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Medium-energy $N\pi\pi$ dynamics. II. Rescattering corrections to the isobar model

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A new parametrization of the isobar-model type is proposed for $N\pi\pi$ final states. It incorporates the major subenergy-dependent variations induced by unitarity (rescattering) corrections to the conventional nonunitary isobar model. These corrections have been calculated in J^P waves $1/2^+$, $1/2^-$, $3/2^+$, and $3/2^-$ using the dynamical theory based on subenergy unitarity and analyticity recently developed by the authors. All the corrections vary relatively smoothly with subenergy. It is argued that this result implies that the conclusions of the existing nonunitary fits will not be seriously modified by the inclusion of the rescattering corrections. There are, however, some significant subenergy variations. In all the important cases the detailed results exhibit one of two distinctive subenergy effects, which are interpreted physically; each effect occurs in a systematic fashion and can be succinctly parametrized. Some of the variations may be detectable with improved statistics.

I. INTRODUCTION

In the usual isobar model,[1] constructed for phenomenological applications,[2] the partial-wave amplitudes are taken to depend only on the total energy W and not on the subenergy variables. The assumption is inconsistent with two-body unitarity in the isobar channels.[3] The search for a model based on the isobar expansion, which is unitary in the subenergy channels and has the right analytic structure, leads to a set of integral equations[4] for new isobar amplitudes, each now dependent on its subenergy variables as well as on W. For the $N\pi\pi$ system specifically, these equations have been derived in a previous paper[5] (to be referred to as I) for J^P states $\frac{1}{2}^+$, $\frac{1}{2}^-$, $\frac{3}{2}^+$, and $\frac{3}{2}^-$. All isobar states likely to be important for, say, $W \lesssim 1.5$ GeV, are included; these are: the $N\pi$ isobars S_{11}, S_{31}, P_{11}, and P_{33}, and the π-π isobars in s-wave $I=0$ and $I=2$, and p-wave $I=1$. The purpose of this paper is to show how the dynamical equations governing this system can be put to phenomenological use.

In Sec. II we explain how the theory developed in I is applied to the problem of calculating unitarity corrections to the conventional $N\pi\pi$ isobar model. These corrections amount to multiplying each subenergy-independent isobar amplitude of the conventional model by a function which depends on the subenergy variable of that isobar, as well as on W, and is obtained from the solution of a set of coupled linear single-variable integral equations. These equations, derived in I, are of the general type commonly encountered in the three-body problem. Their numerical solution is still a formidable task, especially for J^P states in which the number of coupled isobar amplitudes is large, as is the case for the present problem. For the purpose of calculating those unitarity corrections which exhibit the most substantial subenergy variation, and thence providing an improved phenomenological parametrization of the isobar amplitudes, the complete solution of the integral equations can be avoided, at the expense of introducing some additional parameters. This simplification of the problem is discussed in Sec. III. After giving details of our parametrization of the two-body amplitudes in Sec. IV, we present in Sec. V the numerical results of our calculations of the unitarity corrections, and give some physical interpretation of them. Here we note the occurrence of chiefly two systematic effects in the dependence on subenergy. Each effect can be given its own distinctive parametric form; the interpretive parameter is a scattering length for the one kind of effect and a barrier constant for the other. The phenomenological implications are discussed in Sec. VI. An Appendix contains the algebraic details.

II. FORMAL CONSIDERATIONS

In our approach the general form of the unitarity corrections to the isobar model is contained in the formalism developed in I. Let us begin by noting that the "isobar factors" introduced in I play a role, in the partial wave analysis of a three-body state, comparable to that of the partial-wave amplitudes in a similar analysis of a two-body state. When unitarity is applied to the latter problem, one obtains an expansion in terms of phase shifts; this effects, in the elastic case, a reduction in the number of independent quantities, at a given energy, from the two of the original amplitude to the one real phase shift. Since the partial-wave expansion is perfectly general, one can of course contemplate calculating the phase shifts from some dynamical theory; but one can also treat them as parameters in a phenomenological analysis of the data. In the three-body case, the imposition of unitarity (and analyticity) on the isobar factors provides powerful constraints on these quantities, which take the form of sets of coupled linear single-variable integral equations for the isobar factors. These equations, which are set down in detail in Eqs. (I.40a)–(I.43b), have the general structure

$$f_\alpha(s_i, s) = c_\alpha + \sum_\beta \int_{-\infty}^{\hat{z}_j} K_{\alpha\beta}(s_i, z_j, s) f_\beta(z_j, s) dz_j\,, \qquad (1)$$

where $f_\alpha(s_i, s)$ (defined precisely in I) is the isobar factor in the isobar channel α with subenergy s_i, and where $s = W^2$. The s-channel quantum numbers J and T (in the notation of I) are suppressed in (1), and the matrix structure implicit in the equation spans the three different subenergy channels, as well as all the quantum numbers of the isobars in a given subenergy channel which can contribute to the J^P state. The integration over the invariant subenergy z_j sweeps down from $\hat{z}_j$, the upper limit of the Dalitz-plot boundary. The expression indicates the coupling of the isobar channel α to the other isobar channels labelled β. A parametrization based on (1), in which the c_α are arbitrary apart from having no unitarity cut in s_i, will satisfy two-body unitarity and analyticity in all three subenergy variables.

The kernels $K_{\alpha\beta}$ are given in I, and are completely determined (up to possible questions of convergence, which we shall take up below) by the two-body amplitudes in each isobar channel; apart from these, they depend only on certain angular and isospin crossing coefficients which have all been tabulated in I. It is precisely the integral part in (1)—describing isobar coupling—which is responsible for the f_α satisfying two-body unitarity (and, indeed, three-body unitarity,[6] at least approximately[7]). If this part were entirely omitted, the f_α would be given simply by the c_α, about which all we know, in the absence of further input, is that they have no unitarity cut in s_i. The integral in (1) therefore unitarizes any given nonunitary c_α. That this unitarization requires the solution of integral equations makes for greater numerical complexity than in the corresponding two-body partial-wave problem, but it is not at all surprising, since the three-body problem has some resemblance to a coupled-channel two-body problem, in which the "channel labels" contain a continuous variable, the subenergy.

As in the case of a phase-shift calculation in potential theory, we could attempt a full numerical prediction of the f_α, taking the c_α to be the appropriate production Born terms, as indicated in I. In the spirit of the usual isobar model, however, we adopt here a more modest approach, and are prepared to introduce some phenomenological parameters to be determined by fitting the data.

We begin by taking the c_α to be independent of s_i: Since they have no unitarity cut, they may be assumed to be approximately constant in s_i, at least in some limited energy region. The c_α can then be interpreted as the amplitudes of the standard nonunitary isobar model; they are the "bare" amplitudes for producing a certain isobar α, with no unitarity corrections. The observed amplitudes f_α then dress the c_α with the unitarity corrections represented by the integral in (1). This is illustrated in Fig. 1. If we replace the integral by matrix summation by means of a suitable quadrature formula, the solution of Eq. (1) can then be written as

$$f = \left[\, 1 + \frac{K\,\mathrm{adj}(1-K)}{\det(1-K)} \right] c\,. \qquad (2)$$

The quantity in square brackets has to be obtained by numerical inversion; it depends only on K and hence only on the two-body amplitudes and not on the constants c. Thus the c's can be retained as phenomenological fitting parameters, and the unitarity corrections take the form of multiplicative functions obtained by a once-for-all numerical inversion; the integral equations do not have to be solved anew for each choice of c_α.

We note that $\det(1-K)$ is the Fredholm determinant D_F for the system (1), depending only on s. On the other hand, the Fredholm numerator $N_F = K\,\mathrm{adj}(1-K)$ depends on both s_i and s, and of course has the requisite matrix structure. In the nonunitary isobar model, the partial-wave amplitudes are *factorized* into a part (the two-body amplitude) depending only on s_i and a part (the nonunitary isobar factor) depending only on s. It is characteristic of the unitarity corrections contained in (2) that they do not, in general, exhibit such a factorization.

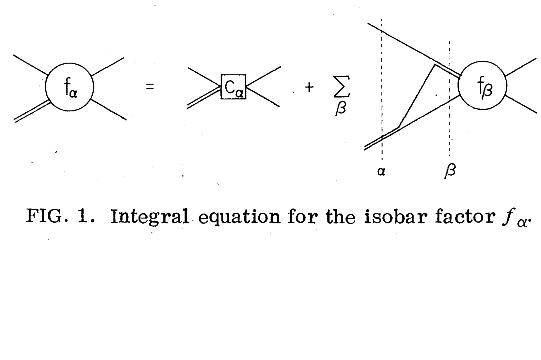

FIG. 1. Integral equation for the isobar factor f_α.

III. APPROXIMATIONS AND SIMPLIFICATIONS

In the present work we have not carried out the full inversion indicated in Eq. (2). We are principally interested in calculating the subenergy variation of the amplitudes f_α, in view of the impact that a very pronounced variation might have on the validity of the established isobar-model analyses.[2] We believe that past experience[8,9] strongly indicates that an excellent picture of this *variation* can be obtained by considering just the first iteration of the full equations. This is obtained by replacing f_β by c_β on the right-hand side of (1), yielding

$$f_\alpha \approx \sum_\beta \left(\delta_{\alpha\beta} + \int_{-\infty}^{\hat{z}_j} K_{\alpha\beta} dz_j\right) c_\beta \,. \tag{3}$$

One then has only to evaluate a finite number of one-dimensional integrals. These integrals certainly contain the important normal threshold (unitarity) branch points in s_i[3], by construction, and also the well-known logarithmic singularities.[10] Higher iterations of Eq. (1) would give rise, in analyticity terms, to singularities progressively further from the physical region, and would serve largely to alter the overall *scale* of the corrections.

The approximation (3) will certainly be good if the iteration series converges rapidly. But this will be the case only if the unitarity corrections are in some sense "small," and therefore probably uninteresting. On the other hand, the 3π calculations showed[9] that even when the corrections were so strong as to generate a three-body resonance, the subenergy *variation* was still largely given by the first iteration. In other words, if we refer to the form of the Fredholm solution, the major s_i variation in N_F is contained in K. The remaining s_i variation in N_F is likely to be minimal, but the s variation in D_F can be substantial, and there will also be some s variation in N_F. We can subsume these unknown s variations into new s-dependent quantities $\bar{c}_\beta(s)$, and rewrite (3) as

$$f_\alpha(s_i, s) \approx c_\alpha(s) + \sum_\beta \left(\int_{-\infty}^{\hat{z}_j} K_{\alpha\beta}(s_i, z_j, s) dz_j\right) \bar{c}_\beta(s) \,. \tag{4}$$

Thus at the expense of introducing the additional parameters $\bar{c}_\beta$, we can hope to apply our approximation even in cases where the corrections are not "small." We emphasize, however, that the introduction of the $\bar{c}_\beta$'s is not a necessary feature of the theory; we could have solved the full integral equations instead. We also note that the $\bar{c}_\beta$'s are proportional to the bare amplitudes for the production of isobar β; one has to recognize, however, that the s dependence of c_α and $\bar{c}_\beta$ in (4) may differ, owing to the s dependence in N_F and, more particularly, in D_F which has been absorbed into $\bar{c}_\beta$.

