrons: Almost by definition, the scale size of a typical hadron lies in the transition region between weak and strong coupling, and no direct information about light-hadron physics is provided by either of the limits. To make progress on either of these problems, it is obviously necessary to understand the details of the transition between weak and strong coupling.

In previous publications[1,2] we have shown that weak-coupling nonperturbative effects (i.e., instantons) cause a rapid increase in $g(d)$ at a rather sharply defined distance scale, d_c. This scale is small enough that one would normally have thought that ordinary perturbation theory would be accurate and g slowly varying. In fact, over a small range in d, $g(d)$ becomes so large that weak-coupling methods of any kind, perturbative or nonperturbative, cease to make sense. The key question is whether at this point g is large enough to put the theory in the strong-coupling limit. If so, dimensional transmutation has occurred, and we may identify the scale at which instanton effects turn on with the hadron scale; if not, a further increase in g arising from instanton-unrelated physics is called for and we do not know how, even approximately, to identify the hadron scale.

In the past we have simply *assumed* that the former is true and shown that a reasonable-looking picture of hadron physics, quite closely related to the bag model, can then be derived. In this paper we will present a series of arguments in support of the assertion that instanton effects carry the theory all the way into the strong-coupling regime. Though they are not without their own problems, we believe the arguments presented here put the semiquantitative picture of the QCD vacuum and of hadron structure which we have advocated on a much sounder footing. We should say at the outset, however, that everything in this paper applies strictly speaking only to the theory with *no* light quarks. Massless fermions and chiral symmetry breaking pose special problems which we do not know how to deal with in the present context. For simplicity we stick to the unrealistic, yet instructive, theory of a pure gauge field with at most static external color charges.

In order to connect the weak- and strong-coupling domains it is particularly useful to construct an effective lattice theory for QCD. Although in this paper we shall not explicitly construct such an effective lattice theory, nor show how a mass gap emerges, we explore, in Sec. II, the general qualitative aspects of this notion. Under the assumption that QCD confines we discuss the structure of the strong-coupling lattice theory, focusing special attention on the Wilson term in the effective action. We translate our continuum QCD bag picture into this lattice theory and indicate how the radius of the flux tube joining quarks can be deduced. Further illustrations of some of the features of the semiclassical bag in terms of the lattice theory are presented in Appendix B. In Sec. III we determine the magnitude of the coefficient, $g(a)$, of the Wilson term in the lattice theory by evaluating the effective coupling using semiclassical methods that include the renormalization due to instantons. We focus in particular on the behavior of the "string tension" and the β function and demonstrate that as the spacing of our effective lattice increases these rapidly approach their strong-coupling values, thus confirming that instantons do indeed bridge the gap, for $g(a)\approx 2-3$, between weak and strong coupling. Furthermore, we estimate σ in terms of Λ. An additional argument that instantons are responsible for the transition to strong coupling is provided in Sec. IV where we compare the string tension calculated by semiclassical extrapolation to that of a strong-coupling flux tube. The agreement of these two independent calculations confirms our contention that the strong-coupling region begins right at the point where weak-coupling methods break down. Finally in Sec. V we offer our conclusions as to the overall picture of the QCD. A discussion of the various coupling-constant definitions used in QCD, and the relation between them, is presented in Appendix A.

II. INSERTING A LATTICE IN THE FUNCTIONAL INTEGRAL

Inserting a lattice into the functional integral is a useful device for extracting some of the physics of instantons. In this section we wish to explore the general, qualitative aspects of this notion. The detailed role played by instantons will be discussed in the following section. In what follows, it will be assumed that the theory confines and that the usual lore about flux tubes, the behavior of strong-coupling lattice theories, and so on is correct.

In principle it is possible to construct an exact lattice theory by doing the continuum functional integration with the link variables $U=P\exp(i\int A\cdot dx)$ held fixed. The remaining integration over the U's defines, of course, an exact lattice theory. A mathematical lattice of this kind could be constructed with any spacing, a, and by a judicious choice of a one might be able to separate the ultraviolet and infrared parts of the theory into, respectively, the continuum and the lattice integrations. In practice, however, such an approach would be plagued with unsolved renormalization problems associated with introducing the constraints into the continuum functional integral. On the other hand, we can avoid this difficulty by starting with the theory defined on a very fine lattice, at the Planck length a_0, say. Then according to the usual lore (i.e., the notion of universality), if the action on this superfine lattice has the Wilson form[3]

$$S_W=\frac{1}{g^2(a_0)}\,\mathcal{L}_W(\{U_i\}) =\frac{1}{g^2(a_0)}\sum_{\text{plaquettes}}\left[\mathrm{Tr}\left(\prod_i U_i\right)+\mathrm{H.c.}\right], \tag{1}$$

with $g^{-2}(a_0)\sim -(11/8\pi^2)\ln a\Lambda$, it will turn out that for small-enough a_0 we will have simply constructed the continuum theory with a particular regularization scheme. Thus for all practical purposes, if a_0 is taken to be small enough, the above Wilson action can be taken to define QCD. Now there is no conceptual problem with the notion of integrating over the basic link variables keeping compound U's on a larger sublattice fixed. The scale of this sublattice, a, can be any integer times a_0, and is for all practical purposes a continuous variable.

The end result of introducing such a lattice would be the computation of an effective action

$$S_a=\frac{1}{g^2(a)}\,\mathcal{L}_W+\text{"higher terms"}, \tag{2}$$

which contains all the information necessary to answer any physical question that can be formulated on this lattice. For reasons which will be apparent later we have explicitly separated out the single-plaquette Wilson term. However, except in the limit $a\to 0$, where the "higher terms" vanish, we do not mean to imply that these extra terms can be dropped. They are not negligible and one of their roles is, in fact, to restore Lorentz invariance on a finite lattice.

On a sufficiently large lattice it is generally believed that (i) the expectation value of a planar Wilson loop becomes proportional to $C^{-N}[1+O(1/g^2(a))]$, where $C>1$ and N is the number of plaquettes in the loop, (ii) the nominal strength of a "higher term" involving n plaquettes is $g^{-2n}(a)$, and (iii) the strong-coupling expansion of the action $e^{-S}=1-S+\frac{1}{2}S^2+\cdots$ can be used to compute arbitrary expectation values. Under these conditions the Wilson term is special in that, to leading order in $g^{-2}(a)$, it completely determines the expectation value of all *planar* Wilson loops. The proof is a standard strong-coupling argument which will not be repeated here. We do wish to emphasize, however, that the restriction to planar loops is essential; for nonplanar observables the "higher terms" can contribute in the same order in $g^{-2}(a)$ as does the Wilson term. The potential between a heavy quark-antiquark pair is a physical example of a problem that can be formulated in terms of a planar (Wilson) loop. (Indeed, it is almost unique in this respect.) On the other hand, the interaction between three or more quarks cannot be deduced from the properties of planar (in space-time) loops.

