Toward a theory of the strong interactions

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A systematic study is made of the relevant degrees of freedom and the dynamics of quantum chromodynamics (QCD). We find that the dynamical properties of QCD are, to a large extent, a consequence of the structure of the vacuum arising from the tunneling between degenerate, classically stable, vacuums, and that the relevant degrees of freedom can be taken to be the Euclidean path histories that can be used to calculate the tunneling in the semiclassical approximation. This nonperturbative vacuum structure appears well suited to the major features of QCD, i.e., the dimensional transmutation that determines the size of the hadrons and the strong-interaction coupling constant, the source of dynamical chiral symmetry breaking, and the mechanism responsible for quark confinement.

I. INTRODUCTION

It is widely believed that the strong interactions are generated by a non-Abelian [SU(3)] gauge theory of quarks and gluons, permanently confined in color-singlet hadronic bound states. This theory is called quantum chromodynamics (QCD). It can be described by the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \prod_{i=1}^{F}\bar{\psi}_i(i\not{D} - m_i)\psi_i\,, \tag{1.1}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu\,,$$

A^a_μ is an SU(3) gauge field, and ψ_i are quark fields with the index i labeling the various quark types, or flavors. The theory is thus parametrized by the one coupling constant g and the values of the quark mass parameters m_i. The total number of quark flavors is so far unknown. In addition to the established up, down ($m_u = m_d \sim 10-100$ MeV), strange ($m_s \simeq 100-300$ MeV), and charmed ($m_c = 1.3-1.4$ GeV) quarks, there might very well exist many heavier quarks with new quantum numbers. Fortunately this is of little relevance to the bulk of hadronic physics, although it is of fundamental importance in understanding the structure of the weak interactions. The properties of light hadrons will not be affected by such heavy quarks. Charmed quarks can be neglected to a very good approximation in describing the properties of noncharmed hadrons.

Our knowledge of the nature of the constituents of hadrons and their interactions derives from their symmetries, the success of various phenomenological quark models, and, most importantly, the observed short-distance behavior of hadronic currents. The success of the SU(3)-symmetry scheme and simple nonrelativistic quark models leads to the picture of hadrons as bound states of spin-$\frac{1}{2}$ triplet colored quarks,[1,2] The absence of colored states leads to the hypothesis of confinement—namely that the only physical states are color singlets.

That the strong interactions are mediated by vector mesons coupled to color is strongly indicated if one is to explain confinement. Even proponents of dynamical schemes outside the framework of QCD invoke colored gauge vector mesons to explain why quark-antiquark bound states occur whereas quark-quark states do not. The observation of scaling in the deep-inelastic scattering of leptons off hadrons singled out non-Abelian gauge theories as the only ones capable of possessing the asymptotic freedom necessary to produce such free-field-like behavior at short distances.[3] In addition these experiments, as well as e^+e^- annihilation to hadrons, allow us to observe directly the quantum numbers of quarks and to derive qualitative predictions that test the validity of the theory.[4,5]

Although one cannot claim that the precise predictions of the theory as to the short-distance structure of hadrons have been experimentally confirmed, the qualitative picture is in remarkable agreement with the data. The asymptotic freedom of QCD has the enormous benefit of allowing one to control the short-distance behavior of the theory. Thus all properties of the theory are calculable in terms of an effective coupling which can be made arbitrarily small by going to short enough distances. As one goes to larger distances, the effective coupling increases, leading to the hypothesis that the increasing coupling in the infrared domain leads to quark confinement—infrared slavery.[4,6]

However, we are still far from possessing a quantitative theory of hadrons in which we could calculate their masses, couplings, and scattering amplitudes. We do not even have a qualitative understanding of the dynamical mechanism for quark confinement. In this paper we shall propose a new approach to QCD, based on an improved understanding of the nature of the QCD vacuum, which might ultimately lead to a quantitative theory of strong interactions.

Before describing the nature of our program we would like to focus on a few of the most difficult and important dynamical problems which must be faced in any serious attempt to solve QCD.

A. Dimensional transmutation

QCD possesses few adjustable parameters. Indeed we shall argue that to a good approximation it has no adjustable parameters. As discussed above the free paramters in the QCD Lagrangian are the coupling g and the various masses, m_i, of the quarks. If we restrict attention to noncharmed hadrons we need three flavors of quark. We should obtain an excellent (10 – 20%) approximation to the real world by setting $m_u = m_d = m_s = 0$. This chiral SU(3)×SU(3)-symmetric approximation to the world is quite reasonable as is evident from the small value of the pion mass. Even if we wish to include the effects of the explicit chiral and SU(3)-symmetry breaking generated by nonvanishing quark masses we do not believe that they would have an important effect on the dynamics and could be treated perturbatively.

Thus, to a good approximation, the theory is described by a single dimensionless parameter g, and contains no relevant dimensional parameter to set the scale of masses. In such a scale-invariant theory the parameter that sets the scale of all dimensional quantities is the renormalization scale parameter μ. This is the arbitrary param eter (with dimension of mass) that is introduced to get the length scale at which the normalization of the quantum fields and the coupling g are defined. Thus all Green's functions will depend on g and μ, although physically measurable quantities $P(g, \mu)$ must depend only on a combination of g and μ invariant under the renormalization, since a change in μ merely indicates a change in what one means by g. Thus

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right)P(g, \mu)=0. \tag{1.2}$$

In particular, masses of physical particles, or any physical parameter with dimensions of masses is given by

$$m(g, \mu)=\mu \exp\left(-\int^{g}\frac{dx}{\beta(x)}\right), \tag{1.3}$$

and all dimensionless physical parameters must be independent of μ and therefore calculable numbers independent of g.[7]

Consequently, except for the overall mass scale of the theory, there are no adjustable parameters. Given some definite renormalization prescription (or definition of g) QCD will be characterized by an effective coupling $\bar{g}(p)$ for a given range of momenta, where

$$p\frac{d\bar{g}}{d(p)}=+\beta(\bar{g}).$$

The value of this coupling for momenta of the order of hadronic masses is a calculable number, $\bar{g}(m_H)$.

This phenomenon of dimensional transmutation has serious implications. It means that ordinary perturbation theory can be of little value in any attempt to calculate the properties of hadrons. This is because Eq. (1.3) implies that hadronic masses behave as

$$m(g, \mu)\sim\mu e^{-b_0/g^2}$$

for small g. Ordinary perturbation theory, or even fancy resummation techniques, will not produce masses that have zero asymptotic expansions in powers of g. Even to construct the vacuum state of QCD, nonperturbative techniques will be required.

One of our major tasks will therefore be to identify nonperturbative mechanisms which could set the mass or length scale of hadronic states, and to attempt to evaluate the magnitude of the hadronic coupling at lengths comparable to the hadronic size.

B. Dynamical symmetry breaking

Owing to the remarkable success of approximate chiral SU(3) symmetry and partial conservation of axial-vector current (PCAC), we believe that the strong interactions possess an almost exact chiral symmetry which is realized in the Goldstone mode. Thus to solve QCD we must not only understand the confinement mechanism but also we must construct the true, chirally asymmetric, ground state. In a theory such as QCD, which does not contain elementary scalar meson fields, the chiral symmetry must be broken dynamically. Thus the fields that acquire nonvanishing expectation values in the true vacuum will be composite (i.e., $\bar{\psi}\psi$), the Goldstone bosons will be quark-antiquark bound states, and the dynamically produced masses will obey Eq. (1.3).

The problem of dynamical symmetry breaking is very difficult, particularly in a gauge theory.

Until now little progress has been made in identifying the mechanism that generates the symmetry breaking, and most attempts to understand confinement have sidestepped the problem by explicitly breaking the chiral symmetry. It is clear that to obtain an accurate description of hadrons it will be necessary to solve this problem. In addition it might very well be the case that the problems of confinement and symmetry breaking are interrelated. In fact, as we shall argue below, the mechanism that we envisage for confinement requires that the quarks acquire a mass dynamically, through a mechanism that we shall identify. The properties of massless quarks might very well differ substantially from those of massive quarks.

C. The U(1) problem

The U(1) problem arises in any quark model which does not contain fundamental scalar fields. It arises due to the fact that in addition to the desirable SU(N) chiral symmetry that obtains for zero quark masses, there also exists a chiral U(1) symmetry generated by the axial baryon number current.[8] If the only explicit breaking of this symmetry is due to the nonvanishing of the quark masses then the dynamical symmetry breaking of chiral U(N) would be expected to generate an isoscalar Goldstone boson with a mass comparable to that of the isovector pion. Weinberg has given plausible arguments which show that in the absence of spontaneous symmetry breaking of SU(3) this would lead to an η particle whose mass would be bounded by $\sqrt{3}\, m_\pi$, in blatant contradiction with experiment.[9]

Consequently any attempt to derive the properties of hadrons and to understand the mechanism of dynamical symmetry breaking in a quark-gluon gauge theory must contain a solution to the U(1) problem or face disaster.

D. Confinement

The central problem in QCD is to understand the mechanism that confines quarks and gluons in color-singlet hadronic bound states. An understanding of this mechanism should then allow one to calculate the properties of the hadronic bound states. At first, however, one wants a simple criterion for confinement. Such a criterion is provided by considering a pure gauge theory (no quarks) and evaluating in such a theory the energy $E(R)$ of a singlet state consisting of two colored external sources separated by a distance R.[10] These sources can be regarded as infinitely massive quarks. A necessary condition for confinement is that $E(R)$ grows with R for large enough separation, so that colored states cannot be produced with a finite expenditure of energy. The advantage of this criterion is that it can be addressed in a gauge-invariant fashion and investigated by means of Euclidean path integrals. Of course, in order to calculate the properties of the hadronic bound states one must contend with real, light, quarks. However, $E(R)$ might be of some physical relevance in treating the low-lying bound states of very massive quarks (e.g. charmonium).

How is one to make progress toward a dynamical understanding of these problems within the framework of QCD? It is clear that a straightforward perturbation theory is useless. Because of infrared slavery, there is no way in which one can ensure a small coupling for low-momentum states. Furthermore, owing to dimensional transmutation and dynamical symmetry breaking, one expects the theory to contain terms which have no asymptotic expansion in powers of g. Such terms, of the form $g^{-p}\exp(-\text{const}/g^2)$, can be large even if g is very small. Of course they will never show up to any order in perturbation theory. It is therefore extremely unlikely that any approach based on perturbation theory, even one that utilizes summation techniques to sum divergent asymptotic series, will be useful in solving QCD.

The most ambitious attempts to calculate within the framework of QCD to date have utilized lattice formulations of the theory.[11] The advantages of this approach are many. By introducing a space-time lattice (or spatial lattice in the Hamiltonian approach) one renders finite the number of degrees of freedom in a finite volume and introduces extra parameters (cutoffs) that can be varied. In addition, lattice QCD has an extremely simple strong-coupling limit which exhibits confinement and many of the qualitative features of hadrons. The systematic approach to lattice QCD envisages the utilization of renormalization-group techniques to proceed from the known dynamics at short distances of the order of the lattice spacing to an effective Hamiltonian appropriate for large (hadronic) distances. At these distances one will presumably be in the strong-coupling regime where other techniques are available (i.e., strong-coupling or high-temperature expansions).

However, the introduction of a space-time lattice has many severe disadvantages. In addition to destroying manifest Lorentz invariance the lattice approach has difficulty in accommodating fermions without explicitly breaking chiral symmetry. Whereas these unphysical features might disappear in the continuum limit, they produce problems at any finite stage of a lattice theory calculation.

In an asymptotically free theory such as QCD there is no need for an ultraviolet cutoff of the type introduced in lattice approximations since the short-distance structure of the theory is completely calculable and under control. Thus it might prove profitable to construct other approximations to the theory which, similar to the lattice models, reduce the number of degrees of freedom but do not spoil the short-distance behavior of the theory or introduce new parameters. Our approach to QCD is an attempt to systematically explore the relevant degrees of freedom, starting from short distances. What we have found is that as one proceeds from short distances physically significant effects are generated by instantons.

Historically the interest in instantons arose because of the discovery of an exact finite-action solution to the classical Yang-Mills equations in Euclidean space-time,[12] and the realization that the existence of such finite-action field configurations indicates that the structure of the vacuum in QCD is much more complicated than one would have surmised from straightforward perturbation theory.[13-15] Thus the classical ground state of QCD is infinitely degenerate and the true quantum-mechanical vacuum is a coherent superposition of these classically degenerate vacuums. For sufficiently weak coupling the true vacuum can be constructed by semiclassical techniques, where the role of multiple-instanton field configurations is to give the dominant contribution in summing over path histories that travel from one classical vacuum to another.

In a scale-invariant theory, such as QCD, there is no way one can adjust the coupling to be small for all distances. Indeed, as we have remarked above QCD has no relevant adjustable parameters. Thus one cannot use semiclassical or weak-coupling methods to determine the structure of the vacuum, which involves all scale lengths. However, all physically relevant questions involve some external length parameter or momentum which sets the scale of the field configurations. Our approach is to explore physical quantities, such as the "potential" between massive quarks $[E(R)]$, characterized by a scale length R, as a function of R. For small enough R, asymptotic freedom will ensure that the effective coupling will be sufficiently small that one can use semiclassical techniques (saddle-point approximations) to evaluate functional integrals. The net effect in this small-distance region, as we shall explain below, is that the path integrals can be replaced by the partition function of a "gas" of instantons characterized by their position, scale size, and SU(3) orientation. The density of instantons of size ρ will be given by

$$D(\rho)\sim\frac{1}{\bar{g}^{12}(\rho)}\exp\left(-\frac{\text{const}}{\bar{g}^2(\rho)}\right),$$

where $\bar{g}^2(\rho)$ is the effective coupling at distance ρ. For small R only instantons of size $\rho\lesssim R$ will matter and since $\bar{g}^2(\rho)\sim\text{const}/\ln(1/\rho)$ as $\rho\to 0$, for small R the analog gas will be sufficiently dilute that one can trust a virial expansion of the partition function in powers of D. As one increases R, the effective size of the relevant instantons increases and the effective coupling increases.

Thus there are two sources of corrections to the free-field asymptotic behavior that occurs for $R\approx 0$. First there are the ordinary perturbation theory corrections to the integration about a given saddle point (including perturbation theory about the ordinary vacuum). These can in principle be calculated by standard perturbation theory (summing Feynman diagrams) and are of order $\bar{g}^2(R)$. In addition, there are the nonperturbative effects due to tunneling, proportional to the density of instantons, $D(R)$. Now one would think that the quantum-mechanical corrections to the ordinary vacuum sector would be much greater than the tunneling effects for small $\bar{g}^2$. This is the case in ordinary quantum mechanics, where tunneling can be neglected for small coupling ($g^2\gg e^{-1/g^2}$), and while for large coupling the tunneling effects may become substantial, they cannot be calculated by semiclassical techniques. In QCD we find, however, that this is not the case. Owing to the large phase space available to instantons, the density of instantons, $D(R)$, becomes large (≈ 1) for distances at which the coupling $\bar{g}^2/8\pi^2$ is still very small ($\approx\frac{1}{10}$). Essentially there exist so many distinct paths to tunnel between the degenerate vacuums, that even though the individual tunneling amplitudes are small, the net amplitude for tunneling can be of order one even for very small coupling. Thus as one proceeds from small distances one first arrives at a region where instanton effects are substantial, yet reliably calculable using the dilute-gas approximation and ordinary perturbative corrections are small.

As one goes to larger distances the density of instantons rises rapidly. One must then take into account the interaction between instantons and anti-instantons. In this highly nonlinear gas the interactions are quite complicated, and at present one can only make semi-quantitative estimates of their effects. Crudely speaking we find that the gas is analogous to a paramagnetic medium of dipolar objects. In this medium there is an effective renormalization of the coupling constant associated with large instantons due to the screening caused by smaller instantons. The result is to cause the coupling $\bar{g}(\rho)$ to grow dramatically with distance, leading to the density rising rapidly to large values at a sharply defined distance, $\rho_c \approx 0.2\mu$.

It is this distance that we associate with confinement. The actual confinement mechanism requires the consideration of field configurations other than instantons. In particular, instantons of scale comparable to ρ_c have a tendency to dissociate into pairs of half-instantons – or "merons." These merons have logarithmic attractive interactions, proportional to $[1/\bar{g}^2(R)]\ln R$, where R is the separation of a meron pair. However, for large enough R the entropy (log of the volume in function space) of a meron pair is proportional to $\ln R$ and can dominate – leading to a phase transition to a plasma of merons. These liberated merons, we shall argue, will confine quarks. To be sure we are still unable to treat the confinement phase very precisely or calculate hadronic masses. However, it seems inevitable that if our mechanism does lead to confinement the size of hadrons will be of order ρ_c and the hadronic mass scale will be of order $1/\rho_c$. Thus the expression of dimensional transmutation in QCD is that the renormalization-group-invariant equation which determines the scale size of hadrons is $D(\bar{g}(\rho_c)) \simeq 1$.

Now one of the consequences of dimensional transmutation is to fix the size of the hadronic coupling constant – i.e., the effective coupling at distances corresponding to hadronic size. This is of course the coupling that is of physical interest. The effective coupling for larger distances is of little interest since as one pulls quarks farther apart than the size of a hadron it will be energetically favorable to produce a quark-antiquark pair from the vacuum, and the stretched hadron will split into two smaller hadrons. Now the phase transition to a meron plasma already occurs for small $\bar{g}^2/8\pi^2$ and it is conceivable (although we certainly are unable to prove this) that confinement will occur on this distance scale and one will never need to go to larger distances where $g^2/8\pi^2$ becomes substantial. We therefore conjecture that *the hadronic coupling is always small.*

Thus we have discovered a small parameter in QCD. This has major consequences. First, the small size of $\bar{g}^2/8\pi^2$ ensures that we can use semiclassical techniques to calculate the properties of hadrons, without having to include quantum corrections beyond one or two orders. This reduces QCD to a semiclassical problem, albeit an extremely difficult one. Second, a small hadronic coupling has many phenomenological consequences, and may shed light on some of the unresolved mysteries of hadronic physics. For example, it is this coupling that determines, in our approach, the probability of producing a quark-antiquark pair from the vacuum. The small magnitude of this probability may explain much of the success of the naive quark and parton models – namely that the hadrons fall into SU(6) multiplets as if they were made of valence quarks alone, that the hadrons as seen in deep-inelastic scattering contain very few quark-antiquark pairs (for $x \neq 0$), the success of the free-field theory Melosh transformation, the rapid approach to scaling, etc. We emphasize that the small value of $\bar{g}^2(\rho_c\mu)/8\pi^2$ does not mean that the strong interactions are weak since nonperturbative instanton and meron effects are large.

Until now we have ignored the problem of generating the dynamical symmetry breaking of chiral symmetry and the U(1) problem. These problems cannot be separated from the problem of confinement. As was originally pointed out by 't Hooft[13] the existence of massless fermions has dramatic effects on instantons. Because of the Adler-Bell-Jackiw anomaly,[16] the conserved axial baryon number, Q_5, is not invariant under gauge transformations. This results in the suppression of tunneling between the classically degenerate vacuums, since they now are eigenstates of Q_5 with different eigenvalues. One must still construct the vacuum as a coherent superposition of the degenerate classical vacuums to satisfy cluster decomposition; however, the only path histories that can now contribute have net topological number zero – i.e., contain an equal number of instantons and anti-instantons.

This phenomenon has two important effects. First, the U(1) problem is eliminated. The axial baryon number charge is no longer a symmetry of the theory constructed about the true vacuum.[13-15] Second, if we try to replace path integrals by a partition function for a gas of instantons we will find strong long-range attractive interactions between instantons and anti-instantons (with a logarithmic dependence on their separation). Consequently in the presence of massless or very light fermions, instantons and antiinstantons will be closely bound in pairs. Such field configurations differ little from the ordinary vacuum and do not have much effect on the dynamics. In short, massless fermions confine instantons, and unless the fermions acquire a mass through dynamical chiral-symmetry breaking, tunneling effects are negligible.

Now we have suggested that the structure of the θ vacuum in QCD is such that there is a natural mechanism for dynamical symmetry breaking,[14] namely the effective determinantal interaction for massless fermions in a θ vacuum. This interaction can be regarded as the source of the U(1) symmetry breaking. Such an interaction, which is of the form $(\bar{\psi}\psi)^2$, if there are but two massless fermions, does not break chiral SU(N). However, it can provide a mechanism for the dynamical breaking of chiral symmetry. Indeed the only known mechanism for dynamical chiral-symmetry breaking utilizes such $(\bar{\psi}\psi)^2$ interactions.[17,7] In a two-dimensional model with two massless fermions we have shown that this type of interaction, generated by instantons, leads to the Goldstone realization of chiral symmetry and gives the fermion a mass.[18]

In the case of QCD the problem is of course much more difficult. To explore the possibility that the true ground state breaks chiral symmetry and to construct this state as well as the quark "mass" requires controlling the physics over arbitrarily large scale sizes. However, we have found it possible, in lieu of this, to demonstrate that the chirally symmetric vacuum is unstable under perturbations that would shift the vacuum expectation value of $\sigma=\bar{\psi}\psi$. To see the instability we calculate the propagator of σ for large momentum. For large enough momentum of course, asymptotic freedom determines this propagator perturbatively in powers of $\bar{g}^2(p)$. As the momentum decreases instanton effects (generated by the effective determinantal interaction) come into play. We find that these are very large for a range of momenta where the instanton gas is still dilute and the effective coupling is still small. In fact they are so large as to generate a tachyon pole in the σ propagator. This tachyon indicates the instability of the θ vacuum under shifts in σ. To be sure we are yet unable to construct the true, chirally asymmetric vacuum state. However, unless the theory is total nonsense, such a ground state will exist. Our calculation indicates that dynamical chiral-symmetry breaking does occur via the mechanism that we have suggested, and that it occurs at rather short distances (compared with what we regard as the confinement scale). This mechanism for chiral-symmetry breaking will not suffer from the U(1) problem. The determinantal interaction which is attractive in the π channel and thus will produce, in the true vacuum, a zero mass pion is repulsive in the η' channel.

The occurence of spontaneous symmetry breaking of chiral symmetry at distances short compared to the confinement scale is crucial to the success of our program. It ensures that as the quarks are pulled apart they acquire a mass at some distance where the dynamics is still manageable. Once this occurs instantons are liberated and, as we proceed to larger distances, begin interacting strongly to produce confinement. Thus, as a function of distance, there are two "phase transitions." At very short distances the vacuum can be described by a gas of tightly bound instanton-anti-instanton pairs, which at distance ρ_A (the asymptotically free chirally symmetric phase) undergo a phase transition to a dilute gas of instantons (the chirally asymmetric phase) due to chiral symmetry breaking. At a somewhat larger distance, ρ_c, the instantons themselves dissociate into meron pairs (the confining phase).

This paper is structured as follows. In Sec. II we present a general introduction to instanton physics. We show that four-dimensional gauge theories possess an infinity of classical vacuum states with a finite-energy barrier separating them, and discuss how instantons can be used to construct the amplitude for tunneling between these states in the semiclassical approximation. The analog gas model of instantons is constructed and the methods used to calculate in the dilute-gas approximation are presented. We give a qualitative picture in real space-time of the effect of tunneling on the interaction between quarks and on the structure of the vacuum wave function. The physical effects produced by massless fermions are discussed including the resolution of the U(1) problem. Finally we discuss briefly the problems associated with going beyond the dilute-gas approximation.

In Sec. III we begin to explore the dynamical effects of instantons in QCD. We first analyze the nature of instanton interactions, show that these give rise to a nonperturbative coupling-constant renormalization. We argue that a sharply defined infrared cutoff on instanton scale size is produced which we identify with the confinement radius or hadronic size.

Section IV is devoted to an evaluation of the instanton contribution to the quark-antiquark "potential," which for small separations can reliably be evaluated (in the absence of massless quarks). We find a large effect but one which is unlikely to produce confinement.

Section V deals with the effects of massless quarks and a discussion of dynamical chiral-symmetry breaking. We argue that the chirally symmetric vacuum is unstable, and identify the mode in which the instability occurs and the mechanism responsible for generating a dynamical quark mass.

Section VI is devoted to a discussion of confinement. Here we describe the phase transition from a gas of instantons to a plasmalike configuration of merons (half-instantons) and argue that this produces a linear potential between quarks.

