NON-CLASSICAL CONFIGURATIONS IN EUCLIDEAN FIELD THEORY

AS MINIMA OF CONSTRAINED SYSTEMS

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ABSTRACT

We study a systematic method of applying semi-classic approximation to Euclidean field theory.

First, we extract generalized collective coordinates which are not in general zero modes. We then apply the semi-class approximation to the other degrees of freedom by minimizing the action with constraints. Hence we are using configurations which are not classical solutions of the original system. After Gaussia integration, we are left with a truncated system, involving only the collective coordinates, with non-trivial dynamics. In particular, this is a clear-cut way to introduce multi-instanton or meron-type configurations. The collective coordinates should be chosen such that their dynamics is a good approximation of the original system for the physical phenomenon considered, a familiar concept in other branches of physics with many degrees of freedom. The formalism leads naturally to the introduction of dynamics in an extra time evolution ; in particular cases, we show that this is a very powerful tool. In this paper, we only discuss general ideas and formalisms. Specific applications are postponed to later publications.

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1 - INTRODUCTION

In the last few years, semi-classical methods have been used to study the Euclidean formulation of field theory.[1-4] These developments are different in two important respects from earlier semi-classical treatments of soliton-like phenomena.[5]

First, contrary to the Minkowskian (real time) case, it is usually not a good approximation to retain only exact Euclidean classical solutions, and one must deal with more general field configurations. Second, the physical result depends crucially upon the statistical properties of the degrees of freedom of these field configurations. Thus, semi-classical Euclidean methods appear essentially as a way of truncating the original system.

Up to now, these two features have not been systematically investigated, even though the saddle point method, which is the basis of the semi-classical approximation, is not applicable when dealing with a general configuration. In this paper, we approach this problem by means of generalized collective coordinates,[6-7] which are not zero modes of the system. The basic idea is the same as in other fields of physics with many degrees of freedom : one extracts from the system the degrees of freedom that one believes to be relevant for the phenomenon considered and one gives them special treatment. The set of the other, less relevant, degrees of freedom is equivalent to the original system with constraints. This set is treated in the semi-classical, gaussian, approximation. This method of constraints allows the introduction of more general field configurations as minima of the constrained action. After the gaussian integration is performed, one is left with the collective coordinates. This defines in a systematic way a truncated system with non trivial dynamics. This can be used to study interactions between field configurations and responses to external sources.

In section 2 , we discuss the general method of introducing collective coordinates in Euclidean field theory while keeping calculations as tractable as possible.

It is customary to speak about the contribution of non-minimal field configurations to the functional integral. We find that this concept is unambiguous only at the classical level. The field configuration is determined by the first functional derivative of the constraint, that is, by the classical external force applied on the system via the constraint. Instead, the first quantum corrections, i.e. the gaussian functional integral, depends upon the second functional derivative of the constraint. A given field configuration can give different contributions because it can be obtained from different constraints. Thus, the important feature is the choice of constraints.

In section 3 , we explicit the analogy of the Euclidea[n] functional integral with the partition function of a classical system at thermal equilibrium. From this point of view, it is natural to introduce the corresponding time evolution and canonic[al] formalism. This time is an additional parameter which should be defined in such a way that the classical system has maximum symmetry. This method may not be practical in all cases, but is extremely powerful ; for instance one can sometimes consider instantons in d dimensions as solitons in d+1 dimensions.

Finally, in section 4 , we discuss specific applications. The results will be described in further publications. Specifically, we are interested in a more rigorous study of the gas of instantons[1-4] in theories without mass scale, and in a determination of possible phase transitions between instantons and other configurations like merons.[6] We shall also describe an application of the method of section 3 to the evaluation and resummation of the singularities of the Borel transform of the perturbation series.[9] More specifically, we shall deal with the ground state energy of the double well anharmonic oscillator.[10]

2 - THE GENERAL METHOD.

If φ is a typical field, the relevant functional integral of Euclidean field theory reads

$$\mathcal{Z} = \int \mathcal{D}\varphi \, e^{-H_0[\varphi]} \tag{2.1}$$

where H_0 is the Euclidean action.

As usual[7], collective coordinates will be introduced in the functional integral by means of constraints. We choose N functionals of φ denoted $H_\alpha[\varphi]$ $\alpha = 1, \ldots, N$ and write

$$1 = \int \prod_\alpha da_\alpha \, \delta(a_\alpha - H_\alpha[\varphi]) \tag{2.2}$$

This expression is inserted into (2.1) which is rewritten as

$$\mathcal{Z} = \int \prod_\alpha da_\alpha \int \mathcal{D}\varphi \prod_\alpha \delta(a_\alpha - H_\alpha[\varphi]) \, e^{-H_0[\varphi]} \tag{2.3}$$

For fixed a_α we look for the minima of the action taking the constraints $a_\alpha = H_\alpha$ into account. The corresponding equation reads

$$\frac{\delta H_0}{\delta \varphi} + \sum_{\alpha=1}^{N} \lambda_\alpha \frac{\delta H_\alpha}{\delta \varphi} = 0 \tag{2.4}$$

where λ_α are Lagrange multipliers to be determined from the condition

$$H_\alpha[\varphi] = a_\alpha \tag{2.5}$$

Let us denote by $\varphi_{cl}(a,x)$ such a solution. In general, that is if at least one λ_α is non zero, φ_{cl} is not a minimum of H_0 without constraint. Hence φ_{cl} is a more general field configuration. Thus, by writing (2.2) we have introduced collective coordinates a_α's defined in terms of the original field $\varphi(x)$ by the equations $a_\alpha = H_\alpha[\varphi]$.

φ_{cl} is a minimum of the action for fixed values of these collective coordinates. We shall make use of it in order to treat the other degrees of freedom of the system semi-classically. This is done by performing the following change of variable into (2.3)

$$\varphi(x) = \varphi_{cl}(a,x) + \chi(x) \tag{2.6}$$

Because of the constraints, we can in (2.3) replace H_0 by $H_0 + \sum_\alpha \lambda_\alpha (H_\alpha - a_\alpha)$. φ_{cl} being a solution of (2.4) is a minimum of this modified action. Hence we do not get terms linear in χ when we perform (2.6) and therefore, we can consistently build the semi-classical development around φ_{cl} Therefore, in the same way as in Minkovski field theory, introducing collective coordinates allows a semi-classical treatment only for a part of the system considered.

If one keeps only the leading terms, one gets

$$\mathcal{Z} = \int \prod_\alpha da_\alpha \, e^{-V(a)/g^2} \int \mathcal{D}\chi \prod_\alpha \delta\left(\int dx \left.\frac{\delta H_\alpha}{\delta \varphi}\right|_{cl} \chi \right) \times \exp\left[-\frac{1}{2} \int dx\, dy\, \Omega(x,y)\, \chi(x)\, \chi(y) \right] \tag{2.7}$$

$$V(a) = g^2 H_0 \left[\varphi_{cl}(a,x) \right]$$

$$\Omega(x,y) = \frac{\delta^2}{\delta\varphi(x)\,\delta\varphi(y)} \left(H_0 + \sum_\alpha \lambda_\alpha H_\alpha \right) \Big|_{cl}$$

g^2 is a coupling constant which is the typical expansion parameter of semi-classical methods. It is introduced to be such that the functionals $H_\alpha[\varphi]$, $\alpha = 0, \ldots N$ can be rewritten as

$$H_\alpha[\varphi] = \frac{1}{g^2} \tilde{H}_\alpha [g\varphi] \tag{2.8}$$

where $\tilde{H}_\alpha$ does not explicitly depend on g. φ_{cl} involves g only through a multiplicative factor g^{-1} . Hence V and Ω do not depend on g.