The fact that, for strong coupling, planar loops are determined by the Wilson term implies that its coefficient $g^{-2}(a)$ satisfies a closed renormalization-group equation. This is most easily derived by evaluating the expectation value of the Wilson loop operator for a planar loop of area $A=Na^2$. In the strong-coupling limit this has the value

$$\langle W\rangle \equiv \exp(-A\sigma)=\left(\frac{1}{3g^2(a)}\right)^N\left[1+O\left(\frac{1}{g^2(a)}\right)\right], \quad (3)$$

where

$$\sigma=\frac{\ln 3g^2(a)}{a^2}+O\left(\frac{1}{g^2(a)}\right) \quad (4)$$

is the energy per unit length of the string connecting the quarks or the string tension. The a dependence of the (strong coupling) lattice coupling $g^2(a)$ is derived by demanding that σ be independent of a. This yields

$$\frac{1}{g(a)}\frac{dg(a)}{d\ln a}=\ln 3g^2(a)+O\left(\frac{1}{g^2(a)}\right). \quad (5)$$

At the other extreme of weak coupling the "higher terms" actually vanish and again one obtains a closed renormalization-group equation for $g^2(a)$. It is of course just the usual asymptotic-freedom result[4] that

$$\frac{1}{g(a)}\frac{dg(a)}{d\ln a}=\frac{11}{16\pi^2}g^2(a)+O(g^4(a)). \quad (6)$$

Because $g(a)$, defined to be the coefficient of the Wilson term in $\mathcal{L}_a$, renormalizes in a simple way for both weak and strong coupling we propose to define β (with opposite sign to the usual convention) as

$$\frac{dg(a)}{d\ln a}\equiv\beta(g(a)). \quad (7)$$

At intermediate values of the coupling there is nothing particularly unique about this definition of β, and it should not be expected to agree in detail with β functions defined in other physically reasonable ways.

The weak- and strong-coupling curves for β/g are shown in Fig. 1. For weak coupling the strength of the (unique definition-independent) order-g^4 term[5] has also been indicated and for strong coupling the size of the next-to-leading term arising from $(1/g^2)\mathcal{L}_W$ is also shown. The latter is less significant since the true correction to the strong-coupling limit involves the effect of unknown "higher terms." Note that there is a region in g, roughly $1<g<4$, where both expansions would appear to be valid but which give quite different values of β.[6] One of the aims of this paper is to resolve this apparent paradox. To anticipate later results, we find that nonperturbative instanton effects cause the weak-coupling β to break sharply away from the perturbative curve at $g\approx 1$ and rise rapidly up to the strong-coupling β. This is indicated schematically by the dashed line in Fig. 1.

The physical manifestation of confinement is the existence of tubes, or "bags," of trapped color flux connecting heavy quark-antiquark pairs. This basically continuum notion has certain consequences for the effective lattice theory. Normally the strong-coupling lattice theory is said to lead to a string picture. This is questionable since within a lattice theory, of spacing a, it is impossible to distinguish an infinitely thin string from a flux tube of diameter a. We shall, however, show that the bag picture has a precise analog within the lattice theory.

In what follows, we will consider a time slice of the lattice and concentrate on the spatial links. Figure 2 shows a flux tube of radius R upon which a lattice with spacing $2R$ has been superimposed. On this lattice the flux tube is represented by the single chain of links running down its center, since all other links are either in the wrong direction or lie outside the tube. The collapse of the entire flux onto a single link is of course just the picture that emerges from the strong-coupling expansion.

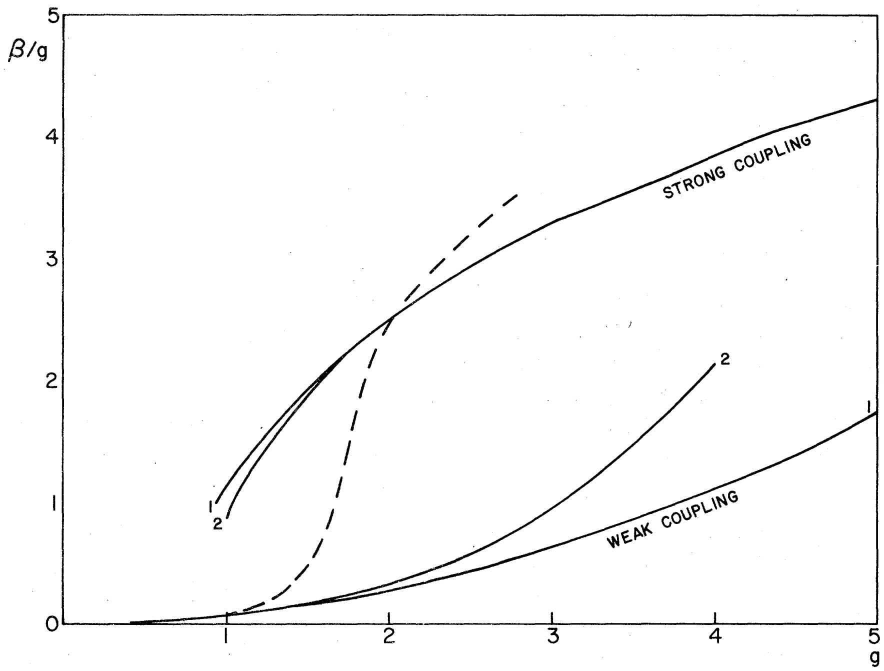

FIG. 1. Plot of $\beta/g \equiv d\ln g(a)/d\ln a$ as a function of g for weak and strong coupling. The solid curves are the weak- and strong-coupling curves to lowest order (1) and to second order (2), respectively. The dashed line is a schematic representation of the effect of instantons.

We can therefore infer that on this lattice the coupling is strong. It is also true that the energy per unit length of the tube is determined entirely by S_W and is in fact just the string tension given by Eq. (4).

The bag picture also implies that the effective action must contain for $a=2R$, non-negligible "higher-order" terms. Consider putting a quark-antiquark pair on diagonal corners of the lattice. Figure 3 shows a flux tube connecting these quarks. Note that there are now no links lying entirely within the tube. Nevertheless the coupling is strong and the apparent paradox is resolved by the fact that for this diagonal flux tube the Wilson term is irrelevant and the entire contribution comes from the "higher terms." (Note that this diagonal flux tube cannot be specified by a space-time loop which is both planar and lies everywhere on the lattice.) Another situation where "higher terms" are important is illustrated by the $qq\bar{q}\bar{q}$ configuration shown in Fig. 4. Here the kinematics are consistent with a simple lattice picture but the two flux tubes are in contact, suggesting that a description in which they are collapsed onto disjoint links will not be adequate. Thus the continuum picture of QCD which leads to the formation of flux tubes of radius R can be translated into an effective strong-coupling lattice theory with a spacing $a \geq 2R$, whose coupling $g^{-2}(a)$ (the coefficient of the Wilson term) can be calculated from the tension of the flux tube, and whose "higher-order" terms can, in principle, be evaluated.

We shall now show that the radius of the flux tube, and the thickness of the tube wall can be related to the spatial dependence of $g^2(a)$. Consider

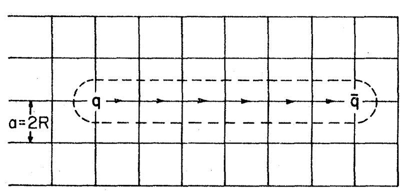

FIG. 2. A flux tube of radius R connecting a quark-antiquark pair overlaid with a lattice of spacing $a=2R$.

