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Sum Rules for Inelastic Functions from the Unitarization of Veneziano Partial-Wave Amplitudes

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Veneziano partial-wave amplitudes are unitarized by the truncated N/D method. The N functions are extrapolated directly from the Veneziano partial-wave amplitudes. Sum rules for inelastic functions are derived from the assumption that the spectrum and the residues of the resonances in the Veneziano model are preserved. Solving these sum rules, the behavior of inelastic functions and the inelastic widths of the resonances can be estimated given a single elastic scattering amplitude of the Veneziano model. An explicit application of this method to the $\pi\pi$ $I=1$ p-wave amplitude is given.

I. INTRODUCTION

The discovery of the Veneziano model[1] and the subsequent extension of the model to many-particle amplitudes[2] is one of the important advances in particle physics in recent years. The model is crossing-symmetric and analytic. It also demonstrates the duality property between resonances and Regge asymptotic behavior. One of the shortcomings of the model is the absence of normal thresholds, so all the resonances of the model lie on the real axis of the complex energy plane, and unitarity is violated explicitly. This difficulty prevents an immediate comparison of the model amplitude with experimental data. Various proposals have been made to overcome this difficulty. These proposals may be roughly classified into three categories: the perturbative approach,[2] the attempts to construct amplitudes with nonlinear Regge trajectories,[3-8] and the attempts to unitarize the partial waves of the Veneziano model.[9-13] The purpose of the first two kinds of approaches is to obtain physically acceptable crossing-symmetric, analytic, and unitary amplitudes, but complete success has not been achieved yet. The third kind of approach is more phenomenology-minded. It tries to extract the maximum possible useful information from the original Veneziano model, and to construct partial-wave amplitudes from this information so that a comparison with experimental partial-wave phase shifts can be made. In the present article a method of the third category is proposed, and sum rules for inelastic functions are derived.

The unitarization of the Veneziano partial-wave amplitudes (the partial-wave amplitudes obtained from partial-wave projection of the original Veneziano amplitude) by the use of K matrices[9,10] is probably the most successful method so far proposed in this category of approaches. This method interprets a Veneziano partial-wave amplitude as the K matrix of that partial-wave amplitude. If a single elastic-scattering amplitude, e.g., $\pi\pi$ elastic scattering, is considered, only elastically unitarized partial-wave amplitudes can be obtained in this method. In order to obtain information about inelastic functions, we must consider simultaneously the amplitudes of quasi-two-body channels which can communicate with the original elastic channel, e.g., $K\bar{K}$, $\pi\omega$, $\rho\rho$, etc. for the case of $\pi\pi$ elastic scattering. On the other hand a single elastic-scattering amplitude of the Veneziano model, e.g., the $\pi\pi$ scattering amplitude, already shows Regge asymptotic behavior which is dual to the superposition of resonances. This implies that some of these resonances of the Veneziano model are inelastic, and a single Veneziano amplitude is already representing the inelastic effects in some average sense. From this point of view we can expect to extrapolate certain kinds of information about inelastic functions from a single elastic-scattering amplitude in the Veneziano model. In this article we consider a method based on the N/D decomposition which allows us to derive sum rules for inelastic functions from a single elastic Veneziano amplitude.

The N/D method has recently been reformulated[14-18] so that the assumption of power-boundedness of the partial-wave amplitude can be dropped. This reformulation of the N/D method justifies its application to the unitarization procedure of Veneziano partial-wave amplitudes, in which partial-wave amplitudes are usually not power-bounded. Some proposals[11-13] have been made to use the N/D method for the unitarization procedure by extrapolating an input potential from the Veneziano partial-wave amplitude. This kind of method has the difficulty that an inelastic function must be guessed arbitrarily and Castillejo-Dalitz-Dyson (CDD) pole parameters must be supplied as well as the input potential in order to solve the N/D integral equation. Another difficulty in some of these proposals is that in general only a finite number of CDD poles can be handled in the N/D method without encountering the question of convergence, whereas a Veneziano partial-wave amplitude contains an infinite number of CDD poles. These difficulties are avoided in the method proposed in this article. We extrapolate an N function instead of a potential from the Veneziano partial-wave amplitude, so the complicated process of solving an integral equation can be avoided. A truncated N/D decomposition of a partial-wave amplitude is used, in which only a finite number of CDD poles are handled at one time. The truncated N function in the region from the threshold to the truncated point is approximated by the N function extrapolated from the Veneziano partial-wave amplitude. This approximation becomes reasonable by choosing the truncated point at specific positions, i.e., on the top of a physical-region zero of the Veneziano partial-wave amplitude. The preservation of the spectrum and residues of the resonances of the original Veneziano model is required, and this fixes all the necessary CDD pole parameters and yields sum rules for inelastic functions. We also show that the truncated D function used in this method is free from ghost zeros.