The integration over χ is formally performed as follows. One writes, in (2.7)

$$\delta\left(\int dx \frac{\delta H_\alpha}{\delta \varphi_{cl}} \chi \right) = \frac{1}{2\pi} \int d\mu_\alpha \, e^{i\mu_\alpha \int dx \left(\frac{\delta H_\alpha}{\delta \varphi_{cl}} \chi \right)} \tag{2.9}$$

One then performs a shift on χ to eliminate the linear term. For this purpose we remark that taking the derivatives of (2.4) with respect to λ_α leads to

$$\int dy\, \Omega(x,y)\, \frac{\delta \varphi_{cl}(y)}{\delta \lambda_\alpha} = - \frac{\delta H_\alpha}{\delta \varphi_{cl}(x)} \tag{2.10}$$

and the relevant shift reads

$$\chi = \tilde{\chi} + i \sum_\alpha \frac{\partial \varphi_{cl}}{\partial \lambda_\alpha} \mu_\alpha \tag{2.11}$$

The integration over μ_α is easily performed using the identity

$$\frac{\partial a_\alpha}{\partial \lambda_\beta} = \int dx\, \frac{\delta H_\alpha}{\delta \varphi_{cl}(x)} \frac{\delta \varphi_{cl}(x)}{\partial \lambda_\beta} \tag{2.12}$$

one obtains

$$\frac{1}{(2\pi)^{N/2}} \det \left[- \partial \lambda_\alpha / \partial a_\beta \right]^{1/2} \tag{2.13}$$

The remaining integration over $\tilde{\chi}$ just gives the inverse square root of the determinant of Ω unconstrained. In general Ω may have negative eigenvalues . In all cases where the constraints lead to a stable minimum, the corresponding i factors in $(\det \Omega)^{1/2}$ will be compensated by the ones appearing in (2.13). The final result of the calculation will be correct even though the gaussian integrals over μ and $\tilde{\chi}$

may not really make sense separately.

At this point it is important to notice that φ_{cl} is entirely specified by $H_\alpha[\varphi_{cl}]$ and $\delta H_\alpha / \delta \varphi_{cl}$ as is obvious from (2.4),(2.5). Indeed these quantities determine the classical force due to the constraints. On the other hand, Ω involves $\delta^2 H_\alpha / \delta \varphi_{cl}(x) \delta \varphi_{cl}(y)$. Hence a given field configuration can be obtained by using different constraints H_α which in general will lead to different quantum correcti Thus the notion of field configuration which is widely used in the current literature is really clear cut only at the classical level. At the quantum level, the unambiguous concept is the choice of constraints, that is of collective coordinates. It uniquely fixes not only the field configuration but also the phase space in a small neighborhood around it.

The previous discussion does not take into account possible symmetries of the system and, in particular, possible zero modes of Ω . The relevant invariance group is as usual the subgroup $\mathcal{G}$ of symmetries of the system which is broken by φ_{cl} We denote by X_β, $\beta = 1, \dots R$ its parameters and by $\varphi_{[x]}$ the transformed of φ . By assumption $\mathcal{G}$ is such that

$$H_0[\varphi_{[x]}] = H_0[\varphi] \quad , \quad \mathcal{D}\varphi_{[x]} = \mathcal{D}\varphi$$

We shall first discuss the case where the constraints H_α are chosen to be invariant by $\mathcal{G}$, so we have in addition

$$H_\alpha[\varphi_{[x]}] = H_\alpha[\varphi] \qquad \alpha = 1, \dots N .$$

In this case Ω has R zero modes and in the same way as for solitons[6-7] they are treated exactly by promoting X_β to additional collective coordinates. This is achieved by using, instead of (2.2),

$$1 = \int da\, dX \prod_\alpha \delta(a_\alpha - H_\alpha[\varphi]) \prod_\beta \delta(F_\beta[\varphi_{[X]}])\, D[\varphi, X]$$

$$D[\varphi, X] = \det \left\{ \frac{\partial F_\beta[\varphi_{[X]}]}{\partial X^\delta} \right\} \tag{2.14}$$

F_β are additional constraints which should be such that $D \neq 0$. Inserting (2.14) into (2.1) we perform the change of variables $\varphi \to \varphi_{[-X]}$ where $[-X]$ means the inverse of $[X]$. The result reads

$$\mathcal{Z} = \int \mathcal{D}\varphi \prod_\alpha da_\alpha\, \delta(a_\alpha - H_\alpha[\varphi]) \prod_\beta dX_\beta\, \delta(F_\beta[\varphi]) \times D[\varphi_{[-X]}, X]\, e^{-H_0[\varphi]} \tag{2.15}$$

It is straightforward to check that

$$D[\varphi_{[-X]}, X] = \mathcal{M}(X)\, J[\varphi] \tag{2.16}$$

where J does not depend on X and $\mathcal{M}(X)$ is such that $\mathcal{M}(X)\,dX$ is the measure invariant by $\mathcal{G}$.

The integration over φ is done as before in the quadratic approximation. In order to obtain a minimum of H_0 taking into account the two types of constraints we remark that the equations (2.4)-(2.5) now have a continuous set of solutions $\varphi_{cl}(a, X_0, x)$ where X_0 is arbitrary and

$$\varphi_{cl}(a, X_0, x) = \varphi_{cl\,[X_0]}(a, 0, x)$$

Since D is non zero, the F constraints completely break the invariance by $\mathcal{G}$. Hence we can satisfy them using φ_{cl} with an appropriate choice of X_0 and we see that the Lagrange multipliers of F constraints vanish. Moreover, in the same way as for Minkovski field theory,[7] the result does not depend on the choice of F.

The zero modes associated to $\mathcal{G}$ are simply $\partial\varphi_{cl}/\partial X_\beta$ and we make the simplest choice of F. We assume here that φ_{cl} breaks $\mathcal{G}$ completely. The general case is discussed at the end of the appendix.

$$F_\beta[\varphi] = \int dx \left[\frac{\partial \varphi_{cl}}{\partial X_\beta} (\varphi - \varphi_{cl}) \right] \tag{2.17}$$

In this formula and hereafter we write $\varphi_{cl}(x)$ and $\partial\varphi_{cl}/\partial X$ instead of $\varphi_{cl}(a, 0, x)$ and $\frac{\partial \varphi_{cl}}{\partial X}\big|_{X=0}$. The choice (2.17) of F leads immediately to $X_0 = 0$ and we thus perform the shift

$$\varphi = \varphi_{cl} + \chi$$

obtaining

$$\mathcal{Z} = \int \prod_\alpha da_\alpha \prod_\beta dX_\beta \, m(X) \, e^{-V(a)/g^2} \int \mathcal{D}\chi \prod_\alpha \delta\left(\int dx \frac{\delta H_\alpha}{\delta \varphi_{cl}} \chi\right)$$

$$\times \delta\left(\int dx \frac{\partial \varphi_{cl}}{\partial X_\alpha} \chi\right) J[\varphi_{cl}] \exp\left[-\tfrac{1}{2}\int dx\, dy\, \Omega(x,y)\, \chi(x)\, \chi(y)\right]$$

(2.18)

Again we use (2.9). Now the shift must be made compatible with the $F=0$ condition. Since $\partial \varphi_{cl} / \partial X$ is a zero mode, we deduce from (2.10) that for arbitrary ρ_α^δ

$$\int dy\, \Omega(x,y) \left[\frac{\partial \varphi_{cl}(y)}{\partial \lambda_\alpha} + \rho_\alpha^\delta \frac{\partial \varphi_{cl}(y)}{\partial X_\delta} \right] = - \frac{\delta H_\alpha}{\delta \varphi_{cl}(x)}$$

(2.19)

We choose ρ_α^δ such that

$$\int dy \frac{\partial \varphi_{cl}}{\partial X_\delta} \left(\frac{\partial \varphi_{cl}}{\partial \lambda_\alpha} + \rho_\alpha^\sigma \frac{\partial \varphi_{cl}}{\partial X_\sigma} \right) = 0$$

(2.20)

Hence if we perform, instead of (2.11), the shift

$$\chi = \tilde{\chi} + i \sum_\alpha \mu_\alpha \left[\frac{\partial \varphi_{cl}}{\partial \lambda^\alpha} + \rho_\alpha^\sigma \frac{\partial \varphi_{cl}}{\partial X_\sigma} \right] ,$$

(2.21)

the $F=0$ condition gives simply

$$\int dx\, \tilde{\chi} \frac{\partial \varphi_{cl}}{\partial X_\alpha} = 0$$

(2.22)

The integration over μ_α is easily performed. The result is again given by (2.13) since from the invariance of H_α by $\mathcal{G}$ one deduces

$$\int dx \frac{\delta H_\alpha}{\delta \varphi_{cl}(x)} \frac{\partial \varphi_{cl}}{\partial X_\alpha} = 0 , \quad \alpha = 1, \dots N$$

(2.23)

The remaining integration over $\tilde{\chi}$ gives the determinan of Ω in the space orthogonal to the zero modes. This completes c discussion of the cases where all H_α's are invariant by $\mathcal{G}$.

