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Small deformations of the Prasad-Sommerfield solution

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I study solutions of the static Euclidean anti-self-dual SU(2) Yang-Mills equations which differ by a small perturbation from the Prasad-Sommerfield solution. I find explicit expressions for two series of perturbation mode functions of angular momentum l and even and odd parity, and classify the modes according to several criteria. There are seven nondilatational modes which have singularities removable by gauge transformation: 3 translations ($l = 1$), 1 gauge mode ($l = 0$), and a family of 3 odd-parity gauge modes ($l = 1$). The translations and $l = 0$ gauge modes have nonvanishing, and normalizable, projections into the background gauge, while the odd-parity $l = 1$ modes have vanishing projection into the background gauge. Among the singular modes, there are an infinite number of modes, irregular at $r = 0$, which nonetheless satisfy the boundary conditions for finite-energy solutions on the sphere at infinity. I show, by discussing the analogous problem of the axially symmetric solutions of the stationary Einstein equations, that non-normalizable modes are relevant in determining whether a spherically symmetric solution of a nonlinear system has axially symmetric extensions. The analysis of perturbations around the Prasad-Sommerfield solution implies that if an axially symmetric extension exists, it cannot be reached by integration out along a tangent vector defined by a nonvanishing, nonsingular small-perturbation mode of the class explicitly constructed.

I. INTRODUCTION

In two recent papers[1,2] I proposed that a static, Euclidean self-dual (or anti-self-dual) SU(2) background-field configuration may act as the quark-confining "bag" in a classical treatment of the quark static force problem. An investigation of the properties of the only explicitly known such configuration, the spherically symmetric Prasad-Sommerfield solution, showed that when introduced as a background solution in the $q-\bar{q}$ force problem it produced an increase in the $q-\bar{q}$ static potential, but that the effect was not strong enough for large $q-\bar{q}$ separations to give confinement. This result raises the question of whether there exist "stretched," axially symmetric extensions of the Prasad-Sommerfield solution, for which the quarks can remain in a strong color-field region for arbitrarily large separations. Such a configuration would be a prime candidate for a confining background field.

One natural way to investigate whether the Prasad-Sommerfield solution can be extended into an axially symmetric family of solutions is to study general Euclidean (anti-) self-dual SU(2) fields which differ from the Prasad-Sommerfield solution by a small perturbation. In Sec. II I obtain an explicit solution for two series of small-perturbation mode functions of angular momentum l and even and odd parity. In Sec. III A I classify these mode functions into two types, according to whether or not their singularities are removable by a gauge transformation. There are seven nondilatational mode functions with removable singularities and I find their projections into the background gauge. In Sec. III B I reclassify the mode functions according to the much weaker criterion of whether they satisfy the boundary conditions for finite-energy solutions on the sphere at infinity. In Sec. IV I discuss the implications of the small-perturbation analysis for the question of whether "stretched" extensions of the Prasad-Sommerfield solution exist. The results of Sec. III provide no positive evidence for the existence of such extensions, but (as I show by several examples) are compatible with the existence of axially symmetric solutions which are not simultaneously analytic functions of their coordinates and their deformation parameters around the Prasad-Sommerfield solution. Gauge transformation and propagator formulas used in the text are given in Appendices A and B, respectively. A procedure for integrating parity-odd first-order perturbations to higher order, in the background gauge, is given in Appendix C.

II. EXPLICIT SOLUTION FOR THE MODE FUNCTIONS

I consider static Euclidean SU(2) fields described by a gauge potential $\vec{b}^\mu = (\vec{b}^0, \vec{b}^j)$ and color electric and magnetic fields $\vec{E}^j$, $\vec{B}^j$, with

$$\begin{aligned}\vec{E}^j &= -D_j \vec{b}^0 ,\\ \vec{B}^j &= \epsilon^{jkl}\left(\frac{\partial}{\partial x^k}\vec{b}^l + \frac{1}{2}\vec{b}^k \times \vec{b}^l\right).\end{aligned} \tag{1}$$

In Ref. 2 I showed that all such field configurations with a finite energy integral

$$\int d^3x(\vec{E}^j\cdot\vec{E}^j + \vec{B}^j\cdot\vec{B}^j) \tag{2}$$

are characterized by two asymptotic parameters, a dimensional parameter κ, and a homotopy index n, defined by

$$\kappa=\lim_{x\to\infty}|\vec{b}^0(x)|\,,\quad 0=\lim_{x\to\infty}\vec{b}^j(x)\,,$$

$$n=\lim_{x\to\infty}\frac{1}{8\pi}\int d^2S_x^i\,\epsilon^{ijk}\epsilon^{abc}\hat{b}^a\frac{\partial}{\partial x^j}\hat{b}^b\frac{\partial}{\partial x^k}\hat{b}^c\,, \tag{3}$$

$$\hat{b}=\vec{b}^0/|\vec{b}^0|\,.$$

I will study in what follows anti-self-dual field configurations, for which

$$\vec{E}^j+\vec{B}^j=0\,, \tag{4}$$

with homotopy index $n=1$. Since Eq. (4) is scale invariant, solutions with general $\kappa\neq 0$ are generated from solutions with $\kappa=1$ by a simple rescaling, and so I will assume $\kappa=1$ henceforth. This restriction implies that the procedure for generating small-fluctuation modes developed below will yield only nondilatational modes.

The only presently known finite-energy solution of Eq. (4) is the Prasad-Sommerfield[3] solution ($\hat{j}$ is the unit vector along the j axis)

$$\begin{aligned}\vec{b}_0^0&=\hat{r}\left(\frac{1}{r}-\coth r\right),\\ \vec{b}_0^j&=(\hat{j}\times\hat{r})\left(\frac{1}{r}-\frac{1}{\sinh r}\right)\,,\\ r&=|\vec{x}|\,,\quad \hat{r}=\vec{x}/r\,.\end{aligned} \tag{5}$$

A useful procedure for generating further solutions to Eq. (4) is provided by the "singular gauge" ansatz[4]

$$\begin{aligned}\vec{b}_{(s)}^0&=-\vec{\nabla}\ln\sigma\,,\\ \vec{b}_{(s)}^j&=i\hat{j}-(\hat{j}\times\vec{\nabla})\ln\sigma,\end{aligned} \tag{6}$$

which on substitution into Eqs. (1) and (4) yields a linear equation for σ

$$\nabla^2\sigma=\sigma\,. \tag{7}$$

Manton[4] has shown that the Prasad-Sommerfield solution of Eq. (5) is obtained by taking $\sigma=\sinh r/r$ in Eq. (6) and then making a complex gauge transformation. Specifically, for the scalar potential, which transforms like a gauge vector, this construction gives

$$\begin{aligned}b_{(s)0}^{a0}&=-\nabla^a\ln\frac{\sinh r}{r}\,,\\ b_0^{a0}&=M^{ab}(x)\,b_{(s)0}^{b0}\,,\end{aligned} \tag{8}$$

with M^{ab} the complex rotation matrix

$$\begin{aligned}M^{ab}(x)=&\cosh r(\delta^{ab}-\hat{r}^a\hat{r}^b)\\ &-i\sinh r\,\epsilon^{abl}\hat{r}^l+\hat{r}^a\hat{r}^b\,,\end{aligned} \tag{9}$$

which satisfies

$$\begin{aligned}&D_0^\mu M^{ab}=M^{ab}D_{(s)0}^\mu\,,\\ &D_0^0=\vec{b}_0^0\times,\quad D_0^j=\frac{\partial}{\partial x^j}+\vec{b}_0^j\times\,,\\ &D_{(s)0}^0=\vec{b}_{(s)0}^0\times,\quad D_{(s)0}^j=\frac{\partial}{\partial x^j}+\vec{b}_{(s)0}^j\times\,.\end{aligned} \tag{10}$$

Unfortunately, the utility of the ansatz of Eq. (6) for generating physically interesting solutions of the anti-self-dual equations is limited by the fact, proved by Manton, that only for the Prasad-Sommerfield case can one find a gauge transformation which makes the imaginary part of the solution vanish.

