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QUASI-PERIODIC BOUNDARY CONDITIONS AND THE VACUUM STRUCTURE IN GAUGE THEORIES *

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The role of the θ angle occurring in the vacuum structure of gauge theories is illustrated by means of a quantum mechanical example.

The purpose of this paper is to develop a quantum mechanical analog for the vacuum structure in gauge theories [1–4], and to illustrate in terms of it the role of the θ angle characterizing the different vacua.

It is well known ** that the self-adjointness of the operator $-\frac{1}{2}d^2/dq^2$ in the interval $0 \leqslant q \leqslant 2\pi$ is compatible with a number of boundary conditions, in particular with the quasi-periodic conditions

$$\psi(2\pi) = e^{-i\theta}\,\psi(0)\,,$$
$$\psi'(2\pi) = e^{-i\theta}\,\psi'(0)\,.$$

The Schrödinger equation corresponding to this operator has eigenfunctions

$$\psi_m(q) = \frac{1}{\sqrt{2\pi}}\, e^{i(m-\theta/2\pi)q}\,, \tag{1a}$$

and eigenvalues

$$E_m = \tfrac{1}{2}\left(m - \frac{\theta}{2\pi}\right)^2\,. \tag{1b}$$

The functions (1a) form a suitable basis for the expansion of quasi-periodic functions satisfying

$$\psi(q+2\pi) = e^{-i\theta}\,\psi(q)\,, \qquad -\infty < q < \infty\,. \tag{2}$$

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** See for instance refs. [5] for a discussion of a closely related problem.

The significance of condition (2) is best seen by considering the variable q as parametrizing the points of a circle. *A priori*, the value $\theta \neq 0$ is perfectly consistent with the basic principles of quantum mechanics on a circle. The situation is quite different if the circle is taken as a restriction of Euclidean space, as in the case of the rigid rotor, where the simple connectedness of the underlying manifold requires $\theta = 0$. However, if we were living in a toroid, we would see no basic reason for choosing $\theta = 0$, in which case θ plays the role of a cosmological constant. In this respect we disagree with the philosophy presented in ref. [6].

The analogy between the angle θ introduced above and the angle characterizing the vacuum of gauge theories becomes explicit by considering the Green function corresponding to the eigenfunctions and eigenvalues (1) in the Feynman path integral representation, which, as we shall demonstrate next, is given by

$$G(q, t) = \sum_{n=-\infty}^{\infty} e^{+in\theta} \int_{q(0)=0}^{q(t)=q+2n\pi} D[q'] \exp[i \int_0^t dt' \tfrac{1}{2}\dot{q}'^2(t')] \ . \tag{3}$$

Each of the functional integrals participating in eq. (3) involves paths from inequivalent topological classes. Already, before evaluation of the right-hand side, we see the quasi-periodic property of $G(q, t)$:

$$G(q + 2\pi, t) = e^{-i\theta} G(q, t) \ . \tag{4}$$

The functional integral is easily performed and leads to

$$G(q, t) = \frac{1}{\sqrt{2\pi i t}} \sum_{n=-\infty}^{\infty} e^{in\theta} e^{(i/2t)(q+2n\pi)^2} \ . \tag{5}$$

In order to convince ourself that this is the correct Green function, we develop (5) in terms of the basis (1a):

$$G(q, t) = \sum_{n=-\infty}^{\infty} a_m(t) \psi_m(q) \ .$$

The coefficients $a_m(t)$ are computed to be

$$a_m(t) = \frac{1}{\sqrt{2\pi i t}} \int_{-\infty}^{\infty} \frac{dq}{\sqrt{2\pi}} e^{-i(m-\theta/2\pi)q} e^{iq^2/2t} \ , \tag{6}$$

so that

$$G(q, t) = \langle qt \mid 00 \rangle = \sum_{m=-\infty}^{\infty} \psi_m(q) \psi_m^*(0) e^{-(i/2)(m-\theta/2\pi)^2 t} \ .$$

In complete analogy to what has been done in gauge theories [2], we may rewrite expression (3) in terms of an effective Langrangian

$$\mathcal{L}_{\rm eff}(\dot{q}, q) = \tfrac{1}{2}\dot{q}^2 + \frac{\theta}{2\pi}\,\dot{q}\,, \tag{7}$$

leading to

$$\langle qt\,|\,00\rangle = e^{-i(\theta/2\pi)q} \int_{q(0)=0}^{q(t)=q} D[q']\,\exp\Big(i\int_0^t \mathcal{L}_{\rm eff}(q',\dot{q}')\,dt'\Big)\,, \tag{8}$$

where the q-dependent normalization constant results from the fact that this $\mathcal{L}_{\rm eff}$ leads to a gauge transform of the free Hamiltonian. In expression (8) the functional integral is performed over paths going around the circle an arbitrary number of times. Although the θ dependence does not affect the classical equations of motion, it evidently plays an important role in (8) to fix the quantum mechanical boundary conditions. In particular, for $\theta \neq 0$ we have a P, T violation, again in complete analogy to the situation discussed by 't Hooft [2].