In practice however it is not always possible, or desirable, to introduce only invariant H_α's . One reason is that the Casimir operators of $\mathcal{G}$ are in general non local functionals of the field φ . Thus, it is necessary to extend the previous discussion to the case where the H_α's are invariant only with respect to a subgroup $\tilde{\mathcal{G}}$ of $\mathcal{G}$. From now on we shall denote by X the parameters of $\tilde{\mathcal{G}}$ and by Y the additional parameters of $\mathcal{G}$. Now, if we go back to Ω we see that only $\partial\varphi_{cl}/\partial X$ will be a zero mode. On the other hand, by assumption, $\mathcal{G}$ is the invariance group broken by φ_{cl}. Hence our final result will not break $\mathcal{G}$ only if we include all configurations $\varphi_{cl[X,Y]}$ obtained from φ_{cl} by applying an arbitrary element of $\mathcal{G}$. This is achieved by promoting both X and Y to collective coordinates. The collective coordinates a_α's will be invariant by $\mathcal{G}$ if, instead of (2.2) or (2.14), we write

$$1 = \int da\, dX\, dY \prod_\alpha \delta(a_\alpha - H_\alpha[\varphi_{[X,Y]}]) \prod_\beta \delta(F_\beta[\varphi_{[X,Y]}]) \times \prod_\gamma \delta(G_\gamma[\varphi_{[X,Y]}])\ \mathcal{D}[\varphi; X, Y] \tag{2.24}$$

$G_\gamma = 0$ are additional constraints associated to the Y_γ's, $\mathcal{D}$ is the Jacobian.

The most appropriate choice of G and the actual computation of the gaussian integral are discussed in the appendix.

Finally, we end up with a result of the form

$$\mathcal{Z} = \mathcal{V} \int da\, \Delta(a)\, e^{-V(a)/g^2}$$

$$\mathcal{V} = \int dX\, dY\, \mathcal{M}(X,Y) \tag{2.25}$$

where $\mathcal{M}(X,Y)\,dX\,dY$ is the measure invariant under $\mathcal{G}$. It is a partition function of a gas at temperature g^2 with potential energy $V(a)$ and density of states $\Delta(a)$. In general, since $\partial V/\partial a \neq 0$, we have interactions between the collective coordinates. The semi-classical method we just described allows us to unambiguously approximate the original system by the one described by (2.21). Note that the density of states $\Delta(a)$ which is crucial to study the physical properties of the truncated system, involves the result of the gaussian integral. It thus depends not only on the field configurations considered but also on the particular choice of H_α.

In practice, in order to get a sufficiently good approximation one may have to include a whole class of constraints with different N and add the corresponding contributions. One ends up with a statistical ensemble in which the number of particles is not fixed.

This viewpoint has already been investigated extensively in the dilute gas approximation of multi-instanton-anti-instanton configurations [1-4]. There one assumes that (2.25) is dominated by the contributions with small density. One neglects the interactions and assumes that $\Delta(a)$ factorizes into a product of one instanton determinants. Then the a_α's are treated as if they were associated to symmetries of the system. The validity of this approximation is unclear if the theory has no mass scale.

3 - CANONICAL FORMALISM

In the previous section we discussed the general method of extracting collective coordinates by means of constraints without assuming any particular property for the system considered. The previous method may not be the most appropriate for systems with a richer structure. In this section we discuss more powerful methods which however are less general.

Following the current literature, we have adopted the language of statistical mechanics in our previous discussion. The functional integral (2.1) is thus considered as the partition function of a classical system. To be complete this picture requires the introduction of classical dynamics. This is equivalent to the choice of families of canonical transformations in phase space which are considered as time evolutions and are Hamiltonian flows. This time is an extra dimension since it has nothing to do with space time.

In this way, we can make use of the fact that instantons in d dimensions are solitons in $d+1$ dimensions and apply the techniques already developed in order to introduce soliton collective coordinates[6-7]. In particular a method has been proposed to deal, at least formally, with multi-soliton states in a time independent formalism. This method will be easily adapted to the present problem. Moreover if the soliton system is completely integrable classically we can hope to get exact results since we are studying its classical partition function.

As before, we start from

$$\mathcal{Z} = \int \mathcal{D}\varphi \, e^{-H_o[\varphi]} \tag{3.1}$$

This formula does not involve the momentum $\pi(x)$ conjugate to φ . π can be introduced in two ways.

First we can consider that it is an independent variable. As is well known, the integration over momentum in a classical partition function is a trivial gaussian if the potential is velocity independent. Hence we can introduce π by considering instead of (3.1)

$$\mathcal{Z} = \int \mathcal{D}\pi \, \mathcal{D}\varphi \, \exp\left[-H_o[\varphi] - \frac{1}{2}\int dx\, \pi^2\right] \tag{3.2}$$

$H_o[\varphi]$ is thus considered as potential energy. Integrating over π , one immediately sees that (3.2) and (3.1) only differ by an irrelevant constant factor. The canonical brackets associated with the symplectic measure of (3.2) are simply the Poisson brackets. We introduce the corresponding time evolution

$$\dot{\pi} = -\frac{\delta H_o}{\delta \varphi} \quad , \quad \dot{\varphi} = \pi \tag{3.3}$$

which, as is well known, is such that the symplectic measure

$$\omega = \mathcal{D}\varphi\,\mathcal{D}\pi \tag{3.4}$$

and the Hamiltonian

$$H = \frac{1}{2}\int dx\, \pi^2 + H_0[\varphi] \tag{3.5}$$

are time independent.

However this method leads to a doubling of phase space and it is better, if possible, to introduce the canonical structure only in terms of φ. Hence we consider a phase space with constraints and write instead of (3.2)

$$\mathcal{Z} = \int \mathcal{D}\varphi\,\mathcal{D}\pi\,\delta\{\pi(x) - f([\varphi],x)\}\, e^{-H_0[\varphi]} \tag{3.6}$$

where $f([\varphi],x)$ is a suitably chosen x - dependent functional.

The constraints $\pi = f$ are to be treated systematically in Dirac's formalism[11].

Introducing

$$\theta(x) = \pi(x) - f([\varphi],x) \tag{3.7}$$

one computes the Poisson bracket

$$-D^{-1}(x,y) = \{\theta(x), \theta(y)\}_0 = \frac{\delta f([\varphi],x)}{\delta\varphi(y)} - \frac{\delta f([\varphi],y)}{\delta\varphi(x)} \tag{3.8}$$

We assume that $\det D \neq 0$ so that the θ constraints are second class.

The Dirac bracket reads

$$\{\varphi(x), \varphi(y)\} = D(x,y) \tag{3.9}$$

In one dimension a typical example of this phase space structure is

$$f([\varphi],x) \equiv \int_{-\infty}^{x} d\xi\,\varphi(\xi) \tag{3.10}$$

There one finds for arbitrary $g_1[\varphi]$, $g_2[\varphi]$

$$\{g_1[\varphi], g_2[\varphi]\} = \int dx\, \frac{\delta g_1}{\delta\varphi}\frac{\partial}{\partial x}\frac{\delta g_2}{\delta\varphi} \tag{3.11}$$

This canonical structure is the relevant one in the study of many integrable systems[12]. (see sect. 4)

Dirac's method is such that it reduces to the ordinary canonical formalism when the system is reexpressed in terms of independent variables. The corresponding measure[13] is

$$\omega = \mathcal{D}\varphi\, \mathcal{D}\pi\, \delta(\theta)\, (\det D)^{-1/2} \tag{3.12}$$

It differs from the one appearing in (3.6). We shall therefore restrict ourselves to constraints θ such that $\det D$ does not depend on φ in such a way that we can redefine $\mathcal{Z}$ up to a factor, as an integral with the measure ω.

We consistently consider time evolution as the Hamiltonian flow of H_o with brackets (3.9). Hence we have

$$\dot{\varphi}(x) = \{\varphi(x), H_o[\varphi]\} = \int dy\, D(x,y) \frac{\delta H_o}{\delta\varphi(y)} \tag{3.13}$$

It is first order contrary to (3.3). As in the previous case, the time evolution is such that the symplectic measure ω and the Hamiltonian $H = H_o$ are time independent.