At this stage we may consider the evaluation of the integrals appearing in (4). The reader may consult Eqs. (I.40a)–(I.43b) to see that the $K_{\alpha\beta}$ are products of three factors: a two-body amplitude denoted by ζ_β in I, an isospin crossing matrix given in Appendix A of I, and a *kernel integral* listed in Eqs. (I.44)–(I.55). For values of z_j in (4) above the two-body threshold, and no doubt for some distance below it also, the ζ_β can be reliably parametrized to fit known two-body data. This parametrization will be given explicitly in Sec. IV. As z_j runs further away from the physical two-body region, this parametrization will become progressively less reliable. One is therefore led to consider truncating the integrals in (4) at some finite lower limit. We have found by numerical calculation that such a truncation corresponds to an approximately *constant* difference in the value of the integrals, for a given s, provided of course that the lower limit selected is not too near the two-body threshold in z_j. This constant difference can be absorbed into the parameter c_α in (4); it in no way affects the calculation of the s_i variation. We have found that the point $z_j = 0$ is a convenient choice of lower limit. We have therefore arrived at the approximate form

$$f_\alpha(s_i, s) \approx c_\alpha(s) + \sum_\beta \left(\int_0^{\hat{z}_j} K_{\alpha\beta}(s_i, z_j, s) dz_j\right) \bar{c}_\beta(s) \,. \tag{5}$$

Equation (5) is our basic formula for calculating, and parametrizing, unitarity corrections to the isobar model; it is illustrated in Fig. 2. The integral over $K_{\alpha\beta}$ corresponds to the *rescattering process* shown in the second term on the right-hand side in the figure: Isobar β is first created (with amplitude $\bar{c}_\beta$) and then decays, the products rescattering to form isobar α. We note that, in terms of the discussion of Sec. VII of I, the truncation of (4) to (5) is equivalent to the statement that the short-range contributions to K are slowly varying in subenergy; the possibility that these contributions generate substantial s dependence in D_F is, however, preserved via the parameters $\bar{c}_\beta(s)$.

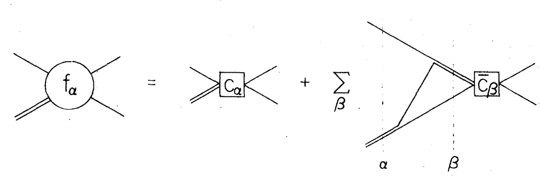

FIG. 2. Isobar factor f_α with rescattering corrections according to Eq. (5).

We now turn to a consideration of the kernel integrals appearing in $K_{\alpha\beta}$; these are listed in Eqs. (I.44)–(I.55). A typical such integral has been expressed there as

$$H_{\alpha\beta}(s_i, z_j, s) = \int_{ij} \frac{\Phi(z_i, z_j, s)}{(z_i - s_i)K_i^n(z_i, s)}\, dz_i \qquad (6)$$

in which the integration $\int_{ij}$ is, as explained in Eqs. (I.56)–(I.58), over a set of traversals T_{ij} of kinematic regions in the $(z_i z_j)$ plane.[5] K_i is the kinematic quantity of Eq. (B2) of Ref. 4, defined such that $K_i/2W$ is the c.m.s. momentum of isobar i. Φ is one of the quantities R_1, R_2, R_{12}, Ω, Υ, and Z, appearing in Eqs. (I.44)–(I.55), whose explicit forms are given in I, Appendix B. The power n in (6) depends on the orbital angular momentum of the final particle + isobar system. It would, of course, be quite possible to evaluate these integrals numerically, and insert the results into (5) for a further numerical integration. But this is unnecessary. Most of the quantities Φ are simple polynomials in the indicated variables; in such cases the integrals in (6) can all be evaluated in terms of the functions Δ_{ij} given in Appendix B of Ref. 4. The exceptions (see I, Appendix B) are R_{12}, Z, and Υ, which involve square roots of the z variables. The exact treatment of $\sqrt{z_i}$ terms in (6) would lead to awkward elliptic integrals,[11] but we have found numerically that the replacement $\sqrt{z_i} \to \sqrt{s_i}$, which completely eliminates this problem, is perfectly satisfactory. The net result of this replacement is that the quantities f_α may differ from the exactly integrated ones by as much as 10%, *but this difference, at a given s, is to an excellent approximation just a (complex) constant.* Hence this replacement merely amounts to absorbing a constant difference into the parameter c_α in those cases where it is made.

There is one further, and important, point to be made about the evaluation of the kernel integrals (6). To illustrate it, we consider one such integral in particular, called $H_{SP}(s_1, z_2)$ in I; with the z_2 integration in (5) truncated at $z_2 = 0$, this integral is

$$H_{SP}(s_1, z_2, s) = \sqrt{2} \int_{D_{12}} \frac{[2z_2(z_1 + z_2 - 2\mu^2) - (z_2 + s - \mu^2)(z_2 + M^2 - \mu^2)]dz_1}{(z_1 - s_1)K_1(z_1, s)} \,. \qquad (7)$$

D_{12} is a traversal at fixed z_2 of the Dalitz region in the $(z_1 z_2)$ plane, as illustrated in I, Fig. 6. If we write

$$z_1 + z_2 - 2\mu^2 = (z_1 - s_1) + (s_1 + z_2 - 2\mu^2)\,,$$

it is clear that we can separate out from (7) an explicitly s_1-independent piece. When this piece is inserted into (5), it will contribute simply a constant in the corresponding $f_\alpha(s_1)$. Hence it also can be absorbed into the appropriate c_α. Such a procedure can be followed in all cases where Φ in (6) is of the same degree in z_i as $(z_i - s_i)K_i^{n-1}$. In cases where Φ is of lower degree than this, the kernel integrals contain no s_i-independent piece. In one case, that of the kernel H_{13}, Eq. (I.53), we have $\Phi = \Upsilon$, which is of one higher degree than this in z_1. The evaluation of H_{13} then leads to an s_1-independent piece, a piece linear in s_1, and a remainder which contains the usual Δ_{13} functions.[11] To reiterate, all s_i-independent contributions to the integrals (6) can be ignored, since they can be absorbed into the c_α.

We close this section with some remarks about convergence and subtractions. The occurrence of s_i-independent (or linear in s_i) pieces in the integrals (6) is of course directly related to divergences in these integrals, since $K_i \sim z_i$ for large z_i; consequently when Φ is of the same degree in z_i as $(z_i - s_i)K_i^{n-1}$, the integral (6) will be logarithmically divergent when taken over an unbounded traversal. If Φ is of one higher degree, the divergence is linear, and so on. Now, if we refer to Eqs. (I.56)–(I.58), we note that the contributions to the kernel integrals (6) involving unbounded traversals enter only for $z_j \leq 0$. Thus we do not need to consider them at all if we cut off the z_j integrations at $z_j = 0$, as we have proposed in (5). This fact, indeed, is an excellent reason for choosing $z_j = 0$ as the truncation point. On the other hand, we are not forced to truncate the z_j integration by the problem of divergences in the $K_{\alpha\beta}$: Since it contributes an s_i-independent piece, a logarithmic divergence in $K_{\alpha\beta}$ can be eliminated by a single subtraction in s_i performed on the original dispersion relation for $f_\alpha(s_i)$, and hence in the integral equations (1), and thence in (5). The consequence of this would be the replacement of the kernel integral $H_{\alpha\beta}(s_i, z_j, s)$ by

$$H_{\alpha\beta}(s_i, z_j, s) - H_{\alpha\beta}(s_{i0}, z_j, s)$$

for a subtraction at $s_i = s_{i0}$, and the introduction of different constants c_α in (1), which would formally be the values of $f_\alpha(s_i)$ at $s_i = s_{i0}$. Since, once again, the c_α are unknown fitting parameters in the present application, such a subtraction is readily accommodated.

In the single case of the kernel H_{13}, there is a linear divergence, which would necessitate two subtractions in s_1. Thus in this case we cannot assert that a truncation of the z_3 integral at $z_3 = 0$ differs by only a constant from a truncation at some lower value, say $z_3 = -A$: In order to specify the kernels over $-A \leq z_3 \leq 0$ we have to make two subtractions in s_1, thereby introducing an uncalculable *linear* variation in s_1. In view of this, the piece of H_{13} explicitly proportional to s_1 [see Eq. (A17) below] may be of doubtful significance.

We present in the Appendix the full content of Eq. (5) for the J^P waves we are considering, with the kernel integrals evaluated after replacing $\sqrt{z_i}$ by $\sqrt{s_i}$ as needed, and with all s_i-independent pieces discarded.

IV. PARAMETRIZATION OF THE ELASTIC TWO-BODY AMPLITUDES

In the present application of the theory, we shall ignore all left-hand cut structure in the two-body amplitudes; such features could be introduced explicitly, but their effect in the present context would be similar to that of subtractions, and consequently is already to a large extent included among the parameters of the theory. In more physical terms, the left-hand structure should not cause substantial s_i variation in the physical region.

We therefore adopt a parametrization of the relativistic effective range type, which is consistent with elastic unitarity, and which is such that the two-body amplitudes have only the normal threshold branch points required by unitarity. Such a parametrization was first introduced by Chew and Mandelstam,[12] for the $\pi\pi$ problem. We adopt the same approach, suitably generalized to the unequal mass $N\pi$ case, and with the inclusion of nonzero orbital angular momentum.