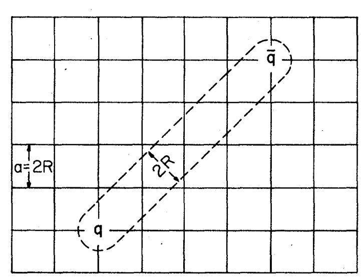

FIG. 3. A flux tube of radius R connecting a quark-antiquark pair on diagonal corners of a lattice of spacing $a=2R$.

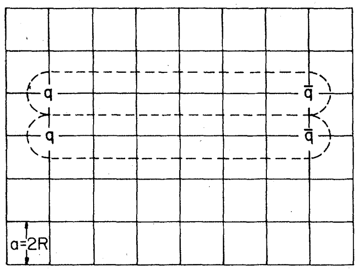

FIG. 4. A two-quark–two-antiquark configuration on a lattice of spacing $a=2R$.

a flux tube, as shown in Fig. 5, overlaid with a lattice of spacing $a=R$ rather than $2R$. There are now five parallel chains of links within the tube, the original one at the center and four (in three dimensions) just inside the boundary. On the lattice the flux is clearly spread out over several links, and it must be that the coupling $g^2(R)$ is weak. In terms of the β function of Fig. 1 we have thus seen that for a lattice spacing equal to the flux tube radius R the coupling has just become weak and one is just below the rise in β, while for a lattice spacing $2R$ the theory is well into strong coupling. This is consistent with the strong-coupling behavior of $g^2(a)$, since

$$g^2(2R)=\tfrac{1}{3}\exp[4\ln 3g^2(R)]\,,$$

and thus if, say, $g(R)=1.5$ then $g(2R)=26$.

If, as we shall argue in the following, instantons are indeed responsible for the sharp transition from weak to strong coupling, we will be able to relate the radius of the flux tube to the renormalization scale parameter Λ, and to compare the string tension of the continuum bag picture with that of the effective lattice theory.

Furthermore, we can also discuss the thickness of the flux tube wall, which is related to the rate at which the coupling rises from weak to strong values. When this transition is rapid (as we shall see it is) the wall will be thin and the flux tube sharply defined.

One can do further calculations which make the connection between the strong-coupling lattice and continuum flux tubes even more explicit. Although interesting in their own right these results are somewhat peripheral to the present paper and have been relegated to an appendix (Appendix B).

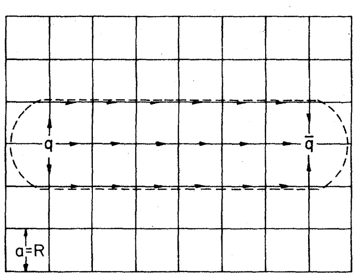

FIG. 5. A flux tube of radius R connecting a quark-antiquark pair overlaid with a lattice of spacing $a=R$.

III. THE EFFECT OF INSTANTONS

Roughly speaking, the form of the lattice action, $\mathcal{L}_a(\{U_l\})$ is determined by the quantum fluctuations of the continuum theory on scales less than a. For small-enough a, the quantum fluctuations are those of perturbation theory and $\mathcal{L}_a$ is just the Wilson action with a coupling constant which varies with a according to asymptotic freedom. Previous work on the Euclidean semiclassical method has shown that the first new effects to appear, as we increase the limiting scale size, are those due to instantons.[1,2] There will therefore be some range of scale sizes over which $\mathcal{L}_a$ is determined by instanton and perturbation theory effects alone, and we should be able to compute the diagnostic quantities β and σ defined in Sec. II and study their deviation from perturbative behavior. If we increase a too far, $\mathcal{L}_a$ will be determined by more complicated configurations than instantons, and we will no longer be able to compute $\mathcal{L}_a$ with any accuracy. Before that happens, however, we shall present evidence that the system has already reached the strong-coupling limit, allowing us to use strong-coupling methods at larger scales.

To actually evaluate $\mathcal{L}_a$ in the manner described in Sec. II would be a monstrous job, which could only be done numerically. Since we are trying to develop insight, we will employ shortcuts whose accuracy can be only imperfectly evaluated, but which should give us a reasonably accurate picture of the physics.

To compute $\mathcal{L}_a$ we hold the link variables $\{U_l\}$ associated with the a lattice fixed and integrate over all the configurations of the primitive lattice consistent with the choice of $\{U_l\}$. Since the primitive lattice is just a special way of introducing a cutoff into the continuum theory, it will have recognizable multi-instanton configurations, and it is precisely over these configurations that we want to integrate. Instantons of scale size significantly larger than a clearly are not relevant since they can be specified by imposing a suitable variation on the link variables of the a lattice. Instanton of scale size smaller than a, as long as they are not too dense, have a simple effect which can be summarized by a modified vacuum permeability, μ, which is greater than one and can be calculated in a dipole gas model. In the present context we make the reasonable assumption that the effect of μ is seen directly in a multiplicative renormalization of the coupling constant associated with the Wilson action: $g^2(a) = g_{AF}^2(a)\mu(a)$ (where g_{AF} is the perturbation-theory, or asymptotic-freedom, coupling constant). In the formulas which were developed in earlier papers,[2] μ is determined by an integration over instanton scale sizes, and we must now decide where to cut this integration off (though it seems obvious that the cutoff should be in the neighborhood of instanton scale size equal to a).

The best information we have on this question comes from our study of instanton effects on the heavy-quark potential.[7] There, rather than fixing link variables, we put a Wilson loop into the functional integral and integrated over dilute-instanton-gas configurations. The interesting point is that if the diameter of the Wilson loop was taken to be more than 1.5 times the instanton scale size, the result of the computation was indistinguishable from that which would have been obtained by ignoring the instantons and renormalizing the perturbation-theory coupling constant by a μ computed according to our dipole-gas arguments. At the same time, if the Wilson loop diameter was taken to be less than about one times the instanton scale size, the result of the computation was very accurately equal to the extrapolation of the zero-size Wilson loop limit. In our context we take these results to mean that, in computing $\mathcal{L}_a$, (a) instantons of scale size less than $\frac{2}{3}a$ may be assumed to renormalize the Wilson action coupling constant by our standard dipole-gas formulas, and (b) instantons of scale size greater than a can be neglected since they are obtained by subsequent integration over the link variables $\{U_l\}$. We have no *a priori* way of knowing how to include the fluctuations on scales intermediate between these two limits. For lack of anything better to do we shall assume that this ignorance can be compensated for by setting the upper limit on the instanton scale-size integration in μ not at $\frac{2}{3}a$ but at some value, a_c, between $\frac{2}{3}a$ and a. We will see that this uncertainty does not materially affect the overall physical picture we are proposing, but does lead to some looseness in our numerical solution of the dimensional-transmutation problem. Presumably this uncertainty could be eliminated by a major lattice calculation in the manner of Wilson.[8]

With these preliminaries out of the way, we are ready to construct $g(a)$, $\beta(g)$, and $\sigma(g)$ in the instanton-dominated regime. The basic formulas, culled from the previous discussion and our earlier papers,[2] are

$$g^2(a) = g_{AF}^2(a)\mu(a)\,, \quad x_{AF}(a) = \frac{8\pi^2}{g_{AF}^2(a)} = 11\ln\frac{1}{\Lambda a}\,,$$
$$\mu(a) = \eta + (1+\eta^2)^{1/2}\,, \quad \eta = \frac{\pi^2}{2}\int_0^{a_c}\frac{d\rho}{\rho}D(\rho)x_{AF}(\rho)\,, \tag{8}$$

where $D(\rho)$ is the instanton density function and Λ is the asymptotic-freedom renormalization scale. Of course these expressions are usable only so long as a_c is small enough that the integrated instanton density,

$$f = \pi^2\int_0^{a_c}\frac{d\rho}{\rho}D(\rho)\,,$$

is less than one—otherwise our instanton-gas arguments would not work. We will shortly determine the limiting values of a and $g(a)$.