In both cases we end up with an expression of the form

$$\mathcal{Z} = \int \omega(\pi,\varphi)\, e^{-H(\pi,\varphi)} \tag{3.14}$$

where ω is invariant by canonical transformations. We can consider φ and π in (3.13) as initial data of the time evolution (3.3) or (3.13) at time t.
In this way we can compute $\mathcal{Z}$ at any fixed time. It is easy to see that the result does not depend on t in agreement with the fact that $\mathcal{Z}$ is supposed to describe a system at statistical equilibrium. One of our motivations for introducing the additional time is that, if the corresponding system is non ergodic, we will naturally find field configurations such as solitons which are more important than others in (3.14).

We can now apply the general method of ref. 7 of introducing collective coordinates which is based on canonical transformations. For completeness we repeat several arguments of this reference. We shall treat at the same time the two types of phase spaces described above since we will not need to specify the Poisson or Dirac bracket used. We shall write separate equations for φ and π. If π is a dependent variable, the equations for π will be redundant as they are consequences of the corresponding equation for φ. This is so because in Dirac's method the Dirac brackets are defined to be compatible with the constraint θ.

Consider a group $\mathcal{G}$ of canonical transformations with parameters X_α. We denote by $\varphi_{[X]}$ and $\pi_{[X]}$ the transformed of φ, π by any finite transformation of $\mathcal{G}$.
If P_α are the infinitesimal generators of $\mathcal{G}$ we have for small $X^\alpha = \varepsilon^\alpha$

$$\varphi_{[\varepsilon]} \simeq \varphi + \sum_\alpha \varepsilon^\alpha \{\varphi, P_\alpha\} + O(\varepsilon^2)$$

$$\pi_{[\varepsilon]} \simeq \pi + \sum_\alpha \varepsilon^\alpha \{\pi, P_\alpha\} + O(\varepsilon^2) \tag{3.15}$$

For general X, $\varphi_{[x]}$ and $\pi_{[x]}$ are to be determined as follows. We compute

$$\frac{\partial \varphi_{[x]}}{\partial X^\alpha} = \lim_{\Delta X \to 0} \left(\varphi_{[x+\Delta x]} - \varphi_{[x]} \right) / \Delta X^\alpha$$

Let us denote by $f_\alpha(X_2^\beta, X_1^\gamma)$ the functions which determine the group multiplication laws of the parameters X. We thus have

$$\left(\varphi_{[x_1]}\right)_{[x_2]} = \varphi_{[f(x_2, x_1)]} \tag{3.16}$$

Introduce ΔY^α such that

$$f^\alpha(\Delta Y, X) = X^\alpha + \Delta X^\alpha + O(\Delta X^2)$$

It is given by

$$\Delta X^\alpha = V^\alpha_\beta(X)\, \Delta Y^\beta \tag{3.17}$$

$$V^\alpha_\beta(X) = \frac{\partial f^\alpha}{\partial Y^\beta}(Y, X)\Big|_{Y=0} \tag{3.18}$$

It is now easy to show that

$$\frac{\partial \varphi_{[x]}}{\partial X^\alpha} = \lim_{\Delta X \to 0} \frac{(\varphi_{[x]})_{[\Delta Y]} - \varphi_{[x]}}{\Delta X^\alpha}$$

and using (3.15) we obtain

$$\frac{\partial \varphi_{[x]}}{\partial X^\alpha} = U^\beta_\alpha(X) \left\{ \varphi_{[x]}, P_\beta\left[\varphi_{[x]}, \pi_{[x]}\right] \right\} \tag{3.19}$$

where U^β_α is defined by

$$U^\beta_\alpha(X)\, V^\gamma_\beta(X) = \delta^\gamma_\alpha \tag{3.20}$$

Similary one can show that $\pi_{[x]}$ is solution of

$$\frac{\partial \pi_{[x]}}{\partial X^\alpha} = U^\beta_\alpha(X) \left\{ \pi_{[x]}, P_\beta\left[\varphi_{[x]}, \pi_{[x]}\right] \right\} \tag{3.21}$$

In (3.19) and (3.21) one can directly compute the Dirac bracket taking $\varphi_{[x]}$ and $\pi_{[x]}$ as variables. This is corre[ct] because the Dirac bracket does not change under a canonical transformation of the variables.

$\varphi_{[x]}$ and $\pi_{[x]}$ are in principle to be determined as solutions of the differential equations (3.19) and (3.21). The integrability condition reads

$$\frac{\partial U_\lambda^\delta}{\partial X^\sigma} - \frac{\partial U_\sigma^\delta}{\partial X^\lambda} = C_{\alpha\beta}^{\delta}\, U_\sigma^\alpha\, U_\lambda^\beta \tag{3.22}$$

where $C_{\alpha\beta}^{\delta}$ are the structure constantsof the group

$$\{P_\alpha , P_\beta\} = C_{\alpha\beta}^{\delta}\, P_\delta \tag{3.23}$$

Equation (3.22) is a well known relation of Lie group theory.

Equation (3.19) and (3.21) may be very difficult to solve. However we shall not need their explicit solutions.

The method of section 2 can now be applied. In this canonical framework it is essentially the same as in ref.7 except that we look at the classical partition function instead of the evolution operator. The starting point is to write

$$1 = \int \prod_{\alpha,\beta} d\, p_\alpha\, d X^\beta\, \delta\big(p_\alpha - P_\alpha[\varphi_{[x]}, \pi_{[x]}]\big)\, \delta\big(Q^\beta[\varphi_{[x]}, \pi_{[x]}]\big)\, D \tag{3.24}$$

where Q^β are arbitrary constraints which are associated to the collective coordinates X^β. At this point the calculation proceeds exactly as in sect. 2 if $\mathcal{G}$ is such that $\{H , P_\alpha\} = 0$ as we assume from now on.

This canonical method is particularly interesting if the dynamical system in $d+1$ dimensions has a large number, or even more an infinite number of conserved quantities. In this case, as already argued in ref. 7 we can make use of time dependent solutions of (3.3) or (3.13), provided they are such that the time evolution can be replaced by a time dependent transformation belonging to $\mathcal{G}$. In this case we have

$$\dot\varphi = \sum_\alpha v^\alpha \{\varphi, P_\alpha\}$$

$$\dot\pi = \sum_\alpha v^\alpha \{\pi, P_\alpha\} \tag{3.25}$$

This equation combined with (3.3) leads to

$$\frac{\delta H}{\delta \varphi} - \sum_\alpha v^\alpha \frac{\delta P_\alpha}{\delta \varphi} = 0$$

$$\frac{\delta H}{\delta \pi} - \sum_\alpha v^\alpha \frac{\delta P_\alpha}{\delta \pi} = 0 \tag{3.26}$$

which is a minimum of $H(\varphi, \pi)$ with constraints $p_\alpha = P_\alpha$ and Lagrange multipliers $\lambda_\alpha = - v_\alpha$. This is precisely a field configuration of the type considered in section 2 .

As an example one can check that all multi-soliton solutions of the sine-Gordon theory are of this type. Hence we get multi-instanton configurations from multi-soliton solutions.

A remarkable fact is that the dynamics of the collective coordinates X is trivial because they are associated to a symmetry of the system. If there is a large or infinite number of constants of motion, we can introduce an arbitrary number of them as constraints and the multi-instanton dynamics simplifies tremendously.

This simplification can be understood in the following way. In general, taking the minimum with constraint will lead to a rather complicated field configuration which is typically a multi-soliton solution at some fixed time. We can however use the fact that (3.14) is invariant under time evolution to compute its contribution at some very large future time when it has split into widely separated solitons, a very simple field configuration. In this way one minimizes in some sense the interaction between instantons.

Finally, this method leads to an approximation of $\mathcal{Z}$ where only multi-soliton configurations at fixed time are included. Neglecting the background can be understood from the following arguments. First it is known, for example in the sine-Gordon model, that this background does not contribute to the spectrum of the semi-classical Hamiltonian, hence it does not contribute to the semi-classical partition function, nor a fortiori to the classical

one. Second, after a long enough time any background term becomes very small and its contribution becomes part of the remaining gaussian integral.