As in I, we introduce $N\pi$ amplitudes $M^{jP}_{t_1}(s_1)$ and $\pi\pi$ amplitudes $M^{l}_{t_3}(s_3)$, and relate them to reduced amplitudes ζ by extracting the threshold behavior and other convenient factors:

$$M^{jp}_{t_1}(s_1) = (32\pi^3/M)(4s_1Q_1^2)^{l'}\zeta^{jp}_{t_1}(s_1)\,, \tag{8}$$

$$M^{l}_{t_3}(s_3) = 128\pi^3(4s_3k_3^2)^{l}\zeta^{l}_{t_3}(s_3)\,, \tag{9}$$

where l' and l are the orbital angular momenta in the $N\pi$ and $\pi\pi$ channels, respectively. We refer to I for definitions of all other kinematical quantities. The amplitudes M satisfy elastic unitarity,

$$\operatorname{Im}(M^{jp}_{t_1})^{-1} = -\pi\rho_1\,, \tag{10}$$

$$\operatorname{Im}(M^{l}_{t_3})^{-1} = -\pi\rho_3\,, \tag{11}$$

where

$$\rho_1 = MQ_1/16\pi^3 w_1 \tag{12}$$

and

$$\rho_3 = k_3/64\pi^3 w_3\,. \tag{13}$$

We discuss first the $N\pi$ amplitudes. Equation (10) is satisfied by setting

$$\zeta^{jp}_{t_1}(s_1) = [A^{jp}_{t_1}(s_1) + (4s_1Q_1^2)^{l'}J(s_1)]^{-1}\,, \tag{14}$$

where, for $s_1 \geq (M+\mu)^2$,

$$J(s_1) = \left(\frac{M^2-\mu^2}{s_1}\right)\ln\left(\frac{M}{\mu}\right) + \frac{2Q_1}{w_1}\left\{\ln\left[\frac{s_1-(M-\mu)^2+2w_1Q_1}{s_1-(M-\mu)^2-2w_1Q_1}\right] - i\pi\right\}. \tag{15}$$

For $s_1 < (M+\mu)^2$, $J(s_1)$ is found by analytic continuation of (15); its imaginary part vanishes below threshold. $J(s_1)$ is constructed so as to be analytic everywhere except for the square-root branch point at $s_1 = (M+\mu)^2$, and so as to have the discontinuity across the associated cut correctly given by unitarity. The functions $A^{jp}_{t_1}(s_1)$ are then meromorphic in s_1. For the S_{11}, S_{31}, and P_{33} $N\pi$ amplitudes, we express the corresponding A's as polynomials in s_1. In the case of the P_{11} amplitude, we require that there be a zero in (8), corresponding to the known zero in the elastic phase shift; $A(s_1)$ then has to contain a pole. Thus we write in general

$$A^{jp}_{t_1}(s_1) = \sum_n A_n s_1^{\,n} + \frac{r_0}{s_1 - s_0}\,, \tag{16}$$

where on the right-hand side the quantum numbers $j^p l_1$ are to be understood, and where the pole only appears in the P_{11} channel.

The A_n are chosen so that the phase shift defined by

$$M^{jp}_{t_1}(s_1) = \frac{e^{i\delta}\sin\delta}{\pi\rho_1} \tag{17}$$

fits the data of Carter, Bugg, and Carter.[13] One has to ensure, for all but the P_{11} case, that $\zeta^{-1} \neq 0$ below threshold, or else spurious poles in (8) will result. The P_{11} amplitude itself, of course, must contain the nucleon pole, and in this case we require that

$$\zeta^{-1}(s_1) \to \frac{4\pi}{3}\,\frac{(4M^2-\mu^2)}{g^2/4\pi}\,(M^2 - s_1) \tag{18}$$

as $s_1 \to M^2$, where $g^2/4\pi \approx 14.5$.

The $\pi\pi$ amplitudes satisfying (11) are parametrized in a similar way, by setting

$$\zeta^{l}_{t_3}(s_3) = [A^{l}_{t_3}(s_3) + (4s_3k_3^2)^{l}J_3(s_3)]^{-1}\,, \tag{19}$$

TABLE I. Parameters of the two-body amplitudes.

Parameters	A_0 ($\mathrm{GeV}^{4l'}$)	A_1 ($\mathrm{GeV}^{4l'-2}$)	A_2 ($\mathrm{GeV}^{4l'-4}$)	A_3 ($\mathrm{GeV}^{4l'-6}$)	r_0 ($\mathrm{GeV}^{4l'+2}$)	s_0 (GeV^2)
$l'=0$ $S_{11}(S_1)$	4.54	0	0	0		
$l'=0$ $S_{31}(S_3)$	−42.01	42.31	−12.25	0		
$l'=1$ $P_{33}(\Delta)$	6.63	−18.79	18.73	−6.23		
$l'=1$ $P_{11}(\mathfrak{N})$	6.55	−18.63	21.21	−7.50	0.75	1.46
	A_0 (GeV^4)	A_1 (GeV^{4l-2})	A_2 (GeV^{4l-4})	A_3 (GeV^{4l-6})		
$l=0,\ t_3=0$ ϵ_0	8	−25	10	0		
$l=0,\ t_3=2$ ϵ_2	−15.0	0	0	0		
$l=1,\ t_3=1$ ρ	1.00	−0.32	3.10	−13.53		

where, for $s_3 \geq 4\mu^2$,

$$J_3(s_3) = \frac{2k_3}{w_3}\left[\ln\left(\frac{w_3+2k_3}{w_3-2k_3}\right) - i\pi\right]. \tag{20}$$

The imaginary part of J_3 vanishes when (20) is continued to $s_3 < 4\mu^2$. We express $A^l_{t_3}(s_3)$ as a polynomial in s_3 as in (16), but without the pole term. We require no poles in $\zeta^l_{t_3}$ below threshold in s_3, and we determine the A_n by fitting the $\pi\pi$ phase shifts defined by

$$M^l_{t_3}(s_3) = \frac{e^{i\delta}\sin\delta}{\pi\rho_3}. \tag{21}$$

For $l=t_3=0$ we use Fig. 9 of Rosselet *et al.*;[14] we are content to parametrize the small, less well-determined, $t_3=2$, $l=0$ phase shift by a scattering length term A_0 only; for $l=t_3=1$ we fit the phase shift given in Table VI of Protopopescu *et al.*[15]

We list in Table I the values of the parameters A_n, r_0, and s_0 that we have used in our calculations. They give a good representation of the phase shifts over the energy ranges considered, and may perhaps be useful in other contexts.

V. NUMERICAL RESULTS AND INTERPRETATION

For the production of isobar α we calculate the rescattering corrections to the nonunitary isobar model, illustrated in Fig. 2, according to the reduced version of our theory given in Eq. (5). The details have been spelled out in the Appendix; for immediate reference we copy Eq. (A1) here

$$f_\alpha^{J^P T}(s_i, s) = c_\alpha^{J^P T}(s) + \sum_\beta W^T_{t_\alpha t_\beta} I^{J^P}_{\alpha\beta}(s_i, s)\bar{c}_\beta^{J^P T}(s). \tag{22}$$

The quantities of interest are the integrals $I^{J^P}_{\alpha\beta}(s_i, s)$, itemized in their entirety in Eqs. (A4)–(A7). In our approach these contain the dominant s_i variation in the rescattering corrections for the processes $\alpha \leftarrow \beta$, appearing in Fig. 2. In these expressions, and henceforth in the text, we adopt the following compact notation for the isobar channels. For the $N\pi$ isobars, instead of using $(\mathcal{J}^P, t_1)$ as in Sec. IV, we denote the $(\frac{3}{2}^+, \frac{3}{2})$ state by Δ, the $(\frac{1}{2}^+, \frac{1}{2})$ state by $\mathfrak{N}$, and the $(\frac{1}{2}^-, \frac{1}{2})$ and $(\frac{1}{2}^-, \frac{3}{2})$ states by S_1 and S_3, respectively. For the $\pi\pi$ isobars, instead of (l, t_3) we shall write ϵ_0 for (0,0), ϵ_2 for (0,2), and ρ for (1,1). Note that in the calculation of $I_{\alpha\beta}$ the isobar isospin of β matters while that of α does not; when S_1 and S_3 appear as α we shorten them to simply S, and likewise ϵ_0 and ϵ_2 become simply ϵ. A guide to all the $\alpha\beta$ combinations considered is as follows:

$$\begin{bmatrix} SS_1 & SS_3 & S\Delta & S\epsilon_0 & S\epsilon_2 \\ \Delta S_1 & \Delta S_3 & \Delta\Delta & \Delta\epsilon_0 & \Delta\epsilon_2 \\ \epsilon S_1 & \epsilon S_3 & \epsilon\Delta & & \end{bmatrix} \text{ for } J^P = \tfrac{1}{2}^+,$$

$$\begin{bmatrix} \mathfrak{N}\mathfrak{N} & \mathfrak{N}\rho \\ \rho\mathfrak{N} & \end{bmatrix} \text{ for } J^P = \tfrac{1}{2}^-,$$

$$[\Delta\Delta] \text{ for } J^P = \tfrac{3}{2}^+,$$

and

$$\begin{bmatrix} \Delta\Delta & \Delta\rho \\ \rho\Delta & \end{bmatrix} \text{ for } J^P = \tfrac{3}{2}^-.$$

As we describe the results of the numerical evaluation of the $I_{\alpha\beta}$, it will be helpful to refer to the orbital angular momentum quantum numbers (L_α, l_α) and (L_β, l_β) in the rescattering, where L and l apply to the overall and to the isobar center-of-mass frames, respectively. We shall discuss first, in Sec. V A, the s_i variation of the $I_{\alpha\beta}$, and then separately, in Sec. V B, the s variation.

A word about the dimensions of $I_{\alpha\beta}$ is in order here because it has some bearing on how we may compare their respective magnitudes in the following discussion. The dimensions differ with the nature of the rescattering $\alpha \leftarrow \beta$ according to whether the isobars α, β are baryonic (B) or mesonic (M). A perusal of Eqs. (A4)–(A20) reveals that each I_{BB} is dimensionless, while each I_{BM} and each I_{MB} have dimensions of mass and mass^{-1}, respectively. We could easily arrange for each $I_{\alpha\beta}$ to be dimensionless by extracting a factor of the nucleon mass from each mesonic isobar factor f. Instead we shall calculate with the mass unit equal to 1 GeV, approximately the same as the nucleon mass. This permits us to ignore units and compare the magnitudes of any two $I_{\alpha\beta}$'s for any α, β. We hasten to note that even though $I_{\alpha\beta}$ has been well defined, its contribution to f_α via Eq. (22) depends on the size of the unknown fitting parameters c_α and $\bar{c}_\beta$. Despite this, the *relative* scale of the various $I_{\alpha\beta}$'s is not without significance. We shall comment on this further in Sec. VI A below.

A. The subenergy variation of $I_{\alpha\beta}$

Perhaps the single most important point to be made about our results is that *none* of the integrals $I_{\alpha\beta}$ shows any really rapid subenergy variation, certainly nothing as dramatic as that shown in Fig. 2 of Ref. 16. It seems clear that the reason for this is essentially that the requirement of analyticity significantly smoothes out the quite rapid fluctuations present in the discontinuities of the f_α (which are determined by unitarity alone). Thus we are led to conclude that it is most unlikely that unitarity corrections to the isobar model for $N\pi \rightarrow N\pi\pi$ will seriously modify s-channel resonance behavior extracted via the nonunitary fits. We shall discuss this question further in Sec. VI.

Considering the results in more detail, however, we can distinguish several main types of subenergy variation that we have observed in the $I_{\alpha\beta}$.

