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Quantum expansion of soliton solutions*

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A systematic quantum expansion of soliton solutions to nonlinear field equations is developed. The method is based on the standard canonical quantization procedure. When the nonlinear coupling g is small, the Hamiltonian is $O(g^{-2})$ and its quantum eigenstates take on a WKB form, giving a direct connection between the quantum description and the corresponding classical soliton solution (which can be in any dimension, either time-dependent or time-independent). Our general method is illustrated by a variety of one-space-dimensional examples, some new and some old.

I. INTRODUCTION

The nonlinear field equations which form the basis of quantum field theory have long been known to possess a rich array of solutions on the classical level, some of those classical solutions having rather remarkable particlelike properties.[1,2] In this paper we describe a general quantization procedure which permits the quantum-mechanical interpretation of these classical solutions and the computation of quantum corrections to them. Because of their particlelike properties, these nonlinear solutions are called solitons.

To see how such solutions may emerge, let us consider any local field theory, consisting of an arbitrary number of fields. It is convenient to characterize their coupling by an overall constant g, with the requirement that when $g=0$ the field equation becomes linear. Let us assume that the theory possesses some conserved quantities, such as charge Q, or baryon number $N, \ldots$, or some boundary condition at infinity. For example, in the special case of a scalar field $\phi(x)$ in one space dimension x and one time dimension t, the conserved quantity can be simply

$$\phi(\infty)-\phi(-\infty)\ ,$$

which is independent of time.

Assuming that the energy density is positive definite (and is chosen to be 0 everywhere in the vacuum state), we may search for the lowest energy state of such a system with some appropriate conserved quantity fixed. Classically, if the energy density in this lowest energy state is finite, nonzero, and of finite extension in space, then the state is defined to be a soliton, or multisoliton, solution. It is clear that as $g\to 0$, these solutions do not satisfy the corresponding linear field equation.

Examples of soliton solutions can be found in any dimension. In one space dimension,[1-8] there are the sine-Gordon equation, the quartic coupling theory, etc. In two or three space dimensions, there are the various magnetic-monopole solutions[9-11]; in addition, there are solutions similar to the abnormal nuclear state.[12] [As an illustration of the latter case, we may assume the system to consist of only scalar fields, say one charged field ϕ and one neutral field χ, with a coupling contained in $(m+g\chi)^2\phi^\dagger\phi+V(\chi)$, where $V(\chi)$ has an absolute minimum at $\chi=0$ and another local minimum at $\chi=-m/g$. It is quite simple to show that when the "charge" Q exceeds a certain limit, soliton (i.e., abnormal) solutions occur in which Q is confined in a finite volume Ω and carries a finite rest mass; $\chi\cong -m/g$ inside Ω but 0, its vacuum expectation value, outside. We note that keeping Q fixed is a constraint that involves both ϕ and $\dot{\phi}$.]

Because of the localization of energy density the number of solitons (or antisolitons) in such a classical solution can be directly counted, at least when $t\to\pm\infty$. If the constraint depends on the time derivative of the field, then in general even the classical single-soliton solution in its rest frame is time-dependent; otherwise, it will be time-independent. Once a single-soliton solution exists, then at least on the classical level, multisoliton (and/or antisoliton) solutions can be readily constructed by forming the appropriate asymptotic states at $t=-\infty$. Such solutions would naturally be time-dependent, since as time develops, these solitons (and/or antisolitons) should undergo scattering and other possible dynamical changes. Our main concern is to begin with such a classical solution, and to develop a systematic quantum expansion in g, through which the corresponding quantum-mechanical solution can be derived;[13] as we shall see, the lowest-order term in this expansion, which will turn out to be $O(g^{-2})$, is essentially the classical solution.

For definiteness, we consider a Lagrangian density depending on an N-component real scalar field ϕ^i.

$$\mathfrak{L} = -\frac{1}{2}\sum_i \left(\frac{\partial \phi^i}{\partial x_\mu}\right)^2 - g^{-2}\, V(g\phi^i)\ . \tag{1.1}$$

If we expand $V(g\phi^i)$ as a power series in ϕ^i about its minimum, then the quadratic (or mass) term is independent of g, the cubic term is linear in g, etc. Hence, in the limit $g \to 0$, at least formally, $\mathfrak{L}$ contains only quadratic terms in ϕ^i; therefore, g plays the role of a coupling constant.

Let

$$[\phi^i(\vec{\mathbf{r}}, t)]_{\rm cl} \equiv g^{-1}\sigma^i(\vec{\mathbf{r}}, t, z_1^0, \ldots, z_K^0)$$

represent a family of classical solutions to the Euler-Lagrange equation determined by the Lagrangian density (1.1). These solutions all have the same energy, and depend on the integration constants $z_1^0, \ldots, z_K^0$. Because of the factor g^{-1} in its definition, the N-component c-number function σ^i satisfies

$$\frac{\partial^2 \sigma^i}{\partial x_\mu{}^2} - \frac{\partial V(\sigma)}{\partial \sigma^i} = 0\ ,$$

where $V(\sigma) = V(\sigma^i(\vec{\mathbf{r}}, t, z_1^0, \ldots, z_K^0))$. Since the above equation is independent of g, so is σ^i. We will always introduce a sufficient number of parameters $z_1^0, \ldots, z_K^0$ so that translation of our solution in space or time can be accomplished by changing the parameters $z_1^0, \ldots, z_K^0$. In particular, there will exist a set of values $z_k(t, z_1^0, \ldots, z_K^0)$ which allow us to write

$$\sigma^i(\vec{\mathbf{r}}, t, z_1^0, \ldots, z_K^0) = \sigma^i(\vec{\mathbf{r}}, 0, z_1(t, z_1^0, \ldots, z_K^0), \ldots, z_K(t, z_1^0, \ldots, z_K^0))\ . \tag{1.2}$$

For the single-soliton solution in a D-dimensional space, we will use at least D integration constants, say $z_1^0, \ldots, z_D^0$, representing the center-of-mass position of the soliton at a particular time. In the case of an l-soliton solution in a D-dimensional space, there should be at least lD integration constants corresponding to the initial positions of the l solitons. It will often be useful to choose the parametrization $z_1, \ldots, z_K$ of our family of classical solutions so that the $z_k(t, z_1^0, \ldots, z_K^0)$ defined by Eq. (1.2) have the simple form

$$z_k \equiv u_k t + z_k^0\ ,$$

where the u_k's are constants ($k = 1, 2, \ldots, K$), so that

$$\sigma^i = \sigma^i(\vec{\mathbf{r}}, z_1, \ldots, z_K)$$

has no explicit dependence on t, and it satisfies

$$\vec{\nabla}^2\sigma^i - \sum_{k,k'} u_k u_{k'} \frac{\partial^2 \sigma^i}{\partial z_k\, \partial z_{k'}} - \frac{\partial V(\sigma)}{\partial \sigma^i} = 0\ . \tag{1.3}$$

For a time-independent solution, one has $u_k = 0$ and therefore $z_k = z_k^0$. That such a simple choice, $z_k = u_k t + z_k^0$, is always possible for any time-dependent solution follows from the time-translational invariance of the original field equation. We can always choose one of the integration constants, say z_K^0, to be related to the time translation, $t \to t +$ constant, and this leads to the special choice that only $u_K \neq 0$, but all other $u_k = 0$. In general, for an l-soliton solution in a D-dimensional space, it is most convenient to choose $u_1, \ldots, u_{lD}$ to be simply the velocity components of these l solitons as $t \to -\infty$, and therefore $z_1, \ldots, z_{lD}$ become their positions at least in the asymptotic region.

For the quantum theory, we expand the field operator $\phi^i(\vec{\mathbf{r}}, t)$ about the classical solution $g^{-1}\sigma^i(\vec{\mathbf{r}}, z_1, \ldots, z_K)$:

$$\phi^i(\vec{\mathbf{r}}, t) = g^{-1}\sigma^i(\vec{\mathbf{r}}, z_1, \ldots, z_K) + \sum_{n=K+1}^{\infty} q_n(t)\, \psi_n^i(\vec{\mathbf{r}}, z_1, \ldots, z_K)\ , \tag{1.4}$$

where $z_1, \ldots, z_K$, $q_{K+1}, q_{K+2}, \ldots$ are treated as coordinates and the N-component c-number functions $\psi_n^i(\vec{\mathbf{r}}, z_1, \ldots, z_K)$ form a complete set of real functions, subject to the constraints

$$\sum_{i=1}^{N} \int \psi_n^i \frac{\partial \sigma^i}{\partial z_k}\, d\tau = 0 \tag{1.5}$$

and the orthonormality relation

$$\sum_{i=1}^{N} \int \psi_n^i \psi_{n'}^i\, d\tau = \delta_{nn'}\ , \tag{1.6}$$

where $d\tau = d^D r$. We now propose to apply the usual methods of canonical quantization to the Lagrangian density (1.1), written in terms of these new coordinates $z_1, \ldots, z_K$, $q_{K+1}, q_{K+2}, \ldots$.

This approach will be developed in the next section. The quantization procedure can be carried out in a completely standard and straightforward way. The Hamiltonian of the system, as a function of the coordinates $z_1, \ldots, z_K$, $q_{K+1}, \ldots$ and their conjugate momenta $(-i\partial/\partial z_1), \ldots, (-i\partial/\partial z_K)$, $(-i\partial/\partial q_{K+1}), \ldots$ can then be explicitly given. By using this Hamiltonian, we can attempt to express the solution of the resulting quantum-mechanical problem as a power series in g. The leading term for small g is $O(g^{-2})$, and it can be derived by solving a K-dimensional quantum-mechanical problem, depending only on the coordinates $z_1, \ldots, z_K$ and their conjugate momenta. It turns out that to $O(g^{-2})$ the solution of this K-dimensional Schrödinger equation is precisely that given by the WKB approximation; it leads to a direct connection between the quantum-mechanical solution and the classical solution of interest. The next-order

solution is $O(g^0)$, and it involves also q_n and $\pi_n = -i\partial/\partial q_n$. Since terms in the Hamiltonian that depend on cubic or higher powers of q_n and π_n carry additional powers in g, to $O(g^0)$ the Hamiltonian is reduced to one that depends, besides on z_k and $p_k = -i\partial/\partial z_k$, only quadratically on q_n and π_n. This then reduces to a system of harmonic oscillators whose frequencies may depend on z_k. Its quantum-mechanical solution is closely tied to the corresponding classical problem, as shall be explicitly exhibited in Sec. II.

In Sec. III, our general method is illustrated by application to field theories in one space dimension and one time dimension. Independently of the detailed form of the Lagrangian, we derive the quantum-mechanical solution of a single soliton, either at rest or in motion. The general form of the characteristic frequencies of the harmonic oscillators in q_n and π_n is discussed, which allows us to examine the inverse problem of starting from the energy spectrum and then deriving the corresponding field theory.

The question of statistics of solitons in one space dimension is fundamentally different from that in two or more space dimensions. In space dimensions other than one, there is a continuous set of points at infinity. Except for some special theories involving gauge vector-meson fields, because of continuity, we can assume $\phi^i(\infty)$ to be of a constant value equal to that in the vacuum state. Thus, we may at least formally expand the soliton state in terms of the usual free-meson creation operators acting on the vacuum state. A soliton can then be viewed as a bound state of an indefinite number of mesons; its statistics are like those of any bound state, determined by its constituents. In one space dimension, the above argument fails since the value of ϕ^i at $x=\infty$ usually differs from that at $x=-\infty$ in a soliton state. On the other hand, as is well known and shall also be analyzed in Sec. III, the statistics of any one-dimensional system of interacting identical particles is entirely a matter of convention, not of principle.

Section IV contains a brief discussion of the quantization of the time-dependent two-soliton and soliton-antisoliton solutions to the sine-Gordon equation. This section serves primarily as an illustration of the use of the coordinates z_k, q_n of Eq. (1.4) in a nontrivial case. For the soliton-soliton or soliton-antisoliton scattering solutions only the terms of order g^{-2} are discussed. The quantization of the periodic breather mode (soliton-antisoliton scattering with imaginary velocity) is carried out through order g^0 and results similar to those of Dashen, Hasslacher, and Neveu[4] are obtained. Finally, in Sec. V, we introduce and quantize a new one-dimensional scalar field theory whose classical solutions include soliton pairs permanently confined in a "bag."