If the classical system is completely integrable one can even attempt an exact computation of $\mathcal{Z}$. Indeed in this case there exists a canonical transformation to free variables ζ_k, η_k such that

$$H = \sum_k h_k(\zeta_k) \tag{3.27}$$

and one obtains

$$\mathcal{Z} = \prod_k \int d\zeta_k \, d\eta_k \, e^{-h_k} \tag{3.28}$$

The integration is still a non trivial quadrature since one must investigate the range of ζ_k, η_k. This problem is currently under investigation.

So far we discussed only the introduction of constraints which commute with H. This is not the only application of the canonical formalism. We can get interesting results if H is part of a non commutative algebra which closes. For instance assume that the theory with extra-time is Lorentz invariant. We then have a boost operation K and a momentum operator P such that

$$\{H, P\} = 0 \qquad \{H, K\} = -P$$

$$\{P, K\} = -H \tag{3.29}$$

Define $\varphi^{[v]}$ as the transformed of φ by K with velocity v. We can verify that

$$H[\varphi^{[v]}, \pi^{[v]}] = (1-v^2)^{-1/2} \left(H[\varphi, \pi] - v\, P[\varphi, \pi] \right) \tag{3.30}$$

Assume that we have a static solution φ_0, π_0 :

$$\left.\frac{\delta H}{\delta \varphi}\right|_{\varphi_0, \pi_0} = \left.\frac{\delta H}{\delta \pi}\right|_{\varphi_0, \pi_0} = 0 \tag{3.31}$$

It follows from (3.30) that $\varphi_0^{[v]}$ is solution of

$$0 = \left.\frac{\delta H}{\delta \varphi}\right|_{\varphi_0^{[v]}, \pi_0^{[v]}} - v \left.\frac{\delta P}{\delta \varphi}\right|_{\varphi_0^{[v]}, \pi_0^{[v]}}$$

with a similar equation for $\pi_0^{[v]}$.

Hence we get a field configuration with $P = p$ as constraint. Moreover the determinants of small fluctuations around φ_0, π_0 and around $\varphi_0^{[v]}$, $\pi_0^{[v]}$ are the same. Hence if we have taken φ_0, π_0 into account in $\mathcal{Z}$, we can also take into account $\varphi_0^{[v]}$ and $\pi_0^{[v]}$ with minimum additional effort. This idea will be important for configurations of the meron type as discussed in next section since a meron configuration will be related by a same sort of boost to an instanton configuration.

In practice, the problem is there to find Poisson (or Dirac) brackets such that the time evolution has an optimal, and / or maximal set of conserved quantities and "boosts": optimal, in the sense that the minima of H with those conserved quantities as constraints are field configurations with physical or topological characteristics that are believed to be relevant to the structure of the quantum field theory ; maximal, because the more constraints one can impose, the larger the number of field configurations one can include explicitly with relatively little effort (see next section and following papers). Needless to say, we do not know how to solve in general this problem of the best time evolution ; we do not even know the properties of the solution if it exists. Nevertheless, in some particular cases, we know atleast a solution (see section 4), and in a more general and interesting case, like Yang-Mills in 4 dimensions, we can still get interesting results.

4 - EXAMPLES

In this section, we present some examples where one can apply the ideas of the previous sections. Actual calculations will be described in forthcoming papers. We shall consider one, two, and four dimensional cases.

Consider the Euclidian functional integral for the pendulum. Here, in the notation of section 2 .

$$H_0[\varphi] = \int dx \left(\frac{1}{2} \varphi'^2 + \frac{1}{2} \sin^2 \varphi \right) \tag{4.1}$$

As we shall see, this is an interesting example because one can introduce the extra time of section 3 in two different ways while keeping complete integrability and an infinite set of commuting constraints. On (4.1), the way of introducing time evolution that comes to mind first, is to replace $H_0[\varphi]$ by

$$H[\pi, \varphi] = \int dx \left[\frac{1}{2} \pi^2 + \frac{1}{2} \varphi'^2 + \frac{1}{2} \sin^2 \varphi \right] \tag{4.2}$$

where π is the canonical momentum conjugate to φ :

$$\{ \varphi(x), \pi(x') \} = \delta(x - x') \tag{4.3}$$

In (4.2), one recognizes the Hamiltonian of the sine-Gordon theory in laboratory coordinates. Hence, we have an infinite set of conserved quantities[15] , which can be used as commuting constraints The minima of $H[\pi, \varphi]$ with these constraints are multi-soliton solutions of the sine-Gordon equation, or a multi- ins-tanton configuration of the pendulum (4.1).

The conserved quantities of the sine-Gordon equation fall into two classes : some are even in π , like H , others are odd in π . By keeping only the set even in π , one gets configurations with $\pi \equiv 0$ at the minimum of H . One then suspects that there should be another way to introduce time evolution without introducing an independent π , of which these would be the conserved quantities. Indeed, consider the modified KdV equation[16] :

$$\dot{u} + u''' + \frac{3}{2} u^2 u' = 0 \tag{4.4}$$

Setting

$$u = \varphi' + \sin \varphi \tag{4.5}$$

and choosing an appropriate boundary condition, one gets the equation for φ :

$$\dot{\varphi} + \varphi''' + \frac{1}{2}\varphi'^3 + \frac{3}{2}\varphi' \sin^2\varphi = 0 \tag{4.6}$$

An infinite set of conserved quantities of (4.6) can be obtained via (4.5) from those of (4.4). One finds that they include all those of the sine-Gordon equation at $\pi = 0$, in particular $H_0[\varphi]$ as given in (4.1). Other conserved quantities are $\int \sin\varphi \, dx$ and $\int \cos\varphi \, dx$. Equation (4.6) is completely integrable, but does not seem to be known in the literature. Note that the functional change (4.5) can be inverted by solving a Riccati equation, much in analogy with the functional change of variables which reduces the modified KdV equation to the ordinary KdV equation[17]. The Poisson bracket, under which two conserved quantities H_i and H_j of (4.6) commute, is found to be

$$\{H_i, H_j\} \equiv \int \frac{\delta H_i}{\delta\varphi} \partial_x \frac{\delta H_j}{\delta\varphi} dx + \int dx \frac{\delta H_i}{\delta\varphi(x)} \varphi'(x) \int^x \varphi'(\xi) \frac{\delta H_j}{\delta\varphi(\xi)} d\xi \tag{4.7}$$

With these Poisson brackets, the Hamiltonian which reproduces (4.6) as the time evolution equation is

$$H[\varphi] = \int \left(\frac{1}{2}\varphi'^2 + \frac{1}{2}\sin^2\varphi + \cos\varphi \right) dx$$

Note that the φ^2 term cancels in this expression between $1/2 \sin^2\varphi$ and $\cos\varphi$. This means long range forces between the instantons of the corresponding quantum theory. It would be interesting to see how the dilute gas approximation is modified in this soluble example.

Let us now consider the anharmonic oscillator with two degenerate minima :

$$H_0[\varphi] = \int \left[\frac{1}{2}\varphi'^2 + \frac{1}{2}(\varphi^2 - 1)^2 \right] dx \tag{4.8}$$

This is a very well-known system, and the first testing ground of new ideas. (4.8) can be considered as the Hamiltonian of a modified KdV equation. The Poisson bracket is simply

$$\{F[\varphi], G[\varphi]\} = \int \frac{\delta F}{\delta\varphi} \partial_x \frac{\delta G}{\delta\varphi} dx \tag{4.9}$$

This Poisson bracket together with (4.8) as Hamiltonian leads to the modified KdV equation :

$$\dot{\varphi} + \varphi''' + 2\varphi' - 6\varphi^2\varphi' = 0$$

Under (4.9), there exists an infinite set of H_α's which commute with (4.8). The first ones are

$$H_1 = \frac{1}{2}\int (\varphi - 1)^2 dx \tag{4.10}$$

$$H_2 = \frac{1}{2}\int \left[\varphi''^2 + 10\varphi^2\varphi'^2 - 6\varphi'^2 + 2(\varphi^2-1)^3\right] dx \tag{4.11}$$

Note that H_1 lifts the degeneracy between the two minima, which will enable us to consider instanton-anti-instanton configurations. In the next paper, we shall consider in some detail the application of (4.8)-(4.11) to the large order behavior of perturbation theory in the anharmonic oscillator and its partial resummation.