To make an explicit computation we must specify $D(\rho)$. In the weak-coupling limit,

$$D(\rho) = Cx_{AF}(\rho)^6 e^{-x_{AF}(\rho)}\,,$$

where the coefficient C depends on the definition of the QCD coupling constant in a manner which has been discussed elsewhere.[2] In our approach, everything is based on a lattice regularization of the field theory and we must be careful to use the corresponding coupling-constant definition to compute C. It is not too hard to show (the arguments are given in Appendix A) that C so computed equals 1.62×10^6. The large value of this coefficient does not have any direct physical content—it merely causes instanton-induced effects of a given size to occur at a smaller value of g than for a coupling-constant definition in which C is not so large (for example, dimensional regularization yields $C \approx 100$). Since g itself does not have any invariant significance, this should not bother us, but we do have to remember to use the same coupling-constant definition consistently throughout our calculation. Given our basic aim, the lattice definition (or regularization) of the coupling is clearly called for, but we must bear in mind that this is not the definition used in most applications of QCD when we wish to compare our numbers with experiment or with other calculations. The relation between the couplings in various regularization schemes is discussed in Appendix A.

It is now a simple matter of numerical computation to compute the diagnostic quantities $\beta(g)$ and $\sigma(g)$. The result for $\beta(g)$ is plotted in Figs. 6(a) and 6(b). The two cases correspond to the choices $a_c = \frac{2}{3}a$ and $a_c = a$ for the instanton scale size cutoff. The solid line is the computed value of $\beta(g)$ and terminates where the integrated density of the instantons responsible for coupling-constant renormalization is order one (the approximations we are making can no longer be trusted beyond this point).[9] The dashed lines are for comparison and show the strong-coupling and weak-coupling limits of β. The salient feature of both plots is the rapid rise of β, at g roughly equal to one, away from the weak-coupling curve. The difference between the two cutoff choices is simply a slight shift in

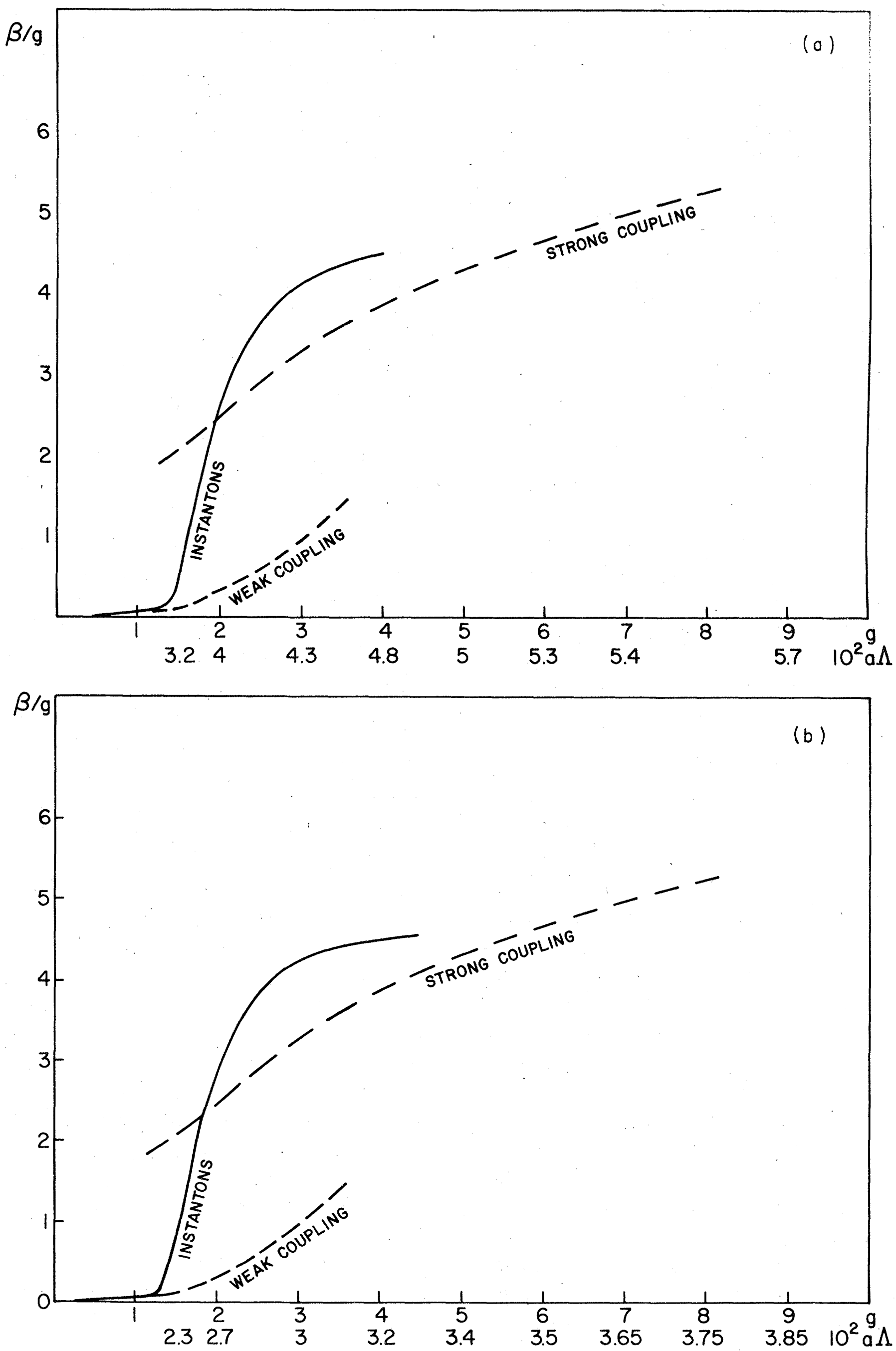

FIG. 6. (a) A plot of β/g due to instantons as a function of g and a for the choice $a_c = \frac{2}{3}a$ The dashed lines represent the strong coupling and the perturbative values of β/g, respectively. (b) A plot of β/g due to instantons as a function of g and a for the choice $a_c = a$. The dashed lines represent the strong coupling and the perturbative values of β/g, respectively.

the value of g at which the nonperturbative increase in β occurs. The value of a at which the transition from weak- to strong-coupling behavior occurs, $\bar{a}$, does depend (linearly) on our choice of a_c. Equally interesting is the fact that in both cases the overall increase in β (before our instanton computation breaks down at $g \sim 3.5$) is enough to bring it up to the strong-coupling curve. We take this as strong evidence that instanton effects carry the system all the way to the strong-coupling regime—after all, the increase in β need not have been large enough to match the strong-coupling curve. To make the argument conclusive, one would have to show that, for $g>3$, β continues to follow the strong-coupling curve. We shall come back to this point in the next section.