II. GENERAL METHOD

Let us now use the usual canonical methods and the coordinates defined in (1.4) to quantize the dynamical system described by the Lagrangian density (1.1). In terms of the coordinates $z_1, \ldots, z_K$, $q_{K+1}, q_{K+2}, \ldots$, the Lagrangian becomes

$$L = \int d\tau \Big\{ \frac{1}{2} \sum_{i=1}^{N} \Big[\sum_{k=1}^{K} \dot{z}_k \Big(g^{-1} \frac{\partial \sigma^i}{\partial z_k} + \sum_{n=K+1}^{\infty} q_n \frac{\partial \psi_n^i}{\partial z_k} \Big) + \sum_{n=K+1}^{\infty} \dot{q}_n \psi_n^i \Big]^2 - \frac{1}{2} \sum_{i=1}^{N} \Big[g^{-1} (\vec{\nabla} \sigma^i) + \sum_{n=K+1}^{\infty} q_n (\vec{\nabla} \psi_n^i) \Big]^2 - g^{-2} V \Big(\sigma^i + g \sum_{n=K+1}^{\infty} q_n \psi_n^i \Big) \Big\} \tag{2.1}$$

in which the derivatives of σ^i and ψ_n^i are taken by regarding $z_1, \ldots, z_K$ and $\vec{r}$ as independent variables. In the following, for convenience we shall adopt the convention that all repeated indices are to be summed over. The superscript i varies from 1 to N, while the subscripts k (or k') and n (or n') are always treated differently; they vary as follows:

$$K \geq k \text{ (or } k') \geq 1 ,$$
$$\text{but} \tag{2.2}$$
$$n \text{ (or } n') \geq K+1 .$$

The momenta conjugate to z_k and q_n are

$$p_k = \frac{\partial L}{\partial \dot{z}_k} = M_{kk'} \dot{z}_{k'} + M_{kn} \dot{q}_n ,$$
$$\pi_n = \frac{\partial L}{\partial \dot{q}_n} = M_{nk} \dot{z}_k + M_{nn'} \dot{q}_{n'} , \tag{2.3}$$

where the different matrix elements of the mass matrix $M(z_k, q_n)$ are given by

$$M_{kk'} = \int d\tau \Big(g^{-1} \frac{\partial \sigma^i}{\partial z_k} + q_n \frac{\partial \psi_n^i}{\partial z_k} \Big) \Big(g^{-1} \frac{\partial \sigma^i}{\partial z_{k'}} + q_{n'} \frac{\partial \psi_{n'}^i}{\partial z_{k'}} \Big) ,$$
$$M_{kn} = M_{nk} = \int q_{n'} \frac{\partial \psi_{n'}^i}{\partial z_k} \psi_n^i \, d\tau , \tag{2.4}$$

and

$$M_{nn'} = \delta_{nn'} .$$

The Hamiltonian for the system is

$$H = \tfrac{1}{2} p_k (M^{-1})_{kk'} p_{k'} + p_k (M^{-1})_{kn} \pi_n + \tfrac{1}{2} \pi_n (M^{-1})_{nn'} \pi_{n'} + g^{-2} \int \overline{V}(\sigma^i + g q_n \psi_n^i)\, d\tau \ , \tag{2.5}$$

where

$$\overline{V}(\xi) \equiv V(\xi) + \tfrac{1}{2} (\vec{\nabla} \xi)^2 \ . \tag{2.6}$$

The coordinates, conjugate momenta, and the Hamiltonian (2.5) can all be identified as operators in the standard way, yielding a well-defined quantum-mechanical theory provided a prescription for ordering the noncommuting factors in H is given. This can be done by viewing the Hamiltonian (2.5) as a differential operator on the Hilbert space wave functions of the variables $z_1, \ldots, z_K$, $q_{K+1}, \ldots$ using the substitution $p_k \to -i(\partial/\partial z_k)$, $\pi_n \to -i(\partial/\partial q_n)$. If we had made the usual expansion of the field operator $\phi^i(x)$ in a *complete* set of orthonormal N-component function, say $f_s^i(\vec{r})$, without any constraint

$$\phi^i(\vec{r}, t) = \sum_s Q_s(t) f_s^i(\vec{r}) \tag{2.7}$$

and used the coefficients $Q_s(t)$ as coordinates, the resulting Hamiltonian would have no ordering ambiguities and if viewed as a differential operator would be of the form

$$H = -\frac{1}{2} \sum_s \frac{\partial^2}{\partial Q_s^2} + U(Q_s) \ . \tag{2.8}$$

The ordering of the operators in Eq. (2.5) can then be determined by requiring that as differential operators the expressions (2.5) and (2.8) are identical, being related by a simple change of variables. We recall that for an arbitrary set of l variables $x_1, \ldots, x_l$ given as functions $x_i(y_1, \ldots, y_l)$ of a second set of variables $y_1, \ldots, y_l$ we can write

$$\sum_{i=1}^{l} \frac{\partial^2}{\partial x_i^2} = \sum_{i,j=1}^{l} \frac{1}{J} \frac{\partial}{\partial y_i} (g^{-1})_{ij} J \frac{\partial}{\partial y_j} \ , \tag{2.9}$$

where the matrix g_{ij} is given by

$$g_{ij} = \sum_{r=1}^{l} \frac{\partial x_r}{\partial y_i} \frac{\partial x_r}{\partial y_j} \ ,$$

while the function J is the square root of the determinant of the matrix g_{ij}. In our case, the matrix g_{ij} is simply M given by (2.4). The matrix M can be written as

$$M = \mathfrak{A} \tilde{\mathfrak{A}} \tag{2.10}$$

in which the tilde denotes the transpose and $\mathfrak{A}$ is of the form

$$\mathfrak{A} = \begin{pmatrix} A & B \\ 0 & 1 \end{pmatrix} \ ,$$

where $A \equiv (A_{kk'})$ is a $(K \times K)$ matrix, whose square is given by [in the notations of (2.2)]

$$(A^2)_{kk'} = M_{kk'} - M_{kn} M_{nk'} \ , \tag{2.11}$$

and B is a $(K \times \infty)$ matrix whose matrix elements are simply

$$B_{kn} = M_{kn} \ .$$

The determinant of the $(\infty \times \infty)$ matrix M is therefore equal to that of the $(K \times K)$ matrix A^2; this leads to

$$J = \det A \ . \tag{2.12}$$

Thus, the precise form of the Hamiltonian operator in terms of our variables z_k, q_n is

$$H = \tfrac{1}{2} J^{-1} [p_k (M^{-1})_{kk'} J p_{k'} + p_k (M^{-1})_{kn} J \pi_n + \pi_n (M^{-1})_{nk} J p_k + \pi_n (M^{-1})_{nn'} J \pi_{n'}] + g^{-2} \int d\tau\, \overline{V}(\sigma^i + g q_n \psi_n^i) \ . \tag{2.13}$$

This Hamiltonian is considerably more complex than that usually encountered in quantum field theory [e.g., Eq. (2.8)]. If it is to be of use we must develop an approximate method of solution, presumably one retaining a connection with the original c-number solution whose properties were the initial motivation of this approach. If we treat the dimensionless coupling constant g as a small parameter, then H contains terms of order $g^{-2}, g^{-1}, g^0, g^1, \ldots$. At least in certain cases it is possible to find explicitly the eigenstates $|\Psi_\alpha\rangle$ of H accurate through zeroth order in g. Their eigenvalues E_α cluster about a large central value $g^{-2}\mathcal{E}$, differing from it by terms of order g^0,

$$E_\alpha = g^{-2} \mathcal{E} + \mathcal{E}_\alpha \ , \tag{2.14}$$

where $\mathcal{E}_\alpha \sim O(g^0)$. As a function of the coordinates z_k, q_n we will write $|\Psi_\alpha\rangle$ as

$$\Psi_\alpha(z_1, \ldots, z_K;\ q_{K+1}, \ldots, q_n, \ldots) = e^{i S(z_1, \ldots, z_K)/g^2} \chi_\alpha(z_1, \ldots, z_K;\ q_{K+1}, \ldots) \ , \tag{2.15}$$

where $S(z_1, \ldots, z_K)$ is independent of g and, to leading order in g, $\chi_\alpha(z_1, \ldots, z_K;\ q_{K+1}, \ldots)$ is independent of g also. The terms of order g^{-2} in the eigenvalue equation

$$(H - E_\alpha) |\Psi_\alpha\rangle = 0 \tag{2.16}$$

involve only the z_k's and can be written

$$\frac{1}{2}\frac{\partial S}{\partial z_k}(M_0^{-1})_{kk'}\frac{\partial S}{\partial z_{k'}}+\frac{1}{2}\int[\vec{\nabla}\sigma^i(\vec{r},z_1,\ldots,z_K)]^2\,d\tau+\int V(\sigma^i(\vec{r},z_1,\ldots,z_K))\,d\tau-\mathcal{E}=0\,, \tag{2.17}$$

where M_0 is a $(K\times K)$ matrix whose matrix elements are

$$(M_0)_{kk'}=\int d\tau\frac{\partial\sigma^i}{\partial z_k}\frac{\partial\sigma^i}{\partial z_{k'}}\,. \tag{2.18}$$

If one derives the Hamilton-Jacobi equation determined by the Lagrangian (2.1), expresses it in terms of the variables z_k, q_n, and then ignores all dependence on the q_n's, Eq. (2.17) is exactly what results. In most cases this Hamilton-Jacobi equation can be solved for a continuous range of the energy $\mathcal{E}$ by a function $S(z_1,\ldots,z_K)$ depending on $\mathcal{E}$ and K integration constants.

In our case, this wide range of possibilities is severely limited because of the requirement that the terms proportional to g^{-1} in Eq. (2.16) must also vanish. More explicitly, these terms are

$$-\frac{\partial S}{\partial z_k}(M_0^{-1})_{kl}\left(q_n\int d\tau\frac{\partial\sigma^i}{\partial z_l}\frac{\partial\psi_n^i}{\partial z_{l'}}\right)(M_0^{-1})_{l'k'}\frac{\partial S}{\partial z_{k'}}+q_n\int d\tau\frac{\partial V(\sigma)}{\partial\sigma^i}\psi_n^i(\tau)+q_n\int d\tau\,\vec{\nabla}\sigma^i\cdot\vec{\nabla}\psi_n^i=0\,, \tag{2.19}$$

where we have simply expanded $(M^{-1})_{kk'}$, $V(\phi)$, and $\vec{\nabla}\phi^i$ to first order in the quantum fluctuation $q_n\psi_n^i(x)$. In the above expression, the repeated indices l and l' are, like k and k', to be summed over from 1 to K. It is convenient to adopt the choice

$$z_k=u_kt+z_k^0 \tag{2.20}$$

discussed in Sec. I. After integrating by parts and using the orthogonality condition (1.5) and the field equation (1.3) satisfied by $\sigma^i(\vec{r},z_1,\ldots,z_K)$

$$-\nabla^2\sigma^i+u_ku_{k'}\frac{\partial^2\sigma^i}{\partial z_k\partial z_{k'}}+\frac{\partial V(\sigma)}{\partial\sigma^i}=0\,, \tag{2.21}$$

Eq. (2.19) becomes

$$\left[\frac{\partial S}{\partial z_k}(M_0^{-1})_{kl}(M_0^{-1})_{l'k'}\frac{\partial S}{\partial z_{k'}}-u_lu_{l'}\right]q_n\int d\tau\psi_n^i\frac{\partial^2\sigma}{\partial z_l\partial z_{l'}}=0\,. \tag{2.22}$$

This is ensured if

$$\frac{\partial S}{\partial z_k}=(M_0)_{kl}u_l. \tag{2.23}$$

[In fact, Eq. (2.19) can be shown to follow from our general parametrization $z_k=z_k(t,z_1^0,\ldots,z_K^0)$ instead of the special choice (2.20), provided that u_l in Eq. (2.23) is replaced by $dz_l(t,z_1^0,\ldots,z_K^0)/dt$, evaluated at $z_k^0=z_k$.] In order for S to exist, we must have

$$\frac{\partial}{\partial z_k}\left(\frac{\partial S}{\partial z_{k'}}\right)=\frac{\partial}{\partial z_{k'}}\left(\frac{\partial S}{\partial z_k}\right), \tag{2.24}$$

or, because of (2.18) and u_k being constant,

$$\int d\tau\frac{\partial\sigma^i}{\partial z_k}\frac{\partial^2\sigma^i}{\partial z_l\partial z_{k'}}u_l-\int d\tau\frac{\partial\sigma^i}{\partial z_{k'}}\frac{\partial^2\sigma^i}{\partial z_l\partial z_k}u_l=0\,. \tag{2.25}$$

We recall that the function $\sigma^i(\vec{r},z_1,\ldots,z_K)$ is derived from a classical time-dependent solution through the substitution (2.20). Thus, we may regard the left-hand side of (2.25) as a function of t, keeping z_k^0 fixed; its time derivative is

$$\int d\tau u_{l'}\frac{\partial}{\partial z_{l'}}\left(\frac{\partial\sigma^i}{\partial z_k}\frac{\partial^2\sigma^i}{\partial z_l\partial z_{k'}}u_l-\frac{\partial\sigma^i}{\partial z_{k'}}\frac{\partial^2\sigma^i}{\partial z_l\partial z_k}u_l\right), \tag{2.26}$$

which is zero on account of (2.21). In any scattering problem between solitons and/or antisolitons, as $t\to-\infty$, $(M_0)_{kl}$ approaches the appropriate limit of free solitons (or antisolitons), which is independent of their positions (i.e., z_k^0). Hence, (2.24) holds in this limit and therefore also at any other time, since it does not change with time.

We are thus led to a unique choice of the function S, obtained by integrating (2.23). It is interesting to note that Eq. (2.23) when combined with Eq. (2.17) implies that the energy $g^{-2}\mathcal{E}$, with $\mathcal{E}$ given by

$$\mathcal{E}=\tfrac{1}{2}u_k(M_0)_{kk'}u_{k'}+\int d\tau[\tfrac{1}{2}(\vec{\nabla}\sigma^i)^2+V(\sigma)]\,, \tag{2.27}$$

is the same as that of the classical solution, and is a constant independent of the parameters z_k.

Next, we examine the terms of order g^0 in the eigenvalue equation (2.16). To do so, we must expand $(M^{-1})_{kk'}$, $V(\phi)$, and $\vec{\nabla}\phi^i$ to second order in

q_n. The final expression assumes a relatively simple form if we first perform a linear transformation among the z_k's so that only one of the u_k's is not zero, say

$$u_K = u \tag{2.28}$$

and

$$u_k = 0 \quad \text{for } k < K . \tag{2.29}$$

The result is

$$\left[-\frac{i}{2}\frac{u}{J}\frac{\partial J}{\partial z_K} - iu\frac{\partial}{\partial z_K} + \tfrac{1}{2}\pi_n \pi_n + \tfrac{1}{2} q_n F_{nn'} q_{n'} + \tfrac{1}{2}(\pi_n G_{nn'} q_{n'} - q_n G_{nn'} \pi_{n'}) - \mathcal{E}_\alpha \right] \chi_\alpha = 0 , \tag{2.30}$$

where $\pi_n = -i(\partial/\partial q_n)$ and $F_{nn'}$ and $G_{nn'}$ are functions of $z_1, \ldots, z_K$ given by

$$F_{nn'} = \int \left[(\vec{\nabla}\psi_n^i)(\vec{\nabla}\psi_{n'}^i) + \psi_n^i \frac{\partial^2 V(\sigma)}{\partial\sigma^i \partial\sigma^j} \psi_{n'}^j \right] d\tau + 3u^2 f_{nk} (M_0^{-1})_{kk'} f_{n'k'} , \tag{2.31}$$

$$f_{nk} = \int \psi_n^i \frac{\partial^2 \sigma^i}{\partial z_k \partial z_K} d\tau ,$$

and

$$G_{nn'} = -G_{n'n} = -u \int \psi_n^i \frac{\partial \psi_{n'}^i}{\partial z_K} d\tau .$$

Before entering the general discussion of how to solve this seemingly complicated eigenvalue equation (2.30), it may be instructive to consider the simplest case in which the classical solution σ^i is time-independent.