In studying the problem of small fluctuations around the Prasad-Sommerfield solution, however, the fact that Eq. (6) leads in general to complex solutions is an advantage. The reason is that given a general complex solution to the equations for a small anti-self-dual fluctuation around a real background, the real and imaginary parts of the solution must *individually* satisfy the small-fluctuation equations. Since the real and imaginary parts of M^{ab} have opposite parity, this leads to a general method for constructing two series of even- and odd-parity small-fluctuation mode functions of arbitrary angular momentum, as follows. Let $i_l(r)$, $k_l(r)$ be the regular and irregular vector spherical harmonics for a complex argument,

$$\begin{aligned}&i_l(r)=(-i)^l j_l(ir)\,,\\ &k_l(r)=(-i)^l h_l^{(1)}(ir)\,,\\ &i_0(r)=\frac{\sinh r}{r}\,,\quad k_0(r)=-\frac{e^{-r}}{r}\,,\\ &i_1(r)=\frac{\cosh r}{r}-\frac{\sinh r}{r^2}\,,\quad k_1(r)=\frac{1+r}{r^2}e^{-r}\,,\\ &i_2(r)=\frac{\sinh r}{r}+\frac{3\sinh r}{r^3}-\frac{3\cosh r}{r^2}\,,\\ &k_2(r)=-\frac{r^2+3r+3}{r^3}e^{-r}\,.\end{aligned} \tag{11}$$

In the singular gauge, a small perturbation about the Prasad-Sommerfield solution can be obtained by taking

$$\begin{aligned}\sigma&=C\left(\frac{\sinh r}{r}-\Lambda\right),\\ \Lambda&=\sum_{l=1}^{\infty}\sum_{m=-l}^{l}a_{lm}i_l(r)Y_{lm}(\hat{r})\\ &\quad+\sum_{l=0}^{\infty}\sum_{m=-l}^{l}b_{lm}k_l(r)Y_{lm}(\hat{r})\,,\end{aligned} \tag{12}$$

since this construction makes σ a general solution of $\nabla^2\sigma = \sigma$ as expressed in spherical coordinates. Writing

$$\begin{aligned}\ln\sigma &= \ln\frac{\sinh r}{r} + \ln\left[1 - \frac{r}{\sinh r}\,\Lambda\right] + \ln C \\ &= \ln\frac{\sinh r}{r} - \frac{r}{\sinh r}\,\Lambda + O(\Lambda^2) + \ln C\,, \end{aligned} \tag{13}$$

a solution to the small-fluctuation problem in the singular gauge is given by

$$\begin{aligned}\vec{b}^{\mu}_{(s)} &= \vec{b}^{\mu}_{(s)0} + \delta\vec{b}^{\mu}_{(s)}\,, \\ \delta\vec{b}^{0}_{(s)} &= \vec{\nabla}\left[\frac{r}{\sinh r}\,\Lambda\right], \\ \delta\vec{b}^{j}_{(s)} &= (\hat{j}\times\vec{\nabla})\left(\frac{r}{\sinh r}\,\Lambda\right)\,. \end{aligned} \tag{14}$$

Since $\delta\vec{b}^{\mu}$ transforms as a vector under gauge transformations of the zeroth-order solution, the small-fluctuation solution in the physical gauge [where the Prasad-Sommerfield solution takes the manifestly real form given in Eq. (5)] is

$$\delta b^{a\mu} = M^{ab}(x)\,\delta b^{b\mu}_{(s)}\,, \tag{15}$$

and by construction satisfies the small-fluctuation equation

$$\begin{aligned}\delta\vec{E}^{j} + \delta\vec{B}^{j} &= 0\,, \\ \delta\vec{E}^{j} &= -D^{j}_{0}\,\delta\vec{b}^{0} + D^{0}_{0}\,\delta\vec{b}^{j}\,, \\ \delta\vec{B}^{j} &= \epsilon^{jkl}D^{k}_{0}\,\delta\vec{b}^{l}\,. \end{aligned} \tag{16}$$

Separating the complex solution of Eq. (15) into real and imaginary parts gives two linearly independent real solutions of the small-fluctuation equation, with even and odd intrinsic parity,[5] respectively, in the original gauge:

$$\begin{aligned}\delta\vec{b}^{0}_{(+)} &= -\sinh r\,(\hat{r}\times\vec{\nabla})\left(\frac{r}{\sinh r}\,\Lambda\right), \\ \delta\vec{b}^{j}_{(+)} &= \sinh r\left(\hat{r}^{j}\vec{\nabla} - \hat{j}\,\frac{\partial}{\partial r}\right)\left(\frac{r}{\sinh r}\,\Lambda\right); \end{aligned} \tag{17a}$$

$$\begin{aligned}\delta\vec{b}^{0}_{(-)} &= \hat{r}\,\frac{\partial}{\partial r}\left(\frac{r}{\sinh r}\,\Lambda\right) + \cosh r\left(\vec{\nabla} - \hat{r}\,\frac{\partial}{\partial r}\right)\left(\frac{r}{\sinh r}\,\Lambda\right), \\ \delta\vec{b}^{j}_{(-)} &= \cosh r\,(\hat{j}\times\vec{\nabla})\left(\frac{r}{\sinh r}\,\Lambda\right) + (\cosh r - 1)\,\hat{r}\,(\hat{r}\times\vec{\nabla})^{j}\left(\frac{r}{\sinh r}\,\Lambda\right). \end{aligned} \tag{17b}$$