It is also useful to plot the string tension, defined to be $\sigma(g)=\ln 3g^2(a)/a^2$. This is not a particularly interesting quantity in the weak-coupling limit, but must approach a constant in the strong-coupling limit, if we are to recover the large-distance properties of the strong-coupling lattice theory. That constant must be of the form $\phi\Lambda^2$, where Λ is the asymptotic-freedom renormalization scale parameter and ϕ is a pure number reflecting the outcome of dimensional transmutation.

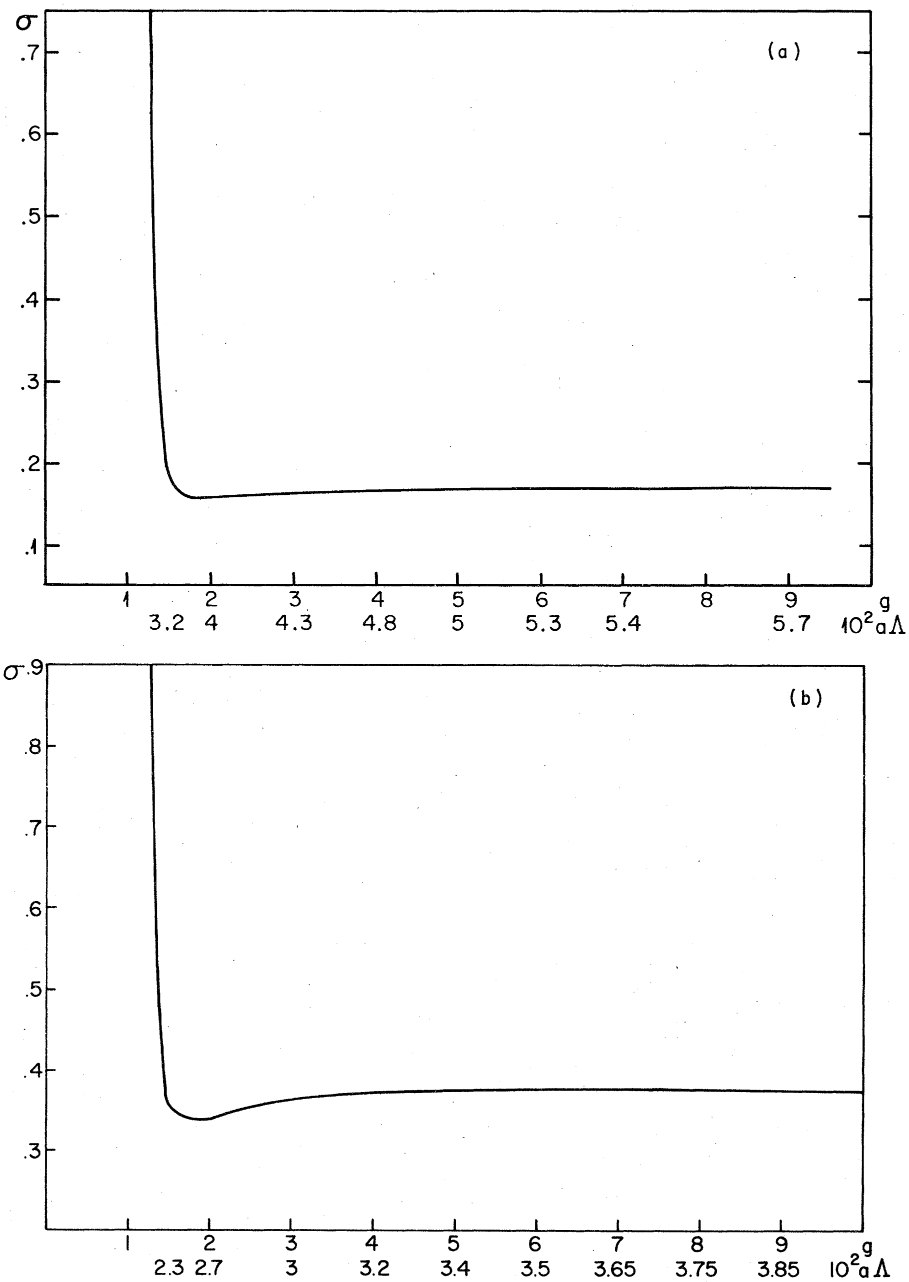

FIG. 7. (a) A plot of the string tension σ, in units of $10^4\ \Lambda^2$, due to instantons as a function of g and a for the choice $a_c=\frac{2}{3}a$. (b) A plot of the string tension σ, in units of $10^4\ \Lambda^2$, due to instantons as a function of g and a for the choice $a_c=a$.

The result of the computation is plotted in Figs. 7(a) and 7(b), the two cases corresponding to the two choices of upper cutoff on instanton scale size. Just as in Fig. 6, the computation is to be regarded as reliable only up to $g\sim 3.5$, although we have plotted σ beyond that point. Note first of all that σ seems to settle down to a constant quite rapidly and to do so within the instanton-dominated regime. Since constant σ is a sign of strong coupling, we take this as another piece of evidence that instanton effects drive the system to strong coupling. Also, in this way of plotting things, we see an apparently large effect of changing the instanton scale size cutoff from $\frac{2}{3}a$ to a—the limiting value of the string tension goes from $1700\Lambda^2$ to $3700\Lambda^2$. This is some measure of our lack of precision in solving the dimensional-transmutation problem—if Λ is thought of as having been determined from scaling-violation measurements, then the predicted value of the string tension is uncertain by a factor of 2 or so. This factor is, of course, simply the square of the uncertainty (1.5) in our choice of a_c.

In this context, it must be remembered that the Λ we are using is that appropriate to lattice regularization, Λ_L, and is not the same as the Λ commonly used by experimentalists in discussing scaling violations. According to the discussion of Appendix A, Λ_L is in fact much smaller than that employed in regularization schemes used to compare with experiment. If, as according to Ref. 10, we take $\Lambda_{\rm exp}$ to be the momentum-space scale parameter Λ_M, then indeed $\Lambda_L=\Lambda_M/14$, and thus in this scheme $\sigma\approx[(3-4.3)\Lambda_M]^2$.

It is difficult to compare this with the measured value of σ [which is related to the Regge slope α' by $\sigma=1/(2\pi\alpha)=(420\text{ MeV})^2$] since we have ignored the effect of light quarks. It seems clear, however, that the inclusion of light quarks will, according to our picture of QCD, lead to a *decrease* in the value of σ. This is because the presence of light quarks increases the scale size at which instanton-induced effects turn on, since these are small until one probes distances over which the quarks acquire a reasonably large dynamical mass.[1,2] A very rough estimate of this effect doubles the value of $\bar{a}$ at which the transition from weak to strong coupling occurs. Since $\sigma\propto\bar{a}^{-2}$, this means that (roughly) in the real world $\sigma\approx[(1.5-2)\Lambda_M]^2$ and thus $\Lambda_M\approx 210-280$ MeV, which is quite reasonable.

