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ON THE VACUUM STRUCTURE OF AN SU(2) YANG–MILLS THEORY

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By using a compactification of the spatial part $\boldsymbol{R}^3$ of Minkowski-space different from the one-point compactification to S^3, we get a new classification of the vacua for an SU(2) gauge theory. It contains, besides the vacua arising in the S^3 compactification, the Gribov vacua as new classes. We discuss the role of pseudoparticle solutions within this framework and comment on the problem of the Coulomb gauge degeneracy.

I. Introduction

In several papers different aspects of the topological structure of the possible vacua in an SU(2) Yang–Mills theory have been discussed [1]–[6]. It turns out that this structure is most easily seen in the $A_0 = 0$ gauge. The fields $\vec{A}(\vec{x})$ which give rise to zero energy configurations are the pure gauge fields $\vec{A}(\vec{x}) = g^{-1}(\vec{x})\vec{\nabla}g(\vec{x})$, with g: $\boldsymbol{R}^3 \to \mathrm{SU}(2)$ standing for any mapping of the spatial part $\boldsymbol{R}^3$ of Minkowski space into the gauge group SU(2). If one allows only continuous mappings g with the property $\lim\limits_{\vec{x}\to\infty} g(\vec{x}) = \text{const}$ then one can classify these mappings via $\pi_3(S^3)$, the third homotopy group of the 3-sphere S^3: every mapping of the above kind induces a mapping $\bar{g}$: $S^3 \to S^3$. Since $\pi_3(S^3) \cong Z$, one gets a sequence $\vec{A}_n$, $n = 0, \pm 1, \ldots$, of vacuum fields. It was shown in [2], that under this restriction, only the field $\vec{A}(x) \equiv 0$ fulfils the Coulomb gauge condition $\vec{\nabla}\cdot\vec{A}(\vec{x}) = 0$.

In an interesting paper [5] Gribov found two additional solutions in the Coulomb gauge. These solutions correspond to nontrivial mappings $g_\pm$: $\boldsymbol{R}^3 \to \mathrm{SU}(2)$, and in fact do not belong to the class of mappings giving rise to mappings from S^3 into S^3. In the literature these vacua are called half-integer valued vacua, because some quantity q, for which the field $\vec{A}_n$ is always an integer, turns out to be $\pm\frac{1}{2}$. The integer valued vacua can be understood from purely topological considerations. So one can ask the question:

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Is there also a topological explanation for the half-integer ones? Or, in other words, does there exist a classification such that the half-integer vacua also appear as homotopy classes?

An affirmative answer to this question would clarify to a large extent the confusion that exists in the literature on this subject (see for instance, [2], [4]). The main reason for the confusion in our opinion is the fact that one has not as yet really analysed the topological structure of the spaces one is working in. Since Gribov's solutions are not mappings from S^3 into S^3, one must be careful in drawing conclusions using topological invariants arising from $\pi_3(S^3)$. It is necessary to specify the spaces on which the mappings are defined: If one takes, for example, the unit 3-ball $\overline{B}_3$ as a compactification of $\boldsymbol{R}^3$, the topological structure is trivial. Every mapping of this space into any other space is always homotopic to the trivial mapping.

From these remarks it is already clear that the problem of classification of the vacua, that is of mappings of $\boldsymbol{R}^3$ into SU(2), is closely related to the manner of compactification of the space $\boldsymbol{R}^3$. By using different compactifications one is led, in general, to completely different classifications.

If one employs, as one did up to now in the literature, the so-called Alexandroff compactification of $\boldsymbol{R}^3$, that is to say the one-point compactification as S^3, one gets the classification via $\pi_3(S^3)$. This compactification can be described also in terms of the extension problem for certain continuous mappings $g\colon \boldsymbol{R}^3 \to S^3$. The one-point compactification allows us to extend continuous mappings $g\colon \boldsymbol{R}^3 \to S^3$ such that $\lim_{\vec{x}\to\infty} g(\vec{x}) = e \in \mathrm{SU}(2)$. In fact, these are the only mappings which can be extended this way.

This suggests that we describe a compactification $\varkappa$ of $\boldsymbol{R}^3$ by giving the set of all mappings $g\colon \boldsymbol{R}^3 \to S^3$ which are to be extended to mappings $\bar{g}\colon \varkappa \to S^3$. The set of all such $\bar{g}$ should be the set of all continuous mappings from $\varkappa$ into S^3.