One still has the freedom to modify Eq. (17) by a gauge transformation of the form

$$\begin{aligned}\delta\vec{b}^{\mu}_{(\pm)} &\to \delta\vec{b}^{\mu}_{(\pm)} + D^{\mu}_{0}\,\vec{\psi}_{(\pm)}\,, \\ \vec{\psi} &= \hat{r}\psi_{r} + \hat{\theta}\psi_{\theta} + \hat{\phi}\psi_{\phi}\,, \end{aligned} \tag{18}$$

with $\psi_{r,\theta,\phi}$ arbitrary scalar functions and with $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$ the unit vectors in spherical coordinates. The explicit form of this gauge transformation, and useful formulas for differentiating spherical unit vectors, are given in Appendix A. It turns out that Eq. (17), as it stands, is in a form convenient for analyzing the irregular eigenmodes (those with $\Lambda \propto Y_{lm}k_{l}$), while to analyze the regular eigenmodes (those with $\Lambda \propto Y_{lm}i_{l}$) it is convenient to regauge Eq. (17) to make $\delta\vec{b}^{0}_{(\pm)}$ purely radial, by choosing

$$\begin{aligned}\vec{\psi}_{(+)} &= \hat{r}\times(\hat{r}\times\vec{\nabla})\left(\frac{\sinh r\,\Lambda}{i_1}\right), \\ \vec{\psi}_{(-)} &= -(\hat{r}\times\vec{\nabla})\left(\frac{\cosh r\,\Lambda}{i_1}\right), \end{aligned} \tag{19}$$

in Eq. (18). This gives the alternative forms in the radial gauge:

$$\begin{aligned}\delta\vec{b}^{0}_{(+)} &= 0\,, \\ \delta\vec{b}^{j}_{(+)} &= \hat{r}\left(\partial^{j} - \hat{r}^{j}\,\frac{\partial}{\partial r}\right)\left(\frac{\Lambda}{i_1}\right) - \left(\vec{\nabla} - \hat{r}\,\frac{\partial}{\partial r}\right)\partial^{j}\left(\frac{\sinh r\,\Lambda}{i_1}\right) + \sinh r\left(\hat{r}^{j}\vec{\nabla} - \hat{j}\,\frac{\partial}{\partial r}\right)\left[\left(\coth r - \frac{1}{r}\right)\frac{\Lambda}{i_1}\right] \\ &\quad + \frac{1}{r}\,(\hat{j} - \hat{r}\,\hat{r}^{j})\,\frac{\partial}{\partial r}\left(\frac{\sinh r\,\Lambda}{i_1}\right); \end{aligned} \tag{20a}$$

$$\delta\vec{b}^{0}_{(-)}=\hat{r}\,\frac{\partial}{\partial r}\left(\frac{r}{\sinh r}\Lambda\right),$$

$$\delta\vec{b}^{j}_{(-)}=\hat{r}(\hat{r}\times\vec{\nabla})^{j}\left(\frac{\Lambda}{ri_1}\right)-(\hat{r}\times\vec{\nabla})\partial^{j}\left(\frac{\cosh r\,\Lambda}{i_1}\right)+\hat{r}^{j}\cosh r(\hat{r}\times\vec{\nabla})\left[\left(\coth r-\frac{1}{r}\right)\frac{\Lambda}{i_1}\right]$$
$$-\hat{r}\times\hat{j}\left\{\cosh r\frac{\partial}{\partial r}\left[\left(\coth r-\frac{1}{r}\right)\frac{\Lambda}{i_1}\right]-\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\cosh r\,\Lambda}{i_1}\right)\right]\right\}. \tag{20b}$$

The radial gauge functions still have an additional gauge arbitrariness of the form

$$\delta\vec{b}^{\mu}_{(\pm)}\to\delta\vec{b}^{\mu}_{(\pm)}+D^{\mu}_{0}[\hat{r}\psi_r]. \tag{21}$$

Some additional formulas which are useful in analyzing the regular mode functions are, in the radial gauge,

$$\hat{r}\cdot\delta\vec{b}^{j}_{(+)}=\left(\partial^{j}-\hat{r}^{j}\frac{\partial}{\partial r}\right)\left(\frac{\Lambda}{i_1}\right),$$

$$\hat{r}\circ\delta\vec{b}^{j}_{(-)}=(\hat{r}\times\vec{\nabla})^{j}\left(\frac{\Lambda}{ri_1}\right), \tag{22}$$

$$\delta\vec{b}^{j}_{(-)}-\hat{r}\times\delta\vec{b}^{j}_{(+)}=\hat{r}(\hat{r}\times\vec{\nabla})^{j}\left(\frac{\Lambda}{ri_1}\right)-(\hat{r}\times\vec{\nabla})\partial^{j}\left(\frac{e^{-r}\Lambda}{i_1}\right)+\hat{r}^{j}e^{-r}(\hat{r}\times\vec{\nabla})\left[\left(\coth r-\frac{1}{r}\right)\frac{\Lambda}{i_1}\right]$$
$$-\hat{r}\times\hat{j}\left\{e^{-r}\frac{\partial}{\partial r}\left[\left(\coth r-\frac{1}{r}\right)\frac{\Lambda}{i_1}\right]-\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{e^{-r}\Lambda}{i_1}\right)\right\}.$$

III. CLASSIFICATION OF MODE FUNCTIONS

A. Classification of mode functions with removable singularities

I turn now to the problem of classifying the mode functions constructed in the preceding section, and begin by analyzing which modes contain only singularities which are removable by an appropriate gauge transformation [the general form of which was given in Eq. (18)]. Since the Prasad-Sommerfield solution is spherically symmetric, it suffices to do the classification only for modes with $m=0$. I define four series of mode functions, obtained by the following substitutions in Eqs. (17) and (20):

$$\begin{aligned}
&\delta\vec{b}^{\mu}_{l\ \text{irregular}\,(+)}:\\
&\quad\Lambda\to k_l(r)P_l(\cos\theta)\ \text{in Eq. (17a)},\\
&\delta\vec{b}^{\mu}_{l\ \text{irregular}\,(-)}:\\
&\quad\Lambda\to k_l(r)P_l(\cos\theta)\ \text{in Eq. (17b)},\\
&\delta\vec{b}^{\mu}_{l\ \text{regular}\,(+)}:\\
&\quad\Lambda\to i_l(r)P_l(\cos\theta)\ \text{in Eq. (20a)},\\
&\delta\vec{b}^{\mu}_{l\ \text{regular}\,(-)}:\\
&\quad\Lambda\to i_l(r)P_l(\cos\theta)\ \text{in Eq. (20b)}.
\end{aligned} \tag{23}$$

As the first step in the classification, I show that for $l\geq L$ these mode functions have singularities that cannot be removed by any gauge transformation, with the L values for the four series given by

$$\begin{aligned}
&L_{\text{irregular}\,(+)}=1,\\
&L_{\text{irregular}\,(-)}=0,\\
&L_{\text{regular}\,(+)}=2,\\
&L_{\text{regular}\,(-)}=2.
\end{aligned} \tag{24}$$

Consider first the irregular modes. Near $r=0$, one has

$$k_l=C_l\frac{1}{r^{l+1}}+O\left(\frac{1}{r^{l}}\right),\quad C_l\neq0, \tag{25}$$

and so from Eq. (17b) we find for the ($-$) modes

$$\delta\vec{b}^{0}_{l\ \text{irregular}\,(-)}=\frac{-C_l}{r^{l+2}}[\hat{r}(l+1)P_l+\hat{\theta}\sin\theta P_l']+O\left(\frac{1}{r^{l+1}}\right). \tag{26}$$

The term proportional to $\hat{r}/r^{l+2}$ cannot be removed by any gauge transformation, and so the irregular ($-$) modes have irremovable singularities at $r=0$ for all $l\geq0$. For the ($+$) modes we have

$$\begin{aligned}
&\delta\vec{b}^{0}_{l\ \text{irregular}\,(+)}=C_l\frac{\hat{\phi}}{r^{l+1}}\sin\theta P_l'+O\left(\frac{1}{r^{l}}\right),\\
&\delta\vec{b}^{j}_{l\ \text{irregular}\,(+)}=O\left(\frac{1}{r^{l+1}}\right).
\end{aligned} \tag{27}$$

Now for $l\geq1$, $P_l'\neq0$, but since $\vec{b}^{0}_{0}\propto\hat{r}r$ for small r, the term $\hat{\phi}/r^{l+1}$ in $\delta\vec{b}^{0}_{l\ \text{irregular}\,(+)}$ can only be removed by a gauge transformation with $\psi_\theta\propto P_l'\sin\theta/r^{l+2}$, which induces new singular terms in $\delta\vec{b}^{j}_{l\ \text{irregular}\,(+)}$ behaving as $1/r^{l+3}$. Hence the irregular (+) modes have irremovable singularities at $r=0$ for all $l \geq 1$.