1. Linear dependence on s_i

The integrals $I^{1/2+}_{\Delta S_1}, I^{3/2-}_{\rho\Delta}, I^{1/2-}_{\rho\mathfrak{N}}$ are of this type, and are shown in Figs. 3(a)–3(c) for $W = 1.5$ GeV. Although some slight curvature can be seen in the imaginary part of $I^{3/2-}_{\rho\Delta}$ and in the real part of $I^{1/2-}_{\rho\mathfrak{N}}$, the amount is very small and these are, besides, the smaller components in each case. We doubt whether the linear variation displayed by these amplitudes would be detectable (see Sec. VI below). Furthermore, the values of the parameters describing such a linear variation of the $I_{\alpha\beta}$ (even if it could be seen in the data) have no significance in the present approach, because they would be effectively absorbed into the unknown fitting parameters c_α and $\bar{c}_\beta$ in Eq. (22). We note in passing that other ignored sources of smooth, possibly linear, behavior (e.g., distant singularities, or

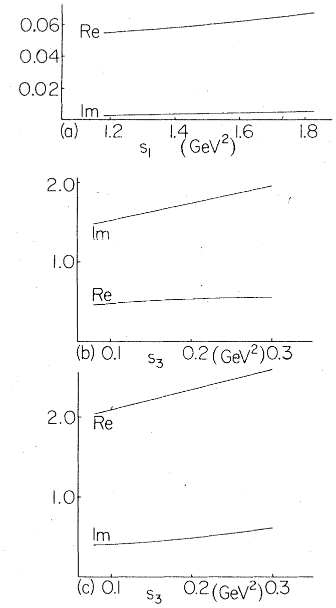

FIG. 3. Rescattering integrals for $W = 1.5$ GeV: (a) $I^{1/2+}_{\Delta S_1}$, (b) $I^{1/2-}_{\rho\mathfrak{N}}$, and (c) $I^{3/2-}_{\rho\Delta}$.

production Born terms[17]) can, by the same token, by regarded as comprehended within the parametrization (22), in the present cases. Thus the significance of our results in these channels is that *they justify a linear parametrization of the corresponding f_α's* in these cases. It is interesting that the dynamical calculations of Aaron *et al.*[17] produced a linear subenergy variation in their isobar amplitudes for producing a ρ and a Δ in the $\frac{3}{2}^-$ wave. (Our calculation cannot be directly compared with theirs because the isobar amplitudes are differently defined). Such a linear parametrization was, indeed, suggested by Aaron and Amado[3] as appropriate for p-wave isobars. That the form should be linear is, however, nontrivial, because *a priori* nonlinear variations, associated for example with the normal threshold branch points[3] and with logarithmic singularities,[10] could be expected. Indeed we shall see directly that in most of the other cases nonlinear variations are found.

2. Curvature near threshold in s_i

An entire category of rescattering corrections consists of those cases which exhibit a curvature at the subenergy threshold, and in addition a striking crossover of the real and imaginary parts. The integrals $I^{1/2+}_{SS_1}$, $I^{1/2+}_{SS_3}$, $I^{1/2+}_{S\epsilon_0}$, $I^{1/2+}_{S\epsilon_2}$, $I^{1/2+}_{\epsilon S_1}$, and $I^{1/2+}_{\epsilon S_3}$ are of this type, and are shown in Figs. 4(a)–4(f). A further remarkable common feature of all these integrals is that they correspond to cases in which all orbital angular momenta (L_α, l_α, L_β, and l_β) are zero. A set of results for which

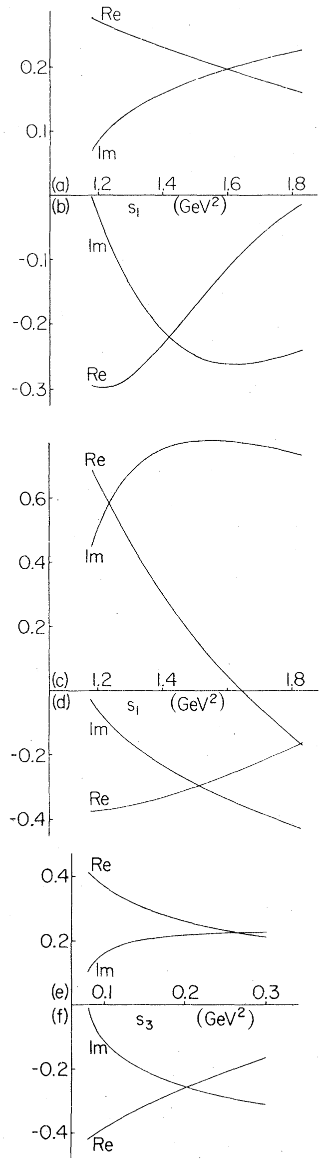

FIG. 4. Rescattering integrals for $W=1.5$ GeV: (a) $I^{1/2^+}_{SS_1}$, (b) $I^{1/2^+}_{SS_3}$, (c) $I^{1/2^+}_{S\epsilon_0}$, (d) $I^{1/2^+}_{S\epsilon_2}$, (e) $I^{1/2^+}_{\epsilon S_1}$, and (f) $I^{1/2^+}_{\epsilon S_3}$.

there are so many strong similarities leads one to seek a simple type of parametrization that may approximately comprehend all of the common features. In fact, we believe we have found such a parametrization, and one that is physically sensible. Consider the function $g(q_i)$ given by

$$g(q_i)=\frac{1}{1-iaq_i}=\frac{1}{1+a^2q_i^2}+\frac{iaq_i}{1+a^2q_i^2}, \tag{23}$$

where q_i (corresponding to s_i) is the momentum of one of the particles in the rest frame of isobar i (i.e., one of the momenta Q_1, Q_2, or k_3, in the notation of I). For $a>0$, the imaginary part of g increases sharply from zero at threshold, while the real part falls smoothly from its maximum value at threshold; and the real and imaginary parts cross at $q_i=a^{-1}$. The intensity $|g|^2$ peaks at threshold, the more sharply the larger a is. These features would appear to describe the structures shown in Fig. 4. If we consider a parametrization of the form

$$I_{\alpha\beta}(s_i,s)\approx c_1(s)+c_2(s)g(q_i), \tag{24}$$

we see that only the single real parameter a is of any significance; the quantities c_1 and c_2 (complex, possibly depending on s) can be absorbed into the c_α and $\bar{c}_\beta$ of (22). The form (23) is, in fact, a specific realization of a general form suggested by Aaron and Amado[3] for s-wave isobars.

What is the physical effect of such corrections? If $g(q_i)$ were a two-body partial-wave amplitude, expression (23) would be the simplest possible effective range expression for it, a being the scattering length. Certainly (23) does have the normal threshold branch point expected from unitarity. In the present case, the full amplitude in channel α is assembled by inserting (24) into (22), and then multiplying the isobar factor f_α by the two-body amplitude M_α. Suppose now that M_α is itself parametrized by a scattering length, a_α, so that $|M_\alpha|^2$ is peaked at threshold. A typical effect of the correction g will then be to sharpen the threshold peaking in $|M_\alpha|^2$, the effect increasing as the a in (23) increases. In particular, we note that, because $\mathrm{Re}I_{\alpha\beta}$ and $\mathrm{Im}I_{\alpha\beta}$ are of the same sign in all our cases, the a in (23) is positive, whatever the sign of a_α might be; if a_α is a positive, the subenergy spectrum in channel α will appear to correspond to a larger a_α than is really the case. Such an effect is well established for the nn channel in the low-energy breakup reaction $nd\to nnp$[18] (our scattering lengths have the opposite sign from the conventional nuclear physics choice).

We have carried out a very rough fit of the type (23) to the curves shown in Figs. 4(a), 4(c)–4(f), to extract a value of a in the different cases. The results (for $W=1.5$ GeV) are given in Table II, where the a's are in fermis. These values are offered only as representative indicators of a recurrent effect—a full application of (22) would utilize the numerical evaluation of the integrals

TABLE II. Values of scattering-length parameter a [Eqs. (23) and (24)].

	$I^{1/2^+}_{SS_1}$	$I^{1/2^+}_{SS_3}$	$I^{1/2^+}_{S\epsilon_0}$	$I^{1/2^+}_{S\epsilon_2}$	$I^{1/2^+}_{\epsilon S_1}$	$I^{1/2^+}_{\epsilon S_3}$
a (fm)	0.34	0.6	0.8	0.3	0.96	0.5

$I_{\alpha\beta}$. On the other hand, the order of magnitude of the a's is by no means unreasonable. Furthermore, such a parametrization as (23) does provide some physical orientation to the bare numerical results. For example, one may compare the magnitudes of the a's for the different channels; these provide a measure of the growth of the corresponding integrals near threshold. If we compare, for example, the $S\epsilon_0$ and $S\epsilon_2$ cases, we see that $a(S\epsilon_0)$ is substantially larger. This can be traced to the fact that the $\pi\pi$ phase shift which determines the two-body function ζ_{ϵ_0} in $I^{1/2^+}_{S\epsilon_0}$ is larger than that which determines ζ_{ϵ_2} in $I^{1/2^+}_{S\epsilon_2}$. In general, the larger ζ is near threshold, the larger is the corresponding a in $g(q_i)$, for transitions to a common final isobar.

The other pure s-wave case, $I^{1/2^+}_{SS_3}$, is shown in Fig. 4 too, even though it exhibits more structure than is contained in (23). This case is qualitatively not too dissimilar, however, and a rough fit of the type (23) and (24) has been applied to yield the value of a given in Table II.

3. *Curvature near the phase-space maximum in s_i*

Another type of subenergy variation we have observed is one in which either the real, or imaginary, or both parts of $I_{\alpha\beta}$ show curvature near the maximum kinematically allowed value of s_i; moreover, the real and imaginary parts have approximately parallel dependence on s_i. We begin with two clear examples of this behavior, $I^{1/2^+}_{\Delta\Delta}$ and $I^{3/2^+}_{\Delta\Delta}$, shown in Figs. 5(a) and 5(b) for $W=1.5$ GeV. If we denote by Q_α the momentum of isobar α in the overall center-of-mass system, we have

$$Q_\alpha=\{[W-(\sqrt{s_i}+m_i)^2][W-(\sqrt{s_i}-m_i)^2]\}^{1/2}/2W\,, \tag{25}$$

where s_i is the invariant mass squared of the isobar, and m_i is the mass of the remaining third particle. Thus a peaking at the upper phase-space limit $s_i=(W-m_i)^2$ corresponds to a peaking at $Q_\alpha=0$. This, combined with the observation that in both the cases under consideration $L_\alpha=L_\beta=1$, strongly suggests an interpretation of the behavior shown in Figs. 5(a) and 5(b) in terms of an angular-momentum barrier effect. In our discussion of threshold behavior in I, we did of course extract factors $Q_\alpha^{L_\alpha}$ in each channel, but that was all; that is to say, we adopted a zero-range barrier factor. There are general arguments[19] which support a range-dependent barrier factor in the *intensity* of the form

$$b(Q_\alpha)=\frac{(Q_\alpha R)^{2L_\alpha}}{1+(Q_\alpha R)^{2L_\alpha}}\,. \tag{26}$$

In the isobar factors as we have defined them we might therefore expect a factor

$$h(Q_\alpha)=\frac{e^{i\theta(Q_\alpha)}}{[1+(Q_\alpha R)^{2L_\alpha}]^{1/2}} \tag{27}$$

to be represented, where the phase θ may in principle depend on Q_α, as indicated. The denominator of (27) produces a curvature at $Q_\alpha=0$, the effect we are seeking to interpret.