One knows [7] that there exists for every reasonable topological space X a compactification βX, called the Stone–Čech compactification, such that every mapping $g\colon X \to K$ where K is a compact space, can be uniquely extended to a mapping $\bar{g}\colon \beta X \to K$.

The space $\beta\boldsymbol{R}^3$ is a very large space, in the sense that it has a large number of points at infinity, which is just $\beta\boldsymbol{R}^3/\boldsymbol{R}^3$. One sees that the one-point compactification is just on the other end of the hierarchy of compactifications, where infinity consists exactly of one point.

We do not known if the space $\beta\boldsymbol{R}^3$ is the one nature has chosen to live in. We propose in this paper another compactification, which has its place somewhere between $\beta\boldsymbol{R}^3$ and S^3. We saw that the choice of a set of mappings, which we want to be extendable in the way explained above is equivalent, in some sense, to embedding the space $\boldsymbol{R}^3$ into a compact space $\varkappa$. We will therefore take a set of mappings and classify these mappings into homotopy classes. This gives then, as a by-product, some useful information about the space $\varkappa$ [7].

To obtain a classification of vacua for the SU(2) Yang–Mills Theory in which the

Gribov vacua can be explained from a topological point of view, we have to choose a set τ of mappings g: $R^3 \to S^3$ such that the Gribov mappings belong to this set. This is the strategy we want to follow in this paper. By choosing the set τ appropriately, we are able to give a complete classification for all the vacua which have been considered in the literature. We show that all these vacua can be classified according to a topological invariant whose value is any positive or negative half-integer. The Gribov vacua turn out to be the generators for all these classes and seem therefore to be the fundamental vacua. This we discuss in the first section of this paper.

Then we discuss the possibility of pseudo-particle solutions in this framework. It turns out that the only pseudo-particle solutions that are possible, interpolate between integer-valued vacua or half-integer ones. Others are not allowed in that framework.

In a final section we comment on the problem of degeneracy of the Coulomb-gauge condition. We show why statements concerning this problem appearing in the literature, as in [2], are not complete. Hence it is pointed out that it is not possible to draw any conclusion about the absence of any other solutions with higher topological numbers in the Coulomb gauge.

In the Appendix we show how our classification for the vacua in the SU(2) theory is connected with an analogous classification in an SO(3) gauge theory.

II. A new "compactification" of the space R^3

The problem we are considering here is to classify the vacua of a SU(2) Yang–Mills theory governed by the Lagrangian

$$L = -\tfrac{1}{4}F_a^{\mu\nu}F_{a\mu\nu} \tag{1}$$

where $F_a^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A_a^\mu + e\,\varepsilon_{abc}A_b^\mu A_c^\nu$.

Introducing the matrices

$$A^\mu = eA_a{}^\mu \frac{\sigma^a}{2i}, \qquad F^{\mu\nu} = eF_a^{\ \mu\nu}\frac{\sigma^a}{2i} \tag{2}$$

where σ^a are the Pauli-matrices, the classical equations of motions for the Lagrangian (1) are

$$\partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0. \tag{3}$$

The vacuum states of the theory are characterized by a vanishing $F^{\mu\nu}$. Furthermore, we work in the gauge $A^0 = 0$. The vector potential $\vec{A}(\vec{x})$ is thus a pure gauge field

$$\vec{A}(\vec{x}) = g^{-1}(\vec{x})\vec{\nabla}g(\vec{x}). \tag{4}$$

In what follows, we work at a fixed time x_0 so that the symbol $g(x)$ in (4) can be looked upon as a mapping g: $R^3 \to$ SU(2).

In order to obtain a canonical classification for these mappings we must impose regularity conditions on g. So the least thing to demand is that they be continuous. As discussed in [1] a physically relevant classification is given by the homotopy classes of the mappings

g: $\boldsymbol{R}^3 \to S^3$. We also assume that the mappings g: $\boldsymbol{R}^3 \to S^3$ are such that the $\lim_{\|\vec{x}\| \to \infty} g(\vec{x})$ exists and is a function of $\hat{x} = \vec{x}/r$ only, where $r = \|\vec{x}\|$.