Consider next the regular modes. In the asymptotic region one has

$$\frac{i_l}{i_1} = 1 - \frac{D_l}{r} + O\left(\frac{1}{r^2}\right), \tag{28}$$

and so from Eq. (20b) we find for the (−) modes

$$\delta\vec{b}^j_{l\ \mathrm{regular}\,(-)} = \frac{1}{2}\frac{e^r}{r^2}\{\hat{\theta}\hat{\phi}^j[P_l(D_l+1) - P_l'\cos\theta] + \hat{\phi}\hat{\theta}^j[P_l'\cos\theta - P_l(D_l+1) - P_l''\sin^2\theta]\} + O(e^r/r^3). \tag{29}$$

Since $P_l/P_l'\cos\theta$ is not a constant for $l \geq 2$, the coefficient of $\hat{\theta}\hat{\phi}^j$ is nonvanishing and reference to Appendix A shows that this term cannot be removed by a gauge transformation without inducing terms in $\delta\vec{b}^0_{l\ \mathrm{regular}\,(-)}$ which grow asymptotically as e^r. Hence the regular (−) modes are asymptotically exponentially growing for all $l \geq 2$. Finally, from Eq. (22) we lear that the difference between $\delta\vec{b}^j_{(-)}$ and $\hat{r}\times\delta\vec{b}^j_{(+)}$, as well as $\hat{r}\cdot\delta\vec{b}^j_{(\pm)}$, cannot grow exponentially in the asymptotic region, and so we have

$$\delta\vec{b}^j_{l\ \mathrm{regular}\,(+)} = -\frac{1}{2}\frac{e^r}{r^2}\{\hat{\phi}\hat{\phi}^j[P_l(D_l+1) - P_l'\cos\theta] - \hat{\theta}\hat{\theta}^j[P_l'\cos\theta - P_l(D_l+1) - P_l''\sin^2\theta]\} + O(e^r/r^3). \tag{30}$$

The coefficient of $\hat{\phi}\hat{\phi}^j$ is again nonvanishing for all $l \geq 2$, and reference to Appendix A shows that this term cannot be removed by a gauge transformation without inducing other exponentially growing terms. Hence the regular (+) modes are also asymptotically exponentially growing for $l \geq 2$.

This analysis leaves as candidates for nonsingular perturbations the following seven modes:

$$\begin{aligned} &l=0\ (m=0)\ \text{irregular}\ (+),\\ &l=1\ (m=0,\pm1)\ \text{regular}\ (-),\\ &l=1\ (m=0,\pm1)\ \text{regular}\ (+).\end{aligned} \tag{31}$$

I discuss their properties in turn.

1. $l=0$ $(m=0)$ irregular (+)

Making the substitution

$$\Lambda = k_0(r) = -e^{-r}/r \tag{32}$$

in Eq. (17a) gives

$$\begin{aligned} \delta\vec{b}^0_{l=0\ \mathrm{irregular}\,(+)} &= 0 = D_0^0(-\hat{r}),\\ \delta\vec{b}^j_{l=0\ \mathrm{irregular}\,(+)} &= -(\hat{j} - \hat{r}\hat{r}^j)\frac{1}{\sinh r}\\ &= -(\hat{\theta}\hat{\theta}^j + \hat{\phi}\hat{\phi}^j)\frac{1}{\sinh r}\\ &= D_0^j(-\hat{r}), \end{aligned} \tag{33}$$

where I have used Eq. (A4), with $\psi_r = -1$, to show that this is a pure gauge mode. Although Eq. (33) is singular at $r=0$, the singularity is clearly removable, and I will show explicitly that it is removed by projection into the background gauge, defined by

$$\begin{aligned} 0 &= D_0^\mu \delta\vec{b}^\mu_{(\mathrm{bg})}\\ &= \vec{b}_0^0 \times \delta\vec{b}^\mu_{(\mathrm{bg})} + \left(\frac{\partial}{\partial x^j} + \vec{b}_0^j\times\right)\delta\vec{b}^j_{(\mathrm{bg})}. \end{aligned} \tag{34}$$

I first give a general procedure for projecting gauge modes into the background gauge, and then apply it to Eq. (33). Consider the general gauge mode

$$\delta\vec{b}^\mu = D_0^\mu\vec{\chi}, \tag{35}$$

and let its covariant divergence be

$$\vec{\Psi} = D_0^\mu\delta\vec{b}^\mu = D_0^\mu D_0^\mu\vec{\chi}. \tag{36}$$

Then in terms of the scalar, isovector propagator $\Delta^{ab}(x,y)$ defined by

$$[D_{0x}^\mu D_{0x}^\mu \Delta(x,y)]^{ab} = -\delta^{ab}\delta^3(x-y), \tag{37}$$

the projection of Eq. (35) into the background gauge is given by

$$\begin{aligned} \delta\vec{b}^\mu_{(\mathrm{bg})} &= D_0^\mu(\vec{\chi} + \vec{\psi}),\\ \psi^a(x) &= \int d^3y\,\Delta^{ab}(x,y)\Psi^b(y), \end{aligned} \tag{38}$$

since Eq. (38) clearly satisfies Eq. (34). Let us now evaluate $\vec{\psi}$ by integration by parts, being careful to keep surface terms. Introducing a natural vector notation for the b-index dependence of $\Delta^{ab}(x,y)$,

$$[\vec{\Delta}^a(x,y)]^b \equiv \Delta^{ab}(x,y), \tag{39}$$

we have

$$\begin{aligned} \psi^a(x) &= \int d^3y\,\vec{\Delta}^a(x,y)\cdot D_{0y}^\mu D_{0y}^\mu\vec{\chi}(y)\\ &= \int d^3y\,D_{0y}^\mu D_{0y}^\mu\vec{\Delta}^a(x,y)\cdot\vec{\chi}(y) + \Sigma^a(x)\\ &= -\chi^a(x) + \Sigma^a(x),\\ \Sigma^a(x) &= \lim_{y\to\infty} y^2\int d\Omega_y\,\hat{y}^j[\vec{\Delta}^a(x,y)\cdot D_{0y}^j\vec{\chi} - D_{0y}^j\vec{\Delta}^a(x,y)\cdot\vec{\chi}]. \end{aligned} \tag{40}$$