Thus we have seen, in a precise quantitative sense, that instantons bridge the gap between weak and strong coupling, by causing the effective lattice coupling to increase sharply at a specific value of the lattice spacing $\bar{a}$ and at that point to behave as a strongly coupled lattice theory. Our solution of the dimensional-transmutation problem consists of calculating the value of $\bar{a}$ and $g(\bar{a})$ in terms of Λ and thus, since we immediately go over to strong coupling, the resulting string tension σ in terms of Λ. Although the answer is not very precise it is quantitatively reasonable.

IV. THE SEMICLASSICAL AND STRONG-COUPLING FLUX TUBES

If QCD is confining (and we *assume* it to be), then, for sufficiently large g, $\beta(g)$ must equal its strong-coupling limit, $\beta_{\rm sc}=\ln 3g^2$, and $\sigma(g)$ must equal a constant, $\sigma_{\rm sc}$. The arguments presented above show that, in the coupling-constant range from $g\simeq 1$ to $g\simeq 3$, β and σ undergo a transition from perturbation-theory behavior to something qualitatively quite different. Moreover, we saw that it was consistent, in a precise quantitative sense, to imagine that β and σ were smoothly joining on to their strong-coupling limits at $g\simeq 3$. If true, this is very important since it means that, by pushing weak-coupling and semiclassical methods to the extreme we can make quantitative calculations of strong-coupling quantities. It is therefore important to give as much support as possible to this view. In this section we will present another, quite independent, consistency argument for the assumption that strong-coupling behavior sets in at $g\approx 3$. Our method will be to use the bag picture to establish some properties of the passage between weak and strong coupling and to show that they fit the picture developed above.

In the bag model, widely separated static color charges are joined by a stationary flux tube of some finite radius, R. Such flux tubes correspond physically to the strings of the strong-coupling limit of a lattice theory. In Sec. II we argued that our effective lattice action, $\mathcal{L}_a$, evaluated at $a=2R$, must be in the strong-coupling phase because only one lattice link at a time can overlap a flux tube. At the same time, $\mathcal{L}_R$, should be just leaving the weak-coupling region since a single flux tube can then overlap two links at a time and there must be reasonably strong correlations between nearest-neighbor links. Therefore, on general grounds *we expect the flux tube diameter to be about twice the lattice spacing at which the instanton effects first turn on strongly.*

We can use these geometrical remarks to generate two alternative, and presumably reasonable, evaluations of the flux tube radius from our plot of $\beta(g)$. If we define R to equal the value of a at which instanton effects first turn on and agree that they have turned on when β/g equals twice its weak-coupling value, we get $R=0.029\Lambda^{-1}$ ($R=0.019\Lambda^{-1}$) from Fig. 6(a) (Fig. 6(b)). If we define R to be *half* the smallest value of a for which

β/g is unambiguously in the strong-coupling region and define that value of a by the point where β/g crosses the strong-coupling curve, we get $R = 0.02\Lambda^{-1}$ ($R = 0.014\Lambda^{-1}$) from Fig. 6(a) (Fig. 6(b)).

The difference between the numbers extracted from Fig. 6(a) and Fig. 6(b) is just a reflection of the uncertainty in our solution of the dimensional-transmutation problem (which we have represented by two possible choices of instanton scale size cutoff.) The difference, for a given choice of cutoff, between the two definitions of R, is presumably a reflection of the fact that the flux tube wall is not infinitely thin. Indeed, our results suggest that its thickness is approximately one-half the flux tube radius, which is thin enough for bag arguments to make sense. For the purposes of the following bag argument, we will take as our estimate of the flux tube radius the average of the values corresponding to the two definitions

$$R = \begin{cases} 0.0245\Lambda^{-1} \\ 0.0165\Lambda^{-1} \end{cases} \text{for } a_c = \begin{cases} \frac{2}{3}a \\ a \end{cases}. \qquad (9)$$

For $a > 2R$, $\mathcal{L}_a$ is definitely in the strong-coupling region and we have the relation

$$\sigma_{sc} = \frac{\ln 3g^2(a)}{a^2}, \qquad (10)$$

where σ_{sc} is the string constant of the flux tube. We have already computed σ_{sc} using our assumption that for $g \gtrsim 3$ the system is in strong coupling. We now wish to compute σ_{sc} from the bag picture and see that we get the same result. This is a test of the contention that instantons do indeed bridge the gap to strong coupling already at $g \approx 3$.

In the bag model, the basic equations governing a color-triplet flux tube are

$$\sigma = \text{energy density per unit length}$$

$$= \frac{1}{g_0^2} E_c^2 \times (\text{flux tube area}), \qquad (11)$$

$$E_c \times (\text{flux tube area}) = Qg_0^2,$$

where E_c is the (critical) electric field in the flux tube, $Q = (\frac{4}{3})^{1/2}$ and g_0 is the perturbation-theory coupling constant describing the coupling of a static charge to the static color field inside the flux tube. Thus

$$\sigma \times (\text{flux tube area}) = \tfrac{4}{3} g_0^2. \qquad (12)$$

The value of g_0 is determined by the size of the flux tube (in units of Λ^{-1}) which is why we went to the trouble to determine the flux tube radius in terms of Λ in the previous paragraphs. The detailed computation of g_0 is given in Appendix A, where we show that $8\pi^2/g_0^2 \sim 30$. Since $8\pi^2/g_0^2$ depends only logarithmically on Λ, our dimensional-transmutation uncertainty of a factor of $\sqrt{2}$ in Λ translates into just a 10% uncertainty in g_0^2.

Since we now know both the flux tube radius and g_0 (computed consistently with that radius), we can use Eq. (12) to evaluate the bag-model string constant. We find that

$$\sigma_{bag} = \begin{cases} 1925\Lambda^2 \\ 3729\Lambda^2 \end{cases} \text{for } a_c = \begin{cases} \frac{2}{3}a \\ a \end{cases}. \qquad (13)$$

These values are very close to the corresponding values of $\sigma_{sc} = 1700\Lambda^2$ or $3700\Lambda^2$ obtained in the preceding section. Since σ is a quadratic function of Λ it is very sensitive to slight changes in scale, and the above agreement is very satisfying. The bag computation is applicable, of course, only in the strong-coupling limit. The fact that it agrees with the semiclassical calculation, extended to this region, gives further support to our contention that the strong-coupling region begins right at the point where weak-coupling methods break down, i.e., at $g \approx 3$.

V. CONCLUSIONS

Let us now take stock of what we have accomplished. The aim of our discussion has not been to *prove confinement* but, *assuming* confinement, to show that everything we know about QCD is consistent in precise numerical sense with the notion that semiclassical effects (i.e., instantons) bridge the gap between weak- and strong-coupling physics.

In our earlier attempts to apply semiclassical wisdom to hadronic physics, we simply assumed that this notion was correct. In particular, in our QCD bag-model computations, we set the bag constant, by hand, equal to the integral over instanton contributions up to a scale size corresponding to integrated instanton density equal to one. This is tantamount to assuming that fluctuations on scales larger than this are safely in the strong-coupling regime, and need not be included in the bag constant. Indeed a simple strong-coupling argument shows that the vacuum energy density, or the bag constant, due to fluctuations on a lattice with coupling $g(a)$ are of order $[1/3g^2(a)]^2$. Although the consequences of this assumption were physically reasonable, we in fact did not have direct evidence for it.