1. Static case

If our original classical solution is static, then $u = 0$, and (2.30) reduces to

$$\tfrac{1}{2}(\pi_n \pi_n + q_n F_{nn'} q_{n'}) \chi_\alpha = \mathcal{E}_\alpha \chi_\alpha . \tag{2.32}$$

This equation can be easily solved once the quadratic form $F_{nn'}$ has been diagonalized. The definition (2.31) shows that $F_{nn'}$ is the n, n' matrix element of the operator

$$-\vec{\nabla}^2 + \frac{\partial^2 V(\sigma)}{\partial\sigma^i \partial\sigma^j} . \tag{2.33}$$

Differentiation of the classical field equations (2.21) with respect to z_k reveals that when $u = 0$ the K functions $\partial\sigma/\partial z_k$ are eigenstates of (2.33) with eigenvalue zero. Thus it is possible to choose the ψ_n^i orthogonal to $\partial\sigma^i/\partial z_k$ and also eigenstates of (2.33) with eigenvalues ω_n^2 so that

$$F_{nn'} = \delta_{nn'} \omega_n^2 . \tag{2.34}$$

The resulting spectrum $\mathcal{E}_\alpha$ of (2.32) is given by

$$\mathcal{E}_\alpha = \sum_n (N_n + \tfrac{1}{2}) \omega_n , \tag{2.35}$$

where the N_n's are occupation numbers $= 0, 1, 2, \ldots$.

The eigenstates χ_α solving Eq. (2.32) will provide eigenstates of our original Hamiltonian accurate through order g^0 only if the eigenvalues ω_n^2 are independent of the variables z_k. In most cases of interest a family of static classical solutions $\sigma^i(\vec{r}, z_1, \ldots, z_K)$ reflects a continuous K-parameter symmetry of the original Hamiltonian. The simplest example is the one-dimensional translational symmetry of the equations studied in Sec. III; similarly for an N-component scalar field in one space dimension, K will be equal to N in the case that the N-component fields are all uncoupled. If such a symmetry does exist, we can define the solution $\sigma^i(\vec{r}, z_1, \ldots, z_K)$ as that obtained by acting on a single fixed solution $\sigma^i(\vec{r}, 0, \ldots, 0)$ by that K-parameter symmetry operation determined by the values $z_1, \ldots, z_K$. In our one-dimensional single-soliton example, the single parameter Z is introduced by translating a solution centered at the origin a distance Z to the right. Likewise, we can choose the functions $\psi_n^i(\vec{r}, z_1, \ldots, z_K)$ to be the result of the same symmetry operation applied to $\psi_n^i(\vec{r}, 0, \ldots, 0)$. In this way the momenta conjugate to the variables z_k become generators of the K-parameter symmetry and commute with H. In particular, the matrix elements $F_{nn'}$ and eigenvalues ω_n^2 are independent of the variables z_k. Thus the products (2.15) are eigenstates of H accurate through order g^0. If a sufficient number of variables z_k has been introduced so that the K functions $\partial\sigma/\partial z_k$ comprise all the eigenvectors of (2.33) with zero eigenvalue, then all the ω_n will be different from zero. Consequently, if we consider states with definite values of the conserved momenta $p_1, \ldots, p_K$, our ground state of H will be nondegenerate to order g^0.

2. Time-dependent case

Next, we consider the general case $u \neq 0$. The eigenstates χ_α can be written down formally as

$$\chi_\alpha(z_k, q_n) = \frac{1}{\sqrt{J}} e^{i(\delta p_K) z_K} U(z_K) \times \chi_\alpha(z_1, \ldots, z_{K-1}, 0, q_n) . \tag{2.36}$$

Here δp_K is a constant, $\chi_\alpha(z_1, \ldots, z_{K-1}, 0, q_n)$ is any function of $z_1, \ldots, z_{K-1}$ and the q_n, while $U(z_K)$ is a "time" development operator which satisfies

$$H_2 U(z_K) = iu \frac{\partial}{\partial z_K} U(z_K) , \tag{2.37}$$

where

$$H_2 = \tfrac{1}{2}(\pi_n \pi_n + q_n F_{nn'} q_{n'} + \pi_n G_{nn'} q_{n'} - q_n G_{nn'} \pi_{n'}) \tag{2.38}$$

and

$$U(0) = 1 \ . \tag{2.39}$$

In H_2, the subscript 2 indicates its quadratic dependence on q_n and π_n. The operator $U(z_K)$ is unitary. If the variable z_K/u is interpreted as the time, Eq. (2.37) can be viewed as a Schrödinger-type equation for an infinite-dimensional harmonic oscillator with time-dependent coefficients. Although not shown explicitly, the operator $U(Z_K)$ will, in general, also depend on the variables $z_1, \ldots, z_{K-1}$. The energy eigenvalue $\mathcal{E}_\alpha$ for the state χ_α specified by (2.36) is

$$\mathcal{E}_\alpha = (\delta p_K)\, u \ . \tag{2.40}$$

The somewhat unusual Hamiltonian H_2 appearing in (2.37) has a very direct and simple interpretation. Consider our original problem expressed in terms of the variables z_k, q_n. A solution to the classical equations lying close to the solution

$$z_k^{\rm cl}(t) = u_k t + z_k^0, \quad p_k^{\rm cl}(t) = g^{-2} \frac{\partial S}{\partial z_k}(z_1^{\rm cl}, \ldots, z_n^{\rm cl}) \ , \quad q_n = 0, \quad \pi_n = 0 \tag{2.41}$$

may be written

$$z_k(t) = z_k^{\rm cl}(t) + \delta z_k(t),$$

$$p_k(t) = g^{-2} \frac{\partial S}{\partial z_k}(z_1^{\rm cl} + \delta z_1, \ldots, z_n^{\rm cl} + \delta z_n) + \delta p_k(t) \ ,$$

$$q_n(t) = \delta q_n(t), \quad \pi_n(t) = \delta \pi_n(t) \ . \tag{2.42}$$

If the classical Hamiltonian is expanded in powers of the fluctuations $\delta z_k, \delta p_k, \delta q_n, \delta \pi_n$, the terms quadratic in these quantities and of leading order in g are

$$H_2^{\rm cl} = \tfrac{1}{2}[(\delta \pi_n)^2 + F_{nn'}(\delta q_n)(\delta q_{n'}) + 2G_{nn'}(\delta \pi_n)(\delta q_{n'})] + \tfrac{1}{2} g^2 \delta p_k (M_0^{-1})_{kk'} \delta p_{k'} - 2g u_k \int \frac{\partial \sigma}{\partial z_k} \frac{\partial \psi_n}{\partial z_l} (M_0^{-1})_{lk'} d\tau \, \delta p_{k'} \delta q_n \ . \tag{2.43}$$

As expected, δz_k does not appear in $H_2^{\rm cl}$, and the choice $\delta p_k = 0$ is thus consistent with Hamilton's equations and eliminates the variables $\delta z_k, \delta p_k$ from $H_2^{\rm cl}$ altogether, leaving simply the Hamiltonian (2.38). Therefore, the operator $U(z_K)$ is precisely the time development operator for the system obtained by quantizing those small oscillations about the classical trajectory $z_k = u_k t + z_k^0$ for which all the δp_k are zero.

Thus the problem of finding eigenstates of the complete Hamiltonian (2.13) accurate to order g^0 has been reduced to that of solving the Schrödinger equation for a system of harmonic oscillators with time-dependent coefficients and an infinite number of degrees of freedom. Since such an equation does not have an explicit general solution, we will limit the remainder of this general discussion to expansions about two specific types of classical solutions: (a) slowing varying and (b) periodic. Examples of other situations can be found in Sec. IV.

(a) Slowly varying. If we assume that u is sufficiently small, so that compared to any characteristic frequency ω_n of H_2 we have

$$\left| u \frac{\partial \sigma^i}{\partial z_K} \right| \ll |\omega_n \sigma^i| \ , \tag{2.44}$$

Eq. (2.37) can then be solved by the usual adiabatic approximation. The resulting eigenfunctions $\chi_\alpha(z_k, q_n)$ are

$$\chi_\alpha(z_k, q_n) = \frac{1}{\sqrt{J}} e^{i(\delta p_K) z_K} \prod_n \Big\{ \exp\left[-\frac{i}{u} (N_n + \tfrac{1}{2}) \int_0^{z_K} \omega_n(z_K')\, dz_K' \right] \times \left(\frac{\omega_n}{\pi} \right)^{1/4} (2^{N_n} N_n!)^{-1/2} h_{N_n}(\omega_n^{1/2} q_n)\, e^{-(1/2)\omega_n q_n^2} \Big\} \ , \tag{2.45}$$

where $h_N(x)$ is a Hermite polynomial of order N and the q_n and ω_n are the coordinates and corresponding eigenfrequencies which diagonalize the quadratic form $F_{nn'}$ obtained from (2.31), by replacing the explicit factor u^2 by zero.

(b) Periodic. If we assume that our classical solution $\sigma^i(\vec{r}, t; z_1, \ldots, z_K)$ is periodic with period T, then

$$\sigma^i(\vec{r}, z_1, \ldots, z_{K-1}, z_K) = \sigma^i(\vec{r}, z_1, \ldots, z_{K-1}, z_K + uT) \ . \tag{2.46}$$

The coordinate points $z_1, \ldots, z_K, q_{K+1}, \ldots$ and $z_1, \ldots, z_K + uT, q_{K+1}, \ldots$ determine through Eq. (1.4) the same configuration of our physical system. Thus in our quantum-mechanical description we will use z_K in the range

$$0 \leq z_K \leq uT \tag{2.47}$$

and require that our wave function have the same value at the physically identical points $z_K = 0$ and $z_K = uT$. This condition will give the familiar quantization of energy. For example, if we make the adiabatic approximation of Eq. (2.45), then for the ground state $N_n = 0$ the entire wave function Ψ of Eq. (2.15) will be periodic if

$$\int_0^{uT} \left[p_K(z_K) - \frac{1}{2u} \sum_n \omega_n(z_K) \right] dz_K + (\delta p_K)\, uT = 2\pi n , \tag{2.48}$$

where δp_K is a constant and $p_K \equiv (\partial S/\partial z_K) = (M_0)_{KK} u$, on account of (2.29). By using (2.40), we have $u = \partial \mathcal{E}_\alpha / \partial(\delta p_K)$, so that (2.48) may be written as

$$\int_0^{uT} \left\{ p_K(z_K) + \frac{\partial(\delta p_K)}{\partial \mathcal{E}_\alpha} \left[\mathcal{E}_\alpha - \frac{1}{2} \sum_n \omega_n(z_K) \right] \right\} dz_K = 2\pi n . \tag{2.49}$$

Thus the eigenenergies $g^{-2}\mathcal{E} + \mathcal{E}_\alpha$ are shifted from those given by the usual WKB values by a sum of the zero-point energies of the harmonic oscillators averaged over one period of the classical motion, $\frac{1}{2}\bar{\omega}_n$, where

$$\bar{\omega}_n \equiv (uT)^{-1} \int_0^{uT} \omega_n(z_K)\, dz_K . \tag{2.50}$$

If Eq. (2.49) is to be satisfied for all values of $z_1, \ldots, z_{K-1}$ with a constant value of $\mathcal{E}_\alpha$, it is sufficient to require that $(M_0)_{KK}$ and $\omega_n(z_K)$ be independent of $z_1, \ldots, z_{K-1}$. This can be achieved if we assume it is possible to choose variables $z_1, \ldots, z_{K-1}$, as described in the paragraph following Eq. (2.35), so that their conjugate momenta $-i(\partial/\partial z_1), \ldots, -i(\partial/\partial z_{K-1})$ commute with H. Again, in most cases of interest, a family of degenerate periodic classical solutions $\sigma^i(x, z_1, \ldots, z_{K-1}, z_K^0 + uT)$ exists because the original Hamiltonian possesses a continuous $(K-1)$-parameter symmetry.

In order to quantize fluctuations about a general periodic orbit, we must know some properties of the operator $U(uT)$ determined by the Schrödinger equation (2.37) and Eq. (2.39). In fact, this operator is completely determined by the solution to the classical problem of small oscillations. It can be easily shown from the classical theory of small oscillations[14] that through canonical transformation our variables $\delta q_n(t)$, $\delta \pi_n(t)$ can be chosen so that

$$a_n \delta q_n(T) + b_n \delta \pi_n(T) = e^{-i\beta_n T} [a_n \delta q_n(0) + b_n \delta \pi_n(0)] , \tag{2.51}$$

$$a_n^* \delta q_n(T) + b_n^* \delta \pi_n(T) = e^{i\beta_n T} [a_n^* \delta q_n(0) + b_n^* \delta \pi_n(0)] ,$$

where β_n is a characteristic exponent (or stability angle) and we have required that our classical orbit be stable so that β_n is real. We will choose a_n and b_n so that

$$i(a_n b_n^* - b_n a_n^*) = 1 , \tag{2.52}$$

and define the quantum-mechanical operator

$$A_n \equiv a_n q_n + b_n (-i\partial/\partial q_n) , \tag{2.53}$$

which obeys

$$[A_n, A_{n'}^\dagger] = \delta_{nn'} .$$

Because the Hamiltonian H_2 in (2.37) is quadratic in the q_n and π_n, the Heisenberg equations of motion for the operators q_n and π_n are identical to the classical equations. Therefore, (2.51) can be written

$$U(Tu)^{-1} A_n U(Tu) = e^{-i\beta_n T} A_n \tag{2.54}$$

which implies that

$$U(Tu) = \exp\left[-i \sum_n (A_n^\dagger A_n + \tfrac{1}{2}) \beta_n T \right] . \tag{2.55}$$