Hence comparing Eqs. (38) and (40) we get

$$\delta\vec{b}^\mu_{(\mathrm{bg})} = D_0^\mu\vec{\Sigma}; \tag{41}$$

that is, the projection of a gauge mode into the background gauge comes entirely from the surface terms at infinity in Eq. (40). If $\vec{\chi}$ is bounded at infinity, then the only terms in $\vec{\Delta}^a$ and $-D^j_{0y}\vec{\Delta}^a$ which contribute to Σ^a are those which behave respectively as y^{-1} and y^{-2} in the asymptotic region. These are readily obtained from the propagator formulas in Appendix B to be

$$\vec{\Delta}^a = \hat{x}^a\left(\coth x - \frac{1}{x}\right)\frac{1}{4\pi}\frac{\hat{y}}{y} + O(y^{-2}),$$
$$-D^j_{0y}\vec{\Delta}^a = \hat{x}^a\left(\coth x - \frac{1}{x}\right)\frac{1}{4\pi}\frac{\hat{y}\hat{y}^j}{y^2} + O(y^{-3}). \tag{42}$$

Returning now to the gauge mode of Eq. (33) and applying this formalism gives

$$\vec{\chi} = -\hat{y},$$
$$D^j_{0y}\vec{\chi} = -(\hat{j} - \hat{y}\hat{y}^j)\frac{1}{\sinh y},$$
$$\hat{y}^j\vec{\Delta}^a \cdot D^j_{0y}\vec{\chi} = 0 + O(y^{-3}),$$
$$\hat{y}^j(-D^j_{0y}\vec{\Delta}^a)\cdot\vec{\chi} = \hat{x}^a\left(\frac{1}{x} - \coth x\right)\frac{1}{4\pi y^2} + O(y^{-3}), \tag{43}$$

giving for the $l=0$, irregular (+) mode in the background gauge

$$\vec{\Sigma} = \hat{r}\left(\frac{1}{r} - \coth r\right) = \vec{b}^0_0,$$
$$\delta\vec{b}^\mu = D^\mu_0\vec{b}^0_0. \tag{44}$$

The fact that Eq. (44) has zero covariant divergence is obvious from the fact that $\vec{b}^0_0$ satisfies the zeroth-order equation $D^\mu_0 D^\mu_0\vec{b}^0_0 = 0$. Equation (44) is also singularity free, and gives the previously known[6] normalizable gauge mode.

2. $l=1$ ($m=0, \pm 1$) regular (−)

Making the substitution

$$\Lambda = i_1(r)P_1(\cos\theta) = i_1(r)\cos\theta \tag{45}$$

into Eq. (20b) and carrying out the differentiations gives for the $l=1, m=0$ regular (−) mode in the radial gauge

$$\delta\vec{b}^0 = \hat{r}\cos\theta\left(\frac{1}{r^2} - \frac{1}{\sinh^2 r}\right),$$
$$\delta\vec{b}^j = -\hat{r}\hat{\phi}^j\frac{\sin\theta}{r^2} - \hat{\phi}\hat{r}^j\frac{\sin\theta}{r\sinh r}$$
$$+(\hat{\phi}\hat{\theta}^j - \hat{\theta}\hat{\phi}^j)\cos\theta\frac{\cosh r}{\sinh^2 r}. \tag{46}$$

To see that this is just the z-axis translation mode in the radial gauge, consider

$$-\frac{\partial}{\partial z}\vec{b}_0 = \hat{r}\left(\frac{1}{r^2} - \frac{1}{\sinh^2 r}\right)\cos\theta$$
$$+\hat{\theta}\frac{1}{r}\left(\frac{1}{r} - \coth r\right)\sin\theta,$$
$$-\frac{\partial}{\partial z}\vec{b}^j_0 = (-\hat{r}\hat{\phi}^j + \hat{\phi}\hat{r}^j)\sin\theta\left(\frac{1}{r^2} - \frac{1}{r\sinh r}\right)$$
$$+(\hat{\phi}\hat{\theta}^j - \hat{\theta}\hat{\phi}^j)\cos\theta\left(\frac{\cosh r}{\sinh^2 r} - \frac{1}{r^2}\right). \tag{47}$$

Comparison with the formulas of Appendix A shows that Eq. (47) can be brought to the form of Eq. (46) by adding $D^\mu_0\vec{\psi}$, with $\vec{\psi} = r^{-1}\sin\theta\hat{\phi}$. Since Eq. (47) is nonsingular at $r=0$, this shows that the singularities of Eq. (46) at $r=0$ are removable. To put the translation modes into background-gauge form, it is not necessary to do a projection using the scalar propagator. Rather, one can use the fact[1,7] that if $\vec{\phi}$ is a solution of $D^\mu_0 D^\mu_0\vec{\phi} = 0$, then

$$\delta\vec{b}^{(0)\mu} = D^\mu_0\vec{\phi},$$
$$\delta\vec{b}^{(a)\mu} = \eta^{(-)}_{\mu\nu a}D^\nu_0\vec{\phi}, \quad a = 1, 2, 3 \tag{48}$$
$$\eta^{(-)}_{\mu\nu a} = -\eta^{(-)}_{\nu\mu a}, \quad \eta^{(-)}_{kla} = \epsilon_{kla}, \quad \eta^{(-)}_{k0a} = -\delta_{ka},$$

are four linearly independent solutions of the small-fluctuation equation in the background gauge. For $\vec{\phi} = \vec{b}^0_0$, one has $D^0_0\vec{\phi} = 0$, $D^j_0\vec{\phi} = -\vec{E}^j_0 \sim 1/r^2$ for large r, and so the mode functions given by Eq. (48) are nonsingular and normalizable. The mode $\delta\vec{b}^{(0)\mu}$ is just the normalizable gauge mode of Eq. (44), while the modes $\delta\vec{b}^{(a)\mu}$, $a=1,2,3$ just give the a axis translations in the background gauge.

3. $l=1$ ($m=0, \pm 1$) regular (+)

From Eq. (22), we see that whenever the regular(−) modes are well behaved at $r=\infty$, the regular(+) modes are also well behaved. Hence we expect the $l=1$ ($m=0, \pm 1$) regular(+) modes to yield small-fluctuation solutions with removable singularities, and this is borne out by explicit calculation. Substituting

$$\Lambda = i_1(r)\cos\theta \tag{49}$$

into Eq. (20a) and carrying out the differentiations gives for the $l=1$, $m=0$ regular (+) mode in the radial gauge

$$\delta\vec{b}^0 = 0 = D_0^0(\hat{r}\cos\theta)\,,$$

$$\delta\vec{b}^j = (\hat{\theta}\hat{\theta}^j + \hat{\phi}\hat{\phi}^j)\frac{\cos\theta}{\sinh r} - \hat{r}\hat{\theta}^j\frac{\sin\theta}{r} \tag{50}$$

$$= D_0^j(\hat{r}\cos\theta)\,,$$

where I have used Eq. (A3), with $\psi_r = \cos\theta$, to write this as a gauge mode. The singularities of Eq. (50) are clearly removable, and the projection into the background gauge is readily found by application of Eqs. (40) and (42), which give

$$\vec{\chi} = \hat{y}\cos\theta_y\,,$$

$$D_{0y}^j\vec{\chi} = (\hat{j} - \hat{y}\hat{y}^j)\frac{\cos\theta_y}{\sinh y} - \hat{y}\hat{\theta}_y^j\frac{\sin\theta_y}{y}\,,$$

$$\hat{y}^j\vec{\Delta}^a \cdot D_{0y}^j\vec{\chi} = 0 + O(y^{-3})\,, \tag{51}$$

$$\hat{y}^j(-D_{0y}^j\vec{\Delta}^a)\cdot\vec{\chi} = \hat{x}^a\left(\coth x - \frac{1}{x}\right)\frac{\cos\theta_y}{4\pi y^2} + O(y^{-3})\,,$$

$$\Sigma^a = \hat{x}^a\left(\coth x - \frac{1}{x}\right)\int d\Omega_y\cos\theta_y = 0\,.$$

Hence the three $l=1$ regular(+) modes vanish identically when projected into the background gauge.