With this by way of motivation, we have attempted a very rough fit to the two I's in Fig. 5 using an expression of the form

$$I_{\alpha\beta}(s_i,s)=c_1(s)+c_2(s)h(Q_\alpha)\,. \tag{28}$$

It is clear by inspection of Figs. 5(a) and 5(b) that $\mathrm{Re}I_{\alpha\beta}$ and $\mathrm{Im}I_{\alpha\beta}$ are approximately parallel, and therefore that θ in (27) does *not* depend on Q_α, to good approximation; this being so, its value can be absorbed into the fitting parameter $\bar{c}_\beta$ of (22). In this circumstance, only the real parameter R in (27) is of any significance. The values of R are listed for these, and several other cases, in Table III. That these values turn out to be about 1 fm may seem almost inevitable; but we should hesitate to expect this necessarily. After all,

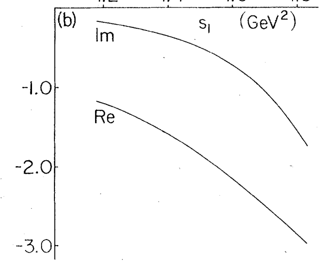

FIG. 5. Rescattering integrals for $W=1.5$ GeV: (a) $I^{1/2^+}_{\Delta\Delta}$ and (b) $I^{3/2^+}_{\Delta\Delta}$.

TABLE III. Values of barrier parameter R [Eqs. (27) and (28)].

	$I^{1/2^+}_{\Delta S_1}$	$I^{1/2^+}_{\Delta\Delta}$	$I^{1/2^+}_{\Delta\epsilon_2}$	$I^{3/2^+}_{\Delta\Delta}$
R (fm)	0.4	1.2–1.6	1.2	1.2–1.6

the processes under consideration are both $\Delta \leftarrow \Delta$ recouplings, and as such involve nucleon exchange, which in Feynman graph terms might suggest a much shorter range. Of course, the precise physical significance of our range parameter is not completely clear, but in any case this interpretation seems to show that the calculations are once again physically sensible, and it is a help in thinking about the numerical results.

Other cases involving $L_\alpha = 1$ are $I^{1/2^+}_{\Delta\epsilon_0}$, $I^{1/2^+}_{\Delta\epsilon_2}$, and $I^{1/2^+}_{\Delta S_1}$. The last has already been discussed in V A 1, the first two are shown in Figs. 6(a) and 6(b) for $W=1.5$ GeV. As regards $I^{1/2^+}_{\Delta S_1}$, it is clear that, for small R, the quantity h in (27) becomes approximately linear in s_i, if θ is independent of Q_α; indeed, we can fit the ΔS_1 case in Fig. 3(a) with an expression of the form (27) and (28) by taking $\theta = 0$ and $R \approx 0.4$ fm. In terms of the present interpretation, we are therefore predicting a small radius parameter for this process. For $I^{1/2^+}_{\Delta\epsilon_2}$ we need a considerably larger R, of order 1.2 fm, also with $\theta = 0$. The case of $I^{1/2^+}_{\Delta\epsilon_0}$ is less clear. We have performed a separate calculation of this integral using a very simple parametrization of ζ_{ϵ_0} in which [referring to Eq. (19)] $A_{\epsilon_0}(s_3)$ is a constant; the corresponding $t_3 = 0 = l$ $\pi\pi$ scattering length is $0.15\ \mu^{-1}$. For this ϵ_0, the pattern is the same as for $I^{1/2^+}_{\Delta\epsilon_2}$ and $I^{1/2^+}_{\Delta S_1}$: the phase θ is zero, and $R \approx 1$ fm. The much larger $\pi\pi$ phase shift we have used in the parametrization of ζ_{ϵ_0} in Table I increases the imaginary part of $I^{1/2^+}_{\Delta\epsilon_0}$, and also apparently precludes a simple parametrization of the form (28).

In all cases with $L_\alpha = 1$, finite range effects should *a priori* be expected in f_α. The impact on f_α of the type of effect we have been discussing for $I_{\alpha\beta}$ will, of course, depend on the relative magnitudes of the parameters $c_\alpha, \bar{c}_\beta$ in (22). Independently of whether our interpretation of these cases is valid, the variation of the $I_{\alpha\beta}$'s which we have calculated still stands, and may be detectable (see Sec. VI).

For three other cases it may also be observed that the real and imaginary parts of $I_{\alpha\beta}$ are qualitatively parallel as functions of the subenergy.

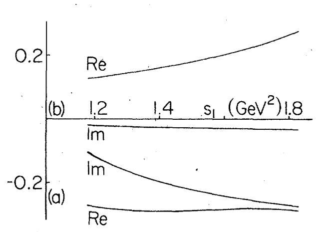

FIG. 6. Rescattering integrals for $W=1.5$ GeV. (a) $I^{1/2^+}_{\Delta\epsilon_0}$ and (b) $I^{1/2^+}_{\Delta\epsilon_2}$.

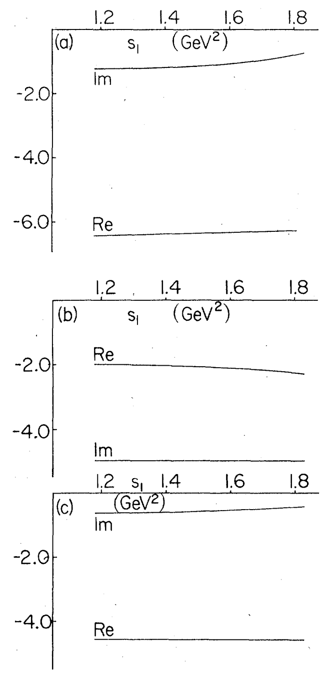

FIG. 7. Rescattering integrals for $W=1.5$ GeV: (a) $I^{1/2^+}_{S\Delta}$, (b) $I^{1/2^-}_{\mathfrak{N}\mathfrak{N}}$, and (c) $I^{3/2^-}_{\Delta\Delta}$.

These are $I^{1/2^+}_{S\Delta}$, $I^{1/2^-}_{\mathfrak{N}\mathfrak{N}}$, and $I^{3/2^-}_{\Delta\Delta}$, plotted in Fig. 7. For each of these $L_\alpha = 0$, so the barrier parametrization (28) is not compelling. Moreover, as one can see from the figure, their most obvious characteristic is that they vary only very slightly with subenergy. For these reasons no effort has been made to supply an interpretive parametrization.

4. *Other cases*

A surprising feature of the preceding results is the absence of marked structure which could be associated with logarithmic singularities.[10] These might be expected in cases where the ζ_β in $I_{\alpha\beta}$ corresponds to a well-defined resonance, such as the Δ and ρ; they produce a peaking at low values of the subenergy variable, but one which (in contrast to those noted in V A 2 above) changes substantially with W, as W varies through an energy region near the particle + resonance mass appropriate to the state β. For $W \lesssim 1.5$ GeV, only $\beta = \Delta$ fulfills this criterion, but of the possible candidates $I^{1/2^+}_{\Delta\Delta}$, $I^{3/2^+}_{\Delta\Delta}$, $I^{1/2^+}_{\epsilon\Delta}$, $I^{3/2^-}_{\Delta\Delta}$, and $I^{3/2^-}_{\rho\Delta}$ only the third shows the singularity clearly. $I^{1/2^+}_{\epsilon\Delta}$ is shown in Fig. 8(a). Presumably kinematic factors present in the other $I_{\alpha\beta}$ suppress this effect. The cases $\beta = \rho$ are expected to produce the effect for $W \approx 1.7$ GeV, and indeed this expectation is confirmed in $I^{3/2^-}_{\Delta\rho}$ (it turns out that $I^{1/2^-}_{\mathfrak{N}\rho} = \sqrt{2}\, I^{3/2^-}_{\Delta\rho}$); this integral is shown in Fig. 8(b). This rescatter-

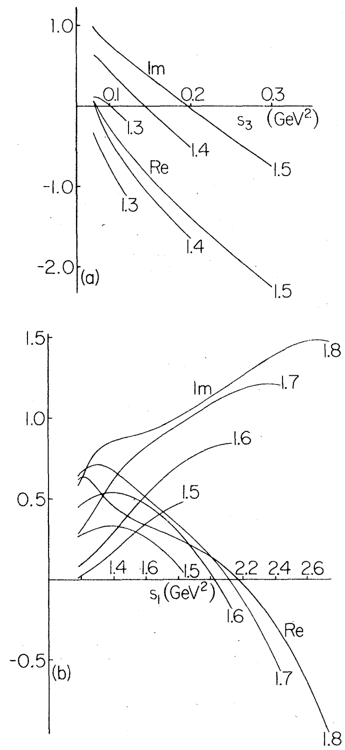

FIG. 8. (a) Rescattering integral $I^{1/2^+}_{\epsilon\Delta}$ for W=1.3, 1.4, and 1.5 GeV. (b) Rescattering integral $I^{3/2^-}_{\Delta\rho}$ for W=1.5, 1.6, 1.7, and 1.8 GeV.

ing correction is large, and exhibits substantial subenergy variation. However, we are not able to give a complete analysis at these values of W, since many other additional amplitudes enter, which we have not included.

The remaining integral not so far categorized is $I^{1/2^+}_{\Delta S_3}$; it is shown in Fig. 9 for W = 1.5 GeV. Though it exhibits some subenergy variation it is numerically small, and we have not ventured any interpretation for it.

B. The s variation of the $I_{\alpha\beta}$'s

It is clearly not feasible to repeat all of the foregoing figures for various values of W. It is also fortunately, and interestingly, largely unnecessary. If we refer to Eq. (22), we see that if a certain

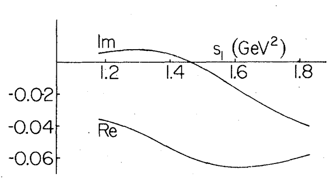

FIG. 9. Rescattering integral $I^{1/2^+}_{\Delta S_3}$ for W=1.5 GeV.

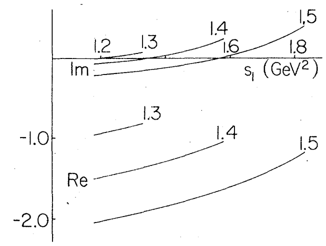

FIG. 10. Behavior of $I^{1/2^+}_{\Delta\Delta}$ for W=1.3, 1.4, and 1.5 GeV.