Finally in Sec. VII we discuss the (many) unsolved problems in our approach and suggest various ways of proceeding to solve these problems.

II. INTRODUCTORY INSTANTON PHYSICS

Our approach to the dynamics of Yang-Mills theories amounts to nothing more than an elaboration of the standard perturbation theory treatment in which the vacuum is regarded as associated with a (typically unique) classical field corresponding to minimum potential energy, and quantum mechanics amounts to including the effects of *Gaussian* fluctuations about this classical vacuum configuration. In Yang-Mills theories there is a new kind of vacuum fluctuation which must be considered. Despite initial appearances, there is a countable infinity of classical vacuum states with only a finite barrier separating them. Consequently there is a finite amplitude for tunneling back and forth *between* such states. Such spontaneous vacuum fluctuations are large in amplitude and potentially much more important than the standard Gaussian zero-point fluctuations. Our first task will be to bring this multiple vacuum structure to light and to show that the tunnelings are conveniently described by certain Euclidean classical solutions of the Yang-Mills equations called instantons.[11] In later sections we will discuss in detail the consequences for physics of including this new class of quantum fluctuation.

A. Tunneling

We consider the pure Yang-Mills theory based on a Lie group G [in practice SU(2) or SU(3)] and described by the Lagrangian

$$\mathcal{L} = \tfrac{1}{4}\sum F^a_{\mu\nu}F^a_{\mu\nu}\,,$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{abc}A^b_\mu A^c_\nu. \tag{2.1}$$

For the purposes we have in mind it is simplest to eliminate the gauge freedom by setting $A^a_0 = 0$. Then the dynamical variables are the space components A^a_i, the canonical momenta are the electric field components $E^a_i = \dot{A}^a_i$, and the Hamiltonian density is

$$\mathcal{H} = \tfrac{1}{2}\left[(E^a_i)^2 + (B^a_i)^2\right],$$
$$B^a_i = \tfrac{1}{2}(\nabla_j A^a_k - \nabla_k A^a_j + f_{abc}A^b_j A^c_k)\epsilon_{ijk}. \tag{2.2}$$

The equations of motion derived from the Hamiltonian are the usual Yang-Mills equation *except* for the analog of Gauss's law

$$C^a(A) = \nabla_i \dot{A}^a_i + f_{abc}A^b_i \dot{A}^c_i = 0\,, \tag{2.3}$$

which is conjugate to A_0 in the usual treatment. On the other hand, $[\mathcal{H}, C] = 0$ so that if the initial configuration satisfies $C(A) = 0$, then so does the time evolution of that configuration. Thus, Gauss's law may be regarded as a *constraint* on the initial p's and q's selecting physical configurations out of a larger manifold.

The quantum-mechanical version of this theory has states $|A^a_i(\vec{x})\rangle$ (eigenstates of the field operator) between which matrix elements of the time evolution operator may be computed by the path-integral technique

$$\langle \tilde{A}^a_i(\vec{x})|e^{-iHT}|A^a_i(\vec{x})\rangle = \int_{A^a_i(\vec{x})}^{\tilde{A}^a_i(\vec{x})} [DA^a_i(\vec{x},t)] \exp\left\{\frac{i}{2g^2}\int_0^T dt\, d^3x [E^a_i(\vec{x},t)^2 - B^a_i(\vec{x},t)^2]\right\}. \tag{2.4}$$

The path integral is over all time histories connecting the configurations $A^a_i(\vec{x})$ to $\tilde{A}^a_i(\vec{x})$ in time T and g is the coupling constant. The quantum-mechanical version of Gauss's law constraint is that $C^a(A)$, regarded as an operator, should annihilate physical states: $C^a(A)|\psi\rangle = 0$. The field eigenstates, $|A^a_i(\vec{x})\rangle$, which are convenient for the path integral do not satisfy this condition and we must characterize the states which do.

At this point it is convenient to write the fields as elements of the Lie algebra of G: $A_i(\vec{x}) = A^a_i(\vec{x})T_a$, $[T_a, T_b] = if_{abc}T_c$. Now the theory is invariant under time-independent gauge transformations

$$A_i \to U_\lambda A_i U_\lambda^{-1} + iU_\lambda \nabla_i U_\lambda^{-1}, \tag{2.5}$$

where

$$U_\lambda = e^{i\lambda^a(\vec{x})T_a}$$

is an arbitrary function of $\vec{x}$ taking values in the group G [we have parametrized it by as many functions of position, $\lambda_a(\vec{x})$, as there are generators]. The above transformation on the fields is implemented by the unitary operator e^{iQ_λ} where

$$Q_\lambda = \int d^3x\, \dot{A}^a_i(\nabla_i\lambda^a + f_{abc}A^b_i\lambda^c)\,. \tag{2.6}$$

If the function $\lambda^a(\vec{x})$ vanishes at spatial infinity we may integrate by parts to find that

$$Q_\lambda = -\int d^3x\, \lambda^a(\vec{x})(\nabla_i\dot{A}^a_i + f_{abc}A^b_i\dot{A}^c_i)\,. \tag{2.7}$$

The term in parentheses is just Gauss's law and annihilates physical states. Thus physical states are also characterized by

$$e^{iQ_\lambda}|\psi\rangle = |\psi\rangle \qquad (2.8)$$

if

$$\lambda^a(\vec{x}) \xrightarrow[|\vec{x}|\to\infty]{} 0\,,$$

which is to say that they are eigenstates with eigenvalue 1 of the subclass of all gauge transformations characterized by $\lambda^a \to 0$ at spatial infinity. Since a field eigenstate transforms under e^{iQ_λ} as

$$\begin{aligned}
&e^{iQ_\lambda}|A_i(\vec{x})\rangle = |A_i(\vec{x})_\lambda\rangle\,,\\
&A_i(\vec{x})_\lambda = U_\lambda A_i U_\lambda^{-1} + iU_\lambda\nabla_i U_\lambda^{-1}\,, \qquad (2.9)\\
&U_\lambda = e^{i\lambda^a(\vec{x})T_a}\,,
\end{aligned}$$

it is apparent that the only way to construct a physical state is to sum over gauge transforms:

$$|A_i(\vec{x})\rangle_{\text{physical}} = \int [D\lambda^a(\vec{x})]e^{iQ_\lambda}|A_i(\vec{x})\rangle\,,$$

where the functional integral is over all λ's which vanish at $|\vec{x}|\to\infty$ and the integration measure is locally gauge invariant.

The notation here is a bit clumsy since we have not identified the variables which actually specify the physical state. In standard perturbation theory treatments it would be convenient to regard the above state as being parametrized by A_i^{tr}, that A_i, among all the A_i's summed over to form the physical state, which satisfies $\vec{\nabla}\cdot\vec{A}=0$. A_i^{tr} has precisely the correct degrees of freedom to describe the massless gluons which are normally thought to express the physical content of the theory. We could adopt the same strategy here, defining a function $U(\vec{x},t)$ by

$$\begin{aligned}
&A_i(\vec{x},t) = UA_i^{\text{tr}}U^{-1} + iU\nabla_i U^{-1}\,,\\
&\nabla_i A_i^{\text{tr}} = 0\,,
\end{aligned}$$

and writing the $A_0=0$ functional integral in terms of A_i^{tr} and U instead of A_i (and regarding the states as functions of A_i^{tr}). Since U turns out to be essentially a cyclic variable (this follows directly from our ability to impose Gauss's law as a constraint, i.e., an equation which when imposed at one time remains true for all time) we could think of eliminating it from the system and writing the path integral entirely in terms of the physical variables, A_i^{tr}. This would amount to casting the theory in the Coulomb gauge. The fact that only restricted classes of gauge transformations U (those which go to a constant at spatial infinity) may be integrated over makes this a rather tricky enterprise and we will not attempt it here.

Although it is not necessary to so limit our attention, we will for convenience now consider physical *vacuum* states $|\omega\rangle$. A vacuum gauge field configuration, corresponding to zero energy density, must have $B_i = 0$ which is possible only if $A_i = ig(\vec{x})\nabla_i g^{-1}(\vec{x})$ [$g(\vec{x})$ takes values in G]. For simplicity of notation let the corresponding state be written $|g(\vec{x})\rangle$. Under a gauge transformation e^{iQ_λ}, $|g(\vec{x})\rangle \to |U_\lambda g(\vec{x})\rangle = |\tilde{g}(\vec{x})\rangle$ and $|g(\vec{x})\rangle_{\text{phys}}$ will be a sum over all $|g(\vec{x})\rangle$ obtainable this way. The equivalence relation $g(\vec{x}) \approx \tilde{g}(\vec{x}) = U_\lambda g(x)$ [$\lambda^a(\vec{x})\to 0$ as $|\vec{x}|\to\infty$] divides the set of all possible elements of G into equivalence classes and possible physical vacuum states are just

$$|\text{vac}\rangle = \sum_{g\in(\text{equivalence class})} |g(\vec{x})\rangle\,. \qquad (2.10)$$

This large class of possible vacuums can be cut down to manageable size, while bringing topological notions into the game, by the following dynamical remarks. Consider a transition amplitude $\langle\overline{\text{vac}}|e^{-iHT}|\text{vac}\rangle$ between two of these possible vacuums. Because the initial and final states may be *simultaneously* gauge transformed without affecting the value of the amplitude, we may without harm choose the initial vacuum to be in the equivalence class of $g(\vec{x})=1$. Clearly, all the states in this equivalence class are characterized by $g(\vec{x})\to g_0$ [or $A_i(\vec{x})\to 0$] as $|\vec{x}|\to\infty$. We would like to argue that there will be a finite transition amplitude from this equivalence class only to other equivalence classes also characterized by $g(\vec{x})\to g_0$ [or $A_i(\vec{x})\to 0$] as $|\vec{x}|\to\infty$. For any other type of final equivalence class $A_i(\vec{x})\not\to 0$ as $|\vec{x}|\to\infty$ and the histories entering the path integral must have $\dot{A}\neq 0$ over an infinite spatial volume. Such inifinite energy transitions must in fact have zero amplitude and we may limit our attention, as far as dynamics is concerned, to vacuum equivalence classes characterized by $g(\vec{x})\to 1$ as $|\vec{x}|\to\infty$.

If $g(\vec{x})\to 1$ as $|\vec{x}|\to\infty$, then the domain of $g(\vec{x})$ may be thought of as three-space with points at infinity identified. This manifold is topologically equivalent to S_3. The vacuum equivalence classes are now seen to be classes of maps from S_3 to G which may be continuously deformed into another: homotopy classes. It is known that the homotopy classes of $G=\text{SU}(n)$ are countable and characterized by a positive and negative integer-valued topological invariant (winding number or Pontryagin class)

$$n = \frac{1}{24\pi^2}\int d^3x\,\epsilon_{ijk}\,\text{tr}[g^{-1}\nabla_i g(\vec{x})g^{-1}\nabla_j g(\vec{x})g^{-1}\nabla_k g(\vec{x})]. \qquad (2.11)$$

A typical element [for $G=\text{SU}(2)$] of the $n=0$ class is $g(\vec{x}) = I$ while a typical element for $n = 1$ is

$$g(\vec{x}) = \exp\left[i\pi \frac{\vec{x}\cdot\vec{\tau}}{(\vec{x}^2+a^2)^{1/2}}\right] = g_1 .$$

For reference, a $g(\vec{x})$ which belongs to *no* homotopy class (and is representative of the vacuums we threw out with our dynamical argument) is

$$g(\vec{x}) = \exp\left[i\alpha \frac{\vec{x}\cdot\vec{\tau}}{(x^2+a^2)^{1/2}}\right]$$

with α not an integer multiple of π. For convenience we will now label the surviving vacuum states by the appropriate winding numbers: $|n\rangle$.

It can be shown that multiplying an element of the n_1 homotopy class by an element of the n_2 class gives an element of the n_1+n_2 class. Therefore if we call R the unitary operator which implements a typical gauge transformation of the $n=1$ class ($g_1 = \exp[i\pi\vec{x}\cdot\vec{\tau}/(x^2+a^2)^{1/2}]$, say) we must have $R|n\rangle = |n+1\rangle$ ($R^{-1}|n\rangle = |n-1\rangle$). Now $R = U_{\lambda_R}$ with $\vec{\lambda}_R = 2\pi\vec{x}/(\vec{x}^2+a^2)^{1/2}$ if we choose R to correspond to g_1. Although R is a gauge transformation, since $\lambda_R \not\to 0$ as $|\vec{x}|\to\infty$ Gauss's law constraint does not specify how physical states behave under R. The physical requirement of gauge invariance is met so long as the physical states are eigenstates of R: $R|\psi\rangle = e^{i\theta}|\psi\rangle$. No physical principle determines what θ must be, although since $[H,R]=0$, neither time evolution nor local gauge-invariant perturbations in general will change θ. In short, θ labels superselection sectors of the theory and the Hamiltonian must be block diagonal in θ. One easily constructs θ eigenstates from n eigenstates by the rule

$$|\theta\rangle = \sum_{-\infty}^{\infty} e^{in\theta}|n\rangle , \tag{2.12}$$

and it is apparent that the θ-vacuum states are nondegenerate. What is not altogether obvious at this stage, although true, is that the theories based on the different $|\theta\rangle$ states are physically different from each other, so that this multiplicity of θ worlds is a nontrivial property of the theory and not some gauge artifact. For $\theta\neq 0$, $|\theta\rangle$ is not an eigenstate of parity or time reversal. Therefore for QCD θ must be very small and in all likelihood is equal to zero. The physical principle (if any) which determines θ is unknown.

We have argued that there will be a finite quantum-mechanical transition amplitude between states $|n\rangle$ and $|n'\rangle$ because the time variation of A_i needed to effect the transition can be localized in space and does not require infinite kinetic energy at any point. On the other hand, if we look at the vacuum $A_i = 0$, say, it is apparent that the classical equations of motion will leave the system in the state $A_i = 0$ forever. Thus transitions from $|n\rangle$ to $|n'\rangle \neq |n\rangle$ look like tunneling processes: classically forbidden but quantum mechanically allowed barrier penetration processes.

To bring the tunneling interpretation into clearer view, it is helpful to pass to the imaginary time picture, i.e., to discuss $\langle n'|e^{-HT}|n\rangle$ instead of $\langle n'|e^{-iHT}|n\rangle$. The reason for this is that we know from experience with ordinary quantum mechanics that imaginary time solutions of the classical equations of motion can be used to obtain a WKB (or small $\hbar$) treatment of barrier penetration problems. In real time a classically forbidden process defines no stationary path which dominates the functional integral and there is no simple way to study tunneling.

The new path integral is

$$\langle n'|\exp(-HT)|n\rangle = \int_n^{n'} [DA_i]\exp\left\{-\int_0^T dt\, d^3x[(\dot{A}_i^a)^2+(B_i^a)^2]\frac{1}{2g^2}\right\}, \tag{2.13}$$

with the implied boundary condition of first computing the path integral between a representative pair $(A_i^a, A_i^{a\prime})$ of functions belonging to the homotopy classes (n,n') and then integrating over all representatives in the (n,n') classes. Since the action is positive-definite, the dominant history is one of *minimum* action consistent with the boundary conditions. Such a path satisfies all the Euclidean Yang-Mills equations—except Gauss's law constraint. Upon varying the end points of the A_i path history, one will finally pick out the path which satisfies the constraint as well. This path is guaranteed to have the *absolute* minimum action consistent with the constraint that it describes a transition $n\to n'$ and satisfies the full set of Euclidean Yang-Mills equations.

A rather large class of Euclidean Yang-Mills solutions are known by now,[19] but we need only discuss the original one of Belavin *et al.*[12] out of which, in a sense, all the others are constructed. In Landau the gauge ($\partial\cdot A=0$) the particular solution is $[G=SU(2)]$

$$A_\mu^a = 2\frac{\eta_{a\mu\nu}x_\nu}{x^2+\rho^2},\quad F_{\mu\nu}^a = \frac{4\eta_{a\mu\nu}\rho^2}{(x^2+\rho^2)^2} , \tag{2.14}$$

where $\eta_{a\mu\nu}$ is a numerical tensor coupling two SU(2)'s to O(4) ($\eta_{a\mu\nu} = \epsilon_{0a\mu\nu} + \frac{1}{2}\epsilon_{abc}\epsilon_{bc\mu\nu}$) and ρ is an arbitrary scale parameter (arising from the scale invariance of the classical theory). The Euclidean action of this solution is $8\pi^2/g^2$ (independent of ρ) so that the magnitude of the path integral dominated by this path is $\exp(-8\pi^2/g^2)$. Remembering that $g^2 \sim \hbar$ one sees that this amplitude vanishes like $\exp(-1/\hbar)$ as $\hbar \to 0$, the sort of essential singularity characteristic of vacuum tunneling in ordinary quantum mechanics. Finally, if we pass to the gauge $A_0 = 0$ in order to make contact with the earlier discussion in this section, we find

$$A_i(x_0 = -\infty) = 0,$$
$$A_i(x_0 = +\infty) = U^{-1}\nabla_i U, \quad U = \exp\left[i\pi \frac{\vec{x}\cdot\vec{\tau}}{(x^2+\rho^2)^{1/2}}\right] \tag{2.15}$$

indicating that this solution describes the vacuum tunneling $|0\rangle \to |1\rangle$. A more convenient and gauge-invariant way of identifying the topological class of a trajectory is

$$\Delta n = \frac{1}{8\pi^2}\int d^4x\, \mathrm{tr}(F_{\mu\nu}\tilde{F}_{\mu\nu}), \tag{2.16}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ is the four-dimensional field strength tensor and $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}F_{\lambda\sigma}$. In general, $\mathrm{tr}(F_{\mu\nu}\tilde{F}_{\mu\nu})$ may be written as a total divergence, and when evaluated in the $A_0 = 0$ gauge the $t = \pm\infty$ surface contributions are seen to be identical to the vacuum winding numbers defined earlier. This solution, called the instanton, exists in a conjugate version, called anti-instanton, describing tunneling from 0 to -1. One simply replaces $\eta_{a\mu\nu}$ by $\bar{\eta}_{a\mu\nu}$ ($\bar{\eta}_{a\mu\nu} = \epsilon_{0a\mu\nu} - \frac{1}{2}\epsilon_{abc}\epsilon_{bc\mu\nu}$). The same solutions describe the basic tunneling event in any theory based on SU(n). In the explicit formulas for A_i, SU(2) generators are simply replaced by an SU(2) subset of the SU(n) generators.[20]

There is a slight conceptual complication that this field-theoretic vacuum tunneling process has over and above its analog in ordinary quantum mechanics, namely, the multiplicity of Euclidean tunneling saddle points. The classical solution may be centered anywhere in space-time and may be of any scale size (small scale size ρ means the tunneling event has large field strengths but happens quickly; vice versa for large scale size).

For weak coupling the detailed tunneling amplitude can be computed by the following device. The matrix element

$$\langle n+1|\exp(-HT)|n\rangle = \langle 1|\exp(-HT)|0\rangle$$

is expressed as a functional integral as in Eq. (2.13). For weak coupling, $g^2 \approx 0$, one then performs the Euclidean functional integral by a saddle-point approximation. This requires a solution to the Euclidean equations of motion satisfying the appropriate boundary conditions—which is just the Belavin-Polyakov-Schwartz-Tyupkin (BPST) one-instanton solution, $A_\mu^a(x - x_I, \rho)$, located at an arbitrary point x_I, with arbitrary scale size ρ. Expanding the quantum field as $A_\mu^a(x - x_I, \rho) + gQ_\mu^a(x)$, one has

$$\langle 1|\exp(-HT)|0\rangle = \exp\left(-\frac{8\pi^2}{g_0^2}\right)\int [DQ_\mu^a] \times \exp\left(-\tfrac{1}{2}\int d^4x\, \mathcal{L}''[A_\mu^a]Q^2\right), \tag{2.17}$$

where $\mathcal{L}''$ is the second functional derivative of the Lagrangian evaluated at the instanton solution, and higher-order terms, proportional to g_0^2 (g_0 is the bare coupling), are neglected.

The normalization of the tunneling amplitude arises from the Gaussian integral around the saddle point. Evaluation of this integral requires calculating the determinant of the operator $\mathcal{L}''(A_\mu^a)$. By exploiting the conformal invariance of the classical theory, 't Hooft has computed this determinant explicitly.[21] The following points are worth mentioning:

(1) For every symmetry of the original Lagrangian there will exist a zero eigenvalue of $\mathcal{L}''(A_\mu^a)$. These zero energy modes can be dealt with by introducing collective coordinates for the degrees of freedom corresponding to the appropriate symmetries, and yield factors of the volume of the corresponding symmetry groups. In the case of QCD the SU(2) instanton possesses 4 translational, 1 scale, and 3 group degrees of freedom. [In the case of an SU(N) instanton, constructed using an SU(2) subgroup, there are $4N - 5$ group degrees of freedom.] For each degree of freedom a factor of $1/g$ will result from the introduction of collective coordinates. Thus the matrix element will take the form [for an SU(N) gauge group]

$$\langle 1|\exp(-HT)|0\rangle = V_N\left(\frac{8\pi^2}{g_0^2}\right)^{2N}\int d^4x\int\frac{d\rho}{\rho^5}\exp\left(-\frac{8\pi^2}{g_0^2}\right)\int [DQ']\exp\left[-\tfrac{1}{2}\int d^4x\, \mathcal{L}''(A^a)Q'^2\right], \tag{2.18}$$

where V_N is a numerical constant and Q' refers to the quantum field with the zero-energy modes removed.

(2) Owing to the standard ultraviolet divergences of ordinary perturbation theory, the remaining determinant (which is simply the exponential of the sum of connected vacuum-to-vacuum diagrams in the background instanton field) will require renormalization. This renormalization is standard since the ultraviolet divergences do not depend on the smooth background field. The net effect is to replace the bare action, $8\pi^2/g_0^2$, with the renormalized value, $8\pi^2/\bar{g}^2(1/\rho\mu)$, where μ is the renormalization mass and $\bar{g}^2$ is the ef-

fective coupling constant of the renormalization group, satisfying

$$\frac{d\bar{g}}{d\ln\rho} = -\beta(\bar{g}) = +b_0\bar{g}^3 + O(\bar{g}^5). \tag{2.19}$$

We thus have

$$\langle 1|\exp(-HT)|0\rangle = VT\int \frac{d\rho}{\rho^5}\left(\frac{8\pi^2}{\bar{g}^2(1/\mu\rho)}\right)^{2N}\exp\left(-\frac{8\pi^2}{\bar{g}^2(1/\rho\mu)}\right)C[1+O(\bar{g}^2(1/\rho\mu))], \tag{2.20}$$

where VT is the volume of space-time, and C is a numerical constant. For SU(2), $C_{SU(2)} = 0.26$ (Ref. 21) whereas for SU(3) we find that $C_{SU(3)} = 0.10$.

The integration over scale sizes is rendered convergent for $\rho \to 0$ by asymptotic freedom, since [for a pure SU(N) gauge theory]

$$8\pi^2/\bar{g}^2(1/\rho\mu) \underset{\rho\to 0}{\sim} (\tfrac{11}{3}N)\ln(1/\rho\mu). \tag{2.21}$$

However, the integration also extends to arbitrarily large scale sizes, where $\bar{g}^2(1/\rho\mu)$ increases. Thus in a scale-invariant theory such as QCD one cannot adjust the coupling to be small, and even constructing the vacuum requires an understanding of the infrared, perhaps strong-coupling, behavior of the theory.

For the moment we shall ignore this problem, and consider the contribution to the tunneling amplitude of instantons of sizes ρ to $\rho + d\rho$, replacing Eq. (2.20) with

$$\langle 1|\exp(-HT)|0\rangle = DVT\exp\left(-\frac{8\pi^2}{\bar{g}^2(1/\rho\mu)}\right), \qquad D = C_N\frac{d\rho}{\rho^5}\left(\frac{8\pi^2}{\bar{g}^2(\rho)}\right)^{2N}. \tag{2.22}$$

This is what one would obtain in a superrenormalizable field theory and will be instructive as to the physical picture of tunneling. In the following section, we shall discuss in great detail the physical consequences that arise from the existence of instantons of all sizes in QCD.