We only consider spinless equal-mass scattering in this article. The truncated N/D decomposition for a partial-wave amplitude is reviewed in Sec. II. The absence of ghost poles from this N/D decomposition is shown in Sec. III. In Sec. IV the procedures of unitarizing a Veneziano partial-wave amplitude and the derivation of the sum rules for inelastic functions are outlined. An explicit application of this method to the $\pi\pi$ $I=1$ p-wave amplitude is given in Sec. V. In the last section we discuss some further developments of this method.

II. TRUNCATED N/D DECOMPOSITION

We review the truncated N/D method in this section. A partial-wave amplitude $A_l(s)$ is parametrized as

$$A_l(s)=|A_l(s)|\exp[i\theta_l(s)]\,.$$

Unitarity requires that $\mathrm{Im}A_l(s)$ be positive, and thus $\theta_l(s)$ must be taken to be

$$2n\pi\leqslant\theta_l(s)\leqslant(2n+1)\pi\quad(n=0,\pm1,\ldots)\,.$$

A function $\tilde{D}_l^T(s)$ is defined as [we choose $0\leqslant\theta_l(s)\leqslant\pi$ for simplicity]

$$\tilde{D}_l^T(s)=\exp\left[-\frac{s}{\pi}\int_{s_0}^{s_2}ds'\,\frac{\theta_l(s')}{s'(s'-s)}\right],\qquad(2.1)$$

where s_2 is the truncated point. We consider the case of one CDD pole here, though the extension to any finite number of CDD poles is straightforward. We assume that the partial-wave amplitude $A_l(s)$ has a zero at $s=s_c$ with $s_0<s_c\leqslant s_2$, and

$$\lim_{\Delta\to0+}\theta_l(s_c-\Delta)=\pi$$

and

$$\lim_{\Delta\to0+}\theta_l(s_c+\Delta)=0\,.$$

The function $\tilde{D}_l^T(s)$ will have a pole at $s=s_c$ which we call a CDD pole. A new function $D_l^T(s)$ free from the pole at $s=s_c$ is defined as

$$D_l^T(s)=c(s_c-s)\tilde{D}_l^T(s)\,,\qquad(2.2)$$

where c is a normalization constant. From the definition of $\tilde{D}_l^T$ and D_l^T, we see that the asymptotic behavior of D_l^T is

$$D_l^T(s)\underset{|s|\to\infty}{\sim}s\,.\qquad(2.3)$$

A truncated N function is defined as

$$N_l^T(s)=A_l(s)D_l^T(s)\,.$$

The partial-wave unitarity relation can be written as

$$\mathrm{Im}\frac{1}{A_l(s)}=-\rho(s)R_l(s)\,,$$

where the phase-space factor $\rho(s)$ is

$$\rho(s)=\left(\frac{s-s_0}{s}\right)^{1/2}$$

and $R_l(s)$ is the inelastic function, which is unity in the elastic region and is larger than or equal to unity above the elastic region. From the definition of N_l^T and the partial-wave unitarity relation, we see that

$$\operatorname{Im}D_l^T(s)=\begin{cases}-\rho(s)R_l(s)N_l^T(s), & s_0\leqslant s\leqslant s_2\\ 0, & s<s_0 \text{ and } s>s_2.\end{cases}$$

From Eq. (2.3) we see that the function $D_l^T(s)$ satisfies a dispersion relation with two subtractions, and it can be written as

$$D_l^T(s)=1+\gamma s-\frac{s^2}{\pi}\int_{s_0}^{s_2}ds'\frac{\rho(s')R_l(s')N_l^T(s')}{s'^2(s'-s)}, \quad (2.4)$$

where $D_l^T(0)$ is normalized to be unity, and where

$$\gamma=\left[\frac{d}{ds}D^T(s)\right]_{s=0}.$$

Further construction of an integral equation for $N_l^T(s)$ will not be considered here, since it is not required for our later application of this N/D decomposition.