The constant $\frac{1}{2}$ in the exponent can be determined from the formula

$$\langle 0 | U(t) | 0 \rangle = \left\{ \det T \left[\exp\left(-i \int_0^t \mathcal{H}(t')\, dt' \right) \right]_{\text{lower}} \right\}^{-1/2} , \tag{2.56}$$

where the matrix $\mathcal{H}$ can be written in four blocks,

$$\mathcal{H} = \begin{pmatrix} \dfrac{\partial^2 H_2}{\partial A_n^\dagger \partial A_{n'}} & \dfrac{\partial^2 H_2}{\partial A_n^\dagger \partial A_{n'}^\dagger} \\ -\dfrac{\partial^2 H_2}{\partial A_n \partial A_{n'}} & -\dfrac{\partial^2 H_2}{\partial A_n \partial A_{n'}^\dagger} \end{pmatrix} ,$$

and only the determinant of the corresponding lower right-hand block is to be taken in evaluating Eq. (2.56). Using Eq. (2.55), we can impose the condition of periodicity in z_K for the wave function

$$\Psi_\alpha(z_k, q_n) = e^{ig^{-2} S(z_1, \ldots, z_K)} \frac{1}{\sqrt{J}} U(z_K) \times \prod_n (A_n^\dagger)^{N_n} \frac{1}{(N_n!)^{1/2}} | 0 \rangle \tag{2.57}$$

and obtain the condition

$$\int_0^{uT} \left\{ p_K(z_K) + \left[\mathcal{E}_\alpha - \sum_n (N_n + \tfrac{1}{2}) \beta_n \right] \frac{\partial(\delta p_K)}{\partial \mathcal{E}_\alpha} \right\} dz_K = 2\pi n \tag{2.58}$$

so that the total energy $(1/g^2)\mathcal{E} + \mathcal{E}_\alpha$ is shifted from the familiar WKB value by

$$\sum_n (N_n + \tfrac{1}{2}) \beta_n \tag{2.59}$$

for a state with the occupation numbers N_n. Thus if the characteristic exponents and corresponding modes of the classical small-oscillation problem can be found, the quantum-mechanical eigenstates and eigenvalues can be found to order g^0. Again we assume that the parameters $z_1, \ldots, z_{K-1}$ can be so defined that their conjugate momenta commute with H guaranteeing that $p_K(z_K)$ and the β_n are independent of $z_1, \ldots, z_{K-1}$ and that Eq. (2.58) can be solved by a constant eigenvalue $\mathcal{E}_\alpha$. If we have introduced sufficient parameters z_k that the requirement $\delta z_k = 0$ eliminates all classical small-oscillation modes with zero characteristic exponent, then, among states with definite values for the conserved quantities $p_1, \ldots, p_{K-1}$, the ground state $N_n = 0$ will be nondegenerate.

This concludes our general discussion. Systematic extension of this procedure to higher order in g may well be possible, as will be considered in the Conclusion.

III. ONE-DIMENSIONAL THEORIES

While the general method that we have developed is applicable to theories in any dimension, explicit examples can be most easily obtained for fields with only one space dimension (besides the time dimension). Considerable attention has already been given in the literature to both the classical solutions[1-3] and some aspects of the quantum solutions,[13] especially of the ϕ^4-coupling theory and the sine-Gordon equation.[4-7] In this and the following two sections, we shall illustrate our general method by examples of various one-dimensional field theories. Although our emphasis is on new developments, naturally some repetition of results that have already been obtained by others is unavoidable.

Let the Lagrangian density of a single Hermitian field ϕ in one space dimension be given by

$$\mathfrak{L} = -\frac{1}{2}\left(\frac{\partial\phi}{\partial x_\mu}\right)^2 - g^{-2}\,V(g\phi)\,, \tag{3.1}$$

where $x_\mu = (x, it)$ and V can be any function of $g\phi$. We assume that V does not contain any derivatives of ϕ, so that the theory is renormalizable. The field equation is

$$\frac{\partial^2\phi}{\partial x_\mu{}^2} - g^{-1}\,V'(g\phi) = 0\,, \tag{3.2}$$

where $V'(\xi) \equiv dV(\xi)/d\xi$. In order to have soliton solutions, the absolute minimum of V is assumed to have degeneracy. Without any loss of generality, we may choose the absolute minimum of $V(g\phi)$ to be zero, which occurs at several different values of $g\phi$, labeled $\alpha_1, \alpha_2, \ldots$ such that

$$V(g\phi) = V(\alpha_j) = 0\,. \tag{3.3}$$

It is convenient to arrange the α_j's in ascending order,

$$\alpha_1 < \alpha_2 < \alpha_3 < \cdots\,. \tag{3.4}$$

Their total number may be either finite or infinite.

A. Classical solution

The time-independent classical solution of a single soliton can be derived by setting $\phi = g^{-1}\sigma(x)$. Equation (3.2) becomes simply

$$\frac{d^2\sigma}{dx^2} - V'(\sigma) = 0\,, \tag{3.5}$$

which implies

$$\frac{1}{2}\left(\frac{d\sigma}{dx}\right)^2 - V(\sigma) = \text{constant}\,. \tag{3.6}$$

Thus, if we regard x as a fictitious "time," the problem becomes identical to one in elementary mechanics, in which there is a point particle at a "position" coordinate σ, moving in a potential $-V(\sigma)$. Because of (3.3), $-V(\sigma)$ has several peaks of equal height at $\sigma = \alpha_1, \alpha_2, \ldots$. In terms of the mechanical analog, the single-soliton (or anti-soliton) solution is one in which the "particle" moves between two neighboring peaks, say α_j and $\alpha_{j\pm1}$, starting from $\sigma = \alpha_j$ at $x = -\infty$ and ending at $\sigma = \alpha_{j+1}$ (or α_{j-1}) at $x = +\infty$. In analytic form, it is given by

$$\int^{\sigma} [2\,V(\sigma)]^{-1/2}\,d\sigma = x\,. \tag{3.7}$$

Because of Lorentz invariance if $g^{-1}\sigma(x)$ is a solution of (3.2), then

$$g^{-1}\sigma(\gamma x - \gamma v t) \tag{3.8}$$

must also satisfy the same field equation, where $\gamma = (1 - v^2)^{-1/2}$.

B. Quantum expansion (c.m. system)

The quantization procedure can be most easily carried out in the center-of-mass system in which the total momentum P is zero. In the notations of Secs. I and II, there is only a single collective coordinate z_k (i.e., $k = K = 1$), which will be labeled Z. The expansion (1.4) now assumes the form

$$\phi = g^{-1}\sigma(x - Z) + q_n(t)\,\psi_n(x - Z)\,, \tag{3.9}$$

where, as in (2.2), the repeated index n is summed over from $n = 2$ to ∞. In accordance with (1.6), the ψ_n's satisfy the orthonormal condition

$$\int_{-\infty}^{\infty} \psi_n(x - Z)\,\psi_{n'}(x - Z)\,dx = \delta_{nn'}\,, \tag{3.10}$$

and the additional orthogonality relation (1.5)

$$\int_{-\infty}^{\infty} [\partial\sigma(x-Z)/\partial Z]\,\psi_n(x-Z)\,dx = 0 \ . \qquad (3.11)$$

Furthermore, according to (2.33) and (2.34), they are the eigenfunctions of a Schrödinger-type equation

$$\left[-\frac{d^2}{dx^2} + V''(\sigma)\right]\psi_n(x-Z) = \omega_n^2\,\psi_n(x-Z) \ , \qquad (3.12)$$

where $V''(\sigma) = d^2V(\sigma)/d\sigma^2$ and $\sigma = \sigma(x-Z)$. This choice (3.12) can also be directly arrived at by simply substituting the expansion (3.9) into the field equation (3.2), setting $q_n(t)$ to be proportional to $\exp(\pm i\omega_n t)$, and then neglecting higher-order terms in q_n. Since $Z \to Z + \delta Z$ represents a translation, the function

$$\psi_1(x-Z) \equiv \frac{\partial}{\partial Z}\sigma(x-Z) \qquad (3.13)$$

must satisfy (3.12) with a zero frequency $\omega_1 = 0$, as can also be readily verified by differentiating (3.5) with respect to Z. From (3.7), it follows that ψ_1 has no node in x; therefore its eigenvalue $\omega_1 = 0$ is the *lowest* of (3.12). Because of (3.11), ψ_1 is excluded in the expansion (3.9).

Because of translational invariance, the Hamiltonian is clearly independent of Z. The conjugate momentum of Z is the total-momentum operator P. In the center-of-mass system, $P = 0$ and therefore (2.13) becomes

$$H = \tfrac{1}{2}J^{-1}[\pi_n(M^{-1})_{nn'}J\pi_n] + g^{-2}\int_{-\infty}^{\infty}\overline{V}(\sigma + gq_n\psi_n)\,dx \ , \qquad (3.14)$$

where $\overline{V}$ is given by (2.6), $\pi_n = -i(\partial/\partial q_n)$, J is the Jacobian of the functional transformation given by (2.12). The mass matrix M is, according to (2.4), given by

$$M_{11} = \int_{-\infty}^{\infty}\left(g^{-1}\frac{\partial\sigma}{\partial Z} + q_n\frac{\partial\psi_n}{\partial Z}\right)^2 dx \ ,$$

$$M_{1n} = M_{n1} = q_{n'}\int_{-\infty}^{\infty}\frac{\partial\psi_{n'}}{\partial Z}\psi_n\,dx \ , \qquad (3.15)$$

and

$$M_{nn'} = \delta_{nn'} \ ,$$

where n and n' are ≥ 2. By using (3.12) and neglecting $O(g)$ corrections, one can readily reduce (3.14) to simply

$$H = \frac{1}{2}\sum_{n=2}^{\infty}(\pi_n^2 + \omega_n^2 q_n^2) + m \ ; \qquad (3.16)$$

the corresponding energy spectrum is given by

$$E = m_r + \sum_{n=2}^{\infty} N_n\,\omega_n \ , \qquad (3.17)$$

where m and m_r are, respectively, the unrenormalized and renormalized masses of the soliton, and N_n is an occupation number, $N_n = 0, 1, 2, \ldots$. Neglecting radiative corrections, one has

$$m = m_r = g^{-2}\int_{-\infty}^{\infty}\left[\frac{\partial\sigma(x)}{\partial x}\right]^2 dx \ . \qquad (3.18)$$

If radiative corrections are included, then $m_r = m + \frac{1}{2}\sum\omega_n + O(g^2)$. Since the difference $m_r - m$ is $O(g^0)$, (3.18) is correct to $O(g^{-2})$.

In general, the energy spectrum (3.17) consists of discrete levels and continuum. We may expand the (renormalized) function $V(\sigma)$ near one of its minima, say $\sigma = \alpha_j$:

$$V(\sigma) = \tfrac{1}{2}\mu_j^2(\sigma - \alpha_j)^2 + O((\sigma - \alpha_j)^3) \ . \qquad (3.19)$$

If $\sigma(x)$ is the classical soliton solution which varies in the region

$$\alpha_j \leq \sigma \leq \alpha_{j+1} \ ,$$

then in (3.12), $V''(\sigma) = \mu_j^2$ at $x = -\infty$ and $V''(\sigma) = \mu_{j+1}^2$ at $x = +\infty$. Thus, the continuum starts at $E = m + \mu$ where μ is the smaller of μ_j and μ_{j+1}. Further discussion of the eigenvalue problem (3.12) will be given in Sec. III D.

C. Quantum expansion (moving system)

Next, we illustrate our general method by carrying out the quantization in a system in which the total momentum P is *not* zero. This also gives the simplest example of performing quantum expansion around a classical time-dependent solution. We begin with the solution (3.8). It is convenient to introduce

$$\theta \equiv \gamma(x - Z) \ , \qquad (3.20)$$

where γ is treated as a *fixed* constant and Z is a coordinate variable. The expansion (1.4) becomes

$$\phi = g^{-1}\sigma(\theta) + q_n(t)\,\psi_n(\theta) \ , \qquad (3.21)$$

where $\sigma(\theta)$ and ψ_n's satisfy

$$\frac{d^2\sigma(\theta)}{d\theta^2} - V'(\sigma) = 0 \ ,$$

$$\int\psi_n(\theta)\,\psi_m(\theta)\,dx = \delta_{nm} \ , \qquad (3.22)$$

and

$$\int\psi_n(\theta)\,\frac{\partial\sigma(\theta)}{\partial Z}\,dx = 0 \ .$$

Throughout our discussion, the integration in x always extends from $x = -\infty$ to $x = +\infty$. As in (3.12), we may choose $\psi_n(\theta)$ to be the eigenfunctions of

$$\left[-\frac{d^2}{d\theta^2} + V''(\sigma)\right]\psi_n(\theta) = \omega_n^2\,\psi_n(\theta) \ . \qquad (3.23)$$

The sum in (3.21) then extends over all such eigenfunctions except the one with the lowest eigenvalue; i.e., those with $\omega_n \neq \omega_1 = 0$. The Hamiltonian H is given by (2.13). Noting that in the present case, $k = K = 1$, $z_1 = Z$, and $p_1 =$ total momentum operator P, we may expand H in powers of q_n and π_n:

$$H = H_0 + H_1 + H_2 + H_3 + \cdots , \tag{3.24}$$

where H_0 is independent of q_n and π_n, H_1 depends on q_n and π_n linearly, H_2 quadratically, H_3 cubically, etc. In explicit form, H_0 and H_1 are given by

$$H_0 = \tfrac{1}{2}(\gamma + \gamma^{-1})m + \tfrac{1}{2}(m\gamma)^{-1} P^2 , \tag{3.25}$$

and

$$H_1 = g^{-1}[1 - \gamma^{-2} - (m\gamma)^{-2} P^2] q_n \int \frac{\partial \psi_n}{\partial Z} \frac{\partial \sigma}{\partial Z} dx , \tag{3.26}$$

where γ is the constant introduced in (3.20), and m is given by (3.18) which is *independent* of γ. In order that H_0 give the correct energy to $O(g^{-2})$, H_1 must be zero, at least to $O(g^{-1})$. [Otherwise, by combining H_1 and H_2 one would be led to $q_n \sim O(g^{-1})$ and consequently also to additional $O(g^{-2})$ terms in energy.] Thus by defining

$$v \equiv (1 - \gamma^{-2})^{1/2} ,$$

we find that

$$H_1 = 0$$

provided

$$P = m\gamma v . \tag{3.27}$$

The same relation also leads to

$$H_0 = m\gamma. \tag{3.28}$$

Now, m is $O(g^{-2})$; the above argument shows that to the same order, P is given by $m\gamma v$ and the energy is $m\gamma$.