This completes the classification of all nondilatational small-fluctuation modes obtained above according to whether or not they have removable singularities. As noted in Sec. II, the restriction to $\kappa=1$ implies that the procedure which has been followed will not yield the dilatational mode, which is associated with changes in the dimensional parameter κ. This mode (which has $l=0$) is easily obtained by scale differentiation of the Prasad-Sommerfield solution of Eq. (5),

$$\delta\vec{b}^0_{\text{dil}} = \frac{\partial}{\partial\kappa}\kappa\vec{b}_0^0(\kappa r)\Big|_{\kappa=1}$$

$$= \hat{r}\left(-\coth r + \frac{r}{\sinh^2 r}\right)\,, \tag{52}$$

$$\delta\vec{b}^j_{\text{dil}} = \frac{\partial}{\partial\kappa}\kappa\vec{b}_0^j(\kappa r)\Big|_{\kappa=1}$$

$$= (\hat{j}\times\hat{r})\left(-\frac{1}{\sinh r} + \frac{r\cosh r}{\sinh^2 r}\right)\,,$$

and is already in background-gauge form.

B. Mode functions which obey the boundary conditions for finite-energy solutions at $r=\infty$

As I will discuss in Sec. IV below, the modes with removable singularities may not be the only ones of interest in studying whether the Prasad-Sommerfield solution has axially symmetric extensions. Another relevant classification of modes is obtained by applying only the boundary condition at infinity,

$$0 = \lim_{x\to\infty}\delta\vec{b}^\mu\,, \tag{53}$$

which is expected[2] to be satisfied by any nondilatational variation of a finite-energy static Euclidean gauge field. Clearly, Eq. (53) is satisfied by the seven modes with removable singularities and in addition by all of the irregular modes, which are singular at $r=0$ but are asymptotically exponentially decreasing. The regular modes with irremovable singularities are asymptotically exponentially increasing, and so do not obey Eq. (53).

IV. DISCUSSION

I turn now to a discussion of the significance of the results of the preceding two sections for the question raised in Sec. I, of whether the spherically symmetric Prasad-Sommerfield solution has nonspherical, axially symmetric extensions. The first point to be made is that while in the analogous instanton classification problem there is a one-to-one correspondence between parameters in the most general solution and *normalizable* zero modes,[8] this need not be the case; a family of solutions to a set of nonlinear equations can contain parameters related to non-normalizable zero modes. A familiar example of this is provided by the Kerr family of solutions to the stationary, axially symmetric Einstein equations. Let us regard the Kerr metric $g^K_{\mu\nu}$, in the limit of small angular momentum/unit mass a, as a perturbation[9] of the spherically symmetric Schwarzschild metric $g^S_{\mu\nu}$,

$$g^K_{\mu\nu} = g^S_{\mu\nu} + h_{\mu\nu}\,,$$

$$g^S_{\mu\nu}dx^\mu dx^\nu = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1}dr^2$$

$$- r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \tag{54}$$

$$h_{t\phi} = h_{\phi t} = \frac{2aM}{r}\sin^2\theta\,,$$

$$h_{\mu\nu} = 0,\quad (\mu\nu) \neq (t\phi), (\phi t)\,.$$

It is easy to verify that $h_{\mu\nu}$ of Eq. (54) is in the transverse, traceless gauge (the general-relativistic analog of the Yang-Mills background gauge) in the Schwarzschild background,

$$g^{S\mu\nu}h_{\mu\nu} = 0\,,$$

$$h^{\mu\nu}{}_{;\nu} = \frac{\partial h^{\mu\nu}}{\partial x^\nu} + \Gamma^{S\mu}_{\lambda\nu}h^{\lambda\nu} + \Gamma^{S\nu}_{\lambda\nu}h^{\mu\lambda} = 0\,. \tag{55}$$

However, in the norm natural to the perturbation problem around the Schwarzschild background,[10] $h_{\mu\nu}$ is non-normalizable,

$$(h,h)=-\int_{2M}^{\infty}dr\int d\theta d\phi(-g^S)^{1/2}h_{\mu\nu}h_{\lambda\sigma}g^{S\mu\lambda}g^{S\nu\sigma}$$
$$=\int_{2M}^{\infty}r^2dr\int \sin\theta\, d\theta d\phi\; 2g^{Stt}(-g^{S\phi\phi})h_{t\phi}{}^2$$
$$=\frac{32\pi a^2M}{3}\ln\left(1-\frac{2M}{r}\right)\Big|_{2M}^{\infty}=\infty\,. \tag{56}$$

In fact, as Gibbons and Perry[10] show, the only normalizable zero modes of the Schwarzschild metric correspond to the infinitesimal translations, which must be taken into account by use of collective coordinates in functional integral quantization around a Schwarzschild background. On the other hand, spin-ups of the Schwarzschild metric do not lead to functional integration collective coordinates, even though they do correspond to a parameter present in the extended Kerr family of solutions. Similarly, if the Prasad-Sommerfield solution is a member of a more general axially symmetric family of (anti-) self-dual solutions, the deformation parameters need not correspond to normalizable zero modes.

To establish a framework for analyzing the results of Secs. II and III, let $\mathfrak{F}$ denote the function space containing all static Euclidean anti-self-dual SU(2) fields, irrespective of boundary conditions which are imposed to specify solutions of physical interest. The Prasad-Sommerfield solution is a point $\mathcal{P}$ in this function space. Although $\mathfrak{F}$ is not a vector space, we can define a tangent vector space $\mathcal{T}_{\mathcal{P}}$ at $\mathcal{P}$ as the space of solutions of the linearized small-fluctuation equations around the Prasad-Sommerfield solution. Each small-fluctuation eigenmode, when projected into the background gauge, gives a tangent vector extending from $\mathcal{P}$ into the function space $\mathfrak{F}$. [In the case of parity-odd perturbations, defined by $\delta\vec{b}^{\mu}(-x)=-\delta\vec{b}^{\mu}(x)$, a simple systematic procedure for integrating in the background gauge out from $\mathcal{P}$ into $\mathfrak{F}$ is given in Appendix C.] Since we have ignored boundary conditions in the small-fluctuation analysis, we can expect that many (perhaps almost all) of these tangent vectors lead to anti-self-dual solutions in $\mathfrak{F}$ which are "near" to $\mathcal{P}$ but which are physically unacceptable, because they are singular and violate the finite-energy condition. In order to relate properties of the tangent vectors to properties of the nearby solutions, an analyticity assumption is needed. I consider two cases:

Case I. Nearby solutions which are analytic functions of their coordinates in the strips $|\mathrm{Im}x^i|\leq\xi$ and which are analytic functions of their deformation parameters in a ball around $\mathcal{P}$.