$I_{\alpha\beta}$ changes by a multiple, or by an additive constant, as s varies, such a change will be absorbed in the fitting parameters $c_\alpha, \bar{c}_\beta$. It is remarkable that in the vast majority of cases this is precisely what happens. An example of a possible W-dependent feature is shown in Fig. 10: the integral $I^{1/2^+}_{\Delta\Delta}$ for W = 1.3, 1.4, and 1.5 GeV. In terms of the form (27), there is a slight suggestion that R may vary with W, being somewhat larger at W = 1.4 GeV than at 1.3 GeV or 1.5 GeV. Similar remarks apply to $I^{3/2^+}_{\Delta\Delta}$. For those cases having a variation of the form (23) and (24), we find that a is remarkably independent of s, once the additative and multiplicative s-dependent terms are allowed, as in (22); Fig. 11 shows $I^{1/2^+}_{SS_1}$ for W = 1.3, 1.4, and 1.5 GeV, as an example. There are suggestions in some cases (e.g., $I^{1/2^+}_{\epsilon S_3}$) of an s dependence in a, but it is not dramatic.

The one example of a noteworthy simultaneous dependence on s_i and s is seen in $I^{1/2^+}_{\epsilon\Delta}$, and has already been discussed in V A 4.

VI. IMPLICATIONS FOR PHENOMENOLOGY

A. Contributing waves and parameters

Up to this point we have presented the discussion in relatively general terms. It is now time to be more realistic. In the first place, even in the elaborate isobar-model analyses of the $N\pi\pi$ sys-

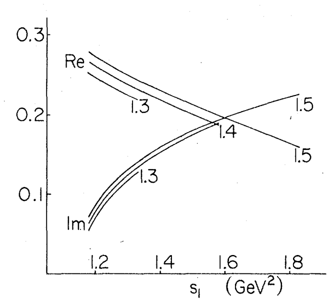

FIG. 11. Behavior of $I^{1/2^+}_{SS_1}$ for W=1.3, 1.4, and 1.5 GeV.

tem done most recently,[2] only the $\Delta\pi$, $N\rho$, and $N\epsilon_0$ channels were included. The $N\pi$ isobars in the S_{11}, S_{31}, and P_{11} waves were omitted, as was the $\pi\pi$ isobar ϵ_2. (There were, however, suggestions[20] that the P_{11}, D_{13}, F_{15}, and D_{15} isobars should be included in any subsequent analysis.) Secondly, in all isobar analyses of the $N\pi\pi$ system carried out to date, only the c_α term in (22) was present; a significant enlargement in the number of channels, coupled with doubling of the number of parameters to include $\bar{c}_\beta$, is unlikely to be a practical proposition.

At this stage it is worth recalling the genesis of (22) [c.f. (4) and (5)]. Although $\bar{c}_\beta$ is not to be identified with the bare (nonunitary) isobar factor for producing channel β, it should nevertheless be proportional to it (with a possbily s-dependent coefficient). The excellent overall fits to the data achieved with the nonunitary model,[2] and their general consistency with each other and with the elastic phase-shift analyses, strongly suggests that the results thus obtained may be taken to be a reliable indication of which are, in fact, the dominant isobar waves. If we adopt this view, then we expect $\beta=\Delta$, ϵ_0, and ρ to be the important set of channels in (22), so that the corresponding $\bar{c}_\beta$ would be large. For consistency, we would hope that rescattering corrections from these channels into isobar channels α *other than* this set would be relatively small in magnitude. A perusal of Figs. 3–9 reveals that this is, indeed, generally true (c.f. the remarks about the magnitudes at the end of the introduction to Sec. V above). There are, however, indications that scattering into the s-wave $N\pi$ channels, and into the P_{11} channel, may be significant. We would also hope that rescattering effects would not build up a minority channel β into an important one. Here, too, we see from Figs. 3–9 that $I_{\alpha\beta}$ is generally smaller for $\beta\neq\Delta$, ϵ_0, and ρ than for $\beta=\Delta$, ϵ_0, and ρ, except that in the case of $\beta=\mathfrak{N}(P_{11})$ large corrections are found; the latter are, however, almost constant in subenergy. Taken together, these observations suggest that the P_{11} channel should be included in subsequent analyses, and that the s-wave $N\pi$ isobars would be worth considering as minority waves. In terms of the existing fits, the most important of our corrections are $I_{\Delta\Delta}^{1/2^+}$, $I_{\Delta\epsilon}^{1/2^+}$, $I_{\epsilon\Delta}^{1/2^+}$, $I_{\Delta\Delta}^{3/2^+}$, $I_{\Delta\Delta}^{3/2^-}$, $I_{\rho\Delta}^{3/2^-}$, and $I_{\Delta\rho}^{3/2^-}$; these rescattering integrals are indeed among those which are relatively large in magnitude.

B. Corrections to existing fits

We shall frame our discussion in terms of the categories of Sec. V A. There is an important general point to be made first, however. We do not anticipate that our corrections will seriously modify the s-channel resonance behavior extracted via the existing fits. In the first place, this is because our corrections do not, after all, vary very rapidly with subenergy. In addition we have seen that in many cases the shape of the s_i variation is very much the same for different values of s, changing only rather smoothly, if at all. This implies that, although there may be some observable corrections to the subenergy spectra, their presence will not seriously distort true s-channel resonance behavior. Thus we believe that our calculations go a long way toward providing a retrospective justification for the nonunitary isobar model, as applied to the extraction of s-channel information.

Turning now to the specific corrections discussed in Sec. V, we begin with the linear subenergy variations of V A 1. These corrections, we recall, describe rescattering into the Δ and ρ isobar channels; therefore, they are to be multiplied by resonant two-body Δ and ρ amplitudes when the full amplitudes are assembled. A linear subenergy variation across a resonance peak produces only a small effect. A computer experiment described by Aaron[21] verifies this numerically, and indicates that, for an event sample of the order of 45 000, a world in which the isobar factors vary linearly with subenergy probably cannot be distinguished from one in which they are independent of subenergy.

The more substantial threshold variation of the type discussed in V A 2, corresponding to a scattering length parameter $a\approx 1$ fm, should in principle be observable. Unfortunately, all such variations occur in amplitudes which either originate from, or lead to, an isobar not included in the standard analyses. The absolute magnitudes of these contributions must therefore be presumed to be small, and we must await a more elaborate analysis in which these minority waves are included.

There is more hope of detecting variation of the type described in V A 3, since several of these rescattering corrections should be associated with large amplitudes. The effect on the subenergy spectrum of introducing a nonzero range parameter R has been noticed by Longacre[22′] (see also Herndon *et al.*,[2] Appendix C). For large R, the function $b(Q_\alpha)$ in (26) rises sharply from the $Q_\alpha=0$ threshold, and saturates quickly, at the value 1; for small R, it rises approximately linearly throughout the physical region. Thus large R will accentuate the region near $Q_\alpha=0$ much more than small R. Consider now a value of W near the threshold for producing a resonant isobar α plus the third particle. The $Q_\alpha^{2L_\alpha}$ factor in $b(Q_\alpha)$ will greatly suppress low momentum Q_α, a region at the upper end of the subenergy phase space.

Therefore, for a finite-width resonance, the lower side of the resonance will be favored over the higher side, by this factor. This effect will, however, be modified by the R-dependent part of $b(Q_\alpha)$, as indicated above. Longacre[22] has shown that the $\pi\pi$ spectrum in $N\pi \to N\pi\pi$ at $W \approx 1.7$ GeV can be interpreted as providing a quite sensitive measure of the effective radius R for producing the $N\rho$ state. He finds that $R \approx \frac{1}{4}$ fm, observing that the value $R \approx 1$ fm gives an unacceptably bad fit. Unfortunately, we have no prediction for R in this case, since the wave which dominates ρ production at this energy has $J^P = \frac{5}{2}^+$, and we have not carried out calculations for this wave.

It is not really possible, without doing a complete re-analysis, to be sure just what effect the variation represented in our $I^{1/2^+}_{\Delta\Delta}$, $I^{3/2^+}_{\Delta\Delta}$, and $I^{1/2^+}_{\Delta\epsilon_0}$ integrals would have on the πN spectrum in the Δ mass region. Such a spectrum is, of course, formed by squaring the coherent sum of *all* contributing amplitudes, and integrating over the other subenergy. All the same, we note that the first two integrals at least have roughly similar R values, $R \approx \frac{1}{4}$ fm; we have therefore calculated the simple intensity (including the ordinary threshold factors)

$$\frac{Q_a^3 \sin^2\delta}{Q_1^3} \times \frac{1}{1+R^2 Q_a^2}, \qquad (29)$$

where δ is the Δ phase shift, Q_a is the momentum of the Δ in the overall c.m.s., and Q_1 is the momentum of the π in the Δ rest frame. We choose $R = 0$ and $R = 1.4$ fm for comparison, and the result is shown in Fig. 12. We cannot say whether existing data are capable of distinguishing between the two cases, but we believe that more accurate data certainly would, and that the result would be interesting. A small value of R, such as Longacre's[22] for the $N\rho$ state ($R \approx \frac{1}{4}$ fm) would justify setting $R = 0$, as was done in the SLAC-Berkeley fits.[2] Our calculations suggest that at least for these Δ channels a bigger R is to be expected.

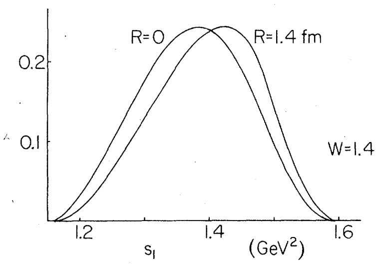

FIG. 12. Δ spectrum from Eq. (29) for $W=1.4$ GeV, with $R=0$ and $R=1.4$ fm (the latter curve has been normalized to the former at $s_1=1.4$ GeV^2).

C. Rescattering in earlier phenomenology

Many years ago Anisovich and Dakhno[23] suggested that the $\pi^+\pi^-$ spectrum in $\pi^- p \to \pi^+\pi^- n$ at $W \sim 1.3$ to 1.5 GeV, in which there is a pronounced paucity of events at low $\pi\pi$ mass, could be explained in terms of a destructive interference between a direct Δ-production term, and a rescattering correction of the type $I_{\epsilon\Delta}$. As we see from Fig. 8, this correction produces a peaking at low $\pi\pi$ mass, but the direct Δ-production term, when projected into the ϵ channel, can also lead to such a peaking, so that a partial cancellation could occur. However, the much more detailed data reported later by Jones, Allison, and Saxon[24] show the same depletion at low $\pi\pi$ mass, for energies W such that the amplitude for Δ production is very small, tending to vitiate $\epsilon \to \Delta$ rescattering. In fact, at these energies the dominant process appears to be direct ϵ_0 production, a term completely absent from the Anisovich-Dakhno model, so that their explanation fails. Morgan and Pennington[25] have since observed that the depletion of events is to be expected because of the presence of the Adler zero near the $\pi\pi$ threshold. Whatever the explanation, it is clear that the dearth of events at low $\pi\pi$ mass will certainly inhibit any attempt to extract low-energy $\pi\pi$ information from the $N\pi \to N\pi\pi$ data, however sophisticated the phenomenological amplitudes may be.