Let $\tau = \{g\colon \boldsymbol{R}^3 \to S^3, g \text{ continuous and } \lim_{\|\vec{x}\| \to \infty} g(\vec{x}) = f(\hat{x}) \text{ exists}\}$. Then we see that the compactification $\varkappa$ of $\boldsymbol{R}^3$ which allows precisely these mappings as continuous mappings from $\varkappa$ into S^3 is given by the unit ball $\bar{B}_3$ of $\boldsymbol{R}^3$, where $\bar{B}_3$ is defined as

$$\bar{B}_3 = \{\vec{x} \in R^3\colon \|\vec{x}\| \leqslant 1\}. \tag{5}$$

The space $\boldsymbol{R}^3$ is mapped into $\bar{B}_3$ via the mapping φ defined as

$$\varphi(\vec{x}) = \frac{\vec{x}}{r+1} \tag{6}$$

and $\varphi(\boldsymbol{R}^3) =$ interior of $\bar{B}_3$.

The points at infinity form a topological 2-sphere. We denote the mapping of $\bar{B}_3 \to S^3$ induced by an element g of τ by $\bar{g}$. Then the classification of these mappings into homotopy classes is trivial because the space $\bar{B}_3$ is contractible and therefore, any mapping can be continuously deformed into the constant mapping. The conclusion would then be that from the topological point of view there exists only one vacuum: the vacuum structure is unique.

To get nontrivial topological structures one has to look for different compactifications. The standard procedure is to identify on the space $\bar{B}_3$ certain points, or, mathematically speaking, introduce equivalence relations and form quotient spaces. The one which has been used so far is given by identifying the boundary $\partial\bar{B}_3$ of $\bar{B}_3$ to one point. Then the quotient space is just S^3. In term of the mapping $g \in \tau$, one considers only g's such that $f(\hat{x}) = \lim_{\vec{x} \to \infty} g(\vec{x}) = e \in S^3$. These mappings form a subset τ_1 of τ. Consider next the subset τ_0 of τ defined by

$$\tau_0 = \{g \in \tau\colon f(\hat{x}) = \pm f(-\hat{x})\}. \tag{7}$$

If we denote by $\bar{\tau}$ the mappings $\bar{g}$: $\bar{B}_3 \to S^3$ induced by the $g \in \tau$, we can describe the set $\bar{\tau}_0$ as:

$$\bar{\tau}_0 = \{\bar{g} \in \bar{\tau}\colon \bar{g}(\vec{x}) = \pm\bar{g}(-\vec{x}) \text{ for } \vec{x} \in \partial\bar{B}_3\}. \tag{8}$$

Let us now try to classify these mappings into homotopy classes. It is immediately clear that the set $\bar{\tau}_0$ falls at least into two classes, $\bar{\tau}_+$, $\bar{\tau}_-$ which are given by

$$\bar{\tau}_+ = \{\bar{g} \in \bar{\tau}_0\colon \bar{g}(\vec{x}) = \bar{g}(-\vec{x}), \vec{x} \in \partial\bar{B}_3\}, \tag{9}$$

$$\bar{\tau}_- = \{\bar{g} \in \bar{\tau}_0\colon \bar{g}(\vec{x}) = -\bar{g}(-\vec{x}), \vec{x} \in \partial\bar{B}_3\}. \tag{10}$$

It is also clear that the mappings in the set τ_1 induce mappings $\bar{g}$: $S^3 \to S^3$ belonging to the class $\bar{\tau}_+$.

II.1. Classification of $\bar{\tau}_+$

LEMMA 1. *The elements of* $\bar{\tau}_+$ *are in one to one correspondence with the continuous mappings of* 3*-dimensional real projective space* P^3 *into* S^3.

This is almost trivial, because 3-dimensional real projective space P^3 can be visualized as the unit-ball $\bar{B}_3$ with antipodal points on the boundary S^2 identified. For a proof see for instance [8].

Then we are left with the homotopy classification of all continuous maps of P^3 into S^3. By definition, this is just the *cohomotopy set* $\pi^3(P^3)$, which is a group. (For definitions see [9].) The Hopf Theorem [10] on the other hand tells us that there is a one-to-one correspondence between $\pi^3(P^3)$ and $H^3(P^3)$, where H^3 denotes the *third cohomology group of* P^3 with integer coefficients. The cohomology groups of all projective spaces are well known and we have $H^3(P^3) = Z$.