The truncated N and D functions, N_l^T and D_l^T, both have branch points at the end point $s=s_2$. The partial-wave amplitude $A_l(s)$ is, of course, free from this spurious branch point. The function N_l^T also has a zero at $s=s_c$. In our later application we will be interested in the limiting case $s_c\to s_2$. Though the function N_l^T still has the branch point at $s=s_2$ in this limit, its over-all behavior near $s=s_2=s_c$ can be calculated by putting

$$\theta_l(s)=\begin{cases}f_1(s)+\pi, & s<s_c\\ f_2(s), & s>s_c\end{cases}$$

where

$$f_1(s_c)=f_2(s_c)=0.$$

Explicit integration gives at the limit $s_c=s_2$

$$D_l^T(s)\sim(s_c-s)^{\pi-\theta_l(s)}\quad[\theta_l(s_c)=\pi],$$

and thus

$$N_l^T(s)\sim(s_c-s)(s_c-s)^{\pi-\theta_l(s)}\quad(s\sim s_c=s_2),$$

where

$$\lim_{s\to s_c}\theta_l(s)=\pi.$$

We have discussed the case with one physical-region zero of $A_l(s)$ (or one CDD pole) in the interval $s_0\leqslant s\leqslant s_2$. The generalization to include any finite number of physical-region zeros of $A_l(s)$ in this interval is straightforward. The change in this generalization is that the dispersion relation of $D_l^T(s)$ requires $n+1$ subtraction constants in order to include n physical-region zeros of $A_l(s)$ in the interval $s_0\leqslant s\leqslant s_2$. Equation (2.4) now becomes

$$D_l^T(s)=1+\sum_{k=1}^{n}\gamma_k s^k-\frac{s^n}{\pi}\int_{s_0}^{s_2}ds'\frac{\rho(s')R_l(s')N_l^T(s')}{s'^n(s'-s)}, \quad (2.5)$$

where

$$\gamma_k=\left[\frac{d^k}{ds^k}D^T(s)\right]_{s=0}.$$

III. ABSENCE OF GHOST ZEROS

We demonstrate in this section the absence of ghost zeros in the $D_l^T(s)$ constructed in the previous section. The function $D_l^T(s)$ calculated from Eq. (2.4) can be parametrized as

$$D_l^T(s)=|D_l^T(s)|\exp[-i\phi(s)]$$

where $\sin\phi(s)$ vanishes for $s<s_0$ and $s>s_2$, and $\phi(s)$ is a real function of s. A function $F(s)$ is constructed as

$$F(s)=\frac{s_c-s}{s_c}\exp\left[-\frac{s}{\pi}\int_{s_0}^{s_2}ds'\frac{\phi(s')}{s'(s'-s)}\right]. \quad (3.1)$$

The asymptotic behavior of $F(s)$ is, from its definition,

$$F(s)\underset{|s|\to\infty}{\sim}s.$$

The asymptotic behavior of $D_l^T(s)$ can be seen from Eq. (2.4) as

$$D_l^T(s)\underset{|s|\to\infty}{\sim}s.$$

The function $D_l^T(s)/F(s)$ is regular on the whole complex s plane and approaches a constant asymptotically in all directions. This implies that

$$D_l^T(s)\equiv F(s),$$

where we have used the condition

$$D_l^T(0)=F(0)=1.$$

Since $F(s)$ defined in Eq. (3.1) is obviously free from any ghost zeros on the complex s plane, so is $D_l^T(s)$.

We note that the required input in this discussion is the value of $N_l^T(s)$ in the interval $s_0\leqslant s\leqslant s_2$. If $N_l^T(s)$ is chosen such that there are no unacceptable singularities at complex positions on the first sheet, the discussion of this section shows that the partial-wave amplitude $A_l(s)$ is free of ghosts.