A somewhat more complicated situation emerges if we want to carry out our calculation to the accuracy $\sim O(1)$. Since P is a constant of motion, its value may be different from $m\gamma v$ by an additional $O(1)$ term. Indeed, for an excited state, moving with velocity v, we should expect

$$P = m\gamma v + N_n \omega_n \gamma v, \tag{3.29}$$

where the repeated index n is to be summed over, and N_n is an occupation number $= 0, 1, 2, \ldots$. Regarding $N_n \sim O(1)$, we find that, after substituting (3.29) into (3.25) and (3.26),

$$H_1 \sim O(g)$$

and

$$H_0 = m\gamma + N_n \omega_n \gamma v^2 + O(g^2). \tag{3.30}$$

To the same order of accuracy, one must include H_2. By using (3.27), or (3.29), one can verify that

$$H_2 = \tfrac{1}{2}(\pi_n \pi_n + q_n F_{nn'} q_{n'} + \pi_n G_{nn'} q_{n'} - q_n G_{nn'} \pi_{n'}), \tag{3.31}$$

where, in agreement with (2.31),

$$F_{nn'} = \int dx \left[\frac{\partial \psi_n}{dx} \frac{\partial \psi_{n'}}{dx} + \psi_n \psi_{n'} \frac{d^2 V(\sigma)}{d\sigma^2} \right] + 3v^2 \left[\int \left(\frac{\partial \sigma}{\partial Z} \right)^2 dx \right]^{-1} f_n f_{n'} , \tag{3.32}$$

$$f_n = \int \frac{\partial \psi_n}{\partial Z} \frac{\partial \sigma}{\partial Z} \, dx, \tag{3.33}$$

$$G_{nn'} = -G_{n'n} = -v \int \psi_n \frac{\partial \psi_{n'}}{\partial Z} dx \tag{3.34}$$

and $\sigma = \sigma(\theta)$. In the present case, because of translational invariance, $F \equiv (F_{nn'})$ and $G \equiv (G_{nn'})$ are constant matrices, independent of Z. The diagonalization of H_2 is straightforward; the details are given in Appendix A. As will be shown there, the spectrum of H_2 is given by, apart from an additive constant related to renormalization,

$$N_n \omega_n / \gamma. \tag{3.35}$$

Together with (3.30), the energy spectrum of the entire Hamiltonian becomes, as expected,

$$(m + N_n \omega_n)\gamma. \tag{3.36}$$

D. Vibrational modes

The vibrational mode ψ_n is determined by (3.12). It is well known that for either the sine-Gordon equation, for which

$$V(\sigma) = \mu^2(1 - \cos\sigma), \tag{3.37}$$

or the quartic coupling theory, for which

$$V(\sigma) = \tfrac{1}{8} \mu^2 (1 - \sigma^2)^2, \tag{3.38}$$

the ψ_n's are related[4] to the associated Legendre functions $P_l^m(y)$, which satisfy the differential equation

$$\left[(1-y^2) \frac{d^2}{dy^2} - 2y \frac{d}{dy} + l(l-1) - \frac{m^2}{1-y^2} \right] P_l^m(y) = 0. \tag{3.39}$$

To see this, we shall follow a slightly more general approach than that in the literature. We shall first try to deduce the appropriate functional relation between the variable y and the space coordinate x. By a change of variable $x \to y(x)$, Eq. (3.12) can be written as

$$\left[\left(\frac{dy}{dx} \right)^2 \frac{d^2}{dy^2} + \left(\frac{d^2 y}{dx^2} \right) \frac{d}{dy} - V''(\sigma) + \omega_n^2 \right] \psi_n = 0. \tag{3.40}$$

In order that this equation should reduce to the associated Legendre equation (3.39), we must have the same ratio between the coefficients of d^2/dy^2 and d/dy in these two equations. Thus,

$$\frac{d^2y/dx^2}{(dy/dx)^2} = -\frac{2y}{1-y^2}. \tag{3.41}$$

A simple integration leads to

$$\frac{dy}{dx} = \kappa(1-y^2), \tag{3.42}$$

where κ is an integration constant. A further integration gives, apart from a trivial additive constant in x,

$$y = \tanh(\kappa x). \tag{3.43}$$

Next, by comparing the (d/dy)-independent term in these two equations, (3.39) and (3.40), we obtain

$$-\frac{d^2V(\sigma)}{d\sigma^2} + \omega_n^2 = \kappa^2[\,l(l-1)(1-y^2) - m^2\,]. \tag{3.44}$$

Let us consider the limit $x \to \pm\infty$; because of (3.43), $y \to \pm 1$, and because of (3.7), (3.37), and (3.38), $(d^2V/d\sigma^2) \to \mu^2$. Thus, (3.44) reduces to

$$\omega_n^2 = \mu^2 - m^2\kappa^2. \tag{3.45}$$

Since the lowest eigenvalue of (3.40) is $\omega_1 = 0$, while the maximum integer value of m is l, this implies that

$$l\kappa = \mu. \tag{3.46}$$

The corresponding eigenfunctions, $d\sigma/dx$ for (3.40) and $P_l^l(y) = \mathrm{sech}^l(\kappa x)$ for (3.39), must be proportional to each other; i.e.,

$$\frac{d\sigma}{dx} = (\lambda\kappa)\mathrm{sech}^l(\kappa x), \tag{3.47}$$

where $\lambda\kappa$ is the proportionality constant. Furthermore, according to (3.7),

$$\frac{1}{2}\left(\frac{d\sigma}{dx}\right)^2 = V(\sigma); \tag{3.48}$$

therefore,

$$V(\sigma) = \tfrac{1}{2}\lambda^2\kappa^2\mathrm{sech}^{2l}(\kappa x). \tag{3.49}$$

From (3.47) and (3.49), one can verify that (3.44) is satisfied, showing the consistency between these expressions. In the following, we discuss a few special cases.

(i) For $l = 1$, from (3.47) we find

$$\sigma = 2\lambda \tan^{-1} e^{\kappa x}, \tag{3.50}$$

and from (3.49)

$$V = \tfrac{1}{2}\lambda^2\kappa^2\sin^2(\sigma/\lambda), \tag{3.51}$$

which is the same as the sine-Gordon potential (3.37), provided $\lambda = 2$ and $\kappa = \mu$. From (3.45), one also sees that the second-lowest eigenvalue of (3.40) is

$$\omega_2 = \mu, \tag{3.52}$$

which is at the beginning of the continuum $\omega_n \geq \mu$.

(ii) Likewise, for $l = 2$, we find

$$\sigma = \lambda \tanh(\kappa x)$$

and

$$V = \tfrac{1}{2}\lambda^2\kappa^2[\,1 - (\sigma/\lambda)^2\,]^2, \tag{3.53}$$

which reduces to the quartic potential (3.38), provided $\lambda = 1$ and $\kappa = \frac{1}{2}\mu$. From (3.45), it follows that the second lowest eigenvalue of (3.40) and its corresponding eigenfunction are

$$\omega_2 = (\tfrac{1}{2}\sqrt{3}\,)\mu \text{ and } \psi_2 \propto P_2^1(y); \tag{3.54}$$

the third lowest eigenvalue and its corresponding eigenfunction are

$$\omega_3 = \mu \text{ and } \psi_3 \propto P_2^0(y), \tag{3.55}$$

which is also at the beginning of the continuum.

(iii) For $l \geq 3$, while V is defined by (3.47) and (3.49), the resulting function $V = V(\sigma)$ contains unphysical singularities in σ. This can be most easily seen by investigating the behavior of $V(\sigma)$ near $x = \infty$. Let us assume $\sigma \to \sigma_0$ as $x \to \infty$. From (3.47) and (3.49), one can verify that in the same limit

$$V(\sigma) \to \tfrac{1}{2}\mu^2(\sigma-\sigma_0)^2 + O\big((\sigma-\sigma_0)^{2+2/l}\big), \tag{3.56}$$

which implies that $V(\sigma)$ has a branch point at $\sigma = \sigma_0$ if l is ≥ 3.

Remarks. Normally, to obtain the vibrational modes, one would start from a Lagrangian. The above discussion allows us to take a different direction from the usual one: We may now explore the inverse problem, in which one starts from the vibrational modes and then constructs the appropriate field theory. For example, if it is given that the vibrational modes ψ_n, expressed in terms of a certain variable y, are known solutions of a differential equation, say

$$\left[\frac{d^2}{dy^2} + P(y)\frac{d}{dy} + Q_n(y)\right]\psi_n = 0, \tag{3.57}$$

what would then be the corresponding field theory?

By comparing (3.40) and (3.57), the functional relation between y and x can be determined. Just as in (3.41), we have

$$\frac{d^2y/dx^2}{(dy/dx)^2} = P(y); \tag{3.58}$$

therefore,

$$\frac{dy}{dx} = \exp\left[\int^{y} P(y)dy\right]. \tag{3.59}$$

The functional relation between σ and x can be obtained by noting that for $n = 1$

$$\psi_1(y) = \frac{d\sigma}{dx}. \tag{3.60}$$

The field theory is then determined, because

$$V(\sigma) = \frac{1}{2}\left(\frac{d\sigma}{dx}\right)^2. \tag{3.61}$$

It turns out that if ψ_n's are hypergeometric functions, all physically interesting examples have already been studied; they are simply the sine-Gordon equation and the quartic-coupling case. However, if we consider differential equations of a more general nature, such as those with four or more regular singularities, then there are new physically interesting possibilities. An example will be given in Sec. V.

E. Statistics

Recently, Coleman[5,6] has shown that the sine-Gordon theory is equivalent to the massive Thirring model, the solitons in the former become the fermions in the latter. Because of this equivalence, the question of the appropriate statistics of solitons in a general case has attracted some attention. In this section, we wish to emphasize once again that the statistics of any one-dimensional particles, and solitons in particular, is entirely a matter of arbitrary choice.

It is instructive to examine first the classical description of a multisoliton system. As discussed in Sec. III A, a single-soliton solution is one in which the classical solution σ varies between two neighboring peaks of $-V(\sigma)$, say $\sigma = \alpha_j$ at $x = -\infty$ and $\sigma = \alpha_{j+1}$ at $x = +\infty$. [See (3.3) for the definition of α_j.] The K-soliton solution is one in which σ varies between $K+1$ neighboring peaks of $-V(\sigma)$, say $\sigma = \alpha_j$ at $x = -\infty$, and $\sigma = \alpha_{j+K}$ at $x = +\infty$. By following a discussion similar to that given in Sec. III A, one can show that for $K \geq 2$ all such solutions are time-dependent; the over-all change in σ

$$\sigma(\infty) - \sigma(-\infty) = \alpha_{j+K} - \alpha_j \tag{3.62}$$

is, of course, time-independent.

Let us assume that at least as $t \to \pm\infty$, the classical solution varies from α_j to α_{j+K} in K steps: At $x \cong z_1(t)$ the solution changes from $\sigma \cong \alpha_j$ to $\cong \alpha_{j+1}$, then at $x \cong z_2(t)$ from $\sigma \cong \alpha_{j+1}$ to $\cong \alpha_{j+2}$, ... until at $x \cong z_K(t)$ from $\sigma \cong \alpha_{j+K-1}$ to $\cong \alpha_{j+K}$. In this solution, the energy density is concentrated at $z_1(t)$, $z_2(t), \ldots, z_K(t)$, which may therefore be regarded as the coordinates of the K solitons. These K solitons are clearly distinguishable, since the field σ near any one is different from that near the other. Yet, because of the way they are defined, these K solitons are always kept in a fixed order:

$$z_1(t) \leq z_2(t) \leq z_3(t) \leq \cdots \leq z_K(t). \tag{3.63}$$

This means that, in one dimension, even classically it is *impossible* to interchange the position of any two solitons. Thus, the problem of their statistics cannot be related to any matter of principle, but rather only to questions of either convention, or convenience.

Next, we recall that in any one-dimensional problem of K identical particles in quantum mechanics, the question of statistics is always a matter of convention. As we shall see, in one dimension if singular potentials are admitted there is *no fundamental difference between fermions and bosons*. A one-dimensional system of fermions can always be viewed as a system of bosons, but with an additional infinite repulsive core between them. Conversely, a one-dimensional system of bosons can always be viewed as a system of fermions, but with an additional short-range attractive potential between them. This additional attractive potential may be viewed as the zero-range limit of a square-well potential with depth u and range r; as $r \to 0$, $u \to \infty$ such that

$$(mu)^{1/2} r = \tfrac{1}{2}\pi, \tag{3.64}$$

where m is the mass of each particle. Thus although the wave function vanishes when the relative coordinate between these two fermions is zero, its derivative vanishes at the edge of this attractive potential, simulating the typical behavior between bosons.

In the case of one-dimensional solitons, this question is further obscured since, in general, the interaction between solitons cannot be exactly described by a two-body potential in the first place.

IV. TWO-SOLITON SOLUTIONS

In this section we will illustrate the general method of Sec. II by briefly discussing the quantization of the time-dependent,[15] two-soliton solutions of the sine-Gordon equation

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \frac{\mu^2}{g^2}\sin g\phi = 0, \tag{4.1}$$

in one space and one time dimension. These solutions, with zero total momentum, are of two forms

$$\sigma_{ss}(x,t)=\frac{4}{g}\tan^{-1}\left(\frac{u\sinh\gamma_u x\mu}{\cosh\gamma_u ut\mu}\right) \tag{4.2}$$

and

$$\sigma_{s\bar{s}}(x,t)=\frac{4}{g}\tan^{-1}\left(\frac{\sinh\gamma_u ut\mu}{u\cosh\gamma_u x\mu}\right), \tag{4.3}$$

where $\gamma_u=(1-u^2)^{-1/2}$. Actually the first solution describes soliton-soliton scattering, containing two regions in each of which the field increases by $2\pi/g$ for increasing x. The second solution represents soliton-antisoliton scattering, since at large $|t|$, for increasing x it increases by $\sim 2\pi/g$ for $x\sim -ut$ but decreases by $\sim 2\pi/g$ for $x\sim +ut$. These solutions both have energy $(16\mu/g^2)\gamma_u$, just twice the energy of a single soliton moving with velocity u. Finally, if an imaginary velocity

$$u=iw \tag{4.4}$$

is substituted in the solution (4.3), the "breather solution" results:

$$\sigma_{\rm br}(x,t)=\frac{4}{g}\tan^{-1}\left(\frac{\sin\gamma_w wt\mu}{w\cosh\gamma_w x\mu}\right), \tag{4.5}$$

where $\gamma_w=(1+w^2)^{-1/2}$. Consideration of these examples will permit a quantum-mechanical discussion of soliton-soliton and soliton-antisoliton scattering as well as a concrete application of our method to a periodic classical solution, $\sigma_{\rm br}$.