In this case, the tangent vectors obtained by differentiating the nearby solutions with respect to parameters will be nonsingular in $0\leq r\leq\infty$. In Sec. III I showed that, when the dilatation mode is included, there are among the modes explicitly constructed eight small-fluctuation eigenmodes with removable singularities. Four of these have even parity [$\delta\vec{b}^{\mu}(-x)=\delta\vec{b}^{\mu}(x)$] and correspond to (i) an $l=0$ gauge automorphism, given by Eq. (44), which leaves $\vec{b}_0^0$ invariant and regauges $\vec{b}_0^j$, and (ii) three $l=1$ ($m=0,\pm1$) translation modes, given by Eq. (48). These four modes are the only normalizable zero modes. The other four modes with removable singularities have odd parity and correspond to (iii) an $l=0$ dilatation mode, given by Eq. (52), and (iv) three $l=1$ ($m=0,\pm1$) gauge modes, the $m=0$ member of which is given in Eq. (50). Since the Prasad-Sommerfield solution has odd parity [from Eq. (5) it is evident that $\vec{b}_0^{\mu}(-x)=-\vec{b}_0^{\mu}(x)$], modes (iv) have just the correct quantum numbers to represent "stretch" distortions of the Prasad-Sommerfield solution; they are analogs of the spin-ups in the Schwarzschild perturbation problem. However, as I also showed in Sec. III, modes (iv) have vanishing projection into the background gauge, and hence do not give tangent vectors to which the integration procedure of Appendix C can be applied. Thus, it is not possible to reach a deformed extension of the Prasad-Sommerfield solution $\mathcal{P}$ by integration out along a tangent vector defined by a nonvanishing, nonsingular small-perturbation mode of the class explicitly constructed.

Case II. Nearby solutions which do not have the simultaneous analyticity in coordinates and parameters postulated in Case I.

In this case, one cannot establish a simple connection between nearby solutions and properties of their tangent vectors. One can have nonsingular nearby solutions with tangent vectors which vanish identically, or with tangent vectors which are singular at $r=0$ (or $r=\infty$). An example of the former would be a perturbation proportional to $e^{-r/a}$, $a>0$; this is bounded and has bounded spatial derivatives in $0\leq r\leq\infty$ but has a vanishing tangent vector $\partial/\partial a|_{a=0}$. Such a behavior might be associated with the $l=1$ odd-parity modes, which we saw degenerated into gauge modes, and vanished in the background gauge, for dynamical rather than kinematical reasons.[11] An example of the latter would be a perturbation proportional to $e^{-(r+a/r)}-e^{-r}$, which again is bounded and has bounded spatial derivatives in $0 \leq r \leq \infty$, but which has a tangent vector $\partial/\partial a|_{a=0}$ which is singular at $r=0$. Such behavior might be associated with any of the infinite number of irregular modes discussed in Sec. III B.

Up to this point I have skirted the question of whether the two series of mode functions obtained above constitute *all* of the anti-self-dual small-fluctuation modes of the Prasad-Sommerfield solution. This question in fact cannot be answered within the framework of the explicit construction I have used. However, there is strong evidence[12,13] for the existence of one additional parity (−) series of mode functions, with regular and irregular eigenmodes behaving at infinity as $r^l, r^{-(l+1)}$ $(l>1)$. It has already been established[12] that the new series can contain no further modes with removable singularities.

To sum up, the results of Sec. III argue against the existence of analytic extensions (in the sense defined above) of the Prasad-Sommerfield solution, and are compatible with (but provide no positive evidence for) the existence of nonanalytic extensions. To do better will require the use of nonperturbative methods.

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APPENDIX A: GAUGE TRANSFORMATION AND SPHERICAL-COORDINATE FORMULAS

Consider a general gauge transformation of the small-fluctuation mode functions of the form

$$\begin{aligned} \delta_g \vec{b}_0^{\mu} &= D_0^{\mu} \vec{\psi}, \\ \vec{\psi} &= \psi_r \hat{r} + \psi_\theta \hat{\theta} + \psi_\phi \hat{\phi}. \end{aligned} \tag{A1}$$

Explicit formulas for the unit vectors $\hat{r}, \hat{\theta}, \hat{\phi}$ in spherical coordinates are

$$\begin{aligned} \hat{r} &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \\ \hat{\theta} &= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \\ \hat{\phi} &= (-\sin\phi, \cos\phi, 0), \\ \hat{r}\times\hat{\theta} &= \hat{\phi}, \quad \hat{\phi}\times\hat{r} = \hat{\theta}, \quad \hat{\theta}\times\hat{\phi} = \hat{r}; \end{aligned} \tag{A2}$$

some useful formulas for computing derivatives in polar coordinates are

$$\begin{aligned} &\vec{\nabla} r = \hat{r}, \quad \vec{\nabla}\theta = \frac{1}{r}\hat{\theta}, \quad \vec{\nabla}\phi = \frac{1}{r\sin\theta}\hat{\phi}, \\ &\frac{\partial \hat{r}}{\partial x^j} = \frac{1}{r}(\hat{j} - \hat{r}\hat{r}^j) = \frac{1}{r}(\hat{\theta}\hat{\theta}^j + \hat{\phi}\hat{\phi}^j), \\ &\frac{\partial \hat{\theta}}{\partial x^j} = \frac{1}{r}(-\hat{r}\hat{\theta}^j + \cot\theta\, \hat{\phi}\hat{\phi}^j), \\ &\frac{\partial \hat{\phi}}{\partial x^j} = \frac{1}{r}(-\hat{r}\hat{\phi}^j - \cot\theta\, \hat{\theta}\hat{\phi}^j). \end{aligned} \tag{A3}$$

Using these, it is easy to compute the gauge transformation generated by Eq. (A1), with the result

$$\begin{aligned} D_0^0\vec{\psi} &= \hat{\theta}\left(\coth r - \frac{1}{r}\right)\psi_\phi + \hat{\phi}\left(\frac{1}{r} - \coth r\right)\psi_\theta, \\ D_0^j\vec{\psi} &= \hat{r}\hat{r}^j \frac{\partial\psi_r}{\partial r} + \hat{r}\hat{\theta}^j\left(\frac{1}{r}\frac{\partial\psi_r}{\partial\theta} - \frac{\psi_\theta}{\sinh r}\right) + \hat{r}\hat{\phi}^j\left(\frac{1}{r\sin\theta}\frac{\partial\psi_r}{\partial\phi} - \frac{\psi_\phi}{\sinh r}\right) + \hat{\theta}\hat{r}^j\frac{\partial\psi_\theta}{\partial r} + \hat{\theta}\hat{\theta}^j\left(\frac{1}{r}\frac{\partial\psi_\theta}{\partial\theta} + \frac{\psi_r}{\sinh r}\right) \\ &\quad + \hat{\theta}\hat{\phi}^j\left(\frac{1}{r\sin\theta}\frac{\partial\psi_\theta}{\partial\phi} - \frac{\cot\theta\,\psi_\phi}{r}\right) + \hat{\phi}\hat{r}^j\frac{\partial\psi_\phi}{\partial r} + \hat{\phi}\hat{\theta}^j\frac{1}{r}\frac{\partial\psi_\phi}{\partial\theta} + \hat{\phi}\hat{\phi}^j\left(\frac{1}{r\sin\theta}\frac{\partial\psi_\phi}{\partial\phi} + \frac{\psi_r}{\sinh r} + \frac{\cot\theta}{r}\psi_\theta\right). \end{aligned} \tag{A4}$$