The scale size ρ can be considered as the time which it takes to tunnel and the coefficient of T, $VD\exp(-8\pi^2/\bar{g}^2)$ can be interpreted as the inverse of the mean time between tunnelings. Then for $\rho \ll T \ll (VD)^{-1}\exp(8\pi^2/\bar{g}^2)$ we can expand the matrix element in Eq. (2.22) according to

$$\begin{aligned}\langle 1|\exp(-HT)|0\rangle &\approx \langle 1|1-HT+\cdots|0\rangle\\ &\cong -\langle 1|H|0\rangle T\\ &\cong TVD\exp\left(-\frac{8\pi^2}{\bar{g}^2}\right).\end{aligned} \tag{2.23}$$

One can then read off the tunneling Hamiltonian which for general $n \to n+1$ is

$$\langle n+1|H|n\rangle = -VD\exp\left(-\frac{8\pi^2}{\bar{g}^2}\right). \tag{2.24}$$

Taking this as the tunneling Hamiltonian it is straightforward to compute the energy of a θ vacuum. If $E(\theta)\delta(\theta-\theta') = \langle\theta'|H|\theta\rangle$ one finds

$$E(\theta) = E_0 - 2\cos\theta\, VD\exp\left(-\frac{8\pi^2}{\bar{g}^2}\right), \tag{2.25}$$

where E_0 is the energy in the absence of tunneling. Note that the energy is decreased and proportional to V as it should be.

It is important to note that there are two steps involved in constructing the θ vacuum. First, the tunneling amplitude $\langle n|H|n+1\rangle$ is determined from a single instanton and then the tunneling Hamiltonian is diagonalized. It is the second step that brings in multi-instanton effects. This is shown schematically in Fig. 1 where the single instanton and anti-instanton (anti-tunneling $n \to n-1$) of Fig. 1(a) are iterated in Fig. 1(b) to produce a θ vacuum. Observe that the θ vacuum looks like a gas of instantons in 4 dimensions. Below we will use this analogy to develop a more powerful method for handling instantons.

B. The analog gas

The above method of constructing a tunneling Hamiltonian gives correct answers for weak coupling but suffers from conceptual difficulties for larger $\bar{g}$ where the mean time between tunnelings is not so small. In particular, it is totally inadequate for QCD.

A systematic approach is based on the long-time, $T \gg (VD)^{-1}\exp(8\pi^2/g^2)$, Euclidean functional integral $\exp(-HT)$,

$$\langle\theta'|\exp(-HT)|\theta\rangle = \sum_{n,n'}\exp[i(n\theta - n'\theta')] \times\langle n'|\exp(-HT)|n\rangle, \tag{2.26}$$

with $\langle n'|\exp(-HT)|n\rangle$ given by Eq. (2.13). A deductive approach to vacuum tunneling would begin with this functional integral. As $T \to \infty$ this be-

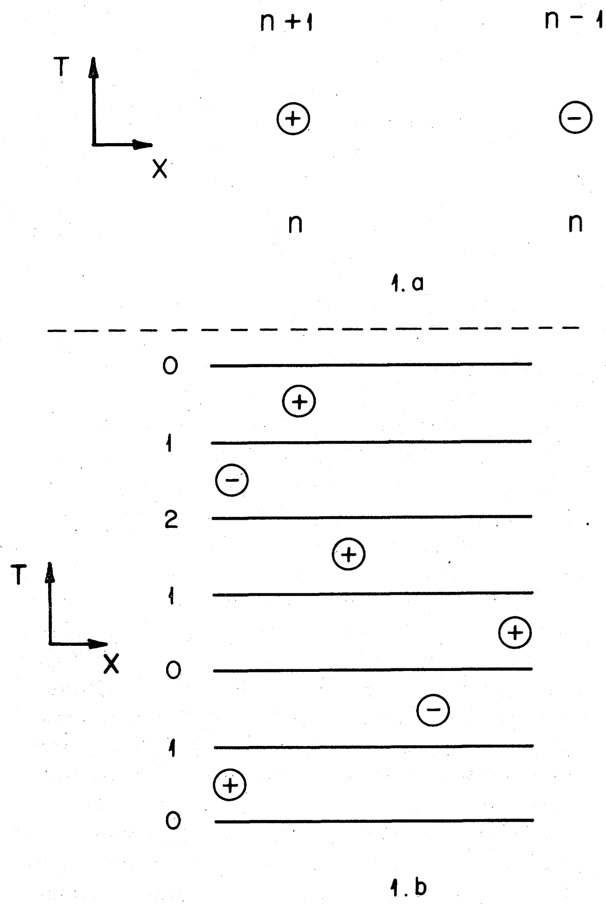

FIG. 1. Construction of a θ vacuum. The single instanton and anti-instanton shown in (a) causes transitions from n to $n+1$ or $n-1$. The θ vacuum is built upon successive transitions back and forth between different n states as in (b). One should imagine integrating over the locations in imaginary time and space of the instantons and anti-instantons.

comes const $\times\,\delta(\theta-\theta')\exp[-E(\theta)T]$. Let us see how this comes about in the weak-coupling limit where the mean time between tunneling is large (but small compared to T). One can then classify the configurations, A^μ, that contribute to $\langle n|\exp(-HT)|0\rangle$ according to the number of well-separated tunnelings (instantons) n_+, and antitunnelings (anti-instantons) n_-, such that $n=n_+-n_-$. We choose the normalization so that the naive vacuum term, $n_+=n_-=0$, is equal to 1. The term with $n_+=1$, $n_-=0$ can be evaluated by the saddle-point method described above yielding

$$VTD\exp\left(-\frac{8\pi^2}{\bar g^2}\right)\exp(i\theta)\,.$$

The term with $n_+=0$, $n_-=1$ is the same with θ replaced by $-\theta$. The first nontrivial term has $n_+=2$, $n_-=0$ and could also be computed by a standard saddle-point technique, since there exist exact two-instanton solutions[19] depending on the right number of parameters to describe two independent tunnelings. The action for these solutions is simply twice $8\pi^2/\bar g^2$, but the determinant D_2 has not yet been calculated for two instantons. However, when the instantons are far apart, as will be the case for small $\bar g^2$, D_2 must reduce to the square of the determinant for $n_+=1$, $n_-=0$, and the dominant contribution to the $n_+=2$, $n_-=0$ sector is

$$\frac{1}{2!}(VTD)^2\exp\left(-\frac{16\pi^2}{\bar g^2}\right)e^{2i\theta}\,,$$

where the $1/2!$ is necessary to avoid double counting. Since the determinant D_2 presumably does depend on the instanton separation when they are close this form is only approximate. There are corrections to it which are proportional to one power of the volume VT, which for small coupling will yield corrections to $E(\theta)$ which are suppressed by $\exp(-8\pi^2/g^2)$. The $n_+=0$, $n_-=2$ contribution is then obtained in the obvious way $(\theta\to-\theta)$.

The $n_+=n_-=1$ term introduces some new physics. This time we cannot evaluate the functional integral by a strict saddle-point method. The minimum action configuration for $n_+-n_-=0$ is simply the naive vacuum $A^a_\mu=0$, corresponding to a tunneling on top of an antitunneling, or no tunneling at all. However, we can certainly imagine configurations corresponding to a well-separated instanton–anti-instanton pair.

Such a configuration is not an exact solution to the Euclidean equations of motion but will nevertheless make a nontrivial contribution equal to $(VTD)^2\exp(-16\pi^2/g^2)$, when

$$VTD\exp\left(-\frac{8\pi^2}{g^2}\right)\gg 1.$$

Generalizing to arbitrary n_+ and n_- then yields

$$\langle\theta|\exp(-HT)|\theta\rangle\approx\sum_{n_+,n_-=0}^{\infty}\frac{1}{n_+!}\,\frac{1}{n_-!}(TVD)^{n_++n_-}\exp\left[-\frac{8\pi^2}{g^2}(n_++n_-)+i\theta(n_+-n_-)\right]$$
$$=\exp\left[2TVD\cos\theta\exp\left(-\frac{8\pi^2}{g^2}\right)\right],\qquad(2.27)$$

and we recover the previous result

$$E(\theta) = (\text{const}) - 2VD\cos\theta\exp(-8\pi^2/g^2).$$

Although it is valid only for weak coupling, Eq. (2.27) contains some important lessons. Specializing for simplicity to $\theta = 0$, we have

$$\langle 0|\exp(-HT)|0\rangle = \sum_{n_+, n_- = 0}^{\infty} \frac{1}{n_+!}\frac{1}{n_-!}(TVD)^{n_+ + n_-}\exp\left[-\frac{8\pi^2}{g^2}(n_+ + n_-)\right]. \tag{2.28}$$

For large T the dominant term in this series is the one for which

$$n_+ = n_- = TVD\exp\left(-\frac{8\pi^2}{g^2}\right), \tag{2.29}$$

and as $T\to\infty$ essentially the entire sum comes from this term alone. Observe that

(i) the dominant term contains both instantons and anti-instantons and cannot be computed by a strict saddle-point method that relies on exact solutions to the Euclidean equations of motion,

(ii) the dominant term is not the one for which the classical action $\exp[-(8\pi^2/g^2)(n_+ + n_-)]$ is minimal.

One conclusion is that although the remarkable exact multi-instanton solutions[19] may indicate a new structure or symmetry to the theory, they are of essentially no relevance when it comes to constructing the vacuum state. In fact the sum over all terms with either n_+ or n_- equal to zero yields

$$2\exp\left[TVD\exp\left(-\frac{8\pi^2}{g^2}\right)\right],$$

which for large T is exponentially small compared to the complete sum,

$$\exp\left\{2\left[TVD\exp\left(-\frac{8\pi^2}{g^2}\right)\right]\right\}.$$

The nature of the long time functional integral is most easily understood in terms of an analog gas. The sum in Eq. (2.28) is precisely the grand partition function for a classical, four-dimensional perfect gas containing two species of particles with equal fugacities $\exp[-8\pi^2/g^2]$ and volume measured in units of D^{-1}. The energy (action) for a configuration with n_+ and n_- members of each species is $(n_+ + n_-)8\pi^2/g^2$ while entropy of the configuration is $\ln[(TVD)^{n_+ + n_-}/n_+!n_-!]$. The dominant term is the one for which the free energy (energy minus entropy) is smallest.

More generally, the entropy of a field configuration can be thought of as the log of the volume in function space occupied by similar configurations. For large coupling the action of a given field configuration decreases like g^{-2} while its entropy is generally less sensitive to g. Thus for moderate or strong coupling the entropy of a field configuration can be a more important consideration than its action. This will become increasingly evident in later sections. It also has its effects at small g: The exact multi-instanton solutions are in a sense uninteresting because they have so little entropy.

When g is small the analog gas is *extremely* dilute and the physics is the same as the tunneling picture discussed above. For larger g when instantons and anti-instantons are closer together the language of tunneling is at best picturesque. However, the idea of an interacting gas where instantons and anti-instantons interfere and the action for an n_+, n_- configuration is not just $(n_+ + n_-)8\pi^2/g^2$ still makes sense.

C. Wave functions and the functional integral

Consider a gauge where the gauge condition ($A^0 = 0$, $A^3 = 0$, or $\vec{\nabla}\cdot\vec{A} = 0$) does not involve time explicitly. One can then consider the wave function $\Phi_\theta[A(\vec{x})]$ of a θ vacuum as a functional of the independent components of A at one fixed time. The modulus of Φ is easily related to the Euclidean functional integral. If the fields at $T/2$ are fixed to be $A(x)$ then

$$\lim_{T\to\infty}\int[dA^\mu]\exp\left[\int_0^T dt\int d^3x\left(-\tfrac{1}{4}\text{tr}F^2 + \frac{i\theta}{8\pi^2}\text{tr}F\tilde{F}\right)\right]\Bigg|_{A(T/2,\vec{x}) = A(\vec{x})} = \text{const}\times\exp[-E(\theta)T]\,|\Phi_\theta[A(\vec{x})]|^2. \tag{2.30}$$

Thus by examining a time slice of the Euclidean functional integral one can determine which field configurations are important in the vacuum. For a dilute instanton gas in the gauge $A^0=0$ most time slices will see only a gauge-rotated-vacuum $\vec{A} = i\Omega^{-1}\vec{\nabla}\Omega$ consistent with the wave function $|\theta\rangle = \sum \exp(in\theta)|n\rangle$ obtained from the tunneling picture. In the dense gas that occurs in QCD most time slices will intersect one or more instantons adding a new component to Φ_θ. For strong enough coupling further objects such as merons (Sec. VI) will also be present.

D. New quark interactions

Until now we have discussed tunneling in a pure Yang-Mills theory. Other fields are easily incorporated without significant effect on the tunneling picture (except for massless fermions as we shall see below). For example, in the presence of massive fermions the action will contain a term $\bar{\psi}(i\not{D}-m)\psi$. In constructing the θ vacuum, however, this will only slightly affect the normalization of the tunneling amplitudes. One will still, in the saddle-point approximation, expand about the multiple-instanton gauge fields; however, the one-loop quantum fluctuation will now include those of the fermion fields in the background instanton field. This will modify the functional integral by a factor of $\{\det[i\not{D}(A_\mu^a) - m]\}^{n_+ + n_-}$ for well-separated instantons.

It is possible to calculate, using the dilute-gas approximation, the Green's functions of any number of gauge-invariant operators. One simply adds to the Lagrangian density a term $\sum_i J_i(x)\theta_i(x)$, where $J_i(x)$ is an external c-number source coupled to the operator in question, $\theta_i(x)$, constructed out of the gauge and matter fields. Euclidean Green's functions of θ_i are then defined as

$$\langle\theta|T(\theta_1(x_1)\cdots\theta_N(x_N))|\rangle = \left(\frac{\delta}{\delta J_1(x_1)}\right)\cdots\left(\frac{\delta}{\delta J_N(x_N)}\right) \times \frac{\int[DA_\mu]\exp[-S(A)+\int\sum J_i(x)\theta_i(x)d^4x]}{\int[DA_\mu]\exp[-S(A)]}\Bigg|_{J_i=0}. \qquad (2.31)$$

In performing the saddle-point integrations for the various multiple-instanton sectors in Eq. (2.31), one treats the term $\sum J_i\theta_i$ as a small perturbation and expands the gauge fields in θ_i about the multiple-instanton configurations.

Of particular concern to us is the effect of tunneling on the quark-antiquark interaction. Let us first give a physical picture of how tunneling modifies the interaction of quarks.

A quark-antiquark state constructed in perturbation theory would be built on one of the n states $|n\rangle$ rather than a θ vacuum. Let $|n\vec{r}\rangle$ be such a state where the $q\bar{q}$ pair is separated by $\vec{r}$. There will be tunneling to other states $|m\vec{r}\rangle$ and the true $q\bar{q}$ state built on a θ vacuum is $|\theta\vec{r}\rangle = \sum_m \exp(im\theta)|m\vec{r}\rangle$. Because the tunneling amplitude $\langle m\vec{r}|H|n\vec{r}\rangle$ will differ from the vacuum amplitude $\langle m|H|n\rangle$ and depend on $\vec{r}$, nonperturbative $q-\bar{q}$ interactions will appear. In a weak-coupling approximation the energy $E(\theta,\vec{r})$ of $|\theta\vec{r}\rangle$ relative to the energy of the θ vacuum is

$$E(\vec{r},\theta) = E_0(\vec{r}) + 2\,\mathrm{Re}[\exp(i\theta)(\langle 1\vec{r}|H|0\vec{r}\rangle - \langle 1|H|0\rangle)], \qquad (2.32)$$

where $E_0(\vec{r})$ includes the perturbative terms, e.g. Coulomb interaction, which are diagonal and the same for each n. For heavy "test quarks" the state $|n\vec{r}\rangle$ can be constructed in a gauge-invariant way to be

$$|n\vec{r}\rangle = \bar{q}(\vec{r})\,U(\vec{r},\vec{0})\,q(\vec{0})|n\rangle, \qquad (2.33)$$

where

$$U(\vec{r},\vec{0}) = T\exp\left(i\int_{\vec{0}}^{\vec{r}}\vec{A}\cdot d\vec{x}\right)$$

is the ordered exponential integrated along a straight line from $\vec{0}$ to $\vec{r}$. The tunneling just shifts $U(\vec{r},\vec{0})$ by a gauge and for a given instanton location (and scale size and orientation) $\langle 1\vec{r}|H|0\vec{r}\rangle$ differs from $\langle 1|H|0\rangle$ only by a factor $\mathrm{tr}[U_1^{-1}(\vec{r},\vec{0})U_0(\vec{r},\vec{0})]$ where the subscripts refer to the values of U in the initial and final n states. Because we are in the gauge $A^0=0$, we can write

$$\mathrm{tr}[U_1^{-1}(\vec{r},\vec{0})U_0(\vec{r},\vec{0})] = T\exp\left(i\oint A^\mu_{\mathrm{inst}}\,dx^\mu\right), \qquad (2.34)$$

where the ordered line integral of the instanton field runs around a rectangular Euclidean loop with corners at $(\vec{0},0)$, $(\vec{r},0)$, $(\vec{r},T)$, $(\vec{0},T)$. This ordered loop integral then has to be integrated over all instanton locations (and scale sizes and orientations) to get $\langle 1\vec{r}|H|0\vec{r}\rangle$ and then from Eq. (2.32) $E(\theta,\vec{r})$ can be obtained.

The tunneling calculation outlined above is, for weak coupling where it is valid, equivalent to the following dilute-instanton-gas calculation. Let $E(\theta,\vec{r})$ be defined by

$$\exp[-E(\theta,\vec{r})T]=\frac{\int[dA^\mu]\mathrm{tr}[T\exp(i\oint A^\mu dx^\mu)]\exp\{\int[-\frac{1}{4}\mathrm{tr}\,F^2+(i\theta/8\pi^2)\mathrm{tr}\,F\tilde{F}]d^4x\}}{\int d[A^\mu]\exp\{\int[-\frac{1}{4}\mathrm{tr}\,F^2+(i\theta/8\pi^2)\mathrm{tr}\,F\tilde{F}]d^4x\}}\,, \tag{2.35}$$

where the Euclidean time T is assumed to be large and the ordered exponential runs around the same Euclidean loop as before. If this ratio of functional integrals is evaluated in exactly the same (dilute gas) approximation as in Eq. (2.32) except that in the numerator each multiple instanton-anti-instanton configuration is multiplied by the loop integral for that configuration (before integrating over locations, scale sizes, etc.) then the resulting approximation to $E(\theta,\vec{r})$ will be equivalent to the tunneling calculation.

A deductive derivation of the effect of instantons on the $q\bar{q}$ interaction would begin with the functional integral in Eq. (2.34). Wilson has argued that the average Euclidean loop integral does in fact yield the interaction energy of a heavy "test quark" pair[10] (not to be confused with the potential between real light quarks). Unlike the tunneling picture the averaged loop integral makes perfectly good sense when the instanton gas is not dilute but rather dense as in QCD.

For the moment we will refrain from commenting on whether or not this new tunneling interaction has anything to do with quark confinement. It is, however, definitely there and will turn out to be non-negligible as we shall see in Sec. IV.

E. Light quarks

The Wilson loop is not directly relevant to the binding of ordinary light quarks. The problem is not just a kinematic one, since, as 't Hooft has emphasized, instantons do qualitatively new things when light fermions are present.[13] It is simplest to discuss the situation in the limit of massless fermions. As before we will develop the physics in a simple tunneling picture and then pass to the long-time Euclidean functional integral.

In a theory with fermions a "classical" state is specified by giving the boson field configuration and saying which fermion states are occupied. A perturbation-theoretic n state is thus one for which the gauge field belongs to class n and all the negative-energy fermion levels are occupied. One can think of tunneling as being an adiabatic process as far as the fermions are concerned. For each value of Euclidean time t, the time-independent Dirac equation in an instanton field will have eigenvalues $\epsilon_i(t)$ and eigenfunctions $\Psi_i(t,\vec{x})$ where t enters only as a parameter. Since the net effect of tunneling is just a gauge transformation the original eigenvalues of $\epsilon_i(0)$ must be in one-to-one correspondence with the final eigenvalues. If the correspondence $\epsilon_i(0)\to\epsilon_i(T)$ is the trivial one $\epsilon_i(0)=\epsilon_i(T)$ then the tunneling can connect two distinct classical vacuums. For an instanton and massless fermions, however, this mapping of the spectrum of the Dirac equation into itself is nontrivial and in particular the highest right-handed negative-energy state crosses zero and becomes the lowest right-handed positive-energy state while the lowest positive-energy left-handed state becomes the highest negative-energy left-handed state. Thus if the initial configuration had all its negative-energy states occupied and all positive-energy states empty, then in the final configuration there will be one right-handed positive-energy state which is occupied and one left-handed negative-energy state which is empty. The tunneling process is then $n\to m+\bar{q}_L\bar{q}_R$ rather than just $n\to m$. Strictly speaking there are then no vacuum tunnelings but only virtual tunnelings $n\to m+\bar{q}q\to n$.

The above picture is easy to demonstrate explicitly in two-dimensional models (the reader can easily work it out, for the two-dimensional Abelian model discussed in Ref. 18) and is known to follow directly from the topology of the instanton field in four dimensions.[22]

A canonical picture of the suppression of tunneling in the presence of massless fermions can also be given. As is well known[8] in QCD the axial baryon number current contains an anomaly, and is not conserved (α labels color, i labels flavor):

$$J_\mu^5=\sum_{\alpha,i}\bar{\psi}_{\alpha i}\gamma_\mu\gamma_5\psi_{\alpha i}\,,\qquad \partial^\mu J_\mu^5=\frac{1}{8\pi^2}\,F_{\mu\nu}^a\tilde{F}_{\mu\nu}^a\,. \tag{2.36}$$

The above current is the gauge-invariant regulated current (defined, say, by a point-separation technique). One can define a gauge-variant current which is conserved:

$$\tilde{J}_\mu^5=J_\mu^5-\frac{g^2N}{32\pi^2}\,\epsilon_{\mu\nu\lambda\sigma}\mathrm{Tr}A_\nu(\partial_\lambda A_\sigma+\tfrac{2}{3}A_\lambda A_\sigma)\,,\qquad Q_5=\int d^3x\tilde{J}_0^5\,,\quad \frac{d}{dt}Q_5=0 \tag{2.37}$$

Q_5 is now conserved but not gauge invariant. It is easy to see that Q_5 is gauge invariant under gauge transformations that vanish at infinity, but not under gauge transformations that change the topological class of A_μ.

If we consider $A_0 = 0$ gauge we find that the unitary operator R which implements a gauge transformation of the $n=1$ class ($R|n\rangle = |n+1\rangle$) has the following effect on Q_5:

$$R\,Q_5 R^\dagger = Q_5 - 2N\,, \tag{2.38}$$

where N is the number of flavors. If the vacuum states of different topology $|n\rangle$ are defined by $|n\rangle = R^n|0\rangle$, with $Q_5|0\rangle = 0$, then $Q_5|n\rangle = 2Nn|0\rangle$. However, Q_5 is conserved, $[Q_5, H] = 0$, so that $\langle n|\exp(-HT)|m\rangle \sim \delta_{n,m}$, namely tunneling is suppressed in the absence of non-chiral-invariant sources.

However, virtual tunneling does occur and again the true vacuum will be the coherent superposition of the n vacuums. Indeed it is only in such a state that one recovers cluster decomposition of operators of chirality $=2N$.[14] However, unlike the case of massive fermions, physical quantities will have no dependence on θ: The vacuum energy will be independent of θ, and $\partial E(\theta)/\partial\theta = \langle\theta|\mathrm{tr}F_{\mu\nu}\tilde{F}_{\mu\nu}|\theta\rangle = 0$. Finally axial baryon number conservation is violated. The operator $\exp(i\alpha Q_5)$ (for $\alpha \neq \pi/N$) is ill defined, and takes one out of the Hilbert space constructed about a θ vacuum. Alternatively the gauge-invariant current, J^5_μ, has a hard divergence, $\mathrm{Tr}F_{\mu\nu}\tilde{F}_{\mu\nu}$, which has nonvanishing matrix elements ($\langle\theta|\bar{\psi}\psi\,\mathrm{tr}F\tilde{F}|\theta\rangle \neq 0$ in the case of one flavor) thus vitiating Goldstone's theorem.