A. Soliton-soliton scattering

The most natural set of coordinates for quantizing the soliton-soliton solution (4.2) are obtained from the family of classical solutions $\sigma_{ss}(x,t)$ by writing

$$\phi(x)=\sigma_{ss}(x-Z,u^{-1}z)+q_n\psi_n(x-Z,z) \tag{4.6}$$

as in Eqs. (1.4)–(1.6). However, the change of coordinates from the usual linear set $\mathcal{Q}_s$ of Eq. (2.7) to Z,z,q_n is not well defined in the region $q_n\simeq 0$ where we wish to use it. Because $\sigma_{ss}(x-Z, u^{-1}z)$ is an even function of z, $\partial\sigma_{ss}/\partial z$ will vanish at $z=0$, causing the Jacobian for change of coordinates J, Eq. (2.12), to vanish at the point $z=q_n=0$. Thus the proposed change of coordinates is singular at this point. One method for avoiding this difficulty uses an alternative variable

$$\zeta=\ln(\cosh\gamma_u z\mu),\quad -\infty\leq\zeta\leq\infty. \tag{4.7}$$

Positive ζ corresponds to physical soliton-soliton separation while the region of negative ζ does not have a simple, physical, two-soliton interpretation. Adopting this coordinate, we write

$$\phi(x)=\sigma(x-Z,\zeta)+q_n\psi_n(x-Z,\zeta), \tag{4.8}$$

where

$$\sigma(x-Z,\zeta)=\frac{4}{g}\tan^{-1}\left[\frac{u\sinh\gamma_u(x-Z)\mu}{e^{\zeta}}\right] \tag{4.9}$$

and the $\psi_n(y,\zeta)$ obey

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{\partial\sigma}{\partial y}(y,\zeta)\psi_n(y,\zeta)dy=0,\\ &\int_{-\infty}^{\infty}\frac{\partial\sigma}{\partial\zeta}(y,\zeta)\psi_n(y,\zeta)dy=0.\end{aligned} \tag{4.10}$$

We can now find those terms in the Hamiltonian H of (2.13) which are of order $1/g^2$ and determine the Hamilton-Jacobi equation (2.17). Straightforward integration yields

$$\begin{aligned}(M_0)_{ZZ}&=\int_{-\infty}^{\infty}\left(\frac{\partial\sigma}{\partial Z}(x-Z,\zeta)\right)^2dx\\ &=\frac{64\mu}{g^2\gamma_u}\frac{e^{2\zeta}}{u^2\sinh^3 a}(\sinh a+a)\left(\frac{e^{2\zeta}}{u^2}-1\right),\\ (M_0)_{\zeta\zeta}&=\int_{-\infty}^{\infty}\left(\frac{\partial\sigma}{\partial\zeta}(x-Z,\zeta)\right)^2dx\\ &=\frac{64}{\mu g^2}\frac{e^{4\zeta}}{\gamma_u u^4\sinh^3 a}(\sinh a-a),\\ \frac{1}{g^2}\int_{-\infty}^{\infty}\bar{V}(\sigma)dx&=\tfrac{1}{2}(M_0)_{ZZ}+\frac{\mu^2}{g^2}\int_{-\infty}^{\infty}[\,1-\cos(\sigma(x-Z,\zeta))\,]\,dx\\ &=\tfrac{1}{2}(M_0)_{ZZ}+\tfrac{1}{2}\mu^2(M_0)_{\zeta\zeta},\\ (M_0)_{Z\zeta}&=0,\end{aligned} \tag{4.11}$$

where

$$\cosh a=\frac{2e^{2\zeta}}{u^2}-1. \tag{4.12}$$

We use the notation of Sec. II, except the variables z_k $(k=1,\ldots,K)$ with $K=2$ are now replaced by Z and ζ. Likewise, when k appears as a subscript it is replaced by the appropriate Z or ζ.

Because our classical solution, $\sigma(x-Z_0,\zeta)$, where $\zeta=\ln\{\cosh[\gamma_u\mu(z_0+ut)]\}$, has zero center-of-mass velocity, vanishing of the order g^{-1} terms of Eq. (2.16) requires that we choose $p_Z=\partial S/\partial Z=0$. The resulting Hamilton-Jacobi equation (2.17),

$$\frac{1}{g^4}\frac{1}{2(M_0)_{\zeta\zeta}}\left(\frac{\partial S}{\partial\zeta}\right)^2+\frac{1}{g^2}\bar{V}(\zeta)-\frac{1}{g^2}\mathcal{E}=0. \tag{4.13}$$

is, of course, the familiar WKB approximation for the terms of order g^{-2} in H. If the specific integrals (4.11) are substituted in Eq. (4.13), it becomes

$$\begin{aligned}\frac{1}{g^4}\frac{1}{2(M_0)_{\zeta\zeta}}\left(\frac{\partial S}{\partial\zeta}\right)^2&-\tfrac{1}{2}(M_0)_{\zeta\zeta}\gamma_u{}^2\mu^2u^2(1-e^{-2\zeta})\\ &+\frac{16\mu\gamma_u}{g^2}-\frac{1}{g^2}\mathcal{E}=0.\end{aligned} \tag{4.14}$$

Again the vanishing of order g^{-1} terms, Eq. (2.19),

requires that $\mathcal{E}$ equal the energy of our initial classical solution (4.2), for which the WKB solution

$$S(\zeta) = \pm \int^{\zeta} d\zeta (M_0)_{\zeta\zeta} u (1 - e^{-2\zeta})^{1/2} \gamma_u \mu g^2 \quad (4.15)$$

has a turning point at $\zeta = 0$. Thus, although (4.13) determines the form of the wave function for $\zeta \ll 0$ and $0 \ll \zeta$ to leading order in g, the vanishing of the order g^{-2} terms in H for $\zeta \simeq 0$ requires us to go to Eq. (2.30) which also contains the q_n to connect the two solutions. The required detailed consideration of small oscillations for the sine-Gordon equation is beyond the scope of this paper. However, we can observe that if the normalization integral over the region $\zeta \ll 0$ is to be finite, then only one of the two solutions (4.15) will be allowed, as is generally the case for a system of two identical particles, either bosons or fermions. Finally, it should be noted that the complexity of the point $\zeta = 0$ where the two solitons coincide effectively obscures the question of which convention of their statistics is the most convenient one. This uncertainty should be expected since the interaction between solitons cannot be exactly described by a two-body potential.

B. Soliton-antisoliton scattering

As in the previous case, expansion about the classical solution $\sigma_{s\bar{s}}(x, t)$ of (4.3) is complicated by the presence of Jacobian singularities. If we write

$$\phi(x) = \frac{4}{g} \tan^{-1}\left[\frac{\sinh\gamma_u z\mu}{u \cosh\gamma_u (x - Z)\mu}\right] + q_n \psi_n(x - Z, z) \quad (4.16)$$

the resulting coordinate change is again singular for $z = q_n = 0$ since at that point $\phi(x)$ is completely independent of Z. Such a situation will always occur when the classical solution vanishes at all points x at a given time. This difficulty can be neatly avoided if we give the classical solution a center-of-mass velocity v. Now the times at which ϕ vanishes for different values of x will no longer be simultaneous and the singularity goes away. Therefore, we use the expansion

$$\phi(x) = \frac{4}{g} \tan^{-1}\left\{\frac{\sinh[\mu\gamma_u(\gamma_v^{-1} z - uv\gamma_v(x - Z))]}{u \cosh[\mu\gamma_u\gamma_v(x - Z)]}\right\} + q_n \psi_n(x - Z, z). \quad (4.17)$$

This new set of coordinates is completely regular in the region $q_n \simeq 0$. The first term on the right-hand side of (4.17) becomes the Lorentz transform of (4.3) for $Z = vt$, $z = ut$.

If we apply the method of Sec. II, we will find a single quantum state with center-of-mass momentum $p_Z = 16\mu v\gamma_u\gamma_v/g^2$ and, for large z, relative momentum $p_z = 16\mu u\gamma_u/g^2\gamma_v^3(1 - u^2v^2)$ plus quantum excitations. Although we are considering two distinct particles, we have found just a single solution with fixed p_Z and p_z. This is because p_z is a signed quantity and to find the solution with relative momentum $-p_z$ we would have to expand about the very different classical solution in which the relative velocity u was replaced by $-u$. Our quantization procedure does not predict transitions between these two distinct quantum states even though they have the same energy. Such is, however, not the case for soliton-soliton scattering. Even if we define coordinates in a moving system as in (4.17)

$$\phi(x) = \frac{4}{g} \tan^{-1}\left\{\frac{u \sinh\mu\gamma_u\gamma_v(x - Z_\mp)}{\cosh\mu\gamma_u[\gamma_v^{-1} z_\pm \mp uv\gamma_v(x - Z_\pm)]}\right\} + q_n^{\pm} \psi_n(x - Z_\pm, z_\pm) \quad (4.18)$$

the quantum states determined using each of these sets of coordinates are identical, being connected by the simple coordinate transformation

$$z_+ = -z_-, \quad Z_+ = Z_-, \quad q_n^+ = q_n^-. \quad (4.19)$$

C. Breather mode

Finally, we consider quantization of the breather solution (4.5). Again a regular set of coordinates can be found by boosting the solution (4.5) to velocity v and writing

$$\phi(x) = \frac{4}{g} \tan^{-1}\left\{\frac{\sin[\gamma_w\mu(\gamma_v^{-1} z - wv\gamma_v(x - Z))]}{w \cosh[\gamma_w\gamma_v\mu(x - Z)]}\right\} + q_n \psi_n(x - Z, z). \quad (4.20)$$

Since the physical field $\phi(x)$ is a periodic function of z with period $2\pi\gamma_v/\mu\gamma_w$, we restrict z to the single interval

$$0 \leq z \leq 2\pi\gamma_v/\mu\gamma_w \quad (4.21)$$

and require that the wave function $\Psi(Z, z, q_3, q_4, \ldots)$ satisfy

$$\Psi(Z, 0, q_3, q_4, \ldots) = \Psi(Z, 2\pi\gamma_v/\mu\gamma_w, q_3, q_4, \ldots). \quad (4.22)$$

The phase iS/g^2 entering Ψ and solving the Hamilton-Jacobi equation (2.17) can be directly determined from Eq. (2.19) which follows from the vanishing of the order g^{-1} terms in the time-independent Schrödinger equation (2.16)

$$\frac{\partial S}{\partial Z} = p_Z = (M_0)_{ZZ} v + (M_0)_{Zz} w, \qquad \frac{\partial S}{\partial z} = p_z = (M_0)_{zZ} v + (M_0)_{zz} w. \quad (4.23)$$

The right-hand side of Eq. (4.23) is simply the value of p_z and p_x, as functions of z, for our classical solution. If we now impose the periodicity requirement (4.22) including the terms of order g^0, following the method of Sec. II, we find that (2.58) becomes

$$\frac{16\mu^2}{g^2}\gamma_w^2\int_0^{2\pi\gamma_v/\mu\gamma_w}dz\left\{\int_{-\infty}^{\infty}dx\left[v\frac{\partial\sigma}{\partial\theta_1}\frac{\partial\sigma}{\partial\theta_2}+w\left(\frac{\partial\sigma}{\partial\theta_1}\right)^2\right]\right\}+\left(\mathcal{E}_\alpha-\sum_n\tfrac{1}{2}\beta_n\right)2\pi\gamma_v/\gamma_w w\mu=2\pi n\ , \tag{4.24}$$

where

$$\sigma(\theta_1,\theta_2)=\tan^{-1}\left(\frac{\sin\theta_1}{w\cosh\theta_2}\right)$$

and (4.25)

$$\theta_1=\gamma_w[\gamma_v^{-1}z-wv\gamma_v(x-Z)]\mu,\quad \theta_2=\gamma_w\gamma_v(x-Z)\mu.$$

If we translate the x integration by $Z-vz/w$ and then change integration variables to

$$x'=\gamma_v(x-vz/w),\quad z'=\gamma_v(z-vwx), \tag{4.26}$$

Eq. (4.24) becomes

$$\frac{16\mu^2}{g^2}\gamma_w^2\int_{-\infty}^{\infty}dx'\int_{-vwx'}^{2\pi(\gamma_w\mu)^{-1}-vwx'}dz'\left[v\frac{\partial\sigma}{\partial\theta_1}\frac{\partial\sigma}{\partial\theta_2}+w\left(\frac{\partial\sigma}{\partial\theta_1}\right)^2\right]+\left(\mathcal{E}_\alpha^{\text{c.m.}}-\sum_n\tfrac{1}{2}\beta_n^{\text{c.m.}}\right)2\pi/\gamma_w w\mu=2\pi n, \tag{4.27}$$

where $\mathcal{E}_\alpha^{\text{c.m.}}$ and $\beta_n^{\text{c.m.}}$ are now center-of-mass quantities: As we saw in (2.35), $\mathcal{E}_\alpha^{\text{c.m.}}=\mathcal{E}_\alpha\gamma_v$, while by definition $\beta_n^{\text{c.m.}}=\gamma_v\beta_n$. In terms of Z' and x', θ_1 and θ_2 become

$$\theta_1=\gamma_w z'\mu,\quad \theta_2=\gamma_w x'\mu. \tag{4.28}$$

Periodicity in the z' variable allows us to rewrite the integral in (4.27) as

$$\frac{16\mu^2}{g^2}\gamma_w\int_{-\infty}^{\infty}dx'\int_0^{2\pi/\gamma_w\mu}dz'\left(\frac{\partial\sigma}{\partial\theta_1}\right)^2=\frac{32\pi}{g^2}\sin^{-1}\gamma_w. \tag{4.29}$$

This can be combined with the value obtained by Dashen *et al.*[4] for the classical sum of characteristic exponents

$$\tfrac{1}{2}\sum_n\beta_n\frac{2\pi}{\gamma_w w\mu}=-\frac{4}{w}+4\sin^{-1}\gamma_w+\delta E\frac{2\pi}{\gamma_w w\mu} \tag{4.30}$$

to yield

$$\frac{32\pi}{g^2}\left(1-\frac{g^2}{8\pi}\right)\sin^{-1}\gamma_w=2\pi n \tag{4.31}$$

if we choose $\mathcal{E}_\alpha=-2\gamma_w\mu/\pi+\delta E$ where δE contains the divergent vacuum energy and a term proportional to the elementary boson's self-energy. Thus

$$\gamma_w=\sin\frac{\gamma'}{16}n,$$

and the mass spectrum of the breather state is given by

$$M_n=\frac{16\mu_r}{g^2}\gamma_w-\delta E+\mathcal{E}_\alpha=\frac{16\mu_r}{\gamma'}\sin\frac{\gamma'}{16}n. \tag{4.32}$$

Here μ_r is the renormalized boson mass and

$$\gamma'=\frac{g^2}{1-g^2/8\pi}.$$

Equation (4.32) gives precisely the mass spectrum for the breather mode found previously by Dashen *et al.*[4] and completes our brief outline of the application of the methods of Sec. II to time-dependent two-soliton solutions of the sine-Gordon equation.