Jones, Allison, and Saxon[24] invoked a rescattering correction of the type $I^{1/2^+}_{\Delta\epsilon}$ to explain a small asymmetry observed between the $\pi^- n$ and $\pi^+ n$ spectra in $\pi^- p \to \pi^+\pi^- n$, for $W = 1.3$ to 1.5 GeV. The feature of this asymmetry was that it was concentrated in a few mass bins at the maximum $\pi^- n$ (or minimum $\pi^+ n$) invariant mass. Obviously such an asymmetry could be explained by direct s-wave Δ production, but the contribution of this wave was in fact strongly limited by the observed angular distributions. Although p-wave Δ production would in principle also produce such an asymmetry, in practice its effect is strongly suppressed at these energies by the angular momentum barrier. Jones, Allison, and Saxon[24] claimed that the rescattering term $I^{1/2^+}_{\Delta\epsilon_0}$ (which they evaluated only very crudely) provided a possible explanation of the asymmetry. Unfortunately, we have to point out that these authors are in error: In calculating the effect of $I^{1/2^+}_{\Delta\epsilon_0}$ on the Δ spectrum, they omitted the phase-space factor Q_α in the amplitude, apparently assuming that as the Δ was produced by rescattering from an $L_\beta = 0$ state no factor of $Q_a^{L_\alpha}$ was necessary. Although, as we have seen [Fig. 6(a)], $I^{1/2^+}_{\Delta\epsilon_0}$ is indeed peaked at the high-subenergy end, this effect is definitely not sharp enough (when the correct Q_a factor is included) to support the Jones, Allison, and Saxon conjecture about the observed asymmetry.[24] To accomplish this, a variation in $I_{\Delta\epsilon_0}$ of something like a factor of 3 as the $N\pi$ mass goes from 1.35 GeV to 1.40 GeV is necessary, and we know of no process which produces such a rapid variation.

VII. CONCLUSION

The integral equations for the isobar factors, derived in I, represent the formal conclusion to the program of formulating an isobar model for the $N\pi\pi$ system which is consistent with subenergy unitarity and analyticity. The kinematics was kept fully relativistic apart from the neglect of the Stapp angle,[26] an approximation commonly made[27] at medium energies. In the present paper we have argued that the major subenergy-dependent corrections to the nonunitary isobar model can be found without solving the full integral equations, but rather by calculating merely a number of one-dimensional *rescattering integrals*; we have calculated these integrals. We have proposed a parametrization [see Eq. (22)] of the isobar factors $f_\alpha(s_i, s)$, in which, to subenergy-independent parameters $c_\alpha(s)$ (analogous to the nonunitary amplitudes), one adds the contributions of the rescattering integrals $I_{\alpha\beta}(s_i, s)$, multiplied by additional parameters $\bar{c}_\beta(s)$. A full data analysis is required to establish conclusively whether the use of the corrected f_α in place of the nonunitary amplitudes alters the s-channel resonance properties extracted in the existing fits; this we have not undertaken.[28] Almost certainly, if we are to judge from the work of Aaron *et al.*,[21] an increase by an order of magnitude in the number of events would be needed before any differences would influence the fits. Indeed the major result of our calculations is that the s_i variations of the $I_{\alpha\beta}$ are relatively smooth.

This result can be used to suggest a reason why the existing fits seem to work so well, when they do not even satisfy unitarity. In discussing this issue, one naturally thinks first of all in terms of the magnitudes of the rescattering corrections, as opposed to the shape of their s_i variation. The question can be addressed in our framework, in spite of the fact that c_α and $\bar{c}_\beta$ in Eq. (22) are unknown *a priori*. If we refer to Eq. (2), we note that when the rescattering corrections are numerically small, f_α differs only by small quantities from the s_i-independent nonunitary amplitudes, and there is no problem with these terms. When the corrections are large in magnitude, significant structure may be built up in the full Fredholm denominator $D_F = \det(1-K)$, *but this quantity is a function of s only*. As an example of such structure, we recall that calculations in the 3π case[9,29] have generated resonances (the ω, for example) in this way. The theory in this case would be providing an interpretation of measured s-channel structure (e.g., an assertion like: "this resonance is caused by $N\rho$ dynamics"), but such structure would still be phenomenologically comprehended within the nonunitary parametrization. The significant point is that there is no denominator function in (2), dependent on s_i, which can build up to produce rapid fluctuations in s_i due to large rescattering effects. The s_i dependence is all in the Fredholm numerator, and its variation is well approximated[9] by the rescattering integrals $I_{\alpha\beta}$. This is true even when $D_F(s)$ exhibits resonance behavior in s.[9] In fact our calculations show that the s_i variation of these integrals is not very rapid. In addition, the s_i variations themselves change only very smoothly with s; any major s variation due to D_F we can attribute to $\bar{c}_\beta(s)$ in Eq. (22). In short, the unitarity corrections do not produce remarkable s_i variation.

Having said this, we are nevertheless keen to see whether such subenergy variations as we do predict can, in fact, be detected. We have found that in many cases there is a correlation between the types of variation found and the orbital angular momentum configurations in the rescattering process. The variations which appear to offer the most hope of being detected show a behavior interpretable as the effect of a finite interaction radius in an angular momentum barrier. The values of the effective radii are such that, with improved data near the $\pi\Delta$ threshold, our corrections should be detectable.

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APPENDIX

We give here explicit expressions for all the rescattering corrections which we have calculated. If we refer to Eqs. (I.40a)–(I.43b), and recall the approximations and simplifications introduced in Sec. III above, we can write the general form of the isobar amplitudes as

$$f_\alpha^{J^P T}(s_i, s) = c_\alpha^{J^P T}(s) + \sum_\beta W^T_{t_\alpha t_\beta} I^{J^P}_{\alpha\beta}(s_i, s)\bar{c}_\beta^{J^P T}(s)\,, \qquad \text{(A1)}$$

where α, β label the isobar channels, and $W^T_{t_\alpha t_\beta}$ is the appropriate isospin crossing matrix ($C, \overline{C}$, or D) of I, Appendix A. The quantities $I^{J^P}_{\alpha\beta}(s_i, s)$ are integrals which we shall specify shortly. Our compact notation for the individual isobars is explained in the introduction to Sec. V above.

To expand (A1) in more detail, let us consider the case $J^P=\frac{1}{2}^+$. We introduce two-component column vectors

$$f_S^{1/2^+T}(s_i, s) = \begin{bmatrix} f_{S_1}^{1/2^+T}(s_i, s) \\ f_{S_3}^{1/2^+T}(s_i, s) \end{bmatrix}, \quad c_S^{1/2^+T}(s) = \begin{bmatrix} c_{S_1}^{1/2^+T}(s) \\ c_{S_3}^{1/2^+T}(s) \end{bmatrix}, \tag{A2}$$

so that, if we refer to Eq. (I.40a), we have

$$f_S^{1/2^+T}(s_1, s) = c_S^{1/2^+T}(s) + \int_0^{\hat{z}_2} dz_2 H_{SS}(s_1, z_2, s) D^T \begin{bmatrix} \zeta_{S_1}(z_2)\overline{c}_{S_1}^{1/2^+T}(s) \\ \zeta_{S_3}(z_2)\overline{c}_{S_3}^{1/2^+T}(s) \end{bmatrix} + \int_0^{\hat{z}_2} dz_2 H_{SP}(s_1, z_2, s) D^T_{3/2\text{col}} \zeta_\Delta(z_2) \overline{c}_\Delta^{1/2^+T}(s)$$
$$+ \int_0^{\hat{z}_3} dz_3 H_{S3}(s_1, z_3, s) \overline{c}^T \begin{bmatrix} \zeta_{\epsilon_0}(z_3)\overline{c}_{\epsilon_0}^{1/2^+T}(s) \\ \zeta_{\epsilon_2}(z_3)\overline{c}_{\epsilon_2}^{1/2^+T}(s) \end{bmatrix} \tag{A3}$$

in which $\hat{z}_2 = (W-\mu)^2$ and $\hat{z}_3 = (W-M)^2$ (and below, $\hat{z}_1 = \hat{z}_2$). The corresponding equation for the remaining amplitudes of the $\frac{1}{2}^+$ system, $f_\Delta^{1/2^+T}(s_1, s)$ and $f_\epsilon^{1/2^+T}(s_3, s)$, are obtained from Eqs. (I.40b) and (I.40c), respectively. Those for the other J^P states follow from Eqs. (I.41a)–(I.43b). The form of the ζ function is given in Sec. IV and Table I. From these equations for the f's we can identify the integrals $I^{J^P}_{\alpha\beta}(s_i, s)$ of (22) and (A1). They read as follows.

For $J^P=\frac{1}{2}^+$,

$$I_{SS_1}^{1/2^+}(s_1, s) = \int_0^{\hat{z}_2} dz_2 H_{SS}(s_1, z_2, s)\zeta_{S_1}(z_2),$$
$$I_{SS_3}^{1/2^+}(s_1, s) = \int_0^{\hat{z}_2} dz_2 H_{SS}(s_1, z_2, s)\zeta_{S_3}(z_2),$$
$$I_{S\Delta}^{1/2^+}(s_1, s) = \int_0^{\hat{z}_2} dz_2 H_{SP}(s_1, z_2, s)\zeta_\Delta(z_2),$$
$$I_{S\epsilon_0}^{1/2^+}(s_1, s) = \int_0^{\hat{z}_3} dz_3 H_{S3}(s_1, z_3, s)\zeta_{\epsilon_0}(z_3),$$
$$I_{S\epsilon_2}^{1/2^+}(s_1, s) = \int_0^{\hat{z}_3} dz_3 H_{S3}(s_1, z_3, s)\zeta_{\epsilon_2}(z_3),$$
$$I_{\Delta S_1}^{1/2^+}(s_1, s) = \int_0^{\hat{z}_2} dz_2 H_{PS}(s_1, z_2, s)\zeta_{S_1}(z_2),$$
$$I_{\Delta S_3}^{1/2^+}(s_1, s) = \int_0^{\hat{z}_2} dz_2 H_{PS}(s_1, z_2, s)\zeta_{S_3}(z_2), \tag{A4}$$
$$I_{\Delta\Delta}^{1/2^+}(s_1, s) = \int_0^{\hat{z}_2} dz_2 H_{PP}(s_1, z_2, s)\zeta_\Delta(z_2),$$
$$I_{\Delta\epsilon_0}^{1/2^+}(s_1, s) = \int_0^{\hat{z}_3} dz_3 H_{P3}(s_1, z_3, s)\zeta_{\epsilon_0}(z_3),$$
$$I_{\Delta\epsilon_2}^{1/2^+}(s_1, s) = \int_0^{\hat{z}_3} dz_3 H_{P3}(s_1, z_3, s)\zeta_{\epsilon_2}(z_3),$$
$$I_{\epsilon S_1}^{1/2^+}(s_3, s) = \int_0^{\hat{z}_1} dz_1 H_{3S}(s_3, z_1, s)\zeta_{S_1}(z_1),$$
$$I_{\epsilon S_3}^{1/2^+}(s_3, s) = \int_0^{\hat{z}_1} dz_1 H_{3S}(s_3, z_1, s)\zeta_{S_3}(z_1),$$
$$I_{\epsilon\Delta}^{1/2^+}(s_3, s) = \int_0^{\hat{z}_1} dz_1 H_{3P}(s_3, z_1, s)\zeta_\Delta(z_1).$$