Let us now proceed to examine the effects of massless fermions on the long-time Euclidean functional integral after making the following observation. In an adiabatic approximation the solutions to the time-dependent Dirac equation in an instanton field will be

$$\psi_i(t,\vec{x}) = \Psi_i^0(t,x)\exp\left[-\int_0^t \epsilon_i(t')dt'\right].$$

Such a solution will be normalizable if ϵ_i crosses zero and ϵ_i is positive for $t\to+\infty$ and negative for $t\to-\infty$. Thus the shift in a right-handed state from negative to positive energy in an instanton field can be expected to produce a normalizable solution to the time-dependent Dirac equation.

The existence of normalizable solutions to the massless Dirac equation in fields with nonzero topological quantum number Q has been demonstrated by a number of authors.[23] This has a dramatic effect on the functional integral. The fermion determinant then vanishes identically for any configuration containing an unequal number of instantons and anti-instantons, in agreement with the above argument that only virtual tunnelings are allowed. In the analog gas picture a virtual tunneling is a closely correlated instanton–anti-instanton pair. We have in fact shown in a previous paper that in the presence of several massless fermions a sufficiently dilute instanton gas is actually a "molecular" gas composed of instantons and anti-instantons permanently bound into "diatomic molecules,"[14] by the exchange of massless fermions. This exchange gives rise to a logarithmic attractive interaction proportional to the number of flavors, of the form $6N\ln R$, where R is the distance between the instanton and the anti-instanton. In QCD it is probable that for strong enough coupling there is a phase transition at which point the "molecules" disassociate, liberating free instantons and anti-instantons. This would lead to a spontaneous breakdown of chiral symmetry and will be discussed in Sec. V. In any case the quasitunneling vac $\to$ fermions and anti-fermions (which crosses to fermions $\to$ fermions) leads to qualitatively new nonperturbative interactions among massless fermions.

F. Beyond the dilute-gas approximation

The analog model developed above of a perfect gas of instantons and anti-instantons is only a valid approximation for very small coupling (or small $\hbar$). As the nonlinear coupling g increases one must improve this approximation. In order to see the effects which emerge when g increases let us consider the contribution to the functional integral of the $n_+ = n_- = 1$ sector. This sector has net topological quantum number equal to zero, i.e., the same as the naive vacuum sector $n_+ = n_- = 0$, and strictly speaking the only saddle point (solution of the classical field equations) is the naive vacuum configuration. However, a superposition of widely separated instanton and anti-instanton configurations is very close to a saddle point.

One way of including such approximate saddle points in a systematic fashion is to introduce constraints into the functional integral. Thus we write (schematically)

$$\int_{n_+=n_-=1}[DA_\mu]\,e^{-S(A)} = \int[DA_\mu]\int \pi da_i^+ C_i^+(A_\mu, a_i^+)\pi da_i^- C_i^-(A_\mu, a_i^-)e^{-S(A)}\,, \tag{2.39}$$

where $\int \pi da_i^\pm C_i^\pm(A_\mu, a_i^\pm) = 1$, the a_i^+ (a_i^-) are "collective coordinates" for the individual instanton (anti-instanton) corresponding to the relevant translational, scale, and group degrees of freedom, and the $C_i^\pm$ are functions of the field that, for given a_i, fix these degrees of freedom. Interchanging orders of integration one now expands, for fixed $a_i^\pm$, about a true saddle point of the constrained functional integral. For values of $a_i^\pm$ corresponding to well-separated instanton configurations the saddle-point configuration will be approximately given by the superposition of the instanton and anti-instanton solution, and the action will be simply $-16\pi^2/g^2$, independent of the $a_i^\pm$. This is what yields, upon integration over $a_i^\pm$, the term proportional to the square of the volume of space-time $(VT)^2$ reproducing, with the correct normalization, the perfect gas approximation.

Moreover, one can now imagine improving on this approximation, taking into account that the action at the saddle point does depend on $a_i^\pm$ when the instantons are at a finite separation. We can interpret

$$S(a_i^\pm) = \frac{16\pi^2}{g^2} + \delta S(a_i^+, a_i^-) \tag{2.40}$$

as consisting of, in the analog gas model, a term corresponding to the chemical potential ($8\pi^2/g^2$ a piece) for the instanton and anti-instanton and an interaction energy, $\delta S(a_i^+, a_i^-)$. A similar procedure can be carried out in the multiple instanton–anti-instanton sectors. The result of performing the saddle-point integrations will be to replace the functional integral by the partition function of a gas of interacting instantons and anti-instantons. Since the Lagrangian is a nonlinear functional of the fields there will in general be multibody interactions.

The nature of these interactions for well separated configurations is easily understood. In general instantons will attract anti-instantons and repel instantons, since in the first case one reduces the action by bringing the configurations together and in the latter the action is a minimum for infinite separation. In special theories, such as QCD, instantons will have no interaction with instantons, but an attractive instanton–anti-instanton interaction will always exist. The interaction energy will also depend on the group orientation and scale size. The interaction will vanish for infinite separation, exponentially in non-scale-invariant superrenormalizable theories, and according to a power law in scale-invariant theories such as QCD.

Since for weak coupling we have shown that the density of instantons is proportional to exp $(-8\pi^2/g^2)$ one can perform a virial expansion of the partition function. To first approximation we have the perfect gas described above, where δS has been neglected. Including the effects of the interactions will yield corrections proportional to the density of instantons. Thus one might hope to set up a systematic virial expansion of the analog gas in powers of $\exp(-8\pi^2/g^2)$. There are, however, severe technical and conceptual problems in attempting to do this. The first problem is that of double counting. A well-separated instanton–anti-instanton pair clearly gives an important contribution to the functional integral. However, it is clearly nonsense to consider both the ordinary vacuum field configuration as well as an instanton–anti-instanton pair close together with equal weight. The second problem is how to systematically sum the quantum fluctuations about a given field configuration when tunneling exists. We now know that in the presence of tunneling the ordinary perturbation theory is not Borel-summable.[24] Thus the perturbation theory about the ordinary vacuum will yield an asymptotic power series $\sum_n C_n g^n$ which we do not know precisely how to define when $g \neq 0$.

These problems are interrelated. The lack of Borel summability arises from the existence of real instantons, and the ambiguity in dealing with overlapping instantons is related to whether such configurations have been included as fluctuations in other sectors. At the moment we lack the solution to both of these problems and do not have a systematic formalism for dealing with tunneling for large coupling.

To illustrate the nature of the dilute-gas approximation as well as the double-counting problem, it is instructive to consider for following model, in which all quantum fluctuations have been suppressed. Consider replacing the field variables $A_\mu^a(\vec{x}_i)$ by discrete spins $\sigma_i = \sigma(t_i)$ on a discrete (Euclidean) time lattice. At each discrete time the field will be constrained to be in one of an infinity of possible $|n\rangle$ states, $\sigma_i = -\infty, \ldots, -1, 0, +1, 2, \ldots$. Thus a Euclidean field configuration is represented by a lattice of spins that take integer values: $(\sigma_1, \sigma_2, \ldots, \sigma_T)$, where T = total time. A static $|n\rangle$ vacuum state is represented by the $\sigma_i = n$, $i = 1, \ldots, T$ configuration $(n, n, \ldots, n)$. Obviously an instanton that effects $|n\rangle \to |n+1\rangle$ at time $t = i$, is represented by $(\sigma_1 = \sigma_2 = \cdots = \sigma_i = n;\ \sigma_{i+1} = \cdots = \sigma_T = n+1)$. The topological quantum number is clearly $\nu = \sigma_T - \sigma_1$. We shall choose the action to be

$$S = \sum_{i=1}^{T} \frac{1}{g^2} |\sigma_i - \sigma_{i+1}| , \tag{2.41}$$

so that a configuration consisting of n_+ instantons and n_- anti-instantons has action $= (n_+ + n_-)g^2$. The θ vacuum to θ vacuum amplitude is then given by

$$\langle\theta|e^{-HT}|\theta\rangle = \sum_{(\sigma_i),\, \sigma_1 = 0} e^{-S(\sigma_i) + i\theta(\sigma_T - \sigma_1)} . \tag{2.42}$$

This is simply the partition function for a one-dimensional system of infinite-component classical spins with nearest-neighbor interactions. For $\theta = 0$ the Hamiltonian is simply $H = \sum_i |\sigma_i - \sigma_{i+1}|$, and the temperature is $kT = 1/g^2$.

For this system

$$E(\theta)=\frac{1}{T}\lim_{T\to\infty}\left(-\ln\langle\theta|e^{-HT}|\theta\rangle\right)$$

can easily be evaluated [by the change of variable $x_i=(\sigma_i-\sigma_{i+1})$]:

$$E(\theta)=\ln\left(\frac{1-2\cos\theta e^{-1/g^2}+e^{-2/g^2}}{1-e^{-2/g^2}}\right)$$
$$\simeq -2\cos\theta e^{-1/g^2}+2\sin^2\theta e^{-2/g^2}+O(e^{-3/g^2}). \tag{2.43}$$

Alternatively one can replace the spin variables with instanton and anti-instanton variables. Every configuration of spins corresponds to a configuration of instantons and anti-instantons (which are simply domain walls in the one-dimensional lattice). Thus the configuration ($\sigma_1=\sigma_2=\ldots=\sigma_a=0$, $\sigma_{a+1}=\sigma_{a+2}=\ldots=\sigma_b=1$, $\sigma_{b+1}=\ldots=\sigma_c=0$, $\sigma_{c+1}=\ldots=\sigma_d=2$) corresponds to an instanton at $t=a$, an anti-instanton at $t=b$, and two instantons at $t=c$. We can then rewrite Eq. (2.42) as the partition function for a lattice gas of instantons and anti-instantons with a chemical potential equal to $1/g^2$. The instantons and anti-instantons may be placed anywhere without affecting the action (energy) except that we cannot allow an instanton to sit on top of an anti-instanton. Thus, in effect, the only interaction is an infinite repulsive core between instanton and anti-instanton; instantons do not interact among themselves.

The perfect-gas approximation ignores the double-counting interaction, yielding

$$\langle\theta|e^{-HT}|\theta\rangle=\sum_{n_+,n_-=0}^{\infty}\frac{1}{n_+!n_-!}\times T^{n_++n_-}e^{-(n_++n_-)/g^2}e^{i\theta(n_+-n_-)}$$
$$=\exp(2T\cos\theta e^{-1/g^2}). \tag{2.44}$$

The correction to this, which gives the first term in Eq. (2.43) can be calculated by taking into account the double-counting interactions of the gas. Thus in the two-instanton sector, a configuration consisting of two instantons on top of each other contributes an amount $Te^{-2/g^2}e^{2i\theta}$, not $\frac{1}{2}Te^{-2/g^2}e^{2i\theta}$ as given by Eq. (2.44). Also, an instanton–anti-instanton cannot sit on top of each other and one must subtract a term Te^{-2/g^2} from Eq. (2.44). Thus to order e^{-2/g^2} we have, including the above two corrections,

$$\langle\theta|e^{-HT}|\theta\rangle=1+2T\cos\theta e^{-1/g^2}+\tfrac{1}{2}(2T\cos\theta)^2e^{-2/g^2}+T\cos2\theta e^{-2/g^2}-Te^{-2/g^2}$$
$$\simeq\exp\{T[2\cos\theta e^{-1/g^2}-\sin^2\theta e^{-2/g^2}+O(e^{-3/g^2})]\}, \tag{2.45}$$

in agreement with Eq. (2.43).

In a continuum theory, such as QCD, there will of course be additional corrections due to long-range interactions between instantons and anti-instantons, as well as to the quantum fluctuations about the saddle points. Fortunately the dominant corrections to the perfect gas in QCD arise from the long-range interactions which can be evaluated in the dilute-gas approximation. Furthermore, as we shall see below, instantons never get too dense nor does the coupling constant get large enough to seriously invalidate the dilute-gas approximation. Thus we shall be able to proceed to include effects of instanton interaction even though we lack a systematic procedure for dealing with the problems of double counting and quantum fluctuations which would arise for large densities and couplings.

Finally let us note that, when the coupling increases, other field configurations in addition to instantons might become important. That instantons determine the vacuum structure for arbitrarily small coupling is due to the fact that although the "energy" necessary to create an instanton in the analog gas is $(1/g^2)S_{cl}$, the "entropy" is even larger [$S\sim\ln V$ (V = volume of space time)] and the "free energy" $F=S_{cl}-g^2\ln V$ is dominated by the entropy term. The "temperature" corresponding to a phase transition from an n vacuum is roughly given by

$$\sim g^2=S_{cl}/\ln V\xrightarrow[V\to\infty]{}0.$$

On the other hand, one might consider other field configurations whose action is in some sense infinite. If such field configurations occupy a volume in function space which is large enough they might be important at some finite g^2. Of particular importance are configurations consisting of pairs of configurations (molecules) whose action depends logarithmically on the separation of the pair. The free energy of such a pair, separated by R, will behave as $F\sim C\ln R-g^2\ln R$. For small g^2 the pair will be close together, while for large g^2 the entropy term will dominate and the pair will separate. We might then expect a phase transition at a finite temperature ($=g^2$) from a gas of tightly bound molecules to a gas of dissociated molecules. Such phase transitions are well known in one- and two-dimensional systems with logarithmic interactions.[25]

In QCD there are two important cases of such molecular field configurations. First, there are instantons themselves in the presence of massless fermions. The fermions effectively bind instantons to anti-instantons with an attractive logarithmic potential, thus suppressing tunneling. The phase transition which is responsible for chiral-symmetry breaking, from a gas of instanton–anti-instanton molecules to a dilute gas of instantons, is discussed in Sec. V. The second case concerns merons. As we shall see, instantons themselves can be regarded as tightly bound pairs of merons, which are localized lumps of one-half topological charge, with independent entropy of position and logarithmic interactions. The phase transition in which instantons dissociate into merons and which may be responsible for confinement is discussed in Sec. VI.

III. INSTANTON INTERACTIONS

In this section we will study some simple features of the instanton gas in QCD. As explained in the Introduction, this theory has no free parameters and we cannot vary the relative importance of instantons by varying some convenient coupling constant. However, by asking questions which emphasize different distance scales one may accomplish much the same effect. At short distances the effective coupling becomes small and the density of instantons is very low. Our strategy will be to start at the small scales and work our way outward. Inevitably, we reach a scale where the instantons are close enough that their interactions become significant. This produces new phenomena which we will discuss in this section. We will find that large new effects begin to appear at scales where the density of instantons is reasonably low and quantitative calculations are possible. Still larger scales where confinement presumably becomes manifest will be probed in Sec. VI.

In order to focus on pure QCD effects, we will discuss an unrealistic theory with no light fermions (the properties of light quarks will be studied in Sec. V). We will also treat SU(2) and SU(3) in parallel since they differ in their behavior in instructive ways. Let us begin with the role of the BPST instanton in the SU(2) vacuum. As shown by 't Hooft its contribution to the functional integral is, to one-loop order,

$$\langle n|H|n+1\rangle = \int d^4z \int \frac{d\rho}{\rho^5}(0.26)\left(\frac{8\pi^2}{g_0^{\,2}}\right)^4 \times \exp\left(-\frac{8\pi^2}{g_0^{\,2}} + \tfrac{22}{3}\ln\mu\rho\right), \tag{3.1}$$

where μ is a renormalization mass introduced by the Pauli-Villars procedure. It of course makes sense to express this in terms of a running coupling constant, $g(\rho)$, but this is ambiguous until some precise definition of g has been adopted. To one-loop order, however, the ambiguity is just an additive constant in $8\pi^2/g^2$:

$$\frac{8\pi^2}{g^2(\rho)} = \frac{8\pi^2}{g_0^{\,2}} + \tfrac{22}{3}\ln\frac{1}{\rho\mu} + C\,. \tag{3.2}$$

For simplicity, we shall adopt the coupling-constant definition which sets $C=0$. This is not the same definition as the dimensional regularization definition of g [which amounts in the SU(2) case to setting $C=-6.9$] but does not appear particularly unnatural—indeed, for the values of g we shall be interested in, it makes two-loop contributions to anomalous dimensions of low-lying twist-two operators rather smaller than does the dimensional regularization definition. This is quite important since we shall claim that interesting strong-interaction effects occur at sufficiently small coupling constant that we may neglect higher-loop effects—but the same physics may correspond to large or small g depending on what definition of g has been adopted. In what follows we shall replace μ and g_0 by $\bar{\mu}$ so that

$$\frac{8\pi^2}{g^2(\rho)} = \tfrac{22}{3}\ln\frac{1}{\bar{\mu}\rho} = \tfrac{22}{3}\ln\frac{1}{\mu\rho} + \frac{8\pi^2}{g_0^{\,2}}\,. \tag{3.3}$$

Eventually we will see that $\bar{\mu}$ can be related to the hadron scale size [we expect this one-loop expression for $g(\rho)$ to be useful as long as $\bar{\mu}\rho$ is small]. To be consistent with the requirements of the renormalization group we must also assume that the determinantal factor of $(8\pi^2/g_0^{\,2})^4$ is actually $[8\pi^2/g^2(\rho)]^4$. To be more precise about this it would be necessary to do a two-loop calculation of the instanton determinant, a worthy exercise which has not yet been carried out. In summary, our expression for the one-instanton contribution to the vacuum functional in the pure SU(2) gauge theory is

$$\langle n|H|n+1\rangle_{SU(2)} = \int d^4z \int \frac{d\rho}{\rho^5}(0.26)\left(\frac{8\pi^2}{g^2(\rho)}\right)^4 \times \exp\left[-\frac{8\pi^2}{g^2(\rho)}\right]. \tag{3.4}$$

A modest extension of 't Hooft's calculation of the SU(2) instanton determinant allows us to conclude that, for SU(3),

$$\langle n|H|n+1\rangle_{SU(3)} = \int d^4z \int \frac{d\rho}{\rho^5}(0.10)\left(\frac{8\pi^2}{g^2(\rho)}\right)^6 \times \exp\left[-\frac{8\pi^2}{g^2(\rho)}\right], \tag{3.5}$$

where this time $8\pi^2/g^2(\rho) = 11\ln(1/\rho\bar{\mu})$. The 12 powers of g^{-1} arise from the zero modes: one dilatation, four translations, and seven gauge modes (the λ_8 generator does not induce a change in the instanton field and does not induce a zero mode).

Next we construct the analog gas. To be systematic, we should invent a set of constraints such that solving the Yang-Mills equations under them yields the desired multi-instanton–anti-instanton configurations, evaluate the functional integral about the saddle point, and then integrate out the constraints. Since we do not know a good way to do this we resort to a crude procedure which should be adequate in the limit of low pseudoparticle density. First, we convert the standard instanton to singular gauge by inversion:

$$A_\mu^a = \frac{2}{g}\,\frac{\rho^2}{x^2+\rho^2}\,\bar{\eta}_{a\mu\nu}\,\frac{x_\nu}{x^2}\,. \tag{3.6}$$

Since A_μ now falls as r^{-3} at infinity it makes sense to construct multi-instanton configurations by superposition:

$$A_\mu^a = \sum_i R_{ab}^{(i)} A_{\mu b}^{(i)}(x - z_i, \rho_i)\,. \tag{3.7}$$

In this expression $A_\mu^{(i)}$ is the basic instanton or anti-instanton solution of Eq. (3.6) (for anti-instanton $\bar{\eta}\to\eta$) and R_{ab} is a matrix from the adjoint representation of the group representing the group orientation degree of freedom. This is a solution of the Yang-Mills equations only in the limit of infinite separation, but should be a decent approximation to the dominant configuration so long as the pseudoparticles do not overlap significantly. In this case the functional determinant should just be the product of the individual instanton determinants, and we know what weight to give these configurations when we integrate over $z_i, \rho_i, R_{ab}^{(i)}$. In what follows, we shall in first approximation neglect interactions, taking the action of N pseudoparticles to be $N8\pi^2/g^2$. After exploring the consequences of this assumption, we will turn to a computation and discussion of the effects of instanton–anti-instanton interactions.

The dilute gas arguments of Sec. II then imply that the dominant contribution to the functional integral comes from configurations where the space-time density of pseudoparticles of scale size between ρ and $\rho + d\rho$ is [for SU(2)]

$$\frac{d\rho}{\rho^5} D(\rho) = \frac{d\rho}{\rho^5}(0.26)\left(\frac{8\pi^2}{g^2(\rho)}\right)^4 e^{-8\pi^2/g^2(\rho)}\,. \tag{3.8}$$

To check the consistency of the dilute-gas approximation we may compute the fraction, $f(\rho)$, of space-time occupied by pseudo-particles of scale size less than ρ. If $f(\rho)$ is less than unity we will have a dilute gas at scale size ρ. We take the "volume" of an instanton of scale size ρ to be that of a sphere of radius $\rho[(\pi^2/2)\rho^4]$ and find

$$f(\rho) = \pi^2 \int_0^\rho \frac{d\rho'}{\rho'} D(\rho') \tag{3.9}$$

(this is the sum of equal contributions from instantons and anti-instantons). In the asymptotic freedom regime (small ρ) we may reexpress f as an integral over $x = 8\pi^2/g^2$ by using the [SU(2)] relation

$$\frac{dx}{d\ln 1/\rho\mu} = \tfrac{22}{3}\,. \tag{3.10}$$

The result is

$$f(x) = \frac{3\pi^2}{22}\int_x^\infty dx\, D(x)\,, \tag{3.11}$$

where now $D(x) = 0.26x^4e^{-x}$. Figure 2 displays f as a function of x as well as $(\rho\mu)$. It is clearly a very rapidly varying function: f increases from 0.01 to 1 as ρ increases by only a factor of 2, from $0.15\mu^{-1}$ to $0.35\mu^{-1}$. We shall find that when f is less than one, but not vanishingly small (greater than 0.1, say) instantons cause significant modifications of vacuum properties, in spite of the smallness of the effective coupling in this region ($x \sim 10$). When f is greater than 1, however, the instanton gas picture must break down and some new vacuum physics must take over. Everything that we will find strongly suggests that this transition is associated with confinement, or at least with the physics which sets the scale size of hadrons, and we will provisionally make that identification. Since the rise in f is so rapid as a function of x or ρ one gets a rather sharp definition of the hadronic coupling constant and scale size: $x_c \sim 8$, $\rho_c\mu \sim 0.3$. Thus, through the equation $f(x(\rho_c\mu)) \sim 1$, one realizes dimensional transmutation, eliminates μ in favor of the hadron scale size, and identifies the hadronic coupling constant as a pure number. By most usual measures $x \sim 8$ corresponds to a rather small coupling constant and measurements which

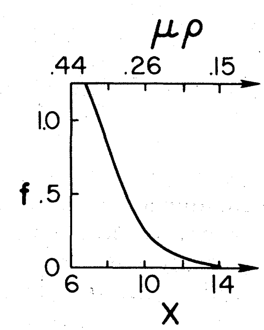

FIG. 2. The fraction f of space-time occupied by instantons smaller than a given scale size ρ in an SU(2) (no quarks) gauge theory. We plot f as a function of ρ and $x(\rho)$.

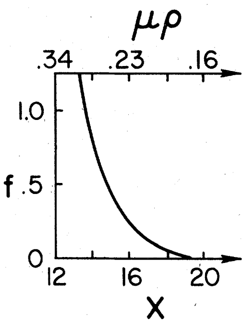

FIG. 3. The fraction f of space-time occupied by instantons smaller than a given scale size ρ in an SU(3) (no quarks) gauge theory. We plot f as a function of ρ and $x(\rho)$.

probe distances smaller than the hadron scale will see even smaller coupling. This picture of what happens in SU(2) requires some modification because it will turn out that at $x \lesssim 11$, the instantons ionize into a new kind of pseudoparticle, dubbed meron, which is very directly related to the dynamics of confinement. This would lead us to revise our estimate of the critical values of ρ and x slightly to $x_c \sim 11$, $\rho_c \sim 0.2\mu^{-1}$. We will amplify this remark in Sec. VI.