V. A ONE-DIMENSIONAL BAG

As a further illustration of the different varieties of extended objects that can be realized in a local quantum field theory, we discuss a simple example of a "bag." The original concept of a "bag" was derived[16,17] from quark models of hadrons. To build such a "bag," several conditions should be satisfied: (i) the total quark number $\mathfrak{N}$ is conserved, (ii) a single quark state ($\mathfrak{N}=1$) has an infinite energy, but (iii) certain multiquark states ($\mathfrak{N}>1$) are stable and of finite mass. Attempts have been made[18] to derive the "bag" structure from a local field theory by taking the infinite limit of certain renormalized mass and coupling constants. Our example differs from these approaches by keeping all renormalized quantities finite, but identifying quarks as solitons.

Again, we shall make the unphysical assumption that there is only one space dimension; i.e., $x_\mu=(x,it)$. Let ϕ be a scalar Hermitian field. The Lagrangian density is

$$\mathfrak{L}=-\frac{1}{2}\left(\frac{\partial\phi}{\partial x_\mu}\right)^2-V(\phi),$$

where $V(\phi)$ is proportional to

$$(g^2\phi^2+\epsilon^2)(1-g^2\phi^2)^2, \tag{5.1}$$

with a proportionality constant $=\frac{1}{8}(\mu/g)^2(1+\epsilon^2)^{-1}$. The constant ϵ is assumed to be real and positive. One sees that the potential function $V(\phi)$ has two absolute minima at $g\phi=+1$ and -1, and a local minimum at $\phi=0$. If we expand ϕ near its absolute minimum $\phi=\pm(g^{-1}+\delta\phi)$, then

$$V(\phi)=\tfrac{1}{2}\mu^2(\delta\phi)^2+\frac{1}{2}\frac{3+\epsilon^2}{1+\epsilon^2}g\mu^2(\delta\phi)^3+\cdots.$$

Therefore, μ denotes the meson mass and g is the dimensionless coupling constant. As ϵ varies, the height of the local minimum at $\phi=0$ changes. When $\epsilon\to\infty$, the local minimum disappears, and the potential V reduces to the special quartic function considered before.

By following the general discussions given in Sec. III, one finds that the field equation

$$\frac{\partial^2\phi}{\partial x_\mu{}^2}-\frac{dV(\phi)}{d\phi}=0$$

admits a time-independent classical solution $\phi=g^{-1}\sigma(x)$ where

$$\sigma(x)=[1+\epsilon^{-2}+\sinh^2(\tfrac{1}{2}\mu x)]^{-1/2}\sinh(\tfrac{1}{2}\mu x). \tag{5.2}$$

A typical shape of σ is given in Fig. 1. The corresponding quantum-mechanical operator $\phi(x,t)$ is then given by

$$\phi(x,t)=g^{-1}\sigma(x-Z)+q_n(t)\psi_n(x-Z), \tag{5.3}$$

where, as in (3.9), the repeated index n is summed over from $n=2$ to ∞. The ψ_n's again satisfy both the orthonormal condition (3.10) and the orthogonality relation (3.11). It is convenient to define the soliton-number operator

$$\mathfrak{N}\equiv g[\phi(\infty)-\phi(-\infty)]. \tag{5.4}$$

The solution (5.2) corresponds to $\mathfrak{N}=2$. As can be seen from Fig. 1, it consists of two kinks, and therefore may be regarded as a two-soliton bound

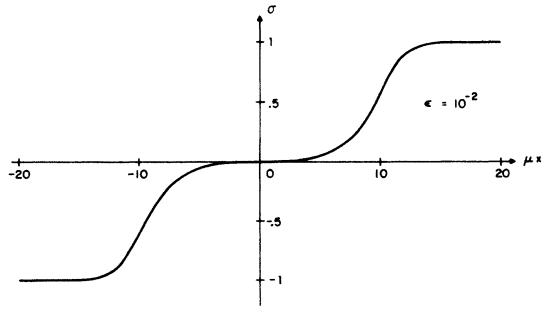

FIG. 1. Two solitons (kinks) in a one-dimensional bag. [See Eq. (5.2).]

state. The interesting feature is that (for arbitrary ϵ finite and nonzero) there is *no single-soliton state*. The state $\mathfrak{N}=1$ has an infinite energy. In accordance with our general discussion, ψ_n's are C-number functions which satisfy the "Schrödinger" equation (3.12)

$$\left[-\frac{d^2}{dx^2}+\frac{d^2V(\sigma)}{d\sigma^2}\right]\psi_n(x)=\omega_n{}^2\psi_n(x). \tag{5.5}$$

As will be shown later, this equation can be transformed into the Heun's equation.[19]

The lowest frequency of (5.5) is

$$\omega_1=0,$$

and its eigenfunction is

$$\psi_1(x)=\frac{d\sigma(x)}{dx}. \tag{5.6}$$

Because of the orthogonality condition (3.11), the sum (5.3) extends over all eigenfunctions of (5.5) except $n=1$. The energy spectrum of the system is given by (assuming the total momentum = 0)

$$E=m+\sum_{n\neq1}N_n\omega_n, \tag{5.7}$$

where the constant m denotes the mass of the two-soliton ground state and the quantum number $N_n=0,1,2,\ldots$. If we neglect the radiative correction, then

$$m=\frac{1}{4}\frac{\mu}{g^2}\left[1-\tfrac{1}{2}\epsilon^2+\frac{2\epsilon^2}{(1+\epsilon^2)^{1/2}}(1+\tfrac{1}{4}\epsilon^2)\ln\frac{1+(1+\epsilon^2)^{1/2}}{\epsilon}\right]. \tag{5.8}$$

As $\epsilon\to0$, $m\to\frac{1}{4}(\mu/g^2)$, and as $\epsilon\to\infty$, $m\to\frac{2}{3}(\mu/g^2)$, which is the same result as that for the quartic-coupling theory.

Let us arrange the characteristic frequencies of (5.5) in ascending order $0=\omega_1\leq\omega_2\leq\omega_3\leq\cdots$. The energy levels (5.7) of the system can be separated into the bound states $E<m+\mu$ and the continuum $E\geq m+\mu$. For the bound states we are only interested in those frequencies that are $\neq0$ and $<\mu$. Each bound state is then characterized by the corresponding set of occupation numbers N_n; its rest mass is

$$\mathfrak{M}\equiv m+\sum_{n\neq1}{}'N_n\omega_n<m+\mu. \tag{5.9}$$

Here, the sum $\sum'$ extends only over those ω_n that satisfy

$$0<\omega_n<\mu. \tag{5.10}$$

(If one wishes, one may extend the definition of the bound states to include also the level $\mathfrak{M}=m+\mu$.) As we shall see, for $\epsilon\ll1$ there are

many bound states with soliton number $\mathfrak{N}=2$; the total number of such bound states is quite large $\sim O(\epsilon^{-1}(-\ln\epsilon))$.

We discuss next the second-lowest frequency ω_2 of the "Schrödinger" equation (5.5). It is easy to verify that

$$\text{as } \epsilon \to \infty, \quad \omega_2 \to \tfrac{1}{2}\sqrt{3}\,\mu \tag{5.11}$$

and apart from a normalization factor

$$\psi_2 \to (1-\sigma^2)^{1/2}\sigma. \tag{5.12}$$

Also,

$$\text{as } \epsilon \to 0, \quad \omega_2 \to \epsilon\mu \tag{5.13}$$

and apart from a normalization factor

$$\psi_2 \to (1-\sigma^2)\sigma. \tag{5.14}$$

The former $\epsilon \to \infty$ limit is simply given by the explicit solution (3.54) for the quartic coupling theory. To derive the latter we note that for ϵ small the two kinks are far apart and therefore behave almost like independent particles. When $\epsilon \to 0$, $\psi_2(x)$ becomes simply $\pm\psi_1(x)$, where $\pm$ depends on the sign of x and $\psi_1(x)$ is the zero frequency eigenfunction given by (5.6). This follows because $\psi_1(x)$ represents a translation of the whole system in which the two kinks move in a parallel direction, while $\psi_2(x)$ is the first vibrational mode in which the two kinks move in an opposite direction.

For arbitrary ϵ, upper bounds on ω_2 can be derived by using the standard variational principle. For example, by assuming a trial function form $\psi_2 \propto (1-\sigma^2)\sigma$, one can readily establish, after some simple manipulation,

$$\omega_2 \leq \mu\epsilon/(1+\epsilon^2)^{1/2}. \tag{5.15}$$

A better upper bound can be obtained by assuming

$$\psi_2 \propto [(1-\sigma^2)+\lambda(1-\sigma^2)^{1/2}]\sigma, \tag{5.16}$$

where λ is a variational parameter. The calculation is straightforward, and the result is given in Fig. 2. This improved upper bound for ω_2 is exact in both the $\epsilon=0$ and $\epsilon=\infty$ limits, and is expected to give a fairly accurate approximation of ω_2 at any ϵ. Because of (5.13), for small ϵ the ($\mathfrak{N}=2$) bound-state levels are close spaced; the corresponding value of the occupation number N_2 can vary from 0, 1, 2, ... up to a maximum value $\cong \epsilon^{-1}$.

For the third-lowest characteristic frequency ω_3, we have again its exact limiting form

$$\text{as } \epsilon \to \infty, \quad \omega_3 \to \mu \tag{5.17}$$

and apart from a normalization factor

$$\psi_3 \to 3\sigma^2 - 1. \tag{5.18}$$

The results are the same as those for the quartic coupling. In addition, as can be readily verified, at $\epsilon^2 = \frac{1}{2}$ we have also the exact result

$$\omega_3 = \tfrac{1}{2}\sqrt{3}\,\mu \tag{5.19}$$

and apart from a normalization factor

$$\psi_3 = (1-\sigma^2)^{1/2}(\sigma^2 - \tfrac{1}{4}). \tag{5.20}$$

In general, the solutions of (5.5) are transcendental functions. Only in some special cases can these solutions be expressed in terms of elementary functions, as given above.

When ϵ is sufficiently small, some approximate knowledge of higher frequencies can be obtained without much effort. Let us examine the behavior of the "potential" term in the "Schrödinger" equation (5.5). As $\epsilon \to 0$,

$$v \equiv \frac{d^2V(\sigma)}{d\sigma^2} \to \tfrac{1}{4}\mu^2(1 - 12\sigma^2 + 15\sigma^4)\,. \tag{5.21}$$

By using (5.2), one sees that over a rather large range of x,

$$-L \leq x \leq L \tag{5.22}$$

where

$$L \cong (2/\mu)(-\ln\epsilon) + O(1), \tag{5.23}$$

the magnitude of σ is $\sim O(\epsilon)$; therefore, v is approximately a constant $= \frac{1}{4}\mu^2$. For $|x|$ beyond the above interval (5.22), v goes through a small dip and then rises rapidly towards its asymptotic value μ^2. Thus, when ϵ is sufficiently small, the eigenvalues of (5.5) that lie between $\frac{1}{4}\mu^2$ and μ^2 can be derived approximately by considering a Schrödinger equation with a simpler potential, one

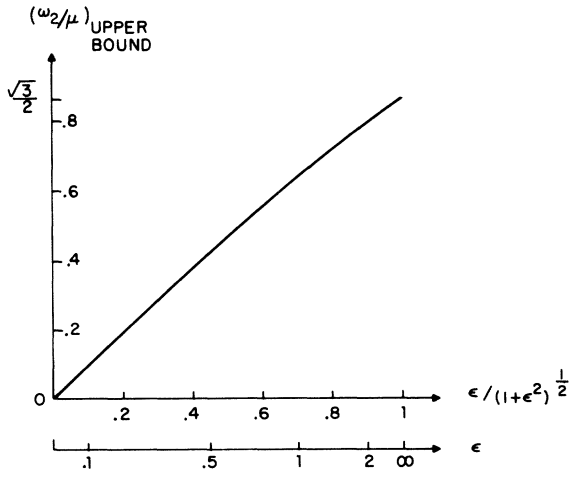

FIG. 2. The second-lowest frequency ω_2 of Eq. (5.5), evaluated by using the trial function (5.16). (The lowest frequency ω_1 is 0.)

that is a constant $\frac{1}{4}\mu^2$ inside $-L \leq x \leq L$ and is μ^2 outside. (The slight variation of the actual v near $x = \pm L$ does not alter the over-all feature of the frequency distribution when ϵ is small.) These eigenvalues are given by

$$\omega_q^2 \cong q^2 + \tfrac{1}{4}\mu^2, \tag{5.24}$$

where the "wave number" q is

$$2qL \cong \pi, 2\pi, \ldots, l\pi. \tag{5.25}$$

The maximum value $l\pi$ is determined by $\omega_q \leq \mu$. Thus, we have, on account of (5.23),

$$l \cong (2\sqrt{3}/\pi)(-\ln\epsilon). \tag{5.26}$$

To construct the bound-state levels, we return to (5.9). Since ω_q is between $\frac{1}{2}\mu$ and μ, its occupation number N_q can only be either 0 or 1. Each bound state is characterized by a set of integers $\{N_q\}$ and N_2, where N_2 denotes the occupation number of the second-lowest frequency, $\omega_2 \cong \epsilon\mu$. When all the N_q's are 0, N_2 can vary from 0 to ϵ^{-1}. When one of the N_q's, say N_p, is 1, then all other N_q's $(q \neq p)$ must be zero, while N_2 can still vary from $0, 1, \ldots$ up to $(\mu - \omega_p)/\epsilon\mu$. Thus, when $\epsilon \to 0$, the total number of bound states Ω is given approximately by

$$\Omega \cong \int (\pi\epsilon\mu)^{-1} 2L(\mu - \omega_q)dq + O(\epsilon^{-1}),$$

where the integration extends from 0 to $\frac{1}{2}\sqrt{3}\,\mu$. By using (5.23), we find

$$\Omega \cong (\pi\epsilon)^{-1}(-\ln\epsilon)[\sqrt{3} - \tfrac{1}{2}\ln(2+\sqrt{3})]. \tag{5.27}$$

When ϵ increases, the number of bound states decreases steadily. As $\epsilon \to \infty$, in the $\mathfrak{N} = 2$ sector, besides the ground state there is only one excited state with $\mathfrak{M} < m + \mu$.