By considering the limit of the propagator for large imaginary times, we obtain, following ref. [7], Feynman path representations for both the ground state wave function and the generating functional for Euclidean correlation functions

$$Z[J] = \int D[q]\,\exp\Big(-\int_{-\infty}^{\infty} (\mathcal{L}_{\rm eff} + Jq)\,dt\Big)$$

having the same topological interpretation as the corresponding functionals considered in gauge field theories [2,4].

On the other hand, the field theoretical analog of the quantum mechanical propagator will be given for such theories, in the gauge $A_0 = 0$, by

$$\langle A_2, t_2\,|\,A_1, t_1\rangle = \sum_{n=-\infty}^{\infty} e^{in\theta} \int_{A_1}^{g_1^n\cdot A_2} D[A]\,\exp\Big(i\int_{t_1}^{t_2} \mathcal{L}\,d^4y\Big)\,, \tag{9}$$

where $g_1^n \cdot A_2$ stands for the gauge transform of A_2 under a gauge transformation carrying winding number n [3]. As in (3), the above propagator possesses the quasi-periodic property

$$\langle g_1 \cdot A_2, t_2\,|\,A_1, t_1\rangle = e^{-i\theta}\,\langle A_2, t_2\,|\,A_1, t_1\rangle\,,$$

implying in turn the quasi-periodicity of the ground state functional [3],

$$\psi_\theta(g_1 \cdot A) = T[g_1]\,\psi_\theta(A) = e^{-i\theta}\,\psi_\theta(A)\,, \tag{10}$$

where $T[g_1]$ is the unitary operator implementing gauge transformations. As in the case of the circle, this means that the wave function evaluated at physically equivalent points in configuration space may differ by a phase.

One particular field theoretical example in which property (10) is realized is the Schwinger model [8]. It is known [9] that the gauge-invariant observables in this model can be entirely expressed in terms of a free massive pseudoscalar field $\Sigma(x)$ of mass $e/\sqrt{\pi}$, and an operator σ carrying chirality 2. On the physical subspace, σ is a constant unitary operator which generates an infinite number of vacua when applied to the original vacuum of the theory:

$$|n\rangle = \sigma^n |0\rangle\,. \tag{11}$$

The irreducible representation of the observables is obtained by considering the θ vacuum

$$|\theta\rangle = \sum_{n=-\infty}^{\infty} e^{in\theta} |n\rangle\,, \tag{12}$$

which corresponds [4] to the different weighting of the topologically inequivalent contributions in expression (9). It has been shown [10] that σ can be related to the operator implementing gauge transformations g_1 carrying winding number one *

$$T[g_1^n] = \sigma^n\,, \qquad \text{on the physical subspace}\,, \tag{13a}$$

with

$$g_1 = e^{i\Lambda(x)}\,, \qquad \Lambda(x^1 = \infty) - \Lambda(x^1 = -\infty) = 2\pi\,. \tag{13b}$$

Eq. (13) means that on the physical subspace T does not depend on the particular gauge function but only on the topological class. Except for the possibility discussed in ref. [10] of exhibiting a finer vacuum structure related to half-winding gauge transformations, it would be redundant to consider gauge transformation with arbitrary winding number, since the corresponding spurions do not participate in Schwinger's solution. From eqs. (11) and (12) we have

$$T[g_1]\,|\theta\rangle = e^{-i\theta}\,|\theta\rangle\,, \tag{14}$$

which expresses the quasi-periodicity property (10) for the Schwinger model. The properties we have just summarized imply the following expression for the vacuum-state functional, which we express in terms of spacial electric field configurations (up to a proportionality factor, $\widetilde{\Sigma}$) and a gauge degree of freedom (η):

$$\psi_\theta[\widetilde{\Sigma}, \eta] = \exp\left(-\int \widetilde{\Sigma}(x^1)\, K(x^1 - y^1)\, \widetilde{\Sigma}(y^1) \mathrm{d}x^1 \mathrm{d}y^1\right) \exp\left(-\frac{i\theta}{2\pi} \int_{-\infty}^{\infty} \partial_1 \eta(z^1)\, \mathrm{d}z^1\right), \tag{15a}$$

* At this point we disagree with Nakanishi's paper [11]. The solution he proposes, eq. (2.7), leads to a physical space corresponding to a particular choice of the θ vacuum, in his notation $\theta = \arg(u_1 u_2^*)$. In such an irreducible representation it is not surprising that he misses the correct relation (13). The spurions introduced by him are indeed spurious since they are not the variable conjugate to θ.

where

$$K(z) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \sqrt{k^2 + \frac{e^2}{\pi}}\, e^{ikz} dk \,. \tag{15b}$$

In obtaining expression (15) we have used the fact that the free field can be decomposed into a collection of independent harmonic oscillators. The η dependence follows from properties (13) and (14) which imply that gauge transformations carrying zero winding ($\Lambda(x^1 = +\infty) = \Lambda(x^1 = -\infty)$) leave the physical states invariant, whereas those carrying winding n transform the state by a phase. The right-hand side of eq. (15a) provides a concrete realization of the quasi-periodicity (10) for the Schwinger model.

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Note added in proof

After the completion of our work some related papers [12] were brought to our attention.

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