This solves the classification of the mappings in $\bar{\tau}_+$. Geometrically, these mappings are again classified according to the way they cover S^3. To find out to which class a given mapping $\bar{g} \in \bar{\tau}_+$ belongs, one has first to deform the mapping continuously so that at every point on the sphere the orientation of the image of P^3 is the same. (Remember P^3 is an orientable space.) Then look how often the image covers the 3-sphere and take the largest integer n smaller than this number. Depending on the orientation of $\mathrm{Im}(P^3)$ relative to the orientation of S^3 the mapping belongs to the homotopy class $\pm n$. From this it is clear that if one takes for P^3 and S^3 the proper orientation, then every mapping $\bar{g} \in \bar{\tau}_1 \subset \bar{\tau}_+$ which belongs to the homotopy class n in $\bar{\tau}_1$ belongs to the homotopy class n in $\bar{\tau}_+$ as well. If $\pi(X, Y)$ denotes the homotopy classes of continuous mappings from the space X into the space Y, then we may state that $\pi(S^3, S^3)$ is in one-to-one correspondence with $\pi(P^3, S^3)$.

II.2. Classification of $\bar{\tau}_-$

It is clear from their definition [5] that the Gribov mappings

$$g_\pm(\vec{x}) = \exp \pm i\left(\alpha(r)\frac{\vec{x}\cdot\vec{\sigma}}{r}\right) \tag{11}$$

where $\alpha(r)$ is a solution of the differential equation

$$\nabla^2\alpha - \frac{1}{r^2}\sin 2\alpha = 0 \tag{12}$$

subject to the boundary condition $\alpha(0) = 0$, and $\alpha(\infty) = \pi/2$, induce mappings $\bar{g}_\pm$ of $\bar{B}_3$ which belong to the set $\bar{\tau}_-$.

In the literature [2], [4], these mappings are said to have half-integer "topological" charge q defined by

$$q = -\frac{1}{2\pi^2}\int_{g_\pm(R^3)} d\Omega \tag{13}$$

where $g_\pm(R^3)$ is the image of R^3 in S^3; q turns out to be $\pm\frac{1}{2}$.

It is clear that this number q is in general not topologically invariant. (See the discussion above). For mappings $g \in \tau$ this is evident, because such a g is homotopic to zero. However, we show below how one can assign a topological meaning to q for the mappings τ_0. To get a feeling of what is going on, consider first the one-dimensional case: the analogous problem is to classify all mappings g of the unit one-ball $[-1, +1]$ into S^1, such that $g(+1) = -g(-1)$. It is clear that these mappings fall into homotopy classes. Given a map g, one first deforms it again in a continuous way into a mapping $\tilde{g}$ such that the orientation of the image $\tilde{g}(\bar{B}_1)$ is at every point the same. One sees that this image covers the unit circle S_1 with half-integral multiplicity. According to the orientation of $\tilde{g}(\bar{B}_1)$ relative to that of S_1 this half-integer (plus or minus) characterizes the homotopy classes. It now follows directly that exactly the same thing happens in higher dimensions. We see that the number q for the three-dimensional case

$$q = -\frac{1}{2\pi^2} \int_{\tilde{g}(\bar{B}_3)} d\Omega \tag{14}$$

is a topological invariant for the mappings in $\bar{\tau}_-$ and takes values in the set

$$\left\{\pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \ldots\right\}. \tag{15}$$

Any element $\bar{g}_{1/2}$ in the class $q = \frac{1}{2}$ is a generator for all other classes in $\bar{\tau}_-$. These are generated by taking odd powers of $\bar{g}_{1/2}$. Even powers of $\bar{g}_{1/2}$ generate the class $\bar{\tau}_+$. In this sense we can say that the Gribov vacuum which belongs to the class $q = +\frac{1}{2}$ is the fundamental vacuum, because it generates all other vacua.

An interesting mathematical question which one naturally asks at this stage is this: Does there exist a compact space $\varkappa$ such that the mappings $\bar{g} \in \bar{\tau}_0$ are the continuous mappings of $\varkappa$ into S^3? From the Hopf Theorem it would then follow that the third cohomology group $H^3(\varkappa)$ is just given by the group $G = \frac{1}{2}\boldsymbol{Z}$. Unfortunately, we could not find the answer to this question. But we show in the Appendix that the set $\bar{\tau}_0$ can be related to the set of continuous mappings of $\boldsymbol{P}^3$ into SO(3) $\simeq \boldsymbol{P}^3$.