Our remaining task is to convert (5.5) to a well-discussed form of transcendental equation. It is convenient to use σ as the independent variable. Equation (5.5) becomes

$$\left(a\frac{d^2}{d\sigma^2} + b\frac{d}{d\sigma} + c + \nu_n^2\right)\psi_n = 0, \tag{5.28}$$

where

$$\nu_n^2 = 4(1+\epsilon^2)\omega_n^2/\mu^2,$$

$$a = (1-\sigma^2)^2(\sigma^2+\epsilon^2),$$

$$b = \sigma(1-\sigma^2)(1-2\epsilon^2-3\sigma^2),$$

and

$$c = -1 + 2\epsilon^2 + 6\sigma^2(2-\epsilon^2) - 15\sigma^4.$$

The equation has five regular singularities at $g\sigma = \pm 1$, $\pm i\epsilon$, and ∞. We define

$$\zeta \equiv \sigma^2$$

and

$$f \equiv (1-\zeta)^{-s}\psi_n,$$

where

$$s \equiv [1 - (\omega_n/\mu)^2]^{1/2}.$$

Equation (5.28) can then be written in the standard form of Heun's equation, one that has only four regular singularities:

$$\frac{d^2f}{d\zeta^2} + \left[\frac{1}{2\zeta} + \frac{1+2s}{\zeta-1} + \frac{1}{2(\zeta+\epsilon^2)}\right]\frac{df}{d\zeta} + \frac{\alpha\beta\zeta - q}{\zeta(\zeta-1)(\zeta+\epsilon^2)}f = 0, \tag{5.29}$$

where

$$\alpha = -\tfrac{5}{2} - s, \quad \beta = \tfrac{3}{2} - s,$$

and

$$q = -\tfrac{1}{4}[1 - 2\epsilon^2 - 4(1-s^2)(1+\epsilon^2) + 2s\epsilon^2].$$

Its solution is usually cast in the standard expression[19]

$$f = P\begin{Bmatrix} 0 & 1 & -\epsilon^2 & \infty \\ 0 & 0 & 0 & \alpha \\ \frac{1}{2} & -2s & \frac{1}{2} & \beta \end{Bmatrix}\zeta\Big\}. \tag{5.30}$$

Detailed properties of these functions can be found in the mathematical literature.

VI. CONCLUSION

In the preceding sections we have developed and applied a quantization procedure which allows the quantum-mechanical description of various classical solutions to nonlinear field equations. Our method is a canonical one in which the resulting quantum-mechanical problem requires the diagonalization of a Hamiltonian operator H (2.13). H is naturally given as a power series in the coupling constant g. If we transform our quantum states Ψ_α by removing the phase factor e^{iS/g^2}, where S obeys Eqs. (2.17) and (2.22), then the transformed Hamiltonian H' which acts on the states

$$\chi_\alpha = e^{-iS/g^2}\Psi_\alpha \tag{6.1}$$

can be written in order of ascending powers of g:

$$H' = H'(-2) + H'(0) + H'(1) + H'(2) + \ldots.$$

The term of order g^{-2} is simply a constant

$$H'(-2) = \frac{1}{g^2}\mathcal{E}, \tag{6.2}$$

the energy of our classical solution. The term of

order g^0 is just the quadratic Hamiltonian

$$H'(0) = -\frac{i}{2}\frac{u}{J}\frac{\partial J}{\partial z_K} - iu\frac{\partial}{\partial z_K} + \tfrac{1}{2}(\pi_n \pi_n + q_n F_{nn'} q_{n'} + \pi_n G_{nn'} q_{n'} - q_n G_{nn'} \pi_{n'}) \quad (6.3)$$

of Eq. (2.30). Since this Hamiltonian contains the q's and π's quadratically, it is closely related to the classical small-oscillation problem and, at least in some cases, it may be possible to diagonalize. If we can diagonalize $H'(0)$, then the effects of the higher-order terms $H'(1)$, $H'(2)$, etc. can be computed to arbitrary order in g by systematic perturbation expansion treating $H'(-2)+H'(0)$ as our diagonal, unperturbed Hamiltonian. Such an expansion may give an accurate description of those quantum-mechanical states lying near our original classical solution. Of course, there are some states, displaced in energy by order μ^2/g^2 from our classical solution, that will be poorly described by any finite order of this perturbation expansion. For example, if we are expanding about a single-soliton solution, our perturbation expansion will not reveal, without some nonperturbative technique, soliton-soliton-antisoliton states, higher in energy by order μ^2/g^2. Of course, if such a three-soliton state is of interest we can always begin by directly expanding about it.

Clearly, in general this perturbation expansion will only be justified, even for those states lying closest to the classical solution, if the parameter g is small. This limited range of applicability seems to be a necessary feature of all expansions about a nontrivial classical solution. Only if the action of the original classical solution, of order g^{-2}, is large can we be assured that it is a reasonable first approximation to the quantum solution.

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APPENDIX A

In this appendix, we discuss the details of how to find the eigenvalues of the Hamiltonian H_2, which is given by (3.31). It is convenient to define

$$\mathcal{K} \equiv \begin{pmatrix} 1 & G \\ \tilde{G} & F \end{pmatrix} \quad \text{and} \quad \zeta = \begin{pmatrix} \pi \\ q \end{pmatrix}, \quad (A1)$$

where

$$\pi \equiv (\pi_n) = \begin{bmatrix} \pi_2 \\ \pi_3 \\ \vdots \end{bmatrix} \quad \text{and} \quad q \equiv (q_n) = \begin{bmatrix} q_2 \\ q_3 \\ \vdots \end{bmatrix}. \quad (A2)$$

The matrices $F = (F_{nn'})$ and $G = (G_{nn'})$ are given, respectively, by (3.32) and (3.33). Thus, H_2 can be written as

$$H_2 = \tfrac{1}{2}\tilde{\xi}\mathcal{K}\xi \ . \quad (A3)$$

Because in this Hamiltonian the coordinate q is coupled to the conjugate momentum π, naturally we are interested in canonical transformations that mix q and π:

$$\xi \to \xi' \equiv \begin{pmatrix} \pi' \\ q' \end{pmatrix} = T \cdot \xi \ . \quad (A4)$$

In order that the new coordinate q'_n and its conjugate momentum π'_n (i.e., the components of q' and π') remain Hermitian, we restrict ourselves only to T real. Furthermore, we require

$$\tilde{T}\rho_2 T = \rho_2 \quad (A5)$$

where

$$\rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (A6)$$

so that the usual canonical commutation relations between the coordinates and the momenta are preserved.

In order to find the transformation matrix T that can diagonalize H_2, it is useful to examine the Heisenberg equation of motion

$$i\dot{\xi} = [\xi, H_2] = (\rho_2\mathcal{K})\xi \ . \quad (A7)$$

The desired transformation matrix T can then be determined by its normal modes

$$\xi(t) = \xi^a(t) \propto \exp(-i\nu_a t);$$

i.e.,

$$(\rho_2\mathcal{K}) \cdot \xi^a = \nu_a \xi^a \ . \quad (A8)$$

As we shall see [in (A19)], the eigenvalues of $\rho_2\mathcal{K}$ are all *real and nonzero*. Since $\rho_2\mathcal{K}$ is purely imaginary, it follows from (A8)

$$(\rho_2\mathcal{K})(\xi^a)^* = -\nu_a(\xi^a)^* \ , \quad (A9)$$

where the asterisk denotes the complex conjugate. Thus, the eigenvalues of $\rho_2\mathcal{K}$ occur in pairs: $\pm\nu_a$. Because $\mathcal{K}$ is real and Hermitian, if ξ^a and ξ^b are both eigenvectors of $(\rho_2\mathcal{K})$ with eigenvalues ν_a and ν_b, respectively, then for $|\nu_a| \neq |\nu_b|$, we have

$$(\xi^a)^\dagger \rho_2 \xi^b = \tilde{\xi}^a \rho_2 \xi^b = 0 \ , \quad (A10)$$

where the dagger denotes the Hermitian conjugate; in addition,

$$\tilde{\xi}^a \rho_2 \xi^a = 0 . \tag{A11}$$

It is convenient to choose the normalization of ξ^a such that

$$(\xi^a)^\dagger \rho_2 \xi^a = 1 . \tag{A12}$$

We may decompose

$$\xi^a = 2^{-1/2}(R^a + iI^a) , \tag{A13}$$

where R^a and I^a are both real, and arrange respectively the eigenvectors and the eigenvalues of $\rho_2 \mathcal{K}$ in a linear order

$$\ldots, \xi^{n-1}, (\xi^{n-1})^*, \xi^n, (\xi^n)^*, \xi^{n+1}, (\xi^{n+1})^* , \ldots$$

and (A14)

$$\ldots, \nu_{n-1}, -\nu_{n-1}, \nu_n, -\nu_n, \nu_{n+1}, -\nu_{n+1}, \ldots .$$

Because of (A10)–(A13), we have

$$\tilde{R}^n \rho_2 R^m = \tilde{I}^n \rho_2 I^m = 0$$

and (A15)

$$\tilde{R}^n \rho_2 I^m = -I^n \rho_2 R^m = -i\delta_{nm} .$$

The transformation matrix T that we are searching for is simply

$$T = (\ldots, R^{n-1}, R^n, R^{n+1}, \ldots, \ldots, I^{n-1}, I^n, I^{n+1}, \ldots) \tag{A16}$$

in which each column of the matrix T is given by the appropriate R^n or I^n. It follows from (A15) that (A5) holds. Furthermore, under (A4) H_2 becomes diagonal:

$$H_2 \to \sum_n \tfrac{1}{2}\nu_n (\pi_n^2 + q_n^2) . \tag{A17}$$

Our remaining task is to solve explicitly the eigenvalue equation (A8). By using (A1), we may express this equation in terms of q^a, the lower component of ξ^a. The resulting expression is

$$[(G + i\nu_a)^2 + F] \cdot q^a = 0 . \tag{A18}$$

As we shall see, the solution is

$$\nu_a = \gamma^{-1}\omega_a , \tag{A19}$$

where $\gamma = (1 - v^2)^{-1/2}$ and ω_a can be any one of the *nonzero* eigenvalues of (3.23); the corresponding q^a is related to the eigenvector $\psi_a(\theta)$ of (3.23) by

$$q_n^a(t) = ce^{-i\omega_a t/\gamma} \int \psi_a(\theta)\psi_n(\theta) e^{i\omega_a v\theta} dx , \tag{A20}$$

where c is a constant, and as in Sec. III the subscript $n = 2, 3, \ldots$.

One can establish (A19) and (A20) either by direct verification or, alternatively by considering the following classical solution of the field equation (3.2) in the c.m. system

$$\phi_{\mathrm{cl}} = \phi^a \equiv g^{-1}\sigma(x^0) + c\psi_a(x^0)\exp(-i\omega_a t^0) . \tag{A21}$$

For clarity the space-time coordinates in the c.m. system are denoted by x^0 and t^0; they are related to those in the moving coordinates in the usual way,

$$x^0 = \gamma(x - vt) \text{ and } t^0 = \gamma(t - vx) . \tag{A22}$$

The solution (A21) can also be written as

$$\phi^a = g^{-1}\sigma(\theta) + q_n^a(t)\psi_n(\theta) , \tag{A23}$$

where as in (3.20), $\theta = \gamma(x - Z)$, and the function $Z = Z(t)$ is defined by the orthogonality relation

$$\int \frac{d\sigma(\theta)}{d\theta}[\phi^a - g^{-1}\sigma(\theta)]\,dx = 0 . \tag{A24}$$

We may expand θ and Z in powers of g:

$$\theta = \theta_0 + g\theta_1 + O(g^2) ,$$

and (A25)

$$Z = Z_0 + gZ_1 + O(g^2) .$$

By substituting (A21) into (A24) and then separating out the $O(g^{-1})$ term, we determine $Z_0 = vt$ and therefore

$$x^0 = \theta_0 . \tag{A26}$$

Because of (A25) and $\theta_1 = -\gamma Z_1$, x^0 is related to θ by

$$x^0 = \theta_0 = \theta + g\gamma Z_1 + O(g^2) . \tag{A27}$$

By using (A27) and $t^0 = \gamma^{-1}t - vx^0$, we may express (A21) in terms of θ and t:

$$\phi^a = g^{-1}\sigma(\theta) + \gamma Z_1 \frac{d\sigma(\theta)}{d\theta} + c\psi_a(\theta)\exp(i\omega_a v\theta - i\gamma^{-1}\omega_a t) + O(g) . \tag{A28}$$

The $O(1)$ term of the orthogonality relation (A24) now gives

$$\left[\gamma \int \left(\frac{d\sigma}{d\theta}\right)^2 dx\right] Z_1(t) = -ce^{-i\omega_a t/\gamma} \int \frac{d\sigma}{d\theta}\psi_a(\theta) e^{i\omega_a v\theta} dx . \tag{A29}$$

Since (A23) = (A28), we find q_n^a to be given by (A20), ν_a by (A19), and therefore the energy spectrum of H_2 by (3.35).

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