It would be of some interest to apply present ideas to two-dimensional soluble field theories, like the non-linear sigma model or the sine-Gordon equation. For such theories, there exists a rather general famework for the introduction of an extra time coordinate while preserving complete integrability. See ref. 18. These two-dimensional field theories are solved by the inverse scattering method, which involves finding two operators L_1 and L_2 in the x variable, such that the operator equation

$$[L_1, L_2] - \partial_y L_1 = 0$$

is equivalent to the non-linear equations for the fields, which appear as potential terms in L_1 and L_2. Zakharov and Shabat[18] have found a general method of introducing an extra time coordinate while preserving integrability by considering instead th equation

$$[L_1, L_2] - \partial_t L_2 - \partial_y L_1 = 0 \tag{4.12}$$

L_1 and L_2 may have to be modified when the variable t is introduced. We show here how this works on the non-linear Schrödinger equation, and how one gets an infinite set of conservation laws in a natural and systematic fashion. Take

$$L_1 = \begin{pmatrix} i\partial_x & -iq \\ -iq^* & -i\partial_x \end{pmatrix} \tag{4.13}$$

$$L_2 = \begin{pmatrix} i\partial_x^2 + ib_3 - b_2 & -iq\partial_x - \frac{i}{2}q_x - \frac{1}{2}q_t \\ -iq^*\partial_x - \frac{i}{2}q_x^* - \frac{1}{2}q_t^* & -i\partial_x^2 - ib_3 + b_2 \end{pmatrix} \tag{4.14}$$

Then equation (4.12) is equivalent to the following set :

$$\begin{cases} -q_y + \frac{i}{2} q_{xx} - \frac{i}{2} q_{tt} + 2i\, \ell_3 q = 0 \\ (\partial_x + i\partial_t)(\ell_3 + i\ell_2) - \frac{1}{2}(\partial_x - i\partial_t)(q^* q) = 0 \end{cases} \tag{4.15}$$

When there is no t dependence, this reduces to $\ell_3 = \frac{1}{2} q^* q$ $\ell_2 = 0$ and to the non-linear Schrödinger equation in x and y. Equations (4.15) can be derived by trivial manipulations from the Lagrangian :

$$\mathcal{L}' = i q^* q_y - \frac{1}{2} q_x^* q_x + \frac{1}{2} q_t^* q_t + 2 \ell_3 q^* q - 2 \ell_{3t} u_t - 2 \ell_{3x} u_x + \frac{1}{2} u_{xx}^2 + \frac{1}{8} v^2 - \frac{1}{2} v_t u_t \tag{4.16}$$

One can then apply on (4.16) the canonical formalism. Introducing canonical momenta π_q , π_{ℓ_3} , π_v , π_u , one finds the Hamiltonian

$$H_0 = \int dx\, dy \left(-i q^* q_y + \frac{1}{2} q_x^* q_x - 2 \ell_3 q^* q + 2 \ell_{3x} u_x - \frac{1}{2} u_{xx}^2 - \frac{1}{8} v^2 + \frac{1}{2} \pi_q^* \pi_q - \pi_u \pi_{\ell_3} \right) \tag{4.17}$$

together with the constraint

$$\pi_v = 2 \pi_{\ell_3} \tag{4.18}$$

The minimum of the Hamiltonian (4.17) naturally satisfies the non-linear Schrödinger equation.

We now outline the derivation of higher conservation laws which could be used as constraints. Consider the set of equations

$$L_2 \psi = -\partial_y \psi \tag{4.19}$$

$$L_1 \psi = \partial_t \psi \tag{4.20}$$

The compatibility condition of (4.19) and (4.20) is (4.15) again. Now, we solve eq. (4.20) in the following form :

$$\psi = \exp(ikx - ik^2y - kt)\begin{pmatrix} 1 + \frac{\varphi_1}{k} + \frac{\varphi_2}{k^2} + \frac{\varphi_3}{k^3} + \cdots \\ \\ \frac{\chi_1}{k} + \frac{\chi_2}{k^2} + \frac{\chi_3}{k^3} + \cdots \end{pmatrix} \tag{4.21}$$

The functions φ_i and χ_i are obtained recursively by quadratures in x and t from eq. (4.20). Imposing that (4.21) also satisfies (4.19) then leads to the infinite set of conservation laws :

$$\frac{d}{dt}\int dx\,dy\left[(\partial_x - i\partial_t)u\,\partial_x\varphi_i - \frac{i}{2}q\,\partial_t\chi_i + \frac{i}{2}q_t\chi_i\right] = 0 \tag{4.22}$$

We note that half of these conservation laws are odd in the canonical momenta, and half are even, so that one may keep only the even set, and take all momenta to be zero, a situation very much analogous to that of the pendulum.

Let us now come briefly to four-dimensional Euclidian $SU(2)$ Yang-Mills fields. We will consider one constraint only and the corresponding collective coordinates of section 2 , but already at this level, we will see that we are probing configurations in phase space where contributions to the functional integral are

very important on physical grounds. In particular, one will be able to discuss the meron contribution. Here, we will outline the method, deferring the details to a subsequent publication. For simplicity, we will not consider scalar or spinor multiplets.

The Yang-Mills action is

$$\mathcal{A}\left[A^a_\mu(x)\right] = -\frac{1}{4}\int d^4x\; G^a_{\mu\nu}G^a_{\mu\nu} \tag{4.23}$$

where

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \varepsilon_{abc}A^b_\mu A^c_\nu \tag{4.24}$$

The instanton configuration is

$$A^{a,0}_\mu(x) = 2\,\eta^a_{\mu\nu}\frac{x_\nu}{x^2+1} \tag{4.25}$$

One can expand the action around configuration (4.25) and in this way obtain the contribution of the instanton.[19] The zero modes of translations and dilatation are handled in the usual way.

Instead, our purpose is to compute the contribution of a larger family of configurations which includes deformations of the instanton. Let us consider the following transformation

$$T(\gamma)A^a_\mu(x) = \left[g_{\mu\nu} + (\gamma-1)\frac{x_\mu x_\nu}{x^2}\right]|x|^{\gamma-1}A_\nu\left(x|x|^{\gamma-1}\right) \tag{4.26}$$

from which it follows that

$$T(\gamma)G^a_{\mu\nu}(x)=\left(g_{\mu\lambda}+(\gamma-1)\frac{x_\mu x_\lambda}{x^2}\right)\left(g_{\nu\rho}+(\gamma-1)\frac{x_\nu x_\rho}{x^2}\right)|x|^{2(\gamma-1)}G_{\lambda\rho}\left(x|x|^{\gamma-1}\right) \tag{4.27}$$

Under this transformation, the instanton (4.25) is transformed into the configuration

$$A^{a,\gamma}_\mu(x)=2\eta_{a\mu\nu}\frac{x_\nu x^{2(\gamma-1)}}{1+x^{2\gamma}}\ , \quad 0\leq\gamma\leq 1 \tag{4.28}$$

which has the property of interpolating between the instanton $(\gamma=1)$ and the meron $(\gamma=0)$.

The transformation $T(\gamma)$ in (4.26)-(4.27) satisfies

$$T(\gamma)T(\gamma')=T(\gamma\gamma') \tag{4.29}$$

and it can be proved that

$$\gamma^{-1}\mathcal{a}\left[T^{-1}(\gamma)A_\mu\right]=\mathcal{a}\left[A_\mu\right]-\frac{1}{2}(\gamma^{-2}-1)\int x^{-2}d^4x\, G^a_{\mu\lambda}x_\lambda G^a_{\mu\rho}x_\rho \tag{4.30}$$

The constraint which will reproduce (4.28) is thus the second term in the right-hand side of (4.30). It is gauge independent. As explained in section 3, the existence of (4.30) greatly simplifies the calculation of the fluctuations around the deformed instanton.