III. Pseudo-particle solutions

For a general discussion of the role these field configurations can play in a quantized version of a gauge theory we refer the reader to the review article by Jackiw [1]. We only repeat here that the pseudoparticle configurations $F_{\mu\nu}(x_4, x)$ are defined as the imaginary time self-dual solutions of the classical equations of motions with zero Euclidean energy. It is known that these solutions can be classified by the Pontryagin index p which is given by

$$p = -\frac{1}{32\pi^2} \int d^4x \, \varepsilon_{\mu\nu\alpha\beta} \, \mathrm{Tr}\,(F_{\mu\nu} F_{\alpha\beta}) \tag{16}$$

and is an integer, provided $F_{\mu\nu}$ vanishes fast enough at infinity and can, therefore, be looked upon as a mapping defined on S^4.

If one assumes sufficient smoothness for the $F_{\mu\nu}$, expression (16) can also be written as [1]:

$$p = q_+ - q_- + q_L \tag{17}$$

where

$$q_\pm = -\frac{1}{24\pi^2}\int_{R^3} d^3\vec{x}\,\varepsilon_{ijk}\,\mathrm{Tr}(A_i A_j A_k)\big|_{x_4=\pm\infty},$$

$$q_L = \frac{1}{24\pi^2}\int_Z dS_\mu\,\varepsilon_{\mu\nu\alpha\beta}\,\mathrm{Tr}(A_\nu A_\alpha A_\beta).$$

In the last expression Z denotes the infinite extended cylinder in $\boldsymbol{R}^4$ with base manifold $\boldsymbol{R}^3$ and dS_μ is the oriented volume element on the cylinder. Because we are only interested in the $A_4 = 0$ gauge, the term q_L vanishes identically so that relation (16) reads

$$p = q_+ - q_-. \tag{18}$$

$F_{\mu\nu} \to 0$ as $||x|| \to \infty$ implies that $\vec{A}$ is a pure gauge, i.e.

$$\lim_{r'\to\infty} \vec{A}(x_4, \vec{x}) = g^{-1}(x_4, \vec{x})\vec{\nabla}g(x_4, \vec{x}) \tag{19}$$

where $r' = \sqrt{x_4^2 + \vec{x}^2}$ and g: $\boldsymbol{R}^4_\infty \to \mathrm{SU}(2)$.

Specifically for $x_4 \to \pm\infty$ we get

$$\lim_{x_4\to\pm\infty} \vec{A}(x_4, \vec{x}) = g^{-1}(\pm, \vec{x})\vec{\nabla}g(\pm, \vec{x}) \tag{20}$$

where $g(\pm, \cdot)$: $\boldsymbol{R}^3 \to \mathrm{SU}(2)$.

Inserting (20) into the definition of q_+ and q_- one finds that these numbers are topological invariants, as long as we take the mappings $g(\pm, \cdot)$: $\boldsymbol{R}^3 \to \mathrm{SU}(2)$ from a certain set τ which we discussed in the first section; Remember the difference $q_+ - q_-$ is always a topological invariant. In other words, only if one stays within a fixed compactification $\varkappa$ of $\boldsymbol{R}^3$, the numbers q_+ and q_- are topological invariants. So, in discussing properties of the pseudo-particle solution by using topological arguments one has to be very careful in specifying exactly what compactification of $\boldsymbol{R}^3$ one is going to use and not to change this compactification from case to case.

After this remark, let us write down what we mean exactly by a pseudo-particle solution $\vec{A}(x_4, x)$ in the $A_4 = 0$ gauge.

(a) $\vec{A}(x_4, \vec{x})$ must satisfy the classical equation of motion on Euclidean space $\boldsymbol{R}^4$ and it must have zero energy in this space.