The situation for SU(3) is significantly different. Now

$$\frac{dx}{d\ln(1/\rho\mu)} = 11 \tag{3.12}$$

and we have [in contrast to Eq. (3.11)]

$$f(x) = \frac{\pi^2}{11}\int_x^\infty dx\, D_{\mathrm{SU}(3)}(x)\,. \tag{3.13}$$

This function, plotted in Fig. 3, attains the critical value at a much larger value of x and grows much more rapidly with ρ than the SU(2) case. The same kind of argument as before would lead us to believe that the hadron coupling constant and scale size are given by $x_c \sim 16$, $\rho_c \sim 0.25\mu^{-1}$. Again, the analysis of various effects arising from merons or instanton interactions will cause us to revise these numbers slightly, but the basic point remains: The dilute instanton gas picture reveals how and where dimensional transmutation occurs and shows, most importantly, that the coupling constant at the hadron scale size is a *small* number which gets smaller as the group gets larger. The value provisionally associated with SU(3), $\alpha \sim 0.4$, is not far from numbers which have been extracted from optimistic studies of scaling in electroproduction.

We now would like to discuss various effects which arise within the dilute-gas picture when the density, while still small, becomes large enough for the pseudoparticles to influence each other. To study this problem, we imagine imposing on the system from the outside a *weak* slowly varying external field, $F_{\mu\nu}^{\text{ext}}$. In the no-instanton vacuum the action of such a configuration is just

$$\delta S = \frac{1}{4g^2}\int d^4x \sum_a (F_{\mu\nu a}^{\text{ext}})^2\,. \tag{3.14}$$

In fact A^{ext} must be regarded as a perturbation on $[A_\mu]^0$, the multiple-instanton configuration which dominates the dilute gas vacuum. The interaction of the external field with the instantons must be included in δS and, as we shall see, the net effect for a weak external field is just a coupling-constant renormalization.

First, we examine the problem of a single instanton in a weak, slowly varying $F_{\mu\nu}^{\text{ext}}$. Thus, $A_\mu = [A_\mu]^0 + \delta A_\mu$ where $[A_\mu]^0$ is the standard single instanton of scale size ρ and δA_μ approaches the potential of a weak constant $F_{\mu\nu}^{\text{ext}}$ at distances large compared to the instanton scale size. In fact we will divide space into two regions by a sphere of radius R, many times ρ, such that inside, in retion I, $[A_\mu]^0$ is larger than δA_μ and outside, in retion II, δA_μ is larger than $[A_\mu]^0$. For $|x|$ comparable to R we may choose $\delta A_\mu = -\frac{1}{2}F_{\mu\nu}^{\text{ext}}x_\nu$, the Landau gauge potential of a weak constant $F_{\mu\nu}$.

Inside R we could find δA_μ explicitly as the solution of the linearized equations of motion in the instanton background field subject to the condition of regularity at the origin and linear growth at large x. On the other hand, if we define

$$S_{\mathrm{I}} = \tfrac{1}{4}\int_{|x|<R} d^4x (F_{\mu\nu}^a)^2\,, \tag{3.15}$$

then

$$S_{\mathrm{I}}([A]^0 + \delta A) - S_{\mathrm{I}}([A]^0) = \int_{|x|<R} d^4x (D_\mu^0\, \delta A_\nu^a) F_{\mu\nu}^{a0} + O(\delta A^2)\,. \tag{3.16}$$

Upon integrating by parts and using the equation of motion for the instanton background field, $D_\mu^0 F_{\mu\nu}^0 = 0$, we can express the interaction energy as a surface term,

$$S_{\mathrm{I}}(A^0 + \delta A) - S_{\mathrm{I}}(A^0) = \int_{|x|=R} d\Omega\, \hat{x}_\mu \delta A_\nu^a (F_{\mu\nu}^a)^0 + O(\delta A^2). \tag{3.17}$$

Now A_μ^0 is the instanton field in singular gauge so that

$$[F_{\mu\nu}^a]^0 = \frac{4}{g} M_{\mu\mu'} M_{\nu\nu'} \frac{\bar{\eta}_{a\mu'\nu'}\rho^2}{(x^2+\rho^2)^2}\,, \tag{3.18}$$

where $M_{\mu\nu} = g_{\mu\nu} - 2\hat{x}_\mu\hat{x}_\nu$. Also, on the surface $|x| = R$ we may set $\delta A_\mu = -\frac{1}{2}F_{\mu\nu}^{\text{ext}}x^\nu$. The angular averages may then be carried out explicitly to evaluate the $O(\delta A)$ surface term, with the result

$$S_I(A^0+\delta A)-S_I(A^0)=-\frac{\pi^2}{g}\rho^2 F^{a\,\text{ext}}_{\mu\nu}\bar{\eta}_{a\mu\nu}+O(\delta A^2). \quad (3.19)$$

The $O(\delta A^2)$ piece can also be seen to reduce to a surface term and turns out to give the integral of $\frac{1}{4}(F^{\text{ext}}_{\mu\nu})^2$ over the region $|x|<R$. A similar analysis of region II, with A^0 now regarded as the perturbing field and A^{ext} as the background field, yields an identical surface term to Eq. (3.19), assuming only that A^{ext} is a solution by itself of the Yang-Mills equations and that it goes to zero at infinity.

The net result of all this is that an instanton in a weak, slowly varying background field may be assigned an interaction energy

$$S_{\text{int}}=-\frac{2\pi^2}{g}\rho^2\bar{\eta}_{a\mu\nu}F^{\text{ext}}_{a\mu\nu}, \quad (3.20)$$

where F^{ext} is the value of background field at the instanton position. Note that if F^{ext} is self-dual, S_{int} vanishes because $\bar{\eta}_{a\mu\nu}$ is an anti-self-dual tensor. If F^{ext} is itself taken to be an anti-instanton field of scale size $\bar{\rho}$ and centered a distance R from the instanton, one finds

$$S_{\text{int}}=+\frac{32\pi^2}{g^2}\frac{\rho^2\bar{\rho}^2}{R^4}R_{ab}\bar{\eta}_{a\mu\nu}\eta_{b\mu\nu'}\hat{R}_\nu\hat{R}_{\nu'}, \quad (3.21)$$

where $\hat{R}$ is the unit vector pointing from the instanton to the anti-instanton and R_{ab} is the matrix describing the relative group orientation of the two pseudoparticles. The maximum value of $-S_{\text{int}}$ (obtained by varying R_{ab}) is $96\pi^2 g^{-2}\rho^2\bar{\rho}^2R^{-4}$, independent of $\hat{R}$. This agrees with Forster's calculation of the action of a pair constrained only as to location, but not as to orientation.[34] Since, in general, S_{int} depends on instanton group orientation and falls with separation like R^{-4}, instantons look like objects carrying a *color magnetic dipole* moment proportional to $\rho^2\bar{\eta}_{\mu\nu}$. [Recall that a magnetic dipole moment is really an antisymmetric tensor $\sim\int(j^\mu x^\nu - j^\nu x^\mu)$.] This interpretation leads us to expect that the dilute instanton gas will behave like a dilute gas of spins with its response to an external field described by a susceptibility. This effect will, among other things, lead to a "classical coupling-constant renormalization" of a rather interesting nature which we will now study.

In the low density limit we may use a virial expansion to find the effect of the medium on an external field: Given the interaction, Eq. (3.20), of a single pseudoparticle with the external field, one first computes

$$-S_{\text{eff}}=\langle e^{-S_{\text{int}}}-1\rangle,$$

averaging over group orientation of the pseudoparticle, and then weights this effective single-particle action with the appropriate pseudoparticle density and integrates over scale size and location to get the full effective action. For weak external fields, one finds from Eq. (3.20) that

$$\langle e^{-S_{\text{int}}}-1\rangle=\left(\frac{F_{\text{ext}}-\tilde{F}_{\text{ext}}}{2}\right)^2\frac{8\pi^2}{g^2}\frac{\pi^2\rho^4}{\Delta(n)}+O(F_{\text{ext}}{}^4), \quad (3.22)$$

where n refers to the SU(n) gauge group, $\Delta(n)$ is 3 (8) for $n=2$ (3), and $\tilde{F}_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}F_{\lambda\sigma}$. The appearance of $F-\tilde{F}$ simply reflects the fact that the instanton interacts only with the anti-self-dual part of F_{ext}. The analog of Eq. (3.22) for an anti-instanton simply replaces $(F-\tilde{F})^2$ by $(F+\tilde{F})^2$. Since instantons and anti-instantons occur with equal probability, the net effect of the medium is proportional to $[(F+\tilde{F})^2+(F-\tilde{F})^2]/4=F^2$. In other words, the effect of the medium is to renormalize the original external field action density by a multiplicative constant K^{-1}, where

$$K^{-1}=1-\int\frac{d\rho}{\rho^5}D_n(\rho)\frac{8\pi^2}{g^2(\rho)}\frac{4\pi^2\rho^4}{\Delta(n)}. \quad (3.23)$$

Since $K>1$, the instantons cause the vacuum to behave like a paramagnetic medium and increase the interaction energy between fixed external sources. In fact, one easily sees from our discussion of the integrated density, $f(\rho)$, that the integral in Eq. (3.20) will be large even in the dilute gas region where f is less than one. If K is large enough, it may even be energetically favorable in the presence of external sources (quarks) to form a flux tube (or bag) in which the flux is expelled from the region of normal vacuum (K large) and confined to a region of abnormal vacuum ($K\simeq 1$) where no instantons are present. Expelling the instantons costs vacuum energy which is made up for by the lowered interaction energy between the quarks. In this picture, the confinement or hadron scale will have directly to do with where K begins to depart significantly from one. This will be different from our earlier criterion based on integrated density, but not dramatically so.

If we focus our attention on the instantons themselves, the above effects can be interpreted as a coupling-constant renormalization. Evidently, the action of an instanton of scale size ρ is decreased by the presence within it of smaller scale instantons. Very crudely

$$\frac{8\pi^2}{\bar{g}^2(\rho)}=\frac{8\pi^2}{g^2(\rho)}-\int_0^\rho\frac{d\rho'}{\rho'}\frac{4\pi^2}{\Delta(n)}\left(\frac{8\pi^2}{g^2(\rho')}\right)^2D_n(\rho'), \quad (3.24)$$

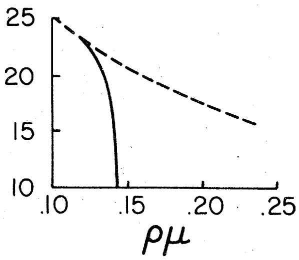

FIG. 4. The dependence of the effective coupling $x = 8\pi^2/g^2$ as a function of distance $\rho\mu$, taking the classical renormalization effects into account for SU(3) (no quarks). The dashed line represents the value of x using asymptotic freedom alone.

where $g(\rho)$ is, at this stage, the standard asymptotic freedom running coupling constant and $\bar{g}(\rho)$ is the running coupling constant including the "classical" renormalization effects of the medium. As expected, the medium amplifies the ordinary perturbative asymptotic freedom effects and causes the effective coupling to increase more rapidly with scale size. One could regard Eq. (3.24) as a self-consistent equation for the effective coupling by replacing $x = 8\pi^2/g^2(\rho)$ under the integral by $\bar{x} = 8\pi^2/\bar{g}^2(\rho)$:

$$\bar{x}(\rho) = x(\rho) - \int_0^\rho \frac{d\rho'}{\rho'}\,\frac{4\pi^2}{\Delta(n)}\,\bar{x}^2(\rho)D_n(\bar{x}(\rho)), \qquad (3.25)$$

where $x(\rho)$ is taken to be the perturbative asymptotic freedom effective coupling. Differentiation with respect to $\ln\rho$ gives a renormalization-group equation for $\bar{x}$,

$$\frac{d\bar{x}}{d\ln(1/\rho\mu)} = C_n + \frac{4\pi^2}{\Delta(n)}\,\bar{x}^2 D_n(\bar{x}) \qquad (3.26)$$

(where $C_2 = \frac{22}{3}$, $C_3 = 11$), which we may integrate numerically. The results for SU(3) are displayed in Fig. 4 and for SU(2) in Fig. 5. In both cases the new effects turn on very sharply at scale sizes where our earlier estimates indicated that the integrated pseudoparticle density was rather small, say 10%. As soon as they are at all significant, the new effects are dominant.

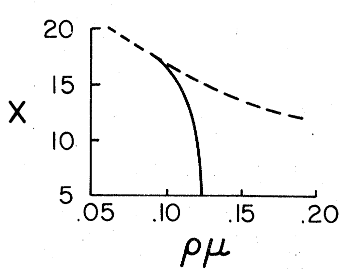

FIG. 5. The dependence of the effective coupling $x = 8\pi^2/g^2$ as a function of distance $\rho\mu$, taking the classical renormalization effects into account for SU(2) (no quarks). The dashed line represents the value of x using asymptotic freedom alone.

A further point is that since $\bar{x}$ increases more rapidly with ρ than required by asymptotic freedom, the integrated density functions Eqs. (3.9) and (3.11) should be modified. Taking account of Eq. (3.25) we have

$$\bar{f}(x) = \pi^2 \int_x^\infty d\bar{x}\,\frac{D_n(\bar{x})}{C_n + [4\pi^2/\Delta(n)]\bar{x}^2 D_n(\bar{x})}\,. \qquad (3.27)$$

Now $\bar{f}(x)$ is less than $f(x)$ and, more importantly, always less than one for interesting values of x ($x \gtrsim 10$): One easily sees that $\bar{f}(x) < x\Delta(n)/4$. Therefore, the coupling-constant renormalization effects appear to reduce instanton densities to a manageable level and change our definition of the critical scale size (where a transition from vacuum physics to confinement physics occurs) to that scale ($\sim 0.1\mu^{-1}$) where the effective coupling begins to increase very rapidly. Therefore we must modify the picture we extracted from the behavior of the noninteracting instanton gas at the beginning of the section. There we said that the onset of new physics is associated with the passage of $f(x)$ through 1, identifying in that way a critical coupling and scale size. Once the effects of interactions are included, the density no longer rises dramatically, but there is still a well-defined scale size and coupling constant at which the renormalized coupling constant (and vacuum susceptibility) begin to rise rapidly. We now identify this transition as setting the hadron scale and find new critical couplings and scale sizes for SU(2) and SU(3) which do not, in fact, differ markedly from the original estimates.

These considerations are probably too crude to be taken very seriously since the rapid rise in $D(x)x^2$ means that most of the renormalization effect on an instanton of a given scale size is coming from instantons of nearly the same size. It probably should be renormalized by instantons of, say, half its size, or smaller, which would delay the onset of sizable renormalization effects to smaller x and integrated densities more nearly equal to one. At the moment we do not know how to translate this notion into manageable mathematics, but do not believe that it would materially change the qualitative conclusions we have reached. The most important of these qualitative effects, let us repeat, is the identification of a well-defined scale size and coupling constant (and a small coupling as well) at which there is a transition between asymptotic freedom behavior and confining behavior. This provides, we believe, the basic explanation for the apparent smallness of the Yang-Mills coupling on the scale of ordinary hadron sizes.

An alternate method of computing the "classical" coupling-constant renormalization is to compute the gauge field propagator $D^{ab}_{\mu\nu}(x-y)$. We now know that the effects of vacuum tunneling may not be negligible and we must include the non-Gaussian fluctuations associated with tunneling. The simplest way to do this is to write A^a_μ as a sum over instanton and anti-instanton fields (in singular gauge) and perform the average by integrating over instanton scale sizes, locations, and group orientations with the appropriate density function $D(\rho)$ [$D(\rho)$ summarizes the effects of *Gaussian* fluctuations about vacuum tunneling]. In the product $A^a_\mu(x)A^b_\nu(y)$, cross terms between different instantons vanish when we perform the independent group orientation averages and we are left with a sum over instantons of the correlation function $\langle A^a_\mu A^b_\nu\rangle$ for a single instanton.

It is best to evaluate the propagator in momentum space (momentum q) and we have ($A^{(0)}_\mu$ is the one-instanton field)

$$\int d^4y\, e^{iq\cdot y}\int d^4x\, e^{iq\cdot x}\int d^4z\, A^{a(0)}_\mu(x-z)A^{b(0)}_\nu(y-z)$$

$$=(2\pi)^4\delta(q'-q)(\tilde{A}^{a(0)}_\mu(q)\tilde{A}^b_\nu(-q)). \quad (3.28)$$

The Fourier transform of the singular gauge instanton field is easily seen to be

$$\tilde{A}^{a(0)}_\mu(q)=\frac{i(4\pi)^2}{g(\rho)}\;\frac{\eta_{a\mu\nu}q_\nu}{q^4}F(\rho q)\,, \quad (3.29)$$

where ρ is the scale size, F has the property

$$F(x)\to 1,\quad x\to\infty$$
$$F(x)\to -\tfrac{1}{4}x^2,\quad x\to 0 \quad (3.30)$$

and $g(\rho)$ is the coupling appropriate to scale size ρ.

To construct the tunneling contribution to the propagator we must form $\tilde{A}^a_\mu\tilde{A}^b_\nu$, average over group orientations ($\eta_{a\mu\nu}\to R_{ab}\eta_{b\mu\nu}$, and average over R), sum over instantons and anti-instantons, and integrate over scale size. The result is

$$D^{ab}_{\mu\nu}(q)_{\text{tunnel}}=(4\pi)^2\delta_{ab}\,\frac{\delta_{\mu\nu}-q_\mu q_\nu q^{-2}}{q^6}\times\int\frac{d\rho}{\rho^5}\;\frac{D(\rho)}{3}\,\frac{F^2(\rho q)}{g^2(\rho)}\,. \quad (3.31)$$

The factor $\frac{1}{3}$ arises from group averaging if the group is SU(2). For SU(3) replace $\frac{1}{3}$ by $\frac{1}{8}$.

If there is an upper cutoff ρ_c on ρ and $q\rho_c<1$, then this result is simplified to

$$D^{ab}_{\mu\nu}(q)_{\text{tunnel}}=(2\pi)^4\delta_{ab}\,\frac{\delta_{\mu\nu}-q_\mu q_\nu/q^2}{q^2}\times\int_0^{\rho_c}\frac{d\rho}{\rho}\;\frac{1}{g^2(\rho)}\;\frac{D(\rho)}{3}\,. \quad (3.32)$$

This is just a numerical multiple of the free propagator, and the result of adding the two effects together is to produce a finite wave-function renormalization

$$Z=1+\frac{\pi^2}{3}\int_0^{\rho_c}\frac{d\rho}{\rho}\left(\frac{8\pi^2}{g^2(\rho)}\right)D(\rho)\,. \quad (3.33)$$

Since the effect of Z on g is $g^2\to g^2Z$ this classical wave-function renormalization increases the effective coupling. Our previous result was

$$\frac{1}{g^2}\to\frac{1}{g^2}\left(1-\frac{\pi^2}{3}\int_0^{\rho_c}\frac{d\rho}{\rho}\,\frac{8\pi^2}{g^2(\rho)}\,D(\rho)\right)\,. \quad (3.34)$$

As long as the renormalization effect is small, the two results are identical. The previous calculation is in fact the more accurate. It was a self-consistent field type of calculation which takes into account instanton–anti-instanton correlations induced by the long-range R^{-4} interactions. Equation (3.27) does not take these correlations into account.

Finally, it is worth emphasizing that the instanton gas is a paramagnetic medium. It is magnetic rather than electric because a Euclidean gauge theory corresponds to static magnetism. It is paramagnetic because the coupling is renormalized upward.

IV. QUARK POTENTIAL

We now turn to the effect of vacuum tunneling on the interaction energy of infinitely massive test particles. As discussed in Sec. II the energy of a quark-antiquark pair separated by a distance R should be given by the Wilson loop in the form

$$e^{-E(R)T}\underset{T\to\infty}{=}\left\langle\left[\text{tr}\exp\left(i\oint dx\cdot A\right)\right]\right\rangle, \quad (4.1)$$

where $\oint$ is taken about the obvious rectangle of length T and width R, P is the symbol for path ordering, and the expectation is taken over gauge fields A with the usual Yang-Mills action as weight. This is of course the famous loop integral one uses as a confinement test, but in this form it is being used to pick out the potential energy of static spinless sources at separation R. This, or some variant of it, is directly relevant to the spectrum of charmonium. Our interest in it, at this stage, is mainly as an indication of how the instanton modifications to vacuum structure have a large and qualitatively important effect on physics.

Our expectation is that $E(R)$ can be computed

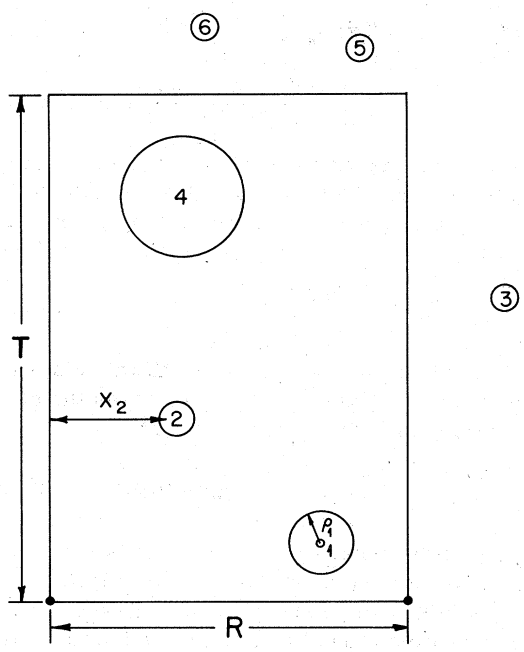

FIG. 6. The contribution of a given instanton–anti-instanton configuration on the quark loop.

reliably in the small R limit since fluctuations in A on a scale large compared to R will cancel out between quark and antiquark. By taking R small enough we can then arrange the loop integral to be sensitive only to the low-density, small-scale-size part of the instanton gas, where calculations can be done reliably.

If this calculation is done in the naive perturbation theory vacuum one will of course find

$$E(R) = -\frac{4}{3}\frac{1}{R}\;\frac{g^2(R)}{4\pi}.$$

The effect is entirely due to one-gluon exchange with a coupling constant varying according to asymptotic freedom. Note that this energy is explicitly $O(g^2)$. But even at small distances the vacuum has nontrivial structure due to vacuum tunneling. The associated vector potentials are $O(1)$ and can have a large effect on $E(R)$ even though the density of small-scale instantons is low.

The computation of this loop integral is in general a rather difficult business, but it simplifies considerably in the dilute-gas approximation. The situation is as illustrated in Fig. 6. In our by now familiar approximation the gauge field is taken to be the sum of the individual instanton gauge fields. Individual instantons are in singular gauge, so that only those that are within their scale size of one or both quarks will influence the loop integral. Since the loop is long in the time direction there is a time-ordered series of instantons (labeled in the figure) which have an effect on the loop. Since the instanton fields fall rapidly at infinity the instantons may be considered in isolation in their local effect on the loop.

Thus the entire loop integral may be written, in this approximation, as

$$\text{Loop} = \text{tr}(U_1^{(+)}U_2^{(+)}\cdots U_n^{(+)}U_n^{(-)}U_{n-1}^{(-)}\cdots U_2^{(-)}U_1^{(-)}) \tag{4.2}$$

where the U_i's are ordered line integrals,

$$P\exp\left(i\oint dx\cdot A^{(i)}\right),$$

associated with individual instantons. For $U^{(+)}$ ($U^{(-)}$) the integral is taken over the increasing (decreasing) time portion of the loop. Since instantons are widely separated and their A's fall rapidly, we assume that we can truly extend the upper and lower limits on the line integrals to infinity.

This expression simplifies when we average over the gauge orientation degree of freedom of the instantons. To average we replace $U_i^{(+)}$ by $g_i U_i^{(+)} g_i^{-1}$ and independently integrate each g_i over the properly normalized group measure. We immediately see that under this sort of averaging,

$$U_n^{(+)}U_n^{(-)} \to \frac{1}{n}\,\text{tr}(U_n^{(+)}U_n^{(-)}) \tag{4.3}$$

[where n is the n of SU(n)]. But then if the averaging is done in reverse order, from n down to 1, the entire loop integral collapses to a product of traces,

$$\text{Loop} \to \prod_{i=1}^{n}\left[\frac{1}{n}\,\text{tr}(U_i^{(+)}U_i^{(-)})\right]. \tag{4.4}$$

So, in the dilute-gas approximation, each instanton contributes a term

$$\delta S_i = \ln\frac{1}{n}[\text{tr}(U_{x_i}^{(+)}U_{x_i}^{(-)})] \tag{4.5}$$

to the action. We label U by the position, x_i, of the instanton to recall that U depends on the remaining free parameters of the instanton. Of course, U depends on the *spatial* position (and scale size) of the instanton because the quark loop has been chosen to be invariant under time translation. δS_i of course vanishes when x_i is far from the loop.