We have not found Poisson brackets which would impleme in a simple fashion the boost-like operator $T(\gamma)$. For this reason, it might be more practical to consider the canonical formalism based on the Lagrangian

$$\mathcal{L}=\frac{1}{2}\varepsilon_{abc}G^a_{\mu\nu}\partial_t A^b_\mu\partial_t A^c_\nu-\frac{1}{4}G^{a\,2}_{\mu\nu} \tag{4.31}$$

where t is the extra time dimension of section 3. This Lagrangian (4.31) is invariant under the $O(5,1)$ conformal group, whose generators are, when acting on scalar fields

$$\left\{\begin{aligned} L_{\mu\nu}&=x_\mu\partial_\nu-x_\nu\partial_\mu\\ P_\mu&=L_{6\mu}+L_{5\mu}=\partial_\mu\\ K_\mu&=L_{6\mu}-L_{5\mu}=2d\,x_\mu+2x_\mu x_\nu\partial_\nu-x^2\partial_\mu\\ D&=L_{65}=d+x_\nu\partial_\nu\end{aligned}\right. \tag{4.32}$$

where d is the dimension of the field (in the sense of naive dimensional analysis).

From (4.31), one obtains the canonical Hamiltonian

$$\mathcal{H} = \frac{1}{2} \Pi_\mu^a \, G^{-1ab}_{\mu\nu} \, \Pi_\nu^b + \frac{1}{4} G^{ab}_{\mu\nu} \tag{4.33}$$

(for the definition of $G^{-1ab}_{\mu\nu}$, see for example ref. 20), Π_μ^a being the canonical momentum conjugate to A_μ^a :

$$\left\{ A_\mu^a(x), \Pi_\nu^b(x') \right\} = \delta_{ab} \delta_{\mu\nu} \delta^4(x-x') \tag{4.34}$$

One then takes the Yang-Mills theory as defined by the functional integral

$$\mathcal{Z} = \int \mathcal{D} A_\mu^a \, \mathcal{D} \Pi_\mu^a \left(\det G^{ab}_{\mu\nu} \right)^{-1/2} e^{-\frac{1}{g^2} \int d^4x \, \mathcal{H}} \tag{4.35}$$

This functional integral is conformally invariant (up, of course, to the regulators and counterterms), and, by integration over Π_μ^a, trivially equal to the usual one. A meron-type configuration can be obtained by using as a constraint the gauge invariant dilatation operator

$$D = \int d^4x \, \Pi_\mu^a \, x_\nu \, G^a_{\mu\nu} \tag{4.36}$$

together with the formalism of section 2 and the appendix.

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APPENDIX

Our starting point is equation (2.24) where D is the Jacobian

$$D[\varphi;X,Y] \equiv \det \frac{\partial\left(H_\alpha[\varphi_{[X,Y]}]-a_\alpha,\ F_\beta[\varphi_{[X,Y]}],\ G_\gamma[\varphi_{[X,Y]}]\right)}{\partial(a_{\alpha'},\ X_{\beta'},\ Y_{\gamma'})} \tag{A.1}$$

Equation (2.20) can be viewed as an implicit definition of a_α , X_β , Y_γ ,as functionals of φ . It is easy to see that the form of (2.24) is such that a is invariant by the group $\mathcal{G}$ if X and Y are taken to transform under $\mathcal{G}$ as parameters of $\mathcal{G}$ itself. The general idea is that the constraints $H_\alpha = a_\alpha$ correspond to constraining forces which violate the symmetry. We reestablish the invariance under $\mathcal{G}$ by attaching them to a "moving frame" with parameters X , Y which we change suitably when a symmetry transformation is performed on the system.

In the same way as in simpler cases, (2.24) is inserted into the original expression of $\mathcal{Z}$ and one performs the change of variables $\varphi \rightarrow \varphi_{[-X,-Y]}$ obtaining

$$\mathcal{Z} = \int da\,dX\,dY\,\mathcal{D}\varphi\ \delta(a_\alpha - H_\alpha[\varphi])\,\delta(F_\beta[\varphi])\,\delta(G_\gamma[\varphi]) \times D[\varphi_{[-X,-Y]};X,Y]\,e^{-\frac{1}{g^2}H_0[\varphi]} \tag{A.2}$$

It is convenient to define G_γ as

$$G_\gamma[\varphi] = \sum_\alpha \lambda_\alpha \frac{\partial H_\alpha}{\partial Y^\gamma} \tag{A.3}$$

In this formula and hereafter we define the derivatives of functionals of φ with respect to parameters of $\mathcal{G}$ as

$$\frac{\partial}{\partial z^\gamma}\mathcal{F}[\varphi] \equiv \frac{\partial}{\partial z^\gamma}\mathcal{F}[\varphi_{[z]}]\Big|_{z=0} \tag{A.4}$$

and we denote by z the parameters of $\mathcal{G}$ $[z]=[X,Y]$ The choice (A.3) is convenient because if φ_{cl} satisfies

$$\frac{\delta H_0}{\delta \varphi_{cl}} + \sum_\alpha \lambda_\alpha \frac{\delta H_\alpha}{\delta \varphi_{cl}} = 0 \tag{A.5}$$

one immediately obtains

$$G_\gamma[\varphi_{cl}] = \int dx \sum_\alpha \lambda_\alpha \frac{\delta H_\alpha}{\delta \varphi_{cl}} \frac{\partial \varphi_{cl}}{\partial Y^\gamma}\Big|_{Y=0}$$
$$= -\int dx \frac{\delta H_0}{\delta \dot{\varphi}_{cl}} \frac{\partial \varphi_{cl}}{\partial Y^\gamma}\Big|_{Y=0} = 0 \tag{A.6}$$

This last expression vanishes because H_0 is invariant. With the choice (A.3), the Lagrange multipliers of G constraints vanish.

Next we discuss the computation of the Jacobian. First applying the same method as in section 3 to derive formula (3.19) one can show that, for any functional of φ

$$\frac{\partial}{\partial z^\alpha} \mathcal{F}[\varphi_{[z]}] = U_\alpha^\beta(z)\, \tilde{\mathcal{F}}_\beta[\varphi_{[z]}] \tag{A.7}$$

where $\tilde{\mathcal{F}}_\beta$ does not depend explicitly on z and where U_α^β is defined as in (3.19) and (3.20). From this we obtain

$$D[\varphi_{[-x;y]}; X, Y] = \det[U(x,y)] \det\left[\frac{\partial \lambda_\sigma}{\partial a_{\sigma'}}\right] J[\varphi] \tag{A.8}$$

$$J[\varphi] \equiv \det \frac{\partial(H_\alpha[\varphi]-a_\alpha, F_\beta[\varphi], G_\gamma[\varphi])}{\partial(\lambda_{\alpha'}, X_{\beta'}, Y_{\gamma'})} \tag{A.9}$$

For later convenience we have also factorized $\det\left[\frac{\partial \lambda}{\partial a}\right]$

Next we show that the expression

$$\mathcal{M}(z)\,dz \equiv \det U_\alpha^\beta \, dz \tag{A.10}$$

is the measure invariant under $\mathcal{G}$. For this purpose, we change variables in (A.7) by letting

$$z = f(z', z_0)$$

where z_0 is an arbitrary fixed element of $\mathcal{G}$ and z' are the new variables. f is the function which specifies the Lie group composition law as introduced in section 3. Hence we have

$$\varphi_{[z]} = \tilde{\varphi}_{[z']} \quad , \quad \tilde{\varphi} = \varphi_{[z_0]}$$

and (A.7) becomes

$$\frac{\partial z'^\gamma}{\partial z^\alpha} \frac{\partial}{\partial z'^\gamma} \mathcal{F}[\tilde{\varphi}_{[z']}] = U_\alpha^\beta(f(z', z_0))\, \tilde{\mathcal{F}}_\beta[\tilde{\varphi}_{[z']}] \tag{A.11}$$

This formula must coincide with (A.7) if φ, Z are replaced by $\tilde{\varphi}, Z'$. Hence we have

$$\frac{\partial f^{\alpha}(Z', Z_0)}{\partial Z'^{\gamma}} U^{\beta}_{\alpha}\left(f(Z', Z_0)\right) = U^{\beta}_{\gamma}(Z') \tag{A.12}$$

and finally

$$\mathcal{M}(Z)\,dZ = \mathcal{M}(Z')\,dZ' \tag{A.13}$$

which is the property we wanted to verify.