(b) If $\lim_{x_4\to\pm\infty} \vec{A}(x_4, \vec{x}) = g^{-1}(\pm, \vec{x})\vec{\nabla}g(\pm, \vec{x})$ then we must have

$$\bar{g}(\pm, \vec{x}) \in \bar{\tau}_0. \tag{21}$$

(c) If $\lim_{\vec{x}\to\infty} \vec{A}(x_4, \vec{x}) = g^{-1}(x_4, \hat{x})\vec{\nabla}g(x_4, \hat{x})$ then the mapping

$$g_{x_4}(\hat{x}) := g(x_4, \hat{x}): S^2 \to S^3$$

must satisfy the condition

$$\bar{g}_{x_4}(\hat{x}) = \pm \bar{g}_{x_4}(-\hat{x}).$$

An immediate consequence of these properties is that the pseudo-particle solution $\vec{A}(x_4, \vec{x})$ induces a homotopy $h_{x_4}(\hat{x}) := \bar{g}(x_4, \hat{x})$ of mappings of the space at infinity into the 3-sphere. Therefore we get the following result:

LEMMA 2. *There cannot be any pseudo-particle solution $\vec{A}(x_4, \vec{x})$ such that $\vec{A}(+\infty, \vec{x}) = g^{-1}(+, \vec{x})\vec{\nabla} g(+, \vec{x})$ with $\bar{g}(+, \vec{x}) \in \bar{\tau}_{\pm}$ and $\vec{A}(-\infty, \vec{x}) = g^{-1}(-, \vec{x})\vec{\nabla} g(-, \vec{x})$ with $\bar{g}(-, \vec{x}) \in \bar{\tau}_{\mp}$ respectively.*

In other words tunnelling between half-integer and integer valued vacua is not possible within our choice of compactification of $\boldsymbol{R}^3$.

On the other hand, there is certainly tunnelling between the integer valued vacua because this is known [1] for the vacua classified via $\pi_3(S^3)$. There remains the problem of tunnelling between half-integer vacua. In fact, such a pseudo-particle solution can be constructed. Take the BPST instant on [11] $A^{\mu}(x_4, \vec{x})$ and transform it into the $A_4 = 0$ gauge, such that it connects the $q = 0$ with the $q = 1$ vacuum [1]. Then make a further gauge transformation induced by the Gribov mapping $g_-(\vec{x})$. Then we get a field $\vec{A}(x_4, \vec{x})$ as

$$\vec{A}(x_4, \vec{x}) = g_-^{-1}(\vec{x})\vec{A}_{\mathrm{BPST}} g_-(\vec{x}) + g_-^{-1}(\vec{x})\vec{\nabla} g_-(\vec{x}). \tag{22}$$

Let us see how this field behaves at infinity:

$$\lim_{x_4 \to +\infty} \vec{A}(x_4, \vec{x}) = (g_1(\vec{x})g_-(\vec{x}))^{-1}\vec{\nabla}(g_1(\vec{x})g_-(\vec{x}))$$

where $g_1(\vec{x})$ is a mapping $\boldsymbol{R}^3 \to S^3$ such that $\lim_{\vec{x} \to \infty} g_1(\vec{x}) = 1$ and has topological number $q = 1$. (See for instance [1]). The mapping $g_1 \cdot g_-$ therefore induces a mapping $\bar{g}_1 \cdot \bar{g}_-$ in the class $\bar{\tau}_-$ with topological number $q = \frac{1}{2}$ and describes a $q = \frac{1}{2}$ vacuum. For $x_4 \to -\infty$ one gets

$$\lim_{x_4 \to -\infty} \vec{A}(x_4, \vec{x}) = g_-^{-1}(\vec{x})\vec{\nabla} g_-(\vec{x})$$

and therefore we are dealing with a vacuum with $q = -\frac{1}{2}$. For fixed x_4 and $\vec{x} \to \infty$ we get finally

$$\lim_{\vec{x} \to \infty} \vec{A}(x_4, \vec{x}) = (g_{x_4}(\hat{x})g_-(\hat{x}))^{-1}\vec{\nabla}(g_{x_4}(\hat{x})g_-(\hat{x})).$$

Because $g_{x_4}(\hat{x}) = g_{x_4}(-\hat{x})$, we find

$$g_{x_4}(\hat{x})g_-(\hat{x}) = -g_{x_4}(-\hat{x})g_-(-\hat{x}).$$

Therefore $\vec{A}(x_4, \vec{x})$ as defined in (22) is an allowed pseudo-particle solution, which connects the two vacua $q = +\frac{1}{2}$ and $q = -\frac{1}{2}$. The above analysis must be modified if one allows for two different compactifications of the space $\boldsymbol{R}^3$ at times $t = +\infty$ and $t = -\infty$.