IV. THE PROCEDURE FOR UNITARIZING A VENEZIANO PARTIAL-WAVE AMPLITUDE

We take a simple Veneziano amplitude

$$A(s,t)=-\beta\frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)}, \quad (4.1)$$

where

$$\alpha_x = ax + b\ , \tag{4.2}$$

and consider its unitarization by the techniques developed in previous sections. A practical and explicit example, the unitarization of the $\pi\pi$ $I=1$ p-wave amplitude, is left to the next section. The partial-wave amplitudes are

$$A_l^V(s) = -\tfrac{1}{2}\beta\Gamma(1-\alpha_s)\int_{-1}^{1} dz_s P_l(z_s)\frac{\Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)} \quad (l=0,1,\ldots)\ . \tag{4.3}$$

The factor $\Gamma(1-\alpha_s)$ contains all the poles of $A_l^V(s)$ on the positive real axis of the complex s plane, and the factor

$$\int_{-1}^{1} dz_s P_l(z_s)\frac{\Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)}$$

contains only the left-hand singularities. Using the relation

$$\Gamma(1-\alpha_s) = \frac{\pi}{\Gamma(\alpha_s)\sin\pi\alpha_s}\ ,$$

we can decompose the Veneziano partial-wave amplitude of Eq. (4.3) into the N/D form by identifying

$$\begin{aligned} D_l^V(s) &= \frac{c}{\pi}\Gamma(\alpha_s)\sin\pi\alpha_s\ , \\ N_l^V(s) &= -\tfrac{1}{2}\beta c\int_{-1}^{1} dz_s P_l(z_s)\frac{\Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)}\ , \end{aligned} \tag{4.4}$$

where c is a normalization constant and is chosen to be

$$c = \Gamma(1-b)$$

to make $D_l^V(0)=1$. In the physical region the Veneziano N function $N_l^V(s)$ has an infinite number of zeros whose positions are denoted as

$$s = \omega_1, \omega_2, \ldots \quad (\omega_1 < \omega_2 < \cdots)\ .$$

The truncated point s_2 of Eq. (2.4) should be chosen as

$$s_2 = \omega_n \quad (n=1,2,\ldots)\ .$$

For the simplest case, we choose $s_2=\omega_1$. The function $N_l^T(s')$ of Eq. (2.4) is approximated as

$$N_l^T(s) = N_l^V(s) \quad \text{for } s_0 \le s \le s_2 = \omega_1\ . \tag{4.5}$$

This approximation is reasonable since $N_l^T(s')$ near $s_2=\omega_1$ has behavior like $(\omega_1-s)^{1+\pi-\theta_l(s)}$, where $\theta_l(\omega_1)=\pi$, and $N_l^V(s')$ has a zero at $s=\omega_1$. Substituting Eq. (4.5) into Eq. (2.4), we obtain an explicit representation for $D_l^T(s)$ which, unlike the Veneziano D function $D_l^V(s)$ of Eq. (4.4), has threshold branch points at $s=s_0$ and $s=s_2$. This representation of $D_l^T(s)$ can be written as

$$D_l^T(s) = 1 + \gamma s - \frac{s^2}{\pi}\int_{s_0}^{s_2} ds'\,\frac{\rho(s')R_l(s')N_l^V(s')}{s'^2(s'-s)}\ . \tag{4.6}$$

The way to determine the constants γ and β and to derive sum rules for $R_l(s')$ is to require the preservation of the spectrum and residues of the resonances in the Veneziano model. For simplicity we consider the case $l=1$ here. The $l=1$ Veneziano partial-wave amplitude has poles at

$$\alpha_s = n \quad (n=1,2,\ldots)\ .$$

The first pole $\alpha_s=1$ is in the interval $s_0<s<s_2=\omega_1$, and is called the ρ meson. The requirement of the existence of this ρ meson in our unitarized partial-wave amplitude implies that we must put

$$\mathrm{Re}D_1^T(m_\rho^2) = 0\ . \tag{4.7}$$

The over-all constant β in the original Veneziano model is fixed by equating the residue of ρ to that of a Breit-Wigner formula and then saturating elastic partial-wave unitarity with the Breit-Wigner formula. The ρ residue of the original Veneziano model is

$$\rho \text{ residue} = N_1^V(m_\rho^2)\Big/\left[\frac{d}{ds}D_1^V(s)\right]_{s=m_\rho^2}\ .$$