The arguments presented in this paper are intended to show that the basic assumption of our QCD bag model meets several consistency checks of a fairly precise nature. The price we have to pay in order to be able to formulate the consistency checks in the first place is the introduction of a comparison lattice theory into the continuum theory. Since the strong-coupling limit of a lattice theory is well understood, we have simple and precise strong-coupling limits against which to compare the behavior of the comparison lattice theory. Our interesting discovery is that the inclusion of weak-coupling semiclassical effects alone is sufficient to make the comparison lattice theory behave in detail in the same way as a strong-coupling lattice theory.

This also lends credence to the point of view that the structure of the QCD vacuum consists of two components; coherent fluctuations on a scale less than $\bar{a}$ which are describable by semiclassical (weak-coupling) techniques and random fluctuations on a scale greater than $\bar{a}$ which are describable by strong-coupling techniques. If this is the case then one need not worry about the quantitative contribution of other coherent fluctuations (such as mesons, vortices, etc.) since they will, most likely, be washed out in the chaotic fluctuations of the strong-coupling phase.

On the other hand, a lattice theory, because it destroys Lorentz invariance, is not useful for discussing detailed hadron physics questions, and our conclusions in this paper are necessarily of an academic nature. The one physical quantity which can easily be studied in the strong-coupling limit is the static string tension, σ, and we were in fact able to compute it in terms of the asymptotic-freedom renormalization mass, Λ. As should be the case, we find essentially the same relation between σ and Λ as we did in the QCD bag-model calculation even though the two methods are superficially quite different. Although this "solution" of the dimensional-transmutation problem is very pleasing, it is not terribly precise, largely because of our uncertainty about the precise manner in which a comparison lattice of a given scale size cuts off the integration over instanton scale size in the continuum theory. This is a very difficult problem, but one which could in principle be solved.

More generally, if we are to make predictions for hadronic properties other than the string tension, we will have to learn how to use what we have found here to refine the QCD bag model. Before even beginning to tackle this problem we will have to deal with the major omission of our discussion—light fermions. With massless or very light fermions in the theory, the passage from weak to strong coupling and the breaking of chiral invariance become inextricably entwined. A comparison lattice method would therefore have to deal correctly with spontaneous fermion mass generation and Nambu-Goldstone bosons. At the moment, we do not know how to do this and therefore cannot make detailed comparisons with the real world.

Let us reiterate that if we have constructed a convincing overall picture of the behavior of QCD, it is because we knew how to characterize the expected large-distance behavior of the theory—we simply assumed it to be that of a lattice gauge theory in the strong-coupling limit. There are variants of QCD for which we do not know how to characterize the large-distance behavior and for which we therefore have no way of establishing an overall picture. We have already mentioned the example of QCD plus massless fermions—although we believe this theory to be confining, because of chiral-invariance difficulties we do not know how to characterize its large-distance behavior. A somewhat more exotic example is the case of QCD without fermions, but with a large vacuum angle, θ. If we take $\theta > \pi/2$, it is not hard to see that weak-coupling semiclassical effects, instead of causing the coupling strength to increase with increasing scale size, actually cause it to *decrease*. This means that instantons, for $\theta > \pi/2$, screen the color fields produced by quarks. It is then far from obvious that the large-distance behavior of the theory is confining. Perhaps for a range of θ the theory chooses the Higgs mechanism. At the moment, we can only speculate.

Finally, we should mention the work of Kogut, Pearson, and Shigemitsu (KPS)[11] from which we drew a certain amount of inspiration. These authors have attempted to construct the β function of QCD by using Padé methods on the strong-coupling expansion of the Wilson action. The aim is to see that the strong-coupling β function smoothly joins the asymptotic-freedom limit, without going through a zero. In fact, they have carried the expansion far enough to see a clear turnover toward the weak-coupling form of β. Furthermore, their results indicate that the passage from weak to strong coupling occurs quite rapidly, over a range of scale sizes of order a factor of 2 or so, and for values of g in the neighborhood of 1. Although there is no reason for the KPS β function to agree with ours in detail in the transition region, it should contain the same information about the nature of the passage from weak to strong coupling. It is apparent that the two methods do agree, on at least a superficial level, and we take this as welcome further evidence that we are on the right track. In the context of our remarks about QCD with nonzero θ (i.e., that qualitatively different physics may emerge if we choose $\theta > \pi/2$) this also suggests that it might be interesting to do this type of calculation for a strong-coupling lattice theory with nonzero θ.

ACKNOWLEDGMENT

This research was sponsored by the National Science Foundation under Grant No. PHY78-01221 and by the Department of Energy under Grant No. EY-76-S-02-2220.

APPENDIX A

There are a number of ways to define the coupling constant in QCD. An example is dimensional regularization with minimal subtractions. Any other coupling g_A^2 can be expressed as a power series in the dimensional coupling g_D^2

$$g_A^2 = g_D^2 + \frac{\alpha}{8\pi^2} g_D^4 + \cdots , \tag{A1}$$

where the coefficients $\alpha, \ldots$ are finite numbers. Equivalently one can write

$$\frac{8\pi^2}{g_A^2} = \frac{8\pi^2}{g_D^2} - \alpha + O(g_D^2) , \tag{A2}$$

or

$$x_A = x_D - \alpha + O\left(\frac{1}{x}\right) . \tag{A3}$$

For the effective coupling constants, $x_A(p) = 11 \ln(p/\Lambda_A)$ and $x_D(p) = 11(p/\Lambda_D)$, Eq. (A3) implies

$$\frac{\Lambda_A}{\Lambda_D} = \exp\left(\frac{\alpha}{11}\right). \tag{A4}$$

Thus to leading order one sees that a change from one definition of g to another is simply equivalent to a change in scale.

In a given calculation it is appropriate to define g so that the physics is as clear as possible. For example, in processes that emphasize a particular (spacelike) momentum p, the "momentum subtracted" coupling g_M^2 is preferable. It is defined in such a way that the gluon propagator satisfies

$$D^{\mu\nu}(p) = g^{\mu\nu} \frac{g_M^2(p)}{p^2} + \text{gauge terms} , \tag{A5}$$

and has the virtue that at p the strength of the (linearized) interaction is known. Celmaster and Gonsalves[10] have shown that

$$\frac{\Lambda_M}{\Lambda_D} \approx 5.83 . \tag{A6}$$

Another useful definition is the Pauli-Villars coupling g_{PV}^2 used by 't Hooft in his calculation of the instanton determinant.[12] 't Hooft showed that

$$\frac{\Lambda_{PV}}{\Lambda_D} \approx 2.75 . \tag{A7}$$

When comparing with lattice theories one should use the bare coupling $g_L^2 = -8\pi^2/(11 \ln a\Lambda_L)$ on a lattice of spacing a. A calculation of the exact value of Λ_L (i.e., its relation to Λ_{PV} or to Λ_D) would be an extremely tedious task and we will content ourselves with an estimate (accurate to a few percent) based on the comparison of the integral $\int d^4p/p^4$ as regulated by a lattice and by the Pauli-Villars scheme. Numerically, we find