IV. The vacuum structure in the Coulomb gauge

Let us discuss in this final section the vacuum structure in the so-called Coulomb gauge:

$$\vec{\nabla}\vec{A} = 0. \tag{23}$$

This problem has been under intensive investigation by several authors since Gribov's [5] work. The possible existence of further solutions more general than Gribov's equation (23) was investigated in [2]. It was shown in this paper that equation (23) does not admit besides the trivial solution $\vec{A} = 0$ any other $\vec{A}(\vec{x})$ which is obtained from a gauge mapping $g(\vec{x})$ with $\lim_{\vec{x}\to\infty} g(\vec{x}) = e \in \mathrm{SU}(2)$.

It is furthermore claimed in that paper that there cannot be any solution $\vec{A}(\vec{x}) = g^{-1}(\vec{x})\vec{\nabla}g(\vec{x})$ such that the mapping $g(\vec{x})$ covers S^3 more than once. Unfortunately, the authors do not say which compactification of $\boldsymbol{R}^3$ they are using. Anyhow, we want to show why their proof of Statement B in the above mentioned paper is not complete and therefore is not proven. The main argument in their proof is the observation that any mapping of $\boldsymbol{R}^3$ which covers the sphere more than once has the property that it sends a certain two-dimensional surface into one point on the sphere. This is only true for mappings g: $\boldsymbol{R}^3 \to S^3$ such that $\lim_{\vec{x}\to\infty} g(\vec{x}) = \mathrm{const} \in \mathrm{SU}(2)$.

Indeed it is very easy to construct mappings g: $\boldsymbol{R}^3 \to S^3$ such that not even a one-dimensional subspace of $\boldsymbol{R}^3$ goes to the same point and nevertheless cover S^3 more than once. Furthermore, g has topological number equal one. One only has to use the fact that S^3 can be obtained in a very simple way as a quotient space of $\bar{B}_3$. Let us define this equivalence relation:

$$\vec{x}, \vec{y} \in \bar{B}_3, \quad \vec{x} = (x_1, x_2, x_3), \quad \vec{y} = (y_1, y_2, y_3).$$

$$\vec{x} \sim \vec{y} \quad \text{iff} \quad \begin{cases} \vec{x}, \vec{y} \notin \partial\bar{B}_3 \text{ and } \vec{x} = \vec{y}, \\ \vec{x}, \vec{y} \in \partial\bar{B}_3 \text{ and } x_i = y_i, \quad i = 1, 2. \end{cases}$$

It is well known that the quotient space $\bar{B}_3/\sim$ is topologically the 3-sphere [12]. So the natural projection π: $\bar{B}_3 \to \bar{B}_3/\sim$ gives a continuous mapping of $\bar{B}_3$ onto S^3 such that S^3 is completely covered and at most two points in $\bar{B}_3$ have the same image in S^3.

Starting from this observation one can now construct continuous mappings of any topological number $|q| \geqslant 1$ belonging for instance to the set τ_0.

So even if one restricts oneself to this set τ_0 of mappings from $\boldsymbol{R}^3$ into S^3 statement B in the above mentioned paper [2] is not proven. So the possible existence of Coulomb gauge vacua with any topological number $q = \pm n/2$ is not excluded by the arguments given there.

V. Conclusions

We have shown in this paper how different classifications of vacua in a SU(2) gauge theory arise from different compactifications of the spatial part $\boldsymbol{R}^3$ of Minkowski space. By using the set τ_0 of mappings g: $\boldsymbol{R}^3 \to S^3$ we obtained a classification in terms of half-integers such that the Gribov vacua belong to the classes $\pm\frac{1}{2}$.

It seems that the half-integer and integer valued vacua are completely separated from each other in the sense that there is no tunnelling between them via pseudo-particle solutions. This result emerges as a consequence of our choice of compactification of $\boldsymbol{R}^3$. If one allows for a larger set of mappings than our set τ_0, then completely different results are to be expected.

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Appendix

In this appendix we want to show how the classification of the set $\bar{\tau}_0$ is related to the classification of the vacua in an SO(3) gauge theory when one compactifies the space $\boldsymbol{R}^3$ to $\boldsymbol{P}^3$.