APPENDIX B: PROPAGATOR FORMULAS

The scalar, isovector propagator $\Delta^{ab}(x,y)$ for the Prasad-Sommerfield solution is given by the formula

$$\begin{aligned} \Delta^{ab}(x,y) &= \frac{1}{4\pi}\frac{x}{\sinh x}\frac{y}{\sinh y}\Sigma^{ab}, \\ \Sigma^{ab} &= \sum_{i=1}^{5}\sigma_i^{ab}(x,y)\lambda_i(x,y); \end{aligned} \tag{B1}$$

$$\begin{aligned} \sigma_1^{ab} &= \delta^{ab} + \frac{\vec{x}\cdot\vec{y}}{x^2y^2}x^a y^b - \frac{x^a x^b}{x^2} - \frac{y^a y^b}{y^2}, \\ \sigma_2^{ab} &= x^a y^b, \\ \sigma_3^{ab} &= x^b y^a - \delta^{ab}\vec{x}\cdot\vec{y}, \\ \sigma_4^{ab} &= \frac{x^a}{x^2}\left(x^b - y^b\frac{\vec{x}\cdot\vec{y}}{y^2}\right), \\ \sigma_5^{ab} &= \frac{y^b}{y^2}\left(y^a - x^a\frac{\vec{x}\cdot\vec{y}}{x^2}\right); \end{aligned} \tag{B2}$$

$$\lambda_1 = \frac{1}{2\Delta}\left[f_2(z_{++}) + f_2(z_{--}) + f_2(z_{+-}) + f_2(z_{-+})\right],$$

$$\lambda_2 = \frac{1}{x^2y^2}\left[\frac{\cosh x\cosh y - e^{-\Delta}}{\Delta} - \frac{\sinh x}{x}\frac{\sinh y}{y}\right],$$

$$\lambda_3=\frac{1}{2xy\Delta}[-f_2(z_{++})-f_2(z_{--})+f_2(z_{+-})+f_2(z_{-+})], \tag{B3}$$

$$\lambda_4=\frac{1}{2x\Delta}\{e^x[f_1(z_{-+})+f_1(z_{--})]-e^{-x}[f_1(z_{+-})+f_1(z_{++})]\},$$

$$\lambda_5=\frac{1}{2y\Delta}\{e^y[f_1(z_{+-})+f_1(z_{--})]-e^{-y}[f_1(z_{-+})+f_1(z_{++})]\};$$

$$\Delta=|\vec{x}-\vec{y}|,$$

$$z_{++}=x+y-\Delta,\quad z_{+-}=x-y-\Delta,$$

$$z_{-+}=-x+y-\Delta,\quad z_{--}=-x-y-\Delta, \tag{B4}$$

$$f_1(z)=\frac{e^z-1}{z},\quad f_2(z)=\frac{e^z-1-z}{z^2}.$$

An inspection of these formulas shows that as $y\to\infty$ with x fixed, the only contribution of order y^{-1} comes from the λ_2 term. This gives

$$\lambda_2=\frac{1}{x^2y^2}\frac{\sinh y}{y}\left(\cosh x-\frac{\sinh x}{x}\right)\times[1+O(y^{-1})], \tag{B5}$$

$$\Delta^{ab}(x,y)=\hat{x}^a\left(\coth x-\frac{1}{x}\right)\frac{1}{4\pi}\frac{\hat{y}^b}{y}+O(y^{-2}),$$

as used in Eq. (42) of the text.

APPENDIX C: BACKGROUND-GAUGE INTEGRATION PROCEDURE FOR PARITY-ODD PERTURBATIONS

To give a procedure for integrating first-order perturbations around the Prasad-Sommerfield solution to finite order, it is convenient to work directly from the Euclidean SU(2) Yang-Mills equations

$$D^\mu\vec{f}^{\mu\nu}=0,$$

$$\vec{f}^{\mu\nu}=\partial^\mu\vec{b}^\nu-\partial^\nu\vec{b}^\mu+\vec{b}^\mu\times\vec{b}^\nu, \tag{C1}$$

$$D^0=\vec{b}^0\times,\quad D^j=\frac{\partial}{\partial x^j}+\vec{b}^j\times,$$

which are automatically satisfied by anti-self-dual fields. Introducing a perturbation expansion about a zeroth-order solution $\vec{b}_0^\mu$ of Eq. (C1) by writing

$$\vec{b}^\mu=\sum_{n=0}^{\infty}\lambda^n\vec{b}_n^\mu, \tag{C2}$$

imposing the zeroth-order background-gauge condition in all orders,

$$D_0^\mu\vec{b}_n^\mu=0,\quad n=1,2,\ldots \tag{C3}$$

and equating like powers of λ, one gets the following set of equations for the perturbation coefficient functions:

$$D_0^\mu D_0^\mu\vec{b}_1^\nu+2\vec{f}_0^{\nu\mu}\times\vec{b}_1^\mu=0, \tag{C4a}$$

$$D_0^\mu D_0^\mu\vec{b}_n^\nu+2\vec{f}_0^{\nu\mu}\times\vec{b}_n^\mu=\vec{j}_n^\nu,\quad n\geq 2$$

$$\vec{j}_n^\nu=\sum_{\substack{k\geq 1,\, l\geq 1\\ k+l=n}}(-2\vec{b}_k^\mu\times D_0^\mu\vec{b}_l^\nu+\vec{b}_k^\mu\times D_0^\nu\vec{b}_l^\mu)-\sum_{\substack{j\geq 1,\, k\geq 1,\, l\geq 1\\ j+k+l=n}}\vec{b}_j^\mu\times(\vec{b}_k^\mu\times\vec{b}_l^\nu). \tag{C4b}$$

For each tangent vector $\vec{b}_1^\nu=\delta\vec{b}^\nu$ satisfying the homogeneous small-fluctuation equation Eq. (C4a), one can iteratively obtain the corresponding higher-order perturbations $\vec{b}_n^\nu$, $n>1$ by using the vector propagator $G^{a\mu,b\nu}$ of Eq. (32) of Ref. 2 to invert Eqs. (C4b). This inversion is well-defined in nth order provided that the current $\vec{j}_n^\nu$ is orthogonal to the normalizable zero modes of the zeroth-order solution. In the case when the zeroth-order solution is the Prasad-Sommerfield solution, the zeroth-order solution $\vec{b}_0^\mu$ has odd parity and its four normalizable zero modes have even parity. Hence in this case a simple sufficient condition for orthogonality is that the tangent vector $\delta\vec{b}^\mu$ have odd parity, since then the $\vec{b}_n^\nu$ and $\vec{j}_n^\nu$ will have odd parity for all $n>1$.

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