Since the instantons are now decoupled, the sum over multiple-instanton contributions to the loop integral simply exponentiates the single-instanton contribution and we have

$$-E(R) = \int d^3x\,\frac{d\rho}{\rho^5}\,D(\rho)\,\frac{1}{n}\,\text{tr}(U_x^{+}U_x^{-} - 1)\,, \tag{4.6}$$

where $D(\rho)$ is the instanton density function [Eq.

(3.5)] and the formula is strictly correct only so long as E is small (density is low). Note that if $R=0$ (the quark and antiquark are at the same place) the paths used to evaluate U^+ and U^- are identical, but traversed in opposite senses so that $U^+U^-=1$. Hence $E(0)=0$ as it should. One can easily convince oneself that the first term in an expansion of $E(R)$ in powers of R is $O(R^2)$ so that the first-order effect of vacuum tunneling on nearby quarks is to provide a harmonic-oscillator potential.

The strength of this potential is easy to evaluate. U is supposed to be evaluated using the singular gauge form of $A_\mu^{\text{instanton}}$. On the other hand, $\text{tr}(U^{(+)}U^{(-)})$ is identically the ordered loop integral for a single instanton and is gauge invariant. It is therefore legal to use the nonsingular gauge form of $A_\mu^{\text{instanton}}$ in evaluating the trace (a more handy gauge for computing). An easy calculation gives

$$U^{\pm}=\exp[\pm i\pi\vec{\tau}\cdot\vec{x}/(x^2+\rho^2)^{1/2}], \tag{4.7}$$

where $\vec{x}$ is the three-vector from the instanton center to the quark (antiquark) position and ρ is the instanton scale size. Simple algebra then shows that

$$\int d\hat{x}\,\text{tr}(U^+U^- -1)=-R^2\left[\frac{\pi^2}{3}\,\frac{\rho^4}{(x^2+\rho^2)^3}+\frac{2}{3x^2}\sin^2\pi\,\frac{x}{(x^2+\rho^2)^{1/2}}\right]+O(R^4), \tag{4.8}$$

where R is the quark-antiquark separation. The sign corresponds to attraction.

Doing the three-dimensional x integration and summing over instanton and anti-instanton contributions gives our final dilute-gas expression for the quark-antiquark potential:

$$E(R)=(33.8)R^2\int_0^{\rho_c}\frac{d\rho}{\rho}\,\frac{1}{\rho^3}\,\frac{1}{n}\,D_n(\rho)\,, \tag{4.9}$$

where D_n is the instanton density function appropriate to SU(n). Now the upper limit on the ρ integration is surely determined by the confinement scale of the theory. We have argued that this scale corresponds to $8\pi^2/\bar{g}^2(\rho_c)\sim 10$ and we will provisionally adopt ρ_c as the upper cutoff in the integral for $E(R)$. However, since $\rho^{-3}D(\rho)$ is a reasonably rapidly decreasing function of ρ [for both SU(2) and SU(3)] for $\rho<\rho_c$, it is apparent that the integral is dominated by the scale sizes near the upper cutoff, and the precise way in which the cutoff is imposed by the physics of confinement becomes important. Nevertheless, our crude evaluation should be good for order-of-magnitude purposes.

If we define a hadronic mass scale by $\mu_h\sim 1/\rho_c$ we can express the unphysical μ in terms of μ_h and evaluate E in terms of physical quantities. For purposes of comparison we consider two cases: (a) SU(2) no flavors and (b) SU(3) three flavors. The results are (a) $E(R)=0.46\,\mu_h\,(\mu_h R)^2$, (b) $E(R)=14.5\mu_h\,(\mu_h R)^2$. Most of the contribution to the integration over scale sizes comes from the range between ρ_c and $\rho_c/2$. The corresponding range in $x=8\pi^2/\bar{g}^2$ is from 10 to ~15, with the integral falling reasonably slowly over this range.

What are the salient features of this result? First, the SU(3) potential is much larger than the SU(2) potential. This is largely due to the fact that for SU(2) the cutoff $x_c=10$ is well within the dilute gas regime (integrated instanton volume = 1 at $x=6.5$) while for SU(3) $x_c=10$ is just about at the limit of the dilute gas regime. Had we extended the SU(2) integral to smaller x we would have gotten a much larger result. What is most interesting is the *size* of our result—the potential energy is of typical strong-interaction magnitude for [at least in the SU(3) case] R's small compared to hadronic size. Further, sizable contributions to this energy come from instantons whose effective coupling constant and density are very small. Once again, the dilute instanton gas produces large effects, just through the vacuum tunneling corrections to the vacuum, even when the coupling constant is small enough that perturbation theory about the naive vacuum would appear to be accurate. Although we will not pursue the matter here, it would appear that this interaction is of phenomenological interest, although when E becomes large, one must do the calculation a bit better.

In this calculation we have ignored a key feature of realistic gauge theories: light quarks. If all quarks had masses larger than μ_{had}, say, then the calculation we have done would be correct. However, we know that massless quarks suppress vacuum tunneling entirely and that light quarks (on the scale of ρ^{-1}) reduce the amplitude for tunneling on a scale ρ. Since there are at least two flavors of quark whose masses are light compared to μ_{had} we must expect $E(R)$ to be considerably reduced. A further, and potentially crucial, complication is that if the bare masses of the two light quarks are zero (or nearly zero) and receive their physical mass by spontaneous symmetry breakdown, then one will expect the effective quark mass to have a scale dependence (at short distances symmetry breaking goes away). We will see shortly that dynamical symmetry breaking occurs at a scale size not too far from ρ_c, the confinement scale. Lesson: Instanton effects are so large that the suppression of their effects by light quarks is crucial to even qualitative understanding of the dynamics.

Finally, we consider briefly the question whether instantons by themselves can confine quarks. To study this we redo the loop calculation, taking R to infinity. Everything goes through as before except that in Eq. (4.6) we may take $U^-=-1$ (or $U^+=-1$) since a single instanton will overlap only the quark or the antiquark. Indeed, the net effect is just a renormalization of the quark mass which turns out to be

$$\Delta E=\left\{64\pi\int_0^\infty du\,u^2\cos^2\left[\frac{\pi}{2}\frac{u}{(1+u^2)^{1/2}}\right]\right\}\times\left[\int_0^{\rho_c}\frac{d\rho}{\rho^2}\frac{1}{n}D_n(\rho)\right]=111\int_0^{\rho_c}\frac{d\rho}{\rho^2}\frac{1}{n}D_n(\rho)\,. \quad (4.10)$$

This of course shows no sign of diverging, no matter what the upper cutoff on ρ. The next term in the large-R expansion of $E(R)$ is proportional to R^{-1} and should be interpreted as a coupling-constant renormalization. An easy calculation shows that

$$\delta\left(\frac{g^2}{8\pi^2}\right)=\frac{\pi^2}{2}\int_0^{\rho_c}\frac{d\rho}{\rho}D_n(\rho)\,, \quad (4.11)$$

a result which coincides with Eq. (3.24) as long as the total coupling constant is small. Once again we see that the effect of instantons is to increase the coupling.

V. MASSLESS FERMIONS AND CHIRAL-SYMMETRY BREAKING

As was realized at the very beginning of the study of the physics of instantons, massless fermions play a very special role, converting the pure vacuum tunneling of the nonfermion theory into combined vacuum tunneling plus emission of one chiral quark-antiquark pair for each flavor of fermion. (In this discussion a fermion is for all practical purposes massless if its mass is small compared to the inverse scale size of the instanton.) This is what underlies the solution of the U(1) problem. In a suitable approximation (to be explained below) an instanton of size ρ provides a nonlocal effective interaction between quarks of the general form

$$\prod_{i=1}^{N}\left[\bar\Psi_i(1-\gamma_5)\Psi_i\right]+(\gamma_5\longrightarrow-\gamma_5)\,, \quad (5.1)$$

where N is the number of massless flavors (the color index as well as the normalization have been suppressed). This vertex is invariant under SU(N) $\times$ SU(N) flavor transformations but *not* under axial U(1), unlike the original Lagrangian of QCD. Since the effective interaction is nonlocal, it includes a short-distance cutoff making it renormalizable. Its numerical magnitude may be estimated in the context of the dilute-gas approximation in much the same way as we evaluated the static quark-antiquark potential. It will turn out that it is of standard strong-interaction magnitude and can be expected to provide a large mass to any would-be η' Goldstone boson.

This effective interaction, however, does not directly break chiral symmetry. Thus in the absence of dynamical symmetry breaking tunneling is suppressed and the evaluation of Green's functions of chirally invariant operators (i.e., that commute with Q_5) will only receive contributions from sectors with net topological quantum number zero. In effect instantons and anti-instantons will be bound together, and have little effect on such Green's functions. To be more specific, consider the one-instanton contribution in the presence of a light quark, whose mass is m. The normalization of the tunneling amplitude is determined by the zero-point fluctuations about the classical solution. The additional term that arises due to the fermions is simply the determinant of the operator $\not{D}_A\equiv(i\not\partial+m-g\not{A}_{cl})$. As 't Hooft discovered,[13] the Dirac equation $\not{D}_A\Psi=E\Psi$ possesses a zero energy normalizable solution, $\Psi_0(x)$, when $m=0$, resulting in $D(0)=\det[\not{D}_A]\,|_{m=0}=0$. This is simply the manifestation, in the path-integral formulation, of the suppression of tunneling. For a light fermion the lowest eigenvalue no longer vanishes, but is proportional to m. Thus

$$D(m)=\det[\not{D}_A]_{m\approx0}\simeq m\rho\,, \quad (5.2)$$

where ρ is the scale size of the instanton. Thus the contribution of an instanton is severely suppressed unless its size is larger than $1/m$. When $m\rho$ is large the fermion determinant, after removing the contribution to the charge renormalization, approaches unity (this is simply a consequence of the decoupling theorem for heavy fermions).

In QCD we believe that at least two of the quarks have very small (bare) mass parameters. Thus in the absence of dynamical mass generation for these quarks instantons of sizes less than $1/m_{up}$ would be totally suppressed. However, if by virtue of the dynamical symmetry breaking of chiral SU(N) the quarks acquire a dynamical mass then instantons of smaller size could be important.

In the above discussion the "quark mass" does not refer to the position of a pole in the quark propagator, but rather to the, in general, momentum-dependent piece of the propagator which commutes with γ_5. Thus we might define a quark mass to be $m(p)=\mathrm{Tr}S^{-1}(p)$. In general this will consist of a true mass term $m_0(p,g)$ and a dynamical mass term $m_D(p,g)$. For weak coupling $m_0(g) \sim m_0$ (m_0 is the renormalized mass parameter in the Lagrangian) and $m_D(g) \sim \exp(-1/g^2)$. The momentum dependence of $m_0(p)$ is calculable from the renormalization group for large p: $m_0(p/\mu) \sim m_0(\ln(p/\mu))^{-a}$ (a is determined by the anomalous dimension of $\bar{\Psi}\Psi$). On the other hand, $m_D(p)$ will behave as $m_D(p) \sim (p^2/\mu^2)^{-G}$ (G depends on the mechanism for dynamical mass generation). In Eq. (5.2) the mass parameter must be evaluated at momenta of order $1/\rho$, i.e.,

$$D(m) \approx \rho\left[m_0\left(\frac{1}{\rho\mu}\right) + m_D\left(\frac{1}{\rho\mu}\right)\right].$$

Thus at very short distances, where only small instantons are relevant, we will see the bare mass which will suppress tunneling. If, however, dynamical symmetry breaking occurs, a dynamical mass will turn on and restore the tunneling at distances d, such that $dm(1/d\mu) \approx 1$. For our program to succeed it is necessary that $d \lesssim \rho_c$.

To illustrate the above picture let us consider the case of a single massless quark (1 flavor). Here the original flavor symmetry of the Lagrangian is $U(1)\times U(1)$, and this is broken down to $U(1)$ by the transition to the θ vacuum. The quark acquires a mass directly by virtue of the nonvanishing θ-vacuum expectation value of $\bar{\Psi}\Psi$. In the presence of an instanton of given size ρ at x_I

$$\langle \bar{\Psi}(x)\Psi(y)\rangle = \bar{\Psi}_0(x - x_I, \rho)\Psi_0(y - x_I, \rho), \qquad (5.3)$$

where $\Psi_0(x,\rho)$ is the normalized zero energy mode of the massless fermion in the instanton field[21]:

$$\Psi_0(x,\rho) = \frac{\rho^{3/2}}{(x^2+\rho^2)^{3/2}}\left(\frac{2}{\pi^2}\right)^{1/2} u. \qquad (5.4)$$

Upon integrating over instanton position and size this yields an effective mass term which is momentum dependent:

$$m(p) = 32\pi^2 \int \frac{d\rho}{\rho} D(\rho)\frac{1}{\rho}e^{-2p\rho}. \qquad (5.5)$$

In effect this mass is generated by considering the fermion propagating in the background instanton field in the dilute gas approximation and only including the modification of the propagator due to the zero-energy mode.

Since one must integrate over all scale sizes, ρ, in Eq. (5.5) $m(p)$ cannot be reliably calculated for all p. If p is large enough the integral over scale sizes will be effectively cut off at $\rho \sim 1/2P$ and thus insensitive to the infrared behavior. As discussed above the instantons interact strongly when $\rho \sim \rho_c \sim 0.2\mu^{-1}$. Thus we can trust the dilute-gas approximation to calculate $m(p)$ only for $p \gtrsim 2.5\mu$. In the case of SU(3) we find from Eq. (5.5)

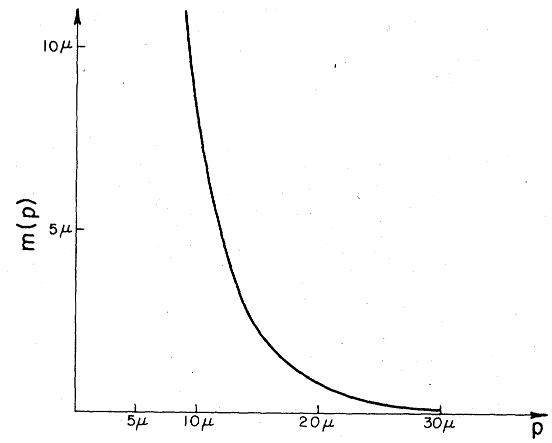

FIG. 7. The p dependence of $m(p)$, in units of the renormalization scale parameter, in the case of one flavor. The gauge group is SU(3).

that $m(p)$ becomes substantial for $p \gtrsim 20\mu$ (see Fig. 7). In fact $m(20\mu) = 0.8\mu$, $m(15\mu) = 2.3\mu$, $m(10\mu) = 8\mu$, and $m(5\mu) = 33\mu$. Thus the quark mass "turns on" at very short distances (compared to $0.2\mu^{-1}$) where our calculation is reliable. This means that instantons will undergo a phase transition—and be liberated once their scale size is roughly $\frac{1}{15}\mu^{-1}$. This transition is certainly in the dilute-gas region and occurs well before the screening effects become substantial.

Unfortunately, it is much more difficult to calculate the quark mass when there are two or more massless quarks, since chiral symmetry is not completely broken by the θ vacuum and additional dynamical symmetry breaking is required. We have suggested that the source of such dynamical symmetry breaking might be the effective determinantal interaction between left-handed and right-handed quarks.[14] Indeed given such an interaction one might attempt to solve self-consistent integral equations for the quark propagator in a chirally asymmetric vacuum. The simplest of such equations (in the case of two massless quarks) is shown in Fig. 8 where the left-hand side is the quark mass operator, the internal line on the right-hand side is the full fermion propagator including the mass operator, and the vertex is the instanton-generated effective fermion interaction.

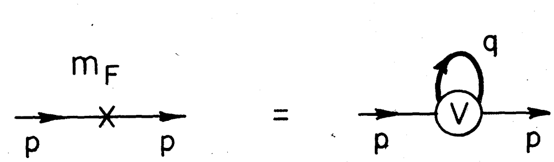

FIG. 8. Graphical expression of the integral equation for the quark mass in the case of two flavors.

The resulting equation is

$$m_F(p)=\int \frac{d^4q}{(2\pi)^4}\frac{m_F(q)}{q^2+m_F{}^2(q^2)}[V(p,q)]\,,$$
$$V(p,q)=\int \frac{d\rho}{\rho}D(\rho)V(\rho,p,q)\,, \tag{5.6}$$

where $V(\rho;p,q)$ is the effective vertex generated by an instanton of size ρ.

In an analogous two-dimensional model we were able to show that such an equation did indeed generate a quark mass,[18] and by exploiting the correspondence between the gas of instantons and the two-dimensional Coulomb gas we were able to verify this result. There we were able to proceed with confidence by adjusting the coupling constant to be arbitrarily small and relying on the infrared instability of the two-dimensional theory to generate the mass. In QCD we have no such adjustable parameters. Furthermore, Eq. (5.6) is not infrared unstable, namely the right-hand side does not tend to diverge as $m_F\to 0$. Thus the existence of a self-consistent solution depends on the magnitude and structure of $V(p,q)$.

The evaluation of $V(p,q)$, however, requires controlling instantons of arbitrary size, and thus in general will be sensitive to the nature of the cutoff on instanton sizes. Since $V(\rho,p,q)$ behaves as $\exp[-2\rho(p+q)]$ one could contemplate using Eq. (5.6) to determine the form of $m_F(p)$ for large p within the framework of the dilute gas approximation, but to actually prove the existence of a self-consistently generated quark mass requires detailed knowledge of $V(\rho,p,q)$ for arbitrarily large ρ. Caldi[26] has investigated Eq. (5.6) using the dilute gas approximation and approximating the kernel by the contribution of an instanton of a given size. He concludes that the interaction is strong enough to generate a quark mass.

In lieu of attempting to construct the quark propagator in the true chirally asymmetric ground state we shall investigate the stability of the chirally symmetric θ vacuum, using as a probe an appropriately chosen Green's function which can be reliably calculated for large momentum. Such a quantitiy is the $\sigma=\sum_{\alpha,i}\overline{\Psi}_{\alpha i}\Gamma_{ij}\Psi_{\alpha j}$ propagator (α = color, i = flavor),

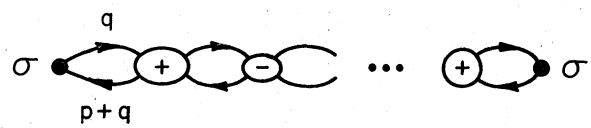

FIG. 9. The structure of the diagrams that produce a tachyon in the σ channel. The + (−) blobs refer to the effective determinantal four-fermion interaction induced by instantons (anti-instantons).

$$D_\sigma(p)=\int \frac{d^4x}{(2\pi)^4}\langle 0|T(\sigma(x)\sigma(0))|0\rangle e^{ipx}\,.$$

The instability of the vacuum will be signaled by the divergence of the σ propagator at positive (Euclidean) momentum squared. This corresponds to a tachyon in the σ channel, leads to the breakdown of cluster decomposition in the θ vacuum, and implies that the chirally symmetric state about which the theory has been expanded is not the state of lowest energy. It strongly suggests that the true vacuum is one with a nonvanishing expectation value of σ.

Now for large enough p, $D_\sigma(p)$ can be reliably calculated, with the corrections to its free-field structure being generated by ordinary perturbation theory plus the effects of instantons in the dilute-gas approximation. In particular, we focus on the effective four-fermion interaction induced by vacuum tunneling, which leads to diagrams such as those displayed in Fig. 9. Large external momentum (compared to the inverse confinement scale) provides a scale size cutoff, restricting us to a regime where the effective coupling is weak and the instanton density is low. As p is decreased we shall find that the instanton interaction increases in magnitude and becomes large enough (for $p\approx 6.5\mu$) to generate a tachyon pole in a region where the calculation is still reliable. This we argue demonstrates that, as in the case of one massless quark, the instantons are the source of chiral-symmetry breaking and that this occurs at a distance short compared to the confinement scale.

We may proceed to a description of the calculation itself. The key ingredient is the instanton four-fermion vertex computed by 't Hooft,[21]

$$\langle\theta|\overline{\Psi}^a_{\alpha i}(w)\Psi^b_{\beta j}(x)\overline{\Psi}^c_{\gamma k}(y)\Psi^d_{\delta l}(z)|\theta\rangle=(\delta_{ab}\delta_{cd}-\delta_{ac}\delta_{bd})\frac{1}{6}\left[(2\delta_{\alpha\beta}\delta_{\gamma\delta}-\delta_{\beta\gamma}\delta_{\alpha\delta})\left(\frac{1-\gamma_5}{2}\right)_{ij}\left(\frac{1-\gamma_5}{2}\right)_{kl}+\begin{pmatrix}\beta\to\delta\\ j\to l\end{pmatrix}\right]$$
$$\times\Psi_0(w)\Psi_0(x)\Psi_0(y)\Psi_0(z)\,, \tag{5.7}$$

where $w,\ldots$ are the locations of the fermion sources (we take the instanton to be centered at the origin), $a,\ldots$ are flavor indices, $\alpha,\ldots$ are color indices, $i,\ldots$ are Dirac indices, and

$$\Psi_0(x)=\left(\frac{2}{\pi^2}\right)^{1/2}\frac{\rho^{3/2}}{(x^2+\rho^2)^{3/2}}$$

is the normalized zero-energy eigenfunction (ρ

is the instanton scale size). As explained by 't Hooft, this simple form is achieved by averaging over the gauge orientation of the instanton.[21] For anti-instantons, replace $(1-\gamma_5)$ by $(1+\gamma_5)$.

To compute the σ propagator we must chain these vertices together as in Fig. 9. In order not to overcount fermion propagators, we must put an inverse massless propagator between each pair of fermion lines which is joined together (Γ is an improper vertex). The resulting γ_5 structure requires instantons to alternate with anti-instantons in the chain.

The explicit flavor and γ_5 structure of the vertices allows one to say something about the corresponding structure of the "σ" propagator. The U(2)⊗U(2) relatives of $\sigma=\sum\overline{\Psi}_i\Psi_i$ are $\eta=\sum\overline{\Psi}_i\gamma_5\Psi_i$, $\pi^a=\sum\overline{\Psi}_i\tau^a_{ij}\Psi_j$, and $\pi^a_s=\sum\overline{\Psi}_i\tau^a_{ij}\gamma_5\Psi_j$, and the same graphical series as in Fig. 9, with appropriate modifications of the terminating vertices, defines their propagators. An $I=1$ projection gives an extra minus for *each* instanton relative to the $I=0$ projection and for γ_5 propagators there is an extra minus sign for terms having an *even* number of instanton vertices. The net result of this is that if the σ propagator is written $A^{(+)}=\sum_{i=1}^{\infty}A_i$, where A_i is the contribution from a chain of i instantons, then the η propagator is written $A^{(-)}=\sum_{i=1}^{\infty}(-)^iA_i$. The general propagator has the symbolic form $A^{(+)}(\sigma^2+\vec{\pi}^2)+A^{(-)}(\eta^2+\vec{\pi}_s{}^2)$. In other words, the σ and $\vec{\pi}$ propagators are identical, as required by SU(2)⊗SU(2) invariance, while the σ and η propagators are *not* the same, as would have been required by U(2)⊗U(2) invariance. The breaking of chiral U(1) comes precisely from terms with instanton number not equal to zero, as expected. This is just another manifestation of the breaking of chiral U(1) by vacuum tunneling.

The signal for vacuum instability will be a singularity in either $A^{(+)}$ or $A^{(-)}$ at some positive (Euclidean) p^2. Whether it occurs in $A^{(+)}$ or $A^{(-)}$ is not significant because the underlying theory is massless and we may freely interchange our definition of η and σ. Because these functions are constructed as a geometric series, they obey an integral equation with a kernel built out of the instanton vertex. The kernel has an eigenvalue spectrum (depending on the momentum, p, flowing through the propagator) and the geometric series will first diverge when the largest eigenvalue of the kernel passes through 1. Our calculation will be trustworthy if this catastrophe happens for sufficiently large p^2.