This problem can be solved by using a Theorem of Wada and Olum [13]: it says that the homotopy classes of all mappings $g\colon \boldsymbol{P}^3 \to \mathrm{SO}(3) \cong \boldsymbol{P}^3$ are in one-to-one correspondence with the following group G

$$G = H^1(\boldsymbol{P}^3, \boldsymbol{Z}_2) \oplus H^3(\boldsymbol{P}^3, \boldsymbol{Z}). \tag{A1}$$

H^1 denotes the first cohomology group of the space $\boldsymbol{P}^3$ with coefficients in the group $\boldsymbol{Z}_2$. H^3 denotes the third cohomology group with integer coefficients. Both these groups are known and are given by

$$H^1(\boldsymbol{P}^3, \boldsymbol{Z}_2) = \boldsymbol{Z}_2, \qquad H^3(\boldsymbol{P}^3, \boldsymbol{Z}) = \boldsymbol{Z}.$$

Therefore G is the direct sum of the two groups

$$G = \boldsymbol{Z}_2 \oplus \boldsymbol{Z}. \tag{A2}$$

Let us now show the connection with the classification of $\bar{\tau}_0$. Let $\bar{g} \in \bar{\tau}_0$. Then we define a mapping $\bar{f}\colon \bar{B}_3 \to \boldsymbol{P}^3$ by

$$\bar{f} = \pi \circ \bar{g}, \tag{A3}$$

where $\pi\colon S^3 \to \boldsymbol{P}^3$ is the natural projection.

Because $\pi(y) = \pi(-y)$ for all $y \in S^3$, we get immediately $\bar{f}(x) = \bar{f}(-x)$ for all $x \in \partial \bar{B}_3$. Therefore $\bar{f}$ induces a mapping $f\colon \boldsymbol{P}^3 \to \boldsymbol{P}^3$. In this sense every mapping $\bar{g} \in \bar{\tau}_0$ can be also looked upon as a mapping $f\colon \boldsymbol{P}^3 \to \boldsymbol{P}^3$. It is also clear that the mappings $\bar{g}$ and $-\bar{g} \in \bar{\tau}_0$ induce the same f.

Let us next show how the sets $\bar{\tau}_+$ and $\bar{\tau}_-$ are defined in terms of mappings of $\boldsymbol{P}^3$ into $\boldsymbol{P}^3$. It can be shown that every mapping $f\colon \boldsymbol{P}^3 \to \boldsymbol{P}^3$ induces an homomorphism $f_*\colon \pi_1(\boldsymbol{P}^3) \to \pi_1(\boldsymbol{P}^3)$ of the fundamental group of $\boldsymbol{P}^3$. Because $\pi_1(\boldsymbol{P}^3) = \boldsymbol{Z}_2$, there exist only two homomorphisms:

$$\begin{aligned} h_+ &= 0 \quad \text{(trivial map)}, \\ h_- &= \mathrm{id} \quad \text{(identity map)}. \end{aligned} \tag{A4}$$

This gives a first classification of all mappings $f\colon \boldsymbol{P}^3 \to \boldsymbol{P}^3$ into two classes:

$$f_* = h_+ \quad \text{or} \quad f_* = h_- .$$

A little reflection shows that every $\bar{g} \in \bar{\tau}_+$ induces a mapping f such that $f_* = h_+$ and every $\bar{g} \in \bar{\tau}_-$ an f with $f_* = h_-$.

If one denotes by $\xi_\pm$ the sets

$$\xi_\pm = \{f\colon \boldsymbol{P}^3 \to \boldsymbol{P}^3 \colon f_* = h_\pm\}, \tag{A5}$$

then one can easily deduce the following lemma from the work in [13]:

LEMMA. *The homotopy classification of the set* $\bar{\tau}_+$ *is in one-to-one correspondence with that of the set* ξ_+. *The classification of the set* $\bar{\tau}_-/\{[+1/2], [-1/2]\}$ *corresponds in a one-to-one manner to that of the set* $\xi_-/[(1/2, 0)]$. *The classes* $[1/2]$ *and* $[-1/2]$ *correspond to the single class* $[(1/2, 0)]$ *in* ξ_-.

Therefore, for instance, the two Gribov vacua belong to the same homotopy class in an SO(3) theory with the space $\boldsymbol{P}^3$ as the compactification of $\boldsymbol{R}^3$.

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