For our unitarized $l=1$, amplitude, the residue of the ρ at $s=m_\rho^2$ is

$$\rho \text{ residue} = N_1^V(m_\rho^2)\Big/\left[\frac{d}{ds}\mathrm{Re}D_1^T(s)\right]_{s=m_\rho^2}\ .$$

Assuming that the ρ width of our unitarized partial wave agrees with the experimental width and that our ρ residue equals that of the original Veneziano model, we have the relation

$$\left[\frac{d}{ds}\mathrm{Re}D_1^T(s)\right]_{s=m_\rho^2} = \left[\frac{d}{ds}D_1^V(s)\right]_{s=m_\rho^2}\ , \tag{4.8}$$

so we can use the same value of β as in the original Veneziano model. Equations (4.7) and (4.8) only contain γ and $R_l(s')$ as unknown quantities. Eliminating γ from these two equations, we obtain a sum rule for $R_l(s')$:

$$1 + m_\rho^2\left[\frac{d}{ds}D_1^V(s)\right]_{s=m_\rho^2} + \frac{m_\rho^4}{\pi}\,\mathrm{P}\int_{s_0}^{\omega_1} ds'\,\frac{\rho(s')s'N_1^V(s')}{s'^2(s'-m_\rho^2)^2}R_1(s') = 0\ . \tag{4.9}$$

This procedure can be extended to any partial-wave amplitude, and s_2 can be chosen to coincide with ω_n for any integer n. Suppose we consider the lth partial wave and choose $s_2=\omega_n$. The dispersion relation for $D_l^T(s)$ is now Eq. (2.5), which contains $n+1$ subtraction constants. One of the $n+1$ subtraction constants is an over-all normalization constant, so we must fix n remaining subtraction constants. Suppose that there are r resonances of the Veneziano partial-wave amplitude in the interval $s_0 < s < s_2 = \omega_n$ and their positions are denoted as

$$s = m^2(l;k) \quad (k = 1, \ldots, r).$$

The requirement of the existence of these resonances in our unitarized partial-wave amplitude and the preservation of the residues of the original Veneziano model imply that

$$\mathrm{Re}D_l^T(s = m^2(l;k)) = 0 \quad (k = 1, 2, \ldots, r) \tag{4.10}$$

and

$$\left[\frac{d\mathrm{Re}D^T(s)}{ds}\right]_{s=m^2(l;k)} = \left[\frac{dD^V(s)}{ds}\right]_{s=m^2(l;k)} \quad (k = 1, \ldots, r). \tag{4.11}$$

Eliminating n subtraction constants in Eq. (2.5) by Eqs. (4.10) and (4.11), we will obtain $2r - n$ sum rules for $R_l(s')$. If $R_l(s')$ is parametrized properly and these sum rules are saturated, we can obtain the partial-wave amplitude $A_l(s)$ in the region $s_0 < s < \omega_n$, and the inelastic width of the resonances can be obtained.

V. UNITARIZATION OF THE $\pi\pi$ $I = 1$ p-WAVE AMPLITUDE

We consider the unitarization of the $\pi\pi$ $I = 1$ p-wave amplitude from the elastic threshold up to the first zero of the Veneziano partial-wave amplitude in this section. The isospin-one s-channel Veneziano amplitude is[19,20]

$$A^{(1)}(s,t,u) = -\beta[A(s,t) - A(s,u)],$$

where

$$A(x,y) = \frac{\Gamma(1-\alpha_x)\Gamma(1-\alpha_y)}{\Gamma(1-\alpha_x-\alpha_y)} \tag{5.1}$$

and

$$\alpha_x = ax + b.$$

The $l = 1$ partial-wave amplitude is

$$a_1^{(1)}(s) = -\beta\Gamma(1-\alpha_s)\int_{-1}^{1} dz_s\, z_s \frac{\Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)},$$