In computing the bubble chain there are three types of integration to do: over the location of each instanton, over the loop momentum in each fermion loop joining two instantons, and over instanton scale sizes. The first simply establishes momentum conservation at each vertex; the second would be the usual trivial fermion-loop integral but for the structure the vertex possesses through the fermion zero energy eigenfunctions—it is in any event an explicit momentum-space integral which we will display; the third is what makes the problem nontrivial—if instantons came in one scale size there would be no integral equation to study, just an explicit geometrical series in an explicit bubble function. We of course must supply the scale-size-dependent instanton density function for each scale-size integration, and we will see that the external momentum provides the desired large scale-size cutoff.

With this preamble we are ready to write down the kernel of our integral equation. The kernel acts on the scale-size variable, ρ, and has the external momentum, p, as a parameter. Its explicit form is

$$K_p(\rho,\rho')=\left(\frac{D(\rho)}{\rho^5}\right)^{1/2}\left(\frac{D(\rho')}{\rho'^5}\right)^{1/2}\left[\left(\frac{2}{\pi^2}\right)^{1/2}4\pi^2\rho^{3/2}\right]^2\left[\left(\frac{2}{\pi^2}\right)^{1/2}4\pi^2\rho'^{3/2}\right]^2\int\frac{d^4q}{(2\pi)^4}e^{-(\rho+\rho')(|q|+|p-q|)}\frac{q\cdot(p-q)}{q^2(p-q)^2}, \tag{5.8}$$

where $D(\rho)$ is the appropriate scale-size density function and $D(\rho)$ appears with a square root since each instanton belongs to two loops. The d^4q integration is the fermion-loop integration. The factors of π, $\rho^{3/2}$, $e^{-\rho|q|}$, and $e^{-\rho|p-q|}$ arise from the Fourier transforms of the zero-energy fermion wave functions in the instanton vertices. The exponential factors provide the large scale-size cutoff mentioned before. The eigenvalue equation is just

$$\epsilon\Psi(\rho)=\int d\rho' K_p(\rho,\rho')\Psi(\rho'). \tag{5.9}$$

Because of the *small* scale-size cutoff built into $D(\rho)$, it is reasonably easy to see that, because of the exponentials, for large p the eigenvalues are all small. Our problem is to find when, as we reduce p, the largest eigenvalue first crosses 1.

The momentum-space integration can be partially performed yielding

$$K_p(\rho,\rho')=32\pi^2\frac{[\rho D(\rho)]^{1/2}[\rho' D(\rho')]^{1/2}}{(\rho+\rho')^2}\times F((\rho+\rho')p), \tag{5.10}$$

where

$$F(p)=\int_1^\infty dy\, e^{-py}p^2 \times\frac{(y^2-1)^{1/2}[3y^2-4-y(y^2-1)^{1/2}]}{y[y+(y^2-1)^{1/2}]}$$
$$=\begin{cases}1, & p\to 0\\ -(\pi p/2)^{1/2}e^{-p}, & p\to\infty .\end{cases} \qquad (5.11)$$

Now we will be interested in $F(p)$ for values of $p\approx 1$, and we find that for these values F is approximately given by (to $\approx 5\%$)

$$F((\rho+\rho')p)\approx e^{-(\rho+\rho')p} \quad \text{for } 0\le p(\rho+\rho')\le 2 . \qquad (5.12)$$

This again makes (as in the case of one massless quark) the large scale cutoff induced by the external momentum very explicit.

The only place where the nature of the gauge group enters is in determining $D(\rho)$. If we study SU(2) color with two light flavors then for small scale sizes

$$D(\rho)=0.26\left(\frac{8\pi^2}{\bar g^2(\rho)}\right)^4\exp\left(-\frac{8\pi^2}{\bar g^2(\rho)}\right)$$

and

$$\frac{8\pi^2}{\bar g^2(\rho)}=6\ln\left(\frac{1}{\mu\rho}\right)$$

so that

$$K_p(\rho,\rho')_{SU(2)}=(0.26)(32\pi^2)6^4\mu^6\,\frac{(\rho\rho')^{3.5}}{(\rho+\rho')^2}\left(\ln\frac{1}{\mu\rho}\ln\frac{1}{\mu\rho'}\right)^2 e^{-p(\rho+\rho')} . \qquad (5.13)$$

In the case of SU(3) color and two massless flavors

$$D(\rho)=0.1\left(\frac{8\pi^2}{\bar g^2(\rho)}\right)^6\exp\left(\frac{-8\pi^2}{\bar g^2(\rho)}\right) \text{ and } \frac{8\pi^2}{\bar g^2(\rho)}=\tfrac{29}{3}\ln\left(\frac{1}{\mu\rho}\right),$$

so that

$$K_p(\rho,\rho')_{SU(3)}=0.1(32\pi^2)(\tfrac{29}{3})^6\mu^8\,\frac{(\rho\rho')^{4.5}}{(\rho+\rho')^2}\left(\ln\frac{1}{\mu\rho}\ln\frac{1}{\mu\rho'}\right)^3 e^{-p(\rho+\rho')} . \qquad (5.14)$$

To find the maximum eigenvalue of K_p we will attempt to maximize its expectation value in a normalized wave function $\Psi(\rho)$. It is convenient to rescale ρ by p and define $x=\rho p$ and $\phi(x)=\sqrt{p}\,\Psi(\rho)$ so that ϕ is normalized with respect to the variable x. The expectation of K is then, for an SU(2) gauge group,

$$\langle\phi|K_p|\phi\rangle=(106421)\left(\frac{\mu}{p}\right)^6\left(\ln\frac{\mu}{p}\right)^4\int_0^\infty dx\,dx'\,\frac{\{\phi(x)x^{3.5}e^{-x}[1+(\ln x)/\ln(\mu/p)]^2\}\{x\to x'\}}{(x+x')^2}$$
$$=\left(\frac{6.9\mu}{p}\right)^6\left(\ln\frac{\mu}{p}\right)^4\int dx\,dx'\phi(x)L_p(x,x')\phi(x') , \qquad (5.15)$$

where $\int dx\,\phi^2=1$. The salient feature of this expression is that the explicit p dependence out front is very rapid while the kernel L_p is rather slowly varying with p. Evidently the critical value of p will be $\sim 6.9\mu$: For p large compared to this, the factor of $(6.9\mu/p)^6$ will drive the maximum eigenvalue to zero, and vice versa for p small compared to 6.9μ. The kernel L_p is rather well behaved and a numerical exploration of this region reveals that the maximum eigenvalue crosses 1 at $p/\mu=5.5$ and the corresponding eigenfunction is reasonably well approximated by $\phi(x)=\theta(x-0.75)\theta(1.75-x)$. Thus the instability arises from instantons whose scale sizes range from $0.14\mu^{-1}$ to $0.32\mu^{-1}$ and the corresponding couplings range from $8\pi^2/g^2=12.0$ to $8\pi^2/g^2=6.9$.

If the gauge group is SU(3) instead of SU(2) the analog of Eq. (5.15) is

$$(\phi,K_p\phi)=\left(\frac{5.84}{p}\right)^{29/3}\left(\ln\frac{p}{\mu}\right)^6 \times\int dx\,dx'\phi(x)L_p(x,x')\phi(x') . \qquad (5.16)$$

Analysis of this equation shows that the maximum eigenvalue crosses 1 at $p\simeq 9.2\mu$ with the corresponding eigenfunction being approximated by $\phi(x)=\phi(x-1.25)\theta(2.25-x)$. Therefore the instantons responsible for the instability range in scale size from 0.14μ to 0.26μ. The corresponding coupling constants range from $8\pi^2/g^2=19$ to 13. In this range the dilute-gas approximation we have employed is reliable.

We therefore conclude that the chirally symmetric θ vacuum is unstable, that in the true vacuum σ will have a nonvanishing vacuum expectation value, and that the quark will possess a dynamical mass. Furthermore, the instability arises at rather short distances, so that we might expect the quark mass to turn on rapidly at rather large momenta, leading to the liberation of the tightly bound instanton-anti-instanton pairs. Clearly additional investigation is required to gain a quantitative control over this phase transition. However, to a first approximation we would imagine that the net effect in calculating Green's functions of chirally invariant operators is that instantons of sizes less than $\rho_A \approx 0.25\mu^{-1}$ or $x_A \sim 15$ are suppressed and instantons of larger sizes are not.

If one goes back now to reevaluate the interaction between instantons as discussed in Sec. III one will find that the screening effects are diminished, since until we get to $x = x_A$ there are no free instantons. Similarly, in the presence of massless fermions, merons will be liberated at a slightly larger scale size, or larger x. Also, the instanton corrections to the quark-antiquark potential will be postponed to distances greater than ρ_A.

Note that because massless fermions suppress the effects of instantons at scale sizes less than ρ_A our calculation of the σ propagator is highly reliable. We need not worry about the effect of instanton interactions in probing for chiral-symmetry breaking, since until one gets to the distance at which the symmetry breaking occurs these interactions are small.

VI. A MECHANISM FOR CONFINEMENT

A. Introduction

We have seen how instantons can make the coupling $\bar{g}$ grow dramatically with distance leading to a strong q-$\bar{q}$ interaction. However, this does not necessarily imply strict confinement in the sense that isolated quarks do not exist. It may be that confinement can only be demonstrated by actually solving the theory, but one would hope that such an important property arises from a simple qualitative mechanism which can be seen at an elementary level. This section is devoted to the discussion of an effect of this kind. It is based on a new kind of field configuration in Euclidean space-time. To motivate the introduction of these configurations it is useful to review some work of Polyakov on (2+1)-dimensional models.[27]

Polyakov studied the Georgi-Glashow model in 2+1 dimensions. This model contains a triplet of heavy Higgs scalars, heavy charged vector bosons W^+ and W^-, and a massless photon. In Euclidean space-time the classical equations of motion for this model are identical to the static equations of motion for the Georgi-Glashow model in 3+1 dimensions. These equations possess a well-known instantonlike configuration: the Polyakov-'t Hooft monopole. Polyakov shows that these instantonlike monopoles confine charged particles in the (2+1)-dimensional theory. We will paraphrase his arguments below, but first it is important to understand that there is an essential difference between the monopole in 2+1 dimensions and the instanton in QCD. For both the monopole and the instanton the vector potential A^μ falls like r^{-1} at large distances. However, in the case of the monopole $F^{\mu\nu}$ falls with its dimensional power, r^{-2}, while for the QCD instanton $F^{\mu\nu}$ falls as r^{-4} and depends on an arbitrary scale size. This difference is fundamental. Because of the slow falloff of $F^{\mu\nu}$, the monopole cannot be interpreted as a vacuum tunneling event. (In fact, it can be shown topologically that n states and θ vacuums as defined in Sec. II do not exist in odd numbers of dimensions.) Conversely, Polyakov's confinement mechanism cannot be straightforwardly applied to instantons in QCD.

The chemical potential of a monopole or antimonopole in Polyakov's theory is $\sim e^{-1/g^2}$ (g is the dimensionless coupling constant). Thus by taking g small the monopole-antimonopole gas can be made as dilute as one likes. Because of the long-range (magnetic) Coulomb interaction between monopoles, the gas does not become noninteracting even in the limit of zero density. However, for sufficiently small density the parameters of the gas are such that the Deybe theory applies. The Euclidean functional integral then describes a Deybe plasma of monopoles and antimonopoles.

To see why charged particles are confined in this model consider the Wilson loop

$$e^{-E(R)T} = \langle \exp(i \oint A \cdot dx) \rangle \tag{6.1}$$

averaged over the vacuum ensemble of monopoles and antimonopoles. The character of this average can be seen as follows. The quantity $E(R)T$ is precisely the free energy of a steady electric current loop in a plasma of magnetic monopoles. (The i appears in the statistical mechanics formula because upon passing to imaginary time the interaction $\int A^\mu \dot{x}^\mu dt$ remains real.) By Ampere's law the current loop has to make a magnetic field, but a magnetic field cannot penetrate into a plasma of magnetic monopoles for the same reason that a static electric field cannot penetrate into an ordinary conductor. As shown in Fig. 10, this conflict will be resolved by the formation of a dipole sheet across the loop. The thickness of the layer will be essentially the Deybe length and when the size of the loop is large compared to the Deybe length the free energy will be proportional to the area. This leads to an energy $\epsilon(R)$

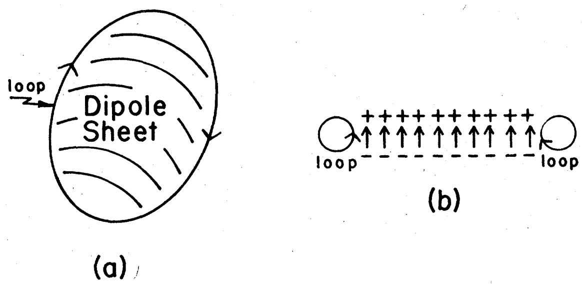

FIG. 10. (a) The Wilson loop with its dipole sheet. (b) A cross section of the sheet. The + (−) signs are monopoles (antimonopoles) and the directed arrows are the magnetic field.

$\sim R$ (but the reader should remember that this linear potential holds only for distances greater than the Deybe length $\sim e^{2/3g^2}$). Also for very large separations the interaction will not be just a static linear potential. It is relatively easy to stretch or bend a large dipole sheet. This will introduce new degrees of freedom and lead to a string-like interaction between charged particles.

The dipole sheet contains a magnetic field $H_k = \epsilon_{kij} F^{ij}$ normal to the sheet. Taking a time slice through the sheet one then finds an electric field $F^{0,\prime\prime} = E''$ lying in the sheet, which means that at a fixed time charged particles are connected by electric flux tubes; see Fig. 11. Thus in a fixed time picture the vacuum would appear to be a magnetic superconductor which expels electric flux. One can in fact verify from Polyakov's dispersion relation for H that the vacuum does in fact expel a static transverse electrical field. There are therefore two ways to look at Polyakov's model, either as an ordinary magnetic conductor in space-time or as a magnetic superconductor at a fixed time.

As far as confinement is concerned, the important property of the monopole is that $F^{\mu\nu}$ goes like r^{-2}, implying $A \sim r^{-1}$ with the r^{-1} term not a pure gauge. Objects with $F^{\mu\nu} \sim r^{-3}$, say, would correspond to permanent magnetic dipoles. A vacuum filled with them would act like a paramagnetic medium (analogous to the dilute in-

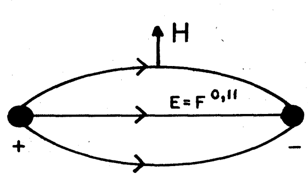

FIG. 11. The sheet at a fixed time showing that the confined Euclidean magnetic field is an electric flux tube in Minkowski language.

stanton gas in QCD) which just renormalizes the logarithmic Coulomb interaction. Still shorter range objects (r^{-n} with $n > 3$) affect the Wilson loop only when they are very close to it, and produce only a mass renormalization. A nontrivial A going like r^{-1} has the property that for a loop of (space-time) radius ρ the integral $\oint A \cdot dx$ will be of order unity as long as the source of the field (e.g. monopole) lies within ρ of the loop. In a plasma there are order ρ^3 such monopoles and if there were no correlations between monopoles the Wilson loop averaged over configurations would behave like $\exp[-(\text{const})\rho^3]$ corresponding, since ρ^3 should be interpreted as R^2T, to a quadratic potential. However, the long-range correlations that exist in a plasma reduce the potential down to linear.

Evidently, if we want to find an analog of Polyakov's mechanism in QCD we need to look for configurations where A goes like r^{-1} and is not a pure gauge. This requires that $F^{\mu\nu} \sim r^{-2}$ and there is an immediate problem: The action of such a configuration will be $\sim g^{-2} \ln(R^4)$, where R^4 is the volume of space-time. It would appear that such configurations cannot strictly exist in an infinite volume, but one has to remember that the entropy of position of such an object (assuming that in some sense it can be considered as localized) is proportional to R^4 and that the probability that it will appear somewhere is

$$R^4 e^{-\ln(R^4)/g^2} .$$

This vanishes as $R \to \infty$ if g is small, indicating that in the infinite volume limit such objects make no finite contribution to the functional integral. On the other hand, for larger g,

$$R^4 e^{-\ln(R^4)/g^2} \to \infty$$

as $R \to \infty$ and isolated configurations of this kind can contribute. This phenomenon is familiar in two-dimensional statistical mechanics and will be discussed in the QCD context below. For dimensions greater than four, $F^{\mu\nu}$ going like r^{-2} implies an action which grows like a power of R and cannot be overcome by entropy. Thus $d = 4$ is the critical dimension for confinement by Polyakov's mechanism.

There is a classical solution to the Euclidean Yang-Mills equations, due to De Alfaro, Fubini, and Furlan,[28] in which A_μ behaves like r^{-1} and $F^{\mu\nu}$ is nontrivial. It is

$$A_a^\mu(x) = \eta_{a\mu\nu} \frac{x^\nu}{x^2} . \tag{6.2}$$

This solution is singular at the origin as well as at infinity. However, since confinement is strictly a long-range problem, we may freely smear the field at short distances (details of the smearing will be given below). One then finds that the $\ln R$ term in the action is multiplied by $3\pi^2/g^2$ and

$$R^4 e^{-3\pi^2/g^2 \ln R}$$

ceases to vanish for large R when $g^2/8\pi^2 > \frac{3}{32}$. It appears therefore that $g^2/8\pi^2 = \frac{3}{32}$ is the critical coupling for the appearance of these objects as significant contributors to the functional integral. Later we will see what this means in terms of the running coupling of Eq. (3.3).

A configuration such as that in Eq. (6.2) will affect a Wilson loop of radius ρ whenever it is centered within ρ of the loop. In an uncorrelated gas of such objects the loop integral would then behave like $\exp(-\text{const} \times \rho^4)$ corresponding to an R^3 $q\bar{q}$ potential. However, as will be explicitly demonstrated below, there are logarithmic interactions between any two of these objects, and strong plasmalike correlations. Our guess, and this is purely conjectural, is that these correlations will produce some analog of Polyakov's dipole sheet and reduce the R^3 potential to a linear one.

Assuming for the moment that some sense can be made out of these rather peculiar objects, it is amusing to ask what their physical interpretation would be. Consider the wave function of the vacuum as discussed in Sec. II: In the gauge $A_0^a = 0$ we take it to be a function of the A_k^a at a fixed time. Taking a time slice of the configuration in Eq. (5.2) at $t=0$ one finds that the magnetic field $H_k^a = \frac{1}{2}\epsilon^{kij} F_a^{ij}$ (which depends only on the A_a^k at one time) is, up to a time-independent gauge rotation,

$$H_k^a = -\frac{x^k x^a}{|\vec{x}|^4} . \tag{6.3}$$

By a gauge transformation which rotates x^a into, say, the third axis H_a^k becomes

$$H_k^a = -\delta_{a3} \frac{x^k}{|\vec{x}|^3} , \tag{6.4}$$

which is just the field of an Abelian magnetic monopole. In particular, except for a lack of smoothing at the origin it is the monopole of the Georgi-Glashow model. Thus when the coupling is large enough so that the configurations in Eq. (6.2) appear in the Euclidean functional integral, the wave function of the vacuum will contain (color) magnetic monopoles (strictly speaking, well-separated monopole-antimonopole pairs; see below). A time slice of the configuration in Eq. (6.2) for $t<0$ ($t>0$) is not quite a monopole but is rather a configuration building up toward (decaying away from) a monopole. Mandelstam has argued that the presence of such objects in the vacuum wave function will lead to a magnetic superconducting state in which (color) electric flux is expelled.[29] Thus we may very well have electric flux tubes between quarks and in general a picture which is qualitatively almost identical to that of the (2+1)-dimensional model.

As was pointed out by De Alfaro *et al.*,[28] the solution in Eq. (6.2) has a half unit of topological

TABLE I. Confinement as a function of dimension. The question marks indicate either unknown or conjectured.

Dimension	$d=2$	$d=3$	$d=4$	$d\geq 5$
Space-time configuration	vortex	monopole	meron	
Naive $q\bar{q}$ potential	linear	quadratic	cubic	
Interaction between configurations	short range	Coulomb (r^{-1})	logarithmic	no confinement
$q\bar{q}$ potential including correlations	linear	linear	linear (?)	no confinement
Vacuum in space-time	?	magnetic conductor	color magnetic conductor (?)	no confinement
Fixed time configuration	?	vortex	monopole	no confinement
Vacuum wave function	?	magnetic superconductor	color magnetic superconductor (?)	

charge Q located at the origin. For this reason we have named these objects merons (from the Greek root $\mu\epsilon\rho o\sigma$ part).[30] Oddly enough, half-integral topological charge seems to be closely related to confinement in the Schwinger model. Following a suggestion of Rothe and Swieca,[31] Nielsen and Schroer[32] have pointed out that certain functional integrals which show confinement in the Schwinger model are dominated by the vortex configuration $A^\mu = \epsilon^{\mu\nu} x^\nu / |x|^2$ which has two-dimensional Abelian topological charge of $\frac{1}{2}$ and which they called a c instanton. The r^{-1} field of these vortices is a pure gauge and the interactions are short range. In fact confinement is essentially kinematic in two-dimensional gauge theories (the Coulomb potential is linear), and we are not sure how much to make of this. It is nonetheless amusing to note that a time slice through the origin of the $(2+1)$-dimensional monopole produces the vortex in the same way that a time slice through a meron yields a monopole. Thus the superconducting vacuum in Polyakov's model has vortices in its wave function.

Note that the confining objects, merons, monopoles, and vortices all come from the configuration in Eq. (6.2) taken in 4, 3, and 2 dimensions (in 3 and 2 dimensions it can be reduced to an Abelian configuration). The physics as a function of dimension is summarized in Table I, where it has been optimistically assumed that merons will confine for $d=4$.

There are clearly sufficient indications that merons are important to warrant a serious study of them. What follows should be considered only as a first step in this direction.

B. Meron kinematics

As mentioned earlier, the meron is characterized by having one half unit of topological charge concentrated at the origin and another half unit at infinity. The logarithmic singularity of the action integral comes precisely from this delta function concentration of topological charge. We will eliminate the singularities by replacing the meron by a different configuration in which the topological charge is spread out around the origin and infinity:

$$A^a_\mu = \begin{cases} \dfrac{2}{x^2+r^2}\,\eta_{a\mu\nu}x_\nu, & x<r \quad \text{I} \\[2ex] \dfrac{1}{x^2}\,\eta_{a\mu\nu}x_\nu, & r<x<R \quad \text{II} \\[2ex] \dfrac{2}{x^2+R^2}\,\eta_{a\mu\nu}x_\nu, & R<x \quad \text{III}. \end{cases} \tag{6.5}$$

Between the inner and outer spheres (whose radii are arbitrary) the field is identical to the meron field. At the inner (outer) radius it joins smoothly onto a standard instanton field whose scale size is chosen such that the net topological charge inside (outside) that radius is one-half unit. This field satisfies the equations of motion everywhere *except* on the two spheres. In fact, it is the solution of the equations of motion under the constraint that there be one-half unit of topological charge both in the inner and outer spheres. None of our qualitative arguments depended on the pointlike distribution of topological charge—all that mattered was the existence of a region where A_μ fell like $1/x$ and was not pure gauge. Consequently, this smeared configuration should be just as interesting from the point of view of confinement.

The action of the new configuration is readily calculated to be

$$S_{\text{meron}} = \frac{8\pi^2}{g^2} + \frac{3\pi^2}{g^2}\ln\frac{R}{r}, \tag{6.6}$$

where the constant term comes from the two half-instantons and the ln term comes from the pure meron region in between. Furthermore, if we let $R \to r$, this configuration becomes the standard instanton and we recognize the meron as the extreme limit of a class of deformations away from the instanton. In a sense that we will eventually make precise, the instanton may be regarded as a bound pair of merons, and there are inescapable circumstances where this new degree of freedom plays an important role in the statistical mechanics of the instanton.

To make this notion clearer it is helpful to invert the configuration about some point a_μ in the region between r and R $[x_\mu \to a_\mu + \rho^2(x-a)_\mu/(x-a)^2$, with ρ an arbitrary scale factor$]$. Because of conformal invariance, this produces an equally good solution of the equations of motion. The geometry before and after inversion is described in Fig. 12. Since topological charge is a conformal invariant, after inversion we have two spherical regions of net topological charge one-half surrounded by an infinite region of zero topological charge density.