$$\left(\frac{a}{2}\right)^4 \int\int\int\int_{-\pi/a}^{\pi/a} \frac{d^4p}{\left[\sum_{i=1}^{4} \sin^2 \frac{p_i a}{2}\right]^2} - \int d^4p\left[\frac{1}{p^4} - \frac{1}{(p^2+\mu^2)^2}\right] = \ln Ma - 1.89, \tag{A8}$$

and thus obtain the estimate

$$\frac{\Lambda_{PV}}{\Lambda_L} \approx \exp(1.89) = 6.6 . \tag{A9}$$

Finally, it remains to specify the weak coupling inside a bag or flux tube. The physics depends on this coupling only through the linearized gluon propagator in the bag, and it follows that we should take it to be $g_M^2(p_0)$, where p_0 is a typical momentum encountered on inverting the Laplacian Δ in a bag. For a flux tube the eigenvalues of Δ can be written as $p_r^2 + p_l^2$ where p_r and p_l are the radial and longitudinal momenta. The longitudinal eigenvalues satisfy $J_1(p_r R) = 0$ and the lowest relevant one is $p_r = 3.85/R$. (The trivial solution $p_r = 0$ corresponds to a constant color electric field and cannot affect a color-singlet state.) To estimate p_l we note that the way g enters into the calculation of the string tension is through the solution of Gauss's law at the end of a flux tube, a roughly spherical calculation. This suggests that we take $p_l^2 \approx \frac{1}{2} p_r^2$ yielding the estimate $4.72/R$ for p_0. The value of g_0 given in the text is obtained by converting the estimate of the bag radius given in Sec. IV in terms of the lattice scale parameter to $R \approx 0.34\Lambda_M^{-1}$ or $0.23\Lambda_M^{-1}$, and using asymptotic freedom

$$\frac{8\pi^2}{g_0^2} \approx 11 \ln \frac{p_0}{\Lambda_M} \approx 11 \ln \frac{4.72}{R\Lambda_M} \tag{A10}$$

to obtain $8\pi^2/g_0^2 \approx 29$ or 33, respectively.

APPENDIX B: FLUX TUBES, IMAGINARY ELECTRIC FIELDS, AND STRONG-COUPLING EXPANSIONS

In this paper we have discussed various properties of color flux tubes in the context of both the continuum theory and the strong-coupling theory. There are some issues, which were a source of possible confusion, with our earlier treatment of the bag model[2] which can be clarified with the help of this new machinery. Most notably in our Euclidean picture there appeared a *coherent* (apart from the averaging needed to form a color singlet) *imaginary* electric field inside the bag, along with an *incoherent real* field outside the bag. Although somewhat outside the line of development of this paper, we would like to show how some of the peculiar features of our continuum bag model appear in a recognizable form in the strong-coupling lattice theory.

To explore the properties of the bag in the lattice theory, we can consider expectation values of an operator O in the presence of a Wilson loop L

$$\langle O\rangle_L=\frac{\int [dU]e^{-S_W}\operatorname{tr}(\Pi_{i\in L}U_i)O}{\int [dU]e^{-S_W}\operatorname{tr}(\Pi_{i\in L}U_i)}\,. \qquad \text{(B1)}$$

We shall choose the lattice so that $g(a)$ is large, and assume that the loop L is large and planar. According to the arguments of Sec. II the lattice manifestation of the continuum bag is the flux sheet across the (planar) surface spanned by L.

We can now study the field strength in the presence of the Wilson loop by evaluating the expectation value of

$$\mathcal{L}_W=\operatorname{tr}\left[\prod_p (U_i)+\text{H. c.}\right]$$

on various plaquettes. It is a straightforward exercise in the strong-coupling expansion to show that

$$\tfrac{1}{3}\langle \mathcal{L}_W\rangle_L=\begin{cases} 3g^2(a) & \text{if the plaquette is coplanar with and inside } L \\ O(1/3g^2(a)) & \text{otherwise}\,. \end{cases} \qquad \text{(B2)}$$

Here one clearly sees the existence of a sheet of flux across L. Note that since $\operatorname{tr}U<3$ the average of $\frac{1}{3}\mathcal{L}_W$ with a positive real weight must satisfy $\langle\frac{1}{3}\mathcal{L}_W\rangle<2$. It is the phase introduced by the Wilson loop that allows $\langle\frac{1}{3}\mathcal{L}_W\rangle_L$ to grow like $3g^2(a)$ for large $g(a)$. This behavior, $\langle\frac{1}{3}\mathcal{L}_W\rangle_L\sim 3g^2(a)$, can equivalently be thought of as due to a coherent imaginary electric field inside the flux sheet that spans the loop L.

It is also interesting to consider $\langle\mathcal{L}_W\rangle_L$ for a plaquette outside of but close to the flux tube. If the quasilinear calculations of Ref. 2 are pushed into a region where they have no reason to be valid, namely into the strong-coupling phase outside the tube, then one would conclude that near the tube there is a coherent imaginary field in addition to the real fluctuating fields. (This is since instantons, by themselves, produce a large but finite μ.) Since there is no trace of this extended imaginary field in the strong-coupling calculation, i.e., Eq. (B2), we conclude that it is an artifact of having pushed a linear calculation into a region where it is not valid.

---

[1]C. Callan, R. Dashen, and D. Gross, Phys. Rev. D 17, 2717 (1978).

[2]C. Callan, R. Dashen, and D. Gross, Phys. Rev. D 19, 1826 (1979).

[3]K. Wilson, Phys. Rev. D 10, 2445 (1975).

[4]D. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973); H. Politzer, *ibid* 30, 1346 (1973).

[5]W. Caswell, Phys. Rev. Lett. 33, 344 (1974); D. R. Jones, Nucl. Phys. B75, 531 (1974).

[6]In fact, the strong-coupling expansion for S_W appears to be extremely good for all $g\gtrsim 1.5$ (see Ref. 10).

[7]C. Callan, R. Dashen, D. Gross, F. Wilczek, and A. Zee, Phys. Rev. D 18, 4684 (1978).

[8]One might imagine introducing the lattice into the continuum theory by inserting

$$1=\int \prod_k dU_k \prod_k \delta\left(U_k-P\exp\left(i\int_k A\cdot dx\right)\right)$$

into a functional integral. If one then expands the δ functions as $\delta(u-v)=\sum_R D_R(u)D_R^\dagger(v)$, one obtains for the continuum functional integral a sum of terms in which one has essentially fixed an *imaginary* background field strength, on each plaquette, proportional to a^{-2}. The suppression of the contribution of large-size instantons (of size $\gtrsim a$) can then be seen to be due to the phenomenon of magnetostriction as discussed in Ref. 2, which leads to a suppression factor, for instantons of scale size ρ, of $\exp[-\text{const}\times(\rho/a)^2]$

[9]On the other hand for $g\lesssim 2.5$ the fraction, f, of space-time occupied by instantons is only a few percent.

[10]W. Celmaster and R. Gonsalves, Phys. Rev. Lett. 42, 1435 (1979).

[11]J. Kogut, R. Pearson, and J. Shigemitsu, Phys. Rev. Lett. 43, 484 (1979).

[12]G. 't Hooft, Phys. Rev. D 14, 3432 (1976).