where

$$z_s = 1 + \frac{2t}{s - 4\mu^2}.$$

The N/D decomposition is achieved by taking

$$\begin{aligned} N_1^V(s) &= -\beta c_1 \int_{-1}^{1} dz_s \frac{\Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)}, \\ D_1^V(s) &= c_1/\Gamma(1-\alpha_s) \\ &= (c_1/\pi)\Gamma(\alpha_s)\sin\pi\alpha_s, \end{aligned} \tag{5.2}$$

where

$$c_1 = \Gamma(1-b).$$

An explicit numerical calculation shows that $N_1^V(s)$ has a zero at $s_c = 1.2$ GeV as well as the threshold zero at $s = 4\mu^2$. The value of $N_1^V(s)$ in the interval $4\mu^2 < s < s_c$ can be reasonably approximated by the expression

$$N_1^V(s) = \kappa(s - 4\mu^2)(s - s_c),$$

where

$$\kappa = -0.44\beta c_1.$$

The value of β in the original Veneziano model is 0.49, where Γ_ρ is taken to be 135 MeV. The derivative of $D_1^V(s)$ at $s = m_\rho^2$ is

$$\left[\frac{d}{ds} D_1^V(s)\right]_{s=m_\rho^2} = -a\Gamma(1-b).$$

The sum rule for $R_1(s)$, Eq. (4.9), becomes

$$1 - am_\rho^2\Gamma(1-b) + \frac{\kappa m_\rho^4}{\pi} \mathrm{P}\int_{4\mu^2}^{s_c} ds' \frac{\rho(s')(s'-4\mu^2)(s'-s_c)}{s'(s'-m_\rho^2)^2} R_1(s') = 0. \tag{5.3}$$

We parametrize $R_1(s)$ as

$$R_1(s) = \begin{cases} 1, & s_0 \le s < s_{K\bar{K}} \\ 1 + d(s - s_{K\bar{K}}), & s_{K\bar{K}} \le s < s_c \end{cases} \tag{5.4}$$

where $s_{K\bar{K}}$ is the $K\bar{K}$ threshold. Substituting the parametrization of Eq. (5.4) into Eq. (5.3), we obtain d by inserting[19]

$$a = 0.885,$$

$$b = 0.483,$$

and

$$\mu = 0.14 \text{ GeV},$$

$$m_\rho = 0.765 \text{ GeV},$$

$$m_K = m_{\bar{K}} = 0.495 \text{ GeV}.$$

The result is

$$d = 4.52 \times 10^2,$$

$$R_1(s_c) = \frac{\sigma_1^{\text{total}}(s_c)}{\sigma_1^{\text{elastic}}(s_c)} \approx 100.$$

We note that in this unitarization procedure the spectrum of zeros of the Veneziano partial-wave amplitude will not be changed, so the unitarized partial-wave amplitude still has a zero at $s = s_c = 1.2$ GeV. This implies that the phase shift $\delta_1(s)$ of this unitarized partial-wave amplitude goes through 180° at $s = s_c$ and the inelasticity $\eta_1(s)$ becomes unity at $s = s_c$. We also note that at the physical-region zero of $A_l(s)$, both σ_l^{total} and $\sigma_l^{\text{elastic}}$ vanish. Therefore the large ratio $R = \sigma_l^{\text{total}}/\sigma_l^{\text{elastic}}$ at this zero is not unrealistic.

VI. DISCUSSION

We have outlined the procedure for unitarizing Veneziano partial-wave amplitudes and derived sum rules for the inelastic functions $R_l(s)$. A simple example, the application of this method to the $\pi\pi$ $I = 1$ p-wave amplitude, is given. For a more realistic application of this method we need to keep the secondary terms of the Veneziano model. By fitting the existing experimental $\pi\pi$ $I = 1$ p-wave phase shift the coefficients of these secondary terms can be determined. Once these coefficients are determined, we can go to other partial waves and unitarize them.

The explicit crossing symmetry of the original Veneziano model is lost in this unitarization procedure. However, the crossing symmetry can be restored by requiring the unitarized partial-wave amplitudes to satisfy the crossing-symmetric constraints.[21-23]

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