For $J^P=\frac{1}{2}^-$

$$I_{\mathfrak{N}\mathfrak{N}}^{1/2^-}(s_1, s) = \int_0^{\hat{z}_2} dz_2 H_{12}(s_1, z_2, s)\zeta_{\mathfrak{N}}(z_2),$$
$$I_{\mathfrak{N}\rho}^{1/2^-}(s_1, s) = \int_0^{\hat{z}_3} dz_3 H_{13}(s_1, z_3, s)\zeta_\rho(z_3), \tag{A5}$$
$$I_{\rho\mathfrak{N}}^{1/2^-}(s_3, s) = \int_0^{\hat{z}_1} dz_1 H_{31}(s_3, z_1, s)\zeta_{\mathfrak{N}}(z_1).$$

For $J^P=\frac{3}{2}^+$,

$$I_{\Delta\Delta}^{3/2^+}(s_1, s) = \int_0^{\hat{z}_2} dz_2 H(s_1, z_2, s)\zeta_\Delta(z_2). \tag{A6}$$

For $J^P=\frac{3}{2}^-$,

$$I_{\Delta\Delta}^{3/2^-}(s_1, s) = \int_0^{\hat{z}_2} dz_2 H'_{12}(s_1, z_2, s)\zeta_\Delta(z_2),$$
$$I_{\Delta\rho}^{3/2^-}(s_1, s) = \int_0^{\hat{z}_3} dz_3 H'_{13}(s_1, z_3, s)\zeta_\rho(z_3) = \frac{1}{\sqrt{2}} I_{\mathfrak{N}\rho}^{1/2^-}, \tag{A7}$$
$$I_{\rho\Delta}^{3/2^-}(s_3, s) = \int_0^{\hat{z}_1} dz_1 H'_{31}(s_3, z_1, s)\zeta_\Delta(z_1).$$

The H functions may be calculated from Eqs. (I.44)–(I.55), and (I.B1)–(I.B6), with the definitions of the integrations as given in Eqs. (I.56)–(I.58). We find

$$H_{SS}=\Delta_{12}(s_1,z_2)\,, \tag{A8}$$

$$H_{SP}\overset{*}{=}\sqrt{2}\,R_2(s_1,z_2)\Delta_{12}(s_1,z_2)\,, \tag{A9}$$

$$H_{S3}=4M\Delta_{13}(s_1,z_3)\,, \tag{A10}$$

$$H_{PS}=\frac{\sqrt{2}}{K_1^{\,2}(s_1)}\left[R_1(s_1,z_2)\Delta_{12}(s_1,z_2)+\frac{s_1+s-\mu^2}{s}K_2(z_2)\right], \tag{A11}$$

$$H_{PP}\overset{*}{=}\frac{2}{K_1^{\,2}(s_1)}\left\{R_2(s_1,z_2)\left[R_1(s_1,z_2)\Delta_{12}(s_1,z_2)+\frac{s_1+s-\mu^2}{s}K_2(z_2)\right]-(s_1z_2)^{1/2}\left[\phi(s_1,z_2)\Delta_{12}(s_1,z_2)-\frac{X(s_1,z_2)}{2s}K_2(z_2)\right]\right\}, \tag{A12}$$

$$H_{P3}=\frac{4\sqrt{2}M}{K_1^{\,2}(s_1)}\left[R_1(s_1,z_3)\Delta_{13}(s_1,z_3)-\frac{s_1+s-\mu^2}{s}K_3(z_3)\right], \tag{A13}$$

$$H_{3S}=(1/2M)\Delta_{31}(s_3,z_1)\,, \tag{A14}$$

$$H_{3P}\overset{*}{=}(1/\sqrt{2}M)R_1(z_1,s_3)\Delta_{31}(s_3,z_1)\,, \tag{A15}$$

$$H_{12}\overset{*}{=}\Omega(s_1,z_2)\Delta_{12}(s_1,z_2)\,, \tag{A16}$$

$$H_{13}\overset{*}{=}4\sqrt{2}M\left[\frac{-2s_1z_3}{(\sqrt{z_3}-M)^2-s}\ln\left(\frac{W-Q_0+Q}{W-Q_0-Q}\right)+\Upsilon(s_1,z_3)\Delta_{13}(s_1,z_3)\right], \tag{A17}$$

$$H_{31}\overset{*}{=}\frac{1}{2\sqrt{2}M}\left[\Upsilon(z_1,s_3)\Delta_{31}(s_3,z_1)-\frac{2K_1(z_1)}{s}\,\frac{Mw_3(z_1+s-\mu^2)+(z_1-\mu^2)(s-M^2)}{(w_3-M)^2-s}-\frac{2|l_1(z_1)|}{M}\,\theta((M-\mu)^2-z_1)\,\frac{w_3(z_1+M^2-\mu^2)+M(s-M^2)}{(w_3-M)^2-s}\right], \tag{A18}$$

$$H\overset{*}{=}\frac{1}{K_1^{\,2}(s_1)}\left\{\tfrac{1}{20}R_2(s_1,z_2)[R_1(s_1,z_2)\Delta_{12}(s_1,z_2)+\frac{s_1+s-\mu^2}{s}K_2(z_2)]+\tfrac{3}{20}\,\Omega(s_1,z_2)[X(s_1,z_2)\Delta_{12}(s_1,z_2)+2K_2(z_2)]+[\tfrac{3}{5}MW-5(s_1,z_2)^{1/2}]\left[\phi(s_1,z_2)\Delta_{12}(s_1,z_2)-\frac{X(s_1,z_2)}{2s}K_2(z_2)\right]\right\}, \tag{A19}$$

$$H_{12}'=H_{12},\quad \sqrt{2}H_{13}'=H_{13},\quad H_{31}'=\sqrt{2}H_{31}\,. \tag{A20}$$

The basic kernel integrals of Ref. 4, Appendix B, appear in these formulas. If we recall that we need only evaluate the H's for $z_j\geq 0$, as prescribed in (5), we have

$$\Delta_{12}=\Delta_{12}^{(1)}\,,\quad \Delta_{13}=\Delta_{13}^{(1)}\,,\quad \Delta_{31}=\Delta_{31}^{(1)}+\theta((M-\mu)^2-z_1)\Delta_{31}^{(2)}\,. \tag{A21}$$

In Eqs. (A8)–(A20), those terms in which an asterisk appears over the equality sign are ones in which an s_i-independent piece has been discarded. In H_{PP} [Eq. (A12)] and in H [Eq. (A19)], we have made the replacement $\sqrt{z_1}\to w_1=\sqrt{s_1}$; in H_{31} [Eq. (A18)], we have made the replacement $\sqrt{z_3}\to w_3=\sqrt{s_3}$. These substitutions have been taken in Eqs. (I.48), (I.55), and (I.54), respectively, in accord with the discussion of this approximation given in Sec. III. We refer the reader to the formulas given in I, Appendix B, for the kinematical quantities R_1, R_2, Ω, Υ, X, and ϕ, and also to the results of Ref. 4. Appendix B, where expressions for the $\Delta_{ij}^{(n)}$ and K_i are given. Additional quantities are $Q_0=(s+M^2-z_3)/2W$ and $Q=(Q_0^{\,2}-M^2)^{1/2}$ in (A17), and

$$l_1(z_1)=\{[z_1-(M+\mu)^2][z_1-(M-\mu^2)]\}^{1/2}$$

in (A18). The quantities $\Delta_{ij}^{(1)}(s_i,z_j)$ in (A21) have imaginary parts given by

$$\mathrm{Im}\Delta_{ij}^{(1)}(s_i,z_j)=\pi/K_i(s_i) \tag{A22}$$

for s_i and z_j in the physical (s_i,z_j) decay region. Outside this region all the H functions are real.

The integrals $I_{\alpha\beta}^{J^PT}$ represent what we believe to be the most important sources of s_i variation in the isobar factors f_α; they correspond to the $\alpha\leftarrow\beta$ rescattering process shown in Fig. 2. They are not completely trivial to evaluate numerically, for a number of reasons. Firstly, the functions $\Delta_{ij}^{(1)}$ have logarithmic singularities on the boundary of the physical (s_i,z_j) decay region; these singularities therefore lie on the integration paths.

Secondly, the imaginary parts of the H functions [which occur via Eq. (A22)] contain inverse powers of the quantities $K_i(s_i)$, which vanish at the end points $\hat{z}_1 = \hat{z}_2 = (W-\mu)^2$ and $\hat{z}_3 = (W-M)^2$; a reasonably powerful quadrature formula must be used to get stable results in this region.

A special problem arises for cases in which the integrand involves the two-body function $\zeta_{\mathfrak{N}}(z)$, corresponding to the P_{11} isobar channel. This quantity has a pole (the nucleon) at $z=M^2$, which lies on the integration path. By considering the derivation of the original integral equations, and the dispositions of the contours therein as described in Ref. 4, Sec. III, it can be shown that the correct prescription for passing the pole at $z=M^2$ is to give M^2 a small negative imaginary part[30] (this is, in fact, the standard Feynman prescription). Near $z=M^2$, therefore, we can write

$$\zeta_{\mathfrak{N}}(z) = \frac{R_M}{\pi}\left(\mathrm{P}\frac{1}{M^2-z} + i\pi\delta(z-M^2)\right), \qquad \text{(A23)}$$

where R_M is the residue of $\zeta_{\mathfrak{N}}$ at the pole, as given in expression (18) in Sec. IV above. Thus, for example, $I^{1/2^-}_{\mathfrak{N}\mathfrak{N}}(s_1,s)$ takes the form

$$I^{1/2^-}_{\mathfrak{N}\mathfrak{N}}(s_1,s) = \mathrm{P}\int_0^{\hat{z}_2} dz_2\, H_{12}(s_1,z_2,s)\zeta_{\mathfrak{N}}(z_2) + iR_M H_{12}(s_1,M^2,s), \qquad \text{(A24)}$$

where P stands for principal value. We have evaluated the principal-value integrals numerically by determining to high accuracy the position of the pole in our $\zeta_{\mathfrak{N}}$ as actually parametrized (we accept a slight departure in the pole position from the exact value of M^2), and by omitting a portion of the integral around this point small enough to ensure stability.

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[1]See, for example, D. J. Herndon, P. Söding, and R. J. Cashmore, Phys. Rev. D 11, 3165 (1975); B. H. Bransden and R. G. Moorhouse, *The Pion-Nucleon System* (Princeton Univ. Press, Princeton, New Jersey, 1973); the latter authors give many earlier references.
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[30]See also footnote 37 of Ref. 7.