We now turn to the study of J . First, the invariance of H_α under the subgroup $\tilde{\mathcal{G}}$ leads to

$$\frac{\partial H_\alpha}{\partial x^\beta} = \frac{\partial G_\gamma}{\partial x^\beta} = 0 \tag{A.14}$$

$$J = J_1 J_2$$

$$J_1 = \det \frac{\partial(H_\alpha - a_\alpha, G_\gamma)}{\partial(\lambda_{\alpha'}, Y_{\gamma'})} \tag{A.15}$$

$$J_2 = \det \frac{\partial F_\beta}{\partial x_{\beta'}} \tag{A.16}$$

As usual, to the order we are working, we only need the Jacobian for $\varphi = \varphi_{cl}$. Since, by construction, φ_{cl} satisfies the constraints, we can rewrite

$$J_1[\varphi_{cl}] = \det \frac{\partial}{\partial(\lambda_{\alpha'}, Y_{\gamma'})}\left(\left(\int dx \frac{\delta H_\alpha}{\delta\varphi_{cl}}(\varphi - \varphi_{cl}), \int dx \frac{\delta G_\gamma}{\delta \varphi_{cl}}(\varphi - \varphi_{cl})\right)\right|_{\varphi = \varphi_{cl}} \tag{A.17}$$

Moreover, in this expression, we can ignore the dependence of $\delta H_\alpha/\delta\varphi_{cl}$ and $\delta G_\gamma/\delta\varphi_{cl}$ on $\lambda_{\alpha'}$ and $Y_{\gamma'}$. The resulting formula is put under a convenient form by means of the following two identities.

First, in the same way as in section 2, taking the derivative of (A.5) with respect to λ_α , we obtain

$$\int dy\, \Omega(x,y) \frac{\partial \varphi_{cl}}{\partial \lambda_\alpha} = -\frac{\delta H_\alpha}{\delta \varphi_{cl}} \tag{A.18}$$

Second, we perform in (A.5) the change of variable $\varphi \to \varphi_{[Z_0]}$ with arbitrary fixed Z_0

$$\frac{\delta H_0[\varphi_{[z_0]}]}{\delta \varphi_{[z_0]}} + \sum_\alpha \lambda_\alpha \frac{\delta H_\alpha[\varphi_{[z_0]}]}{\delta \varphi_{[z_0]}} \Bigg|_{\varphi = \varphi_{cl}[-z_0]} = 0$$

From the chain rule

$$\frac{\delta}{\delta \varphi_{[z_0]}(x)} = \int dy \, \frac{\delta \varphi(y)}{\delta \varphi_{[z_0]}(x)} \frac{\delta}{\delta \varphi(y)}$$

we obtain

$$\frac{\delta H_0[\varphi_{[z_0]}]}{\delta \varphi} + \sum_\alpha \lambda_\alpha \frac{\delta H_\alpha[\varphi_{[z_0]}]}{\delta \varphi} \Bigg|_{\varphi_{cl}[-z_0]} = 0$$

and by differentiation with respect to z_0 at $z_0 = 0$ this leads to

$$\int dy \, \Omega(x,y) \frac{\partial \varphi_{cl}(y)}{\partial Y^\alpha} = - \frac{\delta G_\alpha}{\delta \varphi_{cl}(x)} \tag{A.19}$$

which is the second relation we shall need.

Using (A.19) and (A.17) it is easy to see that $J_1(\varphi_{cl})$ is equal to the determinant of Ω in the subspace generated by $\frac{\partial \varphi_{cl}}{\partial \lambda_\alpha}$ and $\frac{\partial \varphi_{cl}}{\partial Y}$ that is

$$J_1[\varphi_{cl}] = \begin{vmatrix} \Omega_{\alpha\alpha'} & \Omega_{\alpha\gamma'} \\ \Omega_{\gamma\alpha'} & \Omega_{\gamma\gamma'} \end{vmatrix} \tag{A.20}$$

$$\left\{ \begin{aligned} \Omega_{\alpha\alpha'} &= \int dx\,dy \, \frac{\partial \varphi_{cl}}{\partial \lambda_\alpha} \, \Omega \, \frac{\partial \varphi_{cl}}{\partial \lambda_{\alpha'}} \\ \Omega_{\alpha\gamma'} &= \int dx\,dy \, \frac{\partial \varphi_{cl}}{\partial \lambda_\alpha} \, \Omega \, \frac{\partial \varphi_{cl}}{\partial Y_{\gamma'}} \\ \Omega_{\gamma\alpha'} &= \int dx\,dy \, \frac{\partial \varphi_{cl}}{\partial Y_\gamma} \, \Omega \, \frac{\partial \varphi_{cl}}{\partial \lambda_{\alpha'}} \\ \Omega_{\gamma\gamma'} &= \int dx\,dy \, \frac{\partial \varphi_{cl}}{\partial Y_\gamma} \, \Omega \, \frac{\partial \varphi_{cl}}{\partial Y_{\gamma'}} \end{aligned} \right. \tag{A.21}$$

As before we now perform in (A.2) the shift $\varphi = \varphi_{cl} + \chi$, obtaining

$$\mathcal{Z} = \int da\, dX\, dY\, \mathcal{M}(X,Y)\, \mathcal{D}\chi\, \delta\left(\int dx \frac{\delta H_\alpha}{\delta \varphi_{cl}} \chi\right) \delta\left(\int dx \frac{\delta F_\beta}{\delta \varphi_{cl}} \chi\right)$$

$$\times\, \delta\left(\int dx \frac{\delta G_\gamma}{\delta \varphi_{cl}} \chi\right) J_1(\varphi_{cl})\, J_2(\varphi_{cl})$$

$$\times \exp\left[-V(a)/g^2 - \tfrac{1}{2}\int dx\, dy\, \chi\, \Omega\, \chi\right]$$

We exponentiate the first and third setsof constraints

$$\delta\left(\int dx \frac{\delta H_\alpha}{\delta \varphi_{cl}} \chi\right) = (2\pi)^{-1} \int d\mu_\alpha \exp\left[i\mu_\alpha \int dx \frac{\delta H_\alpha}{\delta \varphi_{cl}} \chi\right]$$

$$\delta\left(\int dx \frac{\delta G_\gamma}{\delta \varphi_{cl}} \chi\right) = (2\pi)^{-1} \int d\nu_\gamma \exp\left[i\nu_\gamma \int dx \frac{\delta G_\gamma}{\delta \varphi_{cl}} \chi\right]$$

The additional linear term is eliminated by making the shift

$$\chi = \tilde{\chi} - i\sum_\alpha \mu_\alpha \frac{\delta H_\alpha}{\delta \varphi_{cl}} - i\sum_\beta \rho_\beta \frac{\partial \varphi_{cl}}{\partial X^\beta} - i\sum_\gamma \nu_\gamma \frac{\delta G_\gamma}{\delta \varphi_{cl}}$$

where in the same way as in section 3 the coefficients ρ_β are such that the shift is compatible with the set of F_β constraints which eliminate the zero modes. The gaussian integral over μ and ν is easily performed ; one gets simply $(2\pi)^{-N/2-R/2} \left[J(\varphi_{cl})\right]^{-1/2}$

Integrating over χ one finally obtains

$$\mathcal{Z} = \int da\, dX\, dY\, \mathcal{M}(X,Y)\, e^{-V(a)/g^2} (2\pi)^{-N/2-R/2}$$

$$\times \det\left[\frac{\partial \lambda}{\partial a}\right] \left[J(\varphi_{cl})\right]^{1/2} \left[\det{}'\Omega\right]^{-1/2} \tag{A.22}$$

where $\det{}'\Omega$ is the determinant in the space orthogonal to the zero modes and $J(\varphi_{cl})$ which given by (A.17) is the determin of Ω in the subspace generated by $\partial\varphi_{cl}/\partial Y^\alpha$ and $\partial\varphi_{cl}/\partial$

Finally we comment on the case where the group $\mathcal{G}$ is not completely broken by φ_{cl} .

Let us call $\mathcal{G}'$ the subgroup of $\mathcal{G}$ under which φ_{cl} is invariant. The number of F and G constraints is now smaller than the numbers of parameters of $\mathcal{G}$. In general one will choose the set of all constraints to be invariant by $\mathcal{G}'$ and the left-hand side of (2.24) is equal to the volume of $\mathcal{G}'$ instead of 1. The only change in (A.19) is therefore that one must divide by this volume. This means that one is really integrating over the quotient space $\mathcal{G}/\mathcal{G}'$.

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