This new configuration is shown in Fig. 13. Regions I′ and III′ (the inversions of regions I and III) are again circles whose centers and radii are to be inferred from the coordinates displayed. Since the inversion of an instanton is again an instanton, the field in regions I′ and III′ is an instanton. The center coordinates $x_{\text{I}'}, x_{\text{III}'}$ and scale sizes r', R' of these instantons are indicated on Fig. 13 and it is worth noting that $x_{\text{I}'}$ and $x_{\text{III}'}$ are off center. Region II′ is the inversion of the meron. Explicitly, this field is

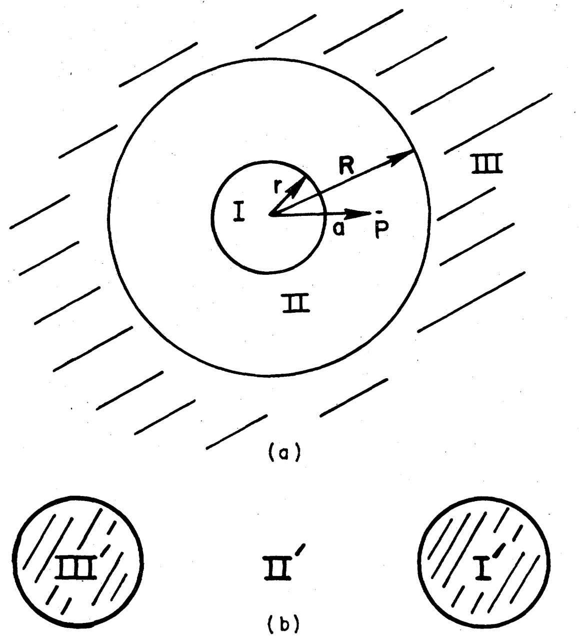

FIG. 12. (a) Concentric sphere geometry for a smeared meron. Region II is the region of zero topological charge. The point a is the center about which we invert to obtain (b). (b) Inversion of (a) containing two localized regions, I′ and III′, of nonzero topological charge.

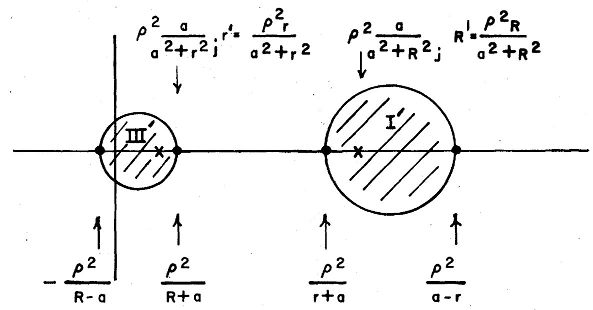

FIG. 13. Detailed specification of the configuration of Fig. 12(b). ρ is the scale parameter of the inversion. The crosses mark the centers of the instanton configurations filling out I′ and III′. r' and R' are their scale sizes.

$$A_\mu^a = \eta_{a\mu\nu}\frac{(x-x_{\mathrm{I}'})_\nu}{(x-x_{\mathrm{I}'})^2} + \eta_{a\mu\nu}\frac{(x-x_{\mathrm{III}'})_\nu}{(x-x_{\mathrm{III}'})^2} \tag{6.7}$$

or just the sum of two merons. The corresponding $F_{\mu\nu}$ falls at infinity as r^{-4}, leading of course to a convergent action integral. So the inverted configuration is a smeared version of two merons at positions $x_{\mathrm{I}'}$ and $x_{\mathrm{III}'}$.

It is very revealing to consider a sequence of such configurations obtained by holding r fixed and increasing R from r to infinity. For definiteness, choose $\rho = a = \sqrt{Rr}$. With these choices, for large R the configuration is as shown in Fig. 14(a): two half instantons of scale size r and separation $\sqrt{Rr} = d$ between the centers of the instanton configurations. The action is

$$S = \frac{8\pi^2}{g^2} + \frac{6\pi^2}{g^2}\ln\frac{d}{r}.$$

On the other hand, in the limit $R \to r$, regions I′ and III′ grow without limit in radius and move toward each other while the centers of the instanton configurations move toward each other. In the limit the configuration is just an instanton of scale size r split in half through the center [Fig. 14(b)].

In other words, the smoothed meron configuration may be thought of as describing various stages in a sequence of deformations of the instanton, leading from the instanton at one extreme to two widely separated smeared merons at the other. In a sense the meron is to be regarded as a constituent of the instanton. This is closely related to the fact that the instanton behaves like a color magnetic dipole—the merons are the configurations into which the dipole can split. We shall see that excitation of the meron pair degree of freedom makes a significant change in such quan-

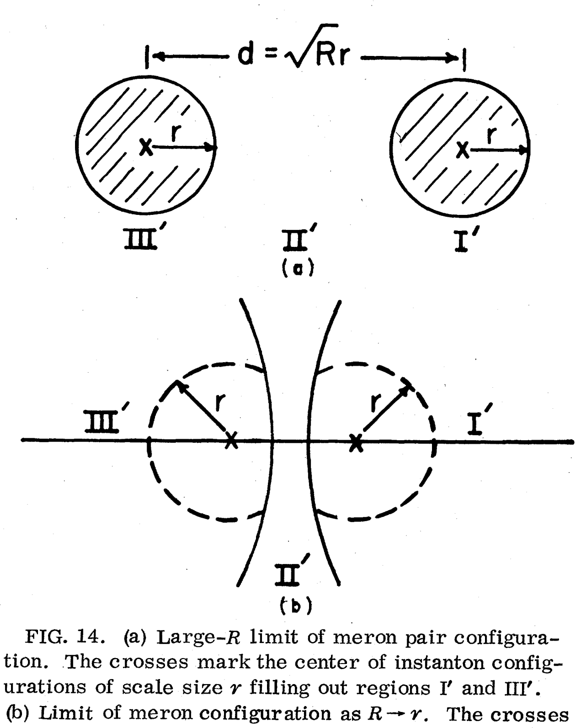

FIG. 14. (a) Large-R limit of meron pair configuration. The crosses mark the center of instanton configurations of scale size r filling out regions I′ and III′. (b) Limit of meron configuration as $R \to r$. The crosses mark the centers of instanton configurations of scale size r filling out I′ and III′. The crosses approach each other and region II′ vanishes as $R \to r$.

tities as vacuum susceptibility. More importantly, there is a phase transition which sets the merons free, which, we believe, leads to quark confinement.

In what follows, we shall elaborate on the properties of the individual meron pair configuration, first showing that it makes a sensible contribution to the functional integral (and has a well-defined entropy) and then evaluating its effect on some interesting physical quantities. In fact, of course, a *single* meron or meron pair makes a negligible contribution to the vacuum functional. What really counts is a *multi*meron configuration in which the mean *density* of merons is chosen to maximize the free energy. In the instanton case it was obvious that a reasonable multi-instanton configuration could be constructed just by superposition. Precisely because of the slow asymptotic falloff of the meron field, it is not obvious how to superpose an arbitrary number of merons and we do not yet have a sensible representation of the sort of configuration which will dominate the functional integral. It would be very helpful to find the N-meron pair generalization of Eq. (6.7), the singular one-meron-pair solution of the Yang-Mills equations, which one could convert to a useful "meron gas" configuration by smearing out the singularities in the way just described for the two-meron configuration. We expect that the problem of making function-theoretic sense of the smearing is basically the same in the $2N$ as in the two-meron case.

We should mention that the spherically symmetric ansatz

$$A_\mu^a = \eta_{\mu\nu}^a \partial_\nu \ln\rho(|x|) \tag{6.8}$$

leads to other infinite-action solutions than the meron. If we define

$$\phi = -1 - \frac{d}{d\tau}\ln\rho\,, \qquad \tau = \ln(x)\,, \tag{6.9}$$

the equations of motion become

$$\frac{d^2}{d\tau^2}\phi = 2\phi(\phi^2 - 1)\,. \tag{6.10}$$

The meron is the solution $\phi = 0$, but this equation also has periodic solutions which would correspond to nested merons. Our belief is that such configurations are special cases of the more general n-meron configuration in which each meron has its own position and scale-size coordinate and will be included once we learn how to deal with the meron "gas."

It is also clear for this ansatz that the only solutions with localizable topological charge are merons or instantons. In fact, for a spherically symmetric configuration, the requirement of localizability of charge for a configuration centered at the origin,

$$x^4 \,\mathrm{Tr}(F_{\mu\nu}\tilde F_{\mu\nu}) \xrightarrow[|x|\to\infty]{} 0\,,$$

requires that

$$\frac{\partial}{\partial\tau}\phi \xrightarrow[\tau\to\infty]{} 0\,.$$

This eliminates all but the meron and instanton solutions. Configurations with other localizable fractions of topological charge will not satisfy the equations of motion anywhere. To obtain these as true saddle points one would have to introduce an infinite number of constraints and one would not obtain for superpositions of such configurations as independent entropy of position for each one.

C. Meron functional determinant

In our qualitative discussion of smeared merons we assumed that they contribute to the functional integral in the same way as instantons, with a chemical potential determined by the action and an entropy determined in an obvious way by the scale size. The only way to be sure of this is to evaluate the Gaussian functional integral about our chosen smeared meron configuration. This is an extremely interesting exercise, because the meron is not, strictly speaking, a solution of the equations of motion and all intuition about Yang-Mills theories so far concerns integrations about *strict* solutions. One may worry that some catastrophe causes the meron functional integral to be undefined.

In fact the situation is not, strictly speaking, much different from that which obtains in the dilute instanton gas. The instanton–anti-instanton configuration is not a solution of the equations of motion either, and we have always argued that it was simply necessary to impose some constraint which picks out the instanton–anti-instanton configuration to the functional integral and then integrate over parameters in the constraints which correspond to the coordinates of the instanton and anti-instanton. (Also, when massless fermions are present the instantons and anti-instantons interact via a long-range logarithmic interaction in precise analogy with a meron pair.) In fact we never carried out this program in detail since it is obvious what the answer is. For the meron, the answer is *not* obvious and we must carefully evaluate the functional integral.

In so doing we will answer some obvious questions. First, why are there half-units of topological charge? The answer is because that is the configuration which you can specify with a finite number of constraints. Second, why is the smearing defined by spherical regions and not some other more complicated shape? When one evaluates the functional integral, one in effect integrates over small deformations about the sphere. The convergence of the integration over small fluctuations indicates that spheres are a locally optimal choice for the smearing constraint. We see no evidence for another nonspherical family of constraints which should be included. At the moment we have no rigorous argument, but only a strong feeling that nothing has been missed.

We will now set up the problem of evaluating the functional integral about the two-meron configuration shown in Fig. 13. In order not to overburden the reader, we will present here only the outline of the calculation as well as the essential features of the results, reserving the details for a future publication. The two-meron configuration is not general enough to tell us directly about confinement, of course, but should suffice to reassure us that the meron makes as well defined a contribution to the functional integral as does the instanton. In fact we shall see the discussion of the meron is *not* optional: Whenever instantons are quantitatively important, merons are more so.

There are several issues to clarify. In the classical action of this configuration,

$$S=\frac{4\pi^2}{g_0^{\,2}}+\frac{3\pi^2}{g_0^{\,2}}\ln\frac{R}{r}+\frac{4\pi^2}{g_0^{\,2}}\,,$$

the three terms arise from the inner instanton, meron, and outer instanton regions, respectively. The renormalization procedure has to turn each g_0 into an appropriate value of the renormalized effective coupling constant. Roughly, one wants the instanton coupling to be renormalized at its scale size and the meron itself to be renormalized at something like the separation of the two merons. This poses the question of how the renormalization and regularization scheme works in detail. On top of this, we must show how to impose and integrate over the constraints needed to pick out the meron configuration.

Let us first tackle the renormalization question by computing the functional determinant of a scalar field of isospin t in meron field background. In this and what follows, our notation conforms to 't Hooft's treatment of the functional determinant of the instanton.[21] We wish to compute

$$D_\phi^{-1/2}=\int[D\phi(x)]\,e^{-A(\phi)}\,, \qquad (6.11)$$

$$A(\phi)=\int d^4x\,\phi\left(-\frac{d^2}{d\rho^2}-\frac{3}{\rho}\frac{d}{d\rho}+\frac{4}{\rho^2}L^2+4\sigma T\cdot L_1+\rho^2\sigma^2T^2\right)\phi\,, \qquad (6.12)$$

where $\rho^2=x^2$ and σ is given by

$$\sigma(\rho)=\begin{cases}\dfrac{2}{\rho^2+r^2}, & 0<\rho<r\\[2mm] \dfrac{1}{\rho^2}, & r<\rho<R\\[2mm] \dfrac{2}{\rho^2+R^2}, & R<\rho<\infty\end{cases}\quad. \qquad (6.13)$$

To fully exploit the conformal invariance of this system it is best to use a dimensionless field variable $\tilde\phi=\rho\phi$. This is unconventional and we will normalize our answer by computing $D_{\text{meron}}/D_{\text{instanton}}$, which is unambiguous, and taking $D_{\text{instanton}}$ from 't Hooft. It is now appropriate to pass to new variables $t=\ln\rho$ and $\Omega=$ solid angle so that the action integral becomes

$$A(\tilde\phi)=\int d\Omega\int_{-\infty}^{\infty}dt\,[\tilde\phi_t^{\,2}+\tilde\phi^2+4L^2\tilde\phi^2+4\sigma\rho^2\tilde\phi\,T\cdot L_1\tilde\phi+(\sigma^2\rho^4)\tilde\phi T^2\tilde\phi]\,. \qquad (6.14)$$

The virtue of this form of the action is that in the meron region $\sigma\rho^2=1$ and the action is a sum of harmonic oscillators. We will regulate this determinant by adding other scalar regulator fields with space-dependent mass terms $\Lambda^2(x)$ proportional to the local magnitude of $F_{\mu\nu}$,

$$\Lambda^2(x)=\lambda^2\times\begin{cases}\dfrac{4r^2}{(r^2+x^2)^2}, & 0<\rho<r\\[2mm] \dfrac{1}{x^2}, & r<\rho<R\\[2mm] \dfrac{4R^2}{(R^2+x^2)^2}, & R<\rho<\infty\,.\end{cases} \qquad (6.15)$$

The virtues of such a choice in the instanton region have been explained by 't Hooft.[21] For us, it suffices to note that the mass term for the $\tilde\phi$ regulator is $\Lambda^2(x)\,x^2$, which is a constant in the meron region ($r<\rho<R$). Therefore, even with regulators, the meron action is still a sum of harmonic oscillators.

The partial-wave decomposition into simultaneous eigenstates of $L_1^{\,2}$ and $(L_1+T)^2=J^2$ completely separates the problem. If

$$\tilde\phi=\sum_{J,L}Y_L^J(\Omega)\,\phi_{JL}(\rho)\,,\qquad \int d\Omega(Y_L^J)^2=1\,, \qquad (6.16)$$

then

$$A(\tilde{\phi}) = \sum_{J,L} \int_{-\infty}^{\infty} dt(\dot{\phi}_{JL}{}^2 + M_{JL}{}^2 \phi_{JL}{}^2), \tag{6.17}$$

where

$$M_{JL}{}^2 = 1 + 4L^2 + 4\sigma x^2 T \cdot L_1 + (\sigma x^2)^2 T^2 + x^2 \Lambda^2(x^2) \tag{6.18}$$

and is independent of x in the meron region. For the moment let us concentrate on a particular partial wave and evaluate the functional integral except for the integration over $\phi_1 = \phi^{JL}(r)$, $\phi_2 = \phi^{JL}(R)$, the values on the boundaries between regions I and II, II and III, respectively. The regions I, II, and III correspond to inner instanton, meron, and outer instanton, respectively. Thus

$$D_{JL}{}^{-1/2} = \int d\phi_1 d\phi_2 F_{\text{I}}^{JL}(0, \phi_1) F_{\text{II}}^{JL}(\phi_1, \phi_2) F_{\text{II}}^{JL}(\phi_2, 0), \tag{6.19}$$

where $F_{\text{I}}^{JL}(\phi, \phi')$ is the JL partial-wave functional integral over the restricted region I with ϕ_{JL} set equal to ϕ_1 at the end points of region I (similarly for regions II and III).

Because we are computing a Gaussian functional integral, it must be true that

$$F_{\text{I}}(0, \phi_1) = F_{\text{I}}(0, 0)\, e^{-m_{\text{I}} \phi_1{}^2/2},$$
$$F_{\text{III}}(\phi_2, 0) = F_{\text{III}}(0, 0)\, e^{-m_{\text{III}} \phi_2{}^2/2}. \tag{6.20}$$

Since regions I and III are instanton regions, which may be transformed into one another by an inversion, we also have $m_{\text{I}} = m_{\text{III}} = m$ and $F_{\text{I}}(0,0) = F_{\text{III}}(0,0)$. Therefore, if we imagine treating the instanton functional determinant in a similar way, writing

$$[D_{JL}{}^{-1/2}]_{\text{instanton}} = \int d\phi\, F_{\text{I}}(0, \phi) F_{\text{III}}(\phi, 0), \tag{6.21}$$

we have

$$[D_{JL}{}^{-1/2}]_{\text{instanton}} = \frac{F_{\text{I}}(0,0) F_{\text{III}}(0,0)}{\sqrt{m}}. \tag{6.22}$$

This will be important when we compare D_{meron} with $D_{\text{instanton}}$.

The function $F_{\text{II}}^{JL}(\phi_1, \phi_2)$ can be computed explicity since the action function in region II is that of a harmonic oscillator of frequency,

$$\mu_{JL}{}^2 = 1 + 4L^2 + 4T \cdot L_1 + T^2 + \lambda^2.$$

Consequently it is easy to show that

$$F_{\text{II}}{}^2(\phi_1, \phi_2) = \frac{\mu_{JL}}{\sinh \mu_{JL} T} \exp[-\mu_{JL}(\phi_1{}^2 + \phi_2{}^2)\coth \mu T + 2\mu_{JL}\phi_1\phi_2 \operatorname{csch} \mu T], \tag{6.23}$$

where $T = t_2 - t_1 = \ln(R/r)$. In the limit of large meron separation $(R/r \to \infty)$ the variables ϕ_1 and ϕ_2 decouple,

$$F_{II}{}^2(\phi_1, \phi_2) \to \mu_{JL} \exp\left(-\mu_{JL} \ln\frac{R}{r}\right) \times \exp[-\mu_{JL}(\phi_1{}^2 + \phi_2{}^2)]. \tag{6.24}$$

The ϕ_1 and ϕ_2 integrations may be done explicitly to yield

$$[D_{JL}{}^{-1/2}]_{\text{meron}} = F_{\text{I}}^{JL}(0,0) F_{\text{III}}^{JL}(0,0) \times \frac{2\sqrt{\mu_{JL}}}{(\mu + m)_{JL}} \exp\left(-\frac{\mu_{JL}}{2} \ln \frac{R}{r}\right) \tag{6.25}$$

or

$$\frac{[D_{JL}{}^{-1/2}]_{\text{meron}}}{[D_{JL}{}^{-1/2}]_{\text{instanton}}} = \frac{2\sqrt{m_{JL}}\sqrt{\mu_{JL}}}{(m+\mu)_{JL}} \times \exp\left(-\frac{\mu_{JL}}{2} \ln \frac{R}{r}\right). \tag{6.26}$$

To compute the full determinant we divide by the vacuum contribution in the same partial wave and then take the product over all partial waves. Under these operations the factor multiplying the exponential can be shown to give a convergent result, a pure number of order one which we have yet to calculate and will denote by c. Thus

$$\prod_{JL} \left(\frac{D_{\text{meron}}}{D_{\text{instanton}}}\right)^{-1/2} = c \exp\left\{-\frac{1}{2}\left[\sum (\mu_{JL} - \mu_{JL}^{\text{vac}}) \ln \frac{R}{r}\right]\right\}, \tag{6.27}$$

and the divergence is entirely contained in $\Sigma(\mu_{JL} - \mu_{JL}^{\text{vac}})$. The vacuum eigenvalues are of course obtained by setting $T = 0$ in Eq. (6.18).

Regulation of this divergence is carried out in the conventional manner described by 't Hooft.[21] If $D^{-1/2}(\lambda)$ is the determinant calculated with regulator mass λ, then the divergences of the partial-wave sum are eliminated by forming

$$D_{(\lambda)}{}^{-1/2} = \prod_{i=1}^{4} [D^{-1/2}(\lambda_i)]^{e_i} [D^{-1/2}(0)], \tag{6.28}$$

where $e_i = \pm 1$, i runs from 1 to (in this case) 4 and

$$\sum_{i=1}^{4} e_i = -1, \quad \sum_{i=1}^{4} e_i \lambda_i = 0, \quad \sum_{i=1}^{4} e_i \lambda_i{}^2 = 0, \tag{6.29}$$

and

$$\sum_{i=1} e_i \ln\lambda_i = -\ln\lambda\,. \qquad (6.30)$$

The parameter λ plays the role of a large cutoff parameter and can only be eliminated by supplying a counterterm for coupling-constant renormalization.

Evidently the eigenvalue sum in Eq. (6.27) serves to renormalize the coefficient $3\pi^2/g_0{}^2$ of $\ln(R/r)$ in the classical action. Since it is precisely the $\ln(R/r)$ term which determines how rapidly the meron determinant falls with increasing separation and therefore how important meron effects are relative to instantons, this eigenvalue sum is what we are mainly interested in. The meron mass matrix may be diagonalized explicitly and the regulated sum computed by a combination of numerical and analytic methods. The result, for $t=1$, is[33]

$$\frac{3\pi^2}{g_0{}^2}\ln\frac{R}{r} \to \left(\frac{3\pi^2}{g_0{}^2}+\tfrac{1}{4}\ln\lambda+0.52\right)\ln\frac{R}{r}\,. \qquad (6.31)$$

The coefficient of $\ln\lambda$ is consistent with the contribution of an $I=1$ scalar to the one-loop renormalization of the coupling.

To complete the renormalization procedure we must add a counterterm which absorbs the divergence in the effective coupling constant and compensates for the space-time dependence of the regulator mass. We make use of 't Hooft's argument that the counterterm must be local and, therefore, locally identical to the fixed-mass counterterm. Thus the counterterm appropriate to defining the renormalized coupling constant at fixed mass μ is

$$\Delta\mathcal{L} = -\frac{1}{96\pi^2}\,\mathrm{tr}(F^2)\,\tfrac{1}{3}\ln\frac{\Lambda^2(x)}{\mu^2}\,. \qquad (6.32)$$

Now $\Lambda^2(x)$ is proportional to λ^2, and it is easy to verify that the $\ln\lambda$ terms cancel between the counterterm and the regulated determinant. What is more interesting is how the resulting finite quantity depends on μ.

In the first instance we are tempted to use

$$\Lambda^2(x) = \lambda^2\times\begin{cases}\dfrac{4r^2}{(r^2+x^2)^2}\,, & 0<x<r\\[2mm] \dfrac{1}{x^2}\,, & r<x<R\\[2mm] \dfrac{4R^2}{(R^2+x^2)^2}\,, & R<x<\infty\,.\end{cases} \qquad (6.33)$$

This is wrong if we are describing the renormalization of the two-meron configuration of Fig. 13. The reason is that we must carry out a conformal transformation (inversion, to be more accurate) to bring the two-meron configuration into spherical form and Λ^2 is not conformally invariant—it transforms like a density of dimension 2. Thus, if the inversion taking us from the spherical configuration to the two-meron configuration is carried out around $x=a$ with scale parameter ρ, then the proper counterterm is obtained by setting

$$\Lambda^2(x) = \frac{(x-a)^4}{\rho^4}\lambda^2\times\begin{cases}\dfrac{4r^2}{(r^2+x^2)^2}\,, & 0<x<r\\[2mm] \dfrac{1}{x^2}\,, & r<x<R\\[2mm] \dfrac{4R^2}{(R^2+x^2)^2}\,, & R<x<\infty\,.\end{cases} \qquad (6.34)$$

Let us choose $a^2=Rr=\rho^2$ so that, as explained earlier, the two-meron configuration consists, at least for large R/r, of two half instantons of radius r separated by a distance $d=\sqrt{Rr}$. In what follows, except when explicitly noted, we take the special case of equal-size merons.

Let us first consider the total action, including the counterterm, coming from the region $0<x<r$. Since $a\gg r$, we may neglect the factor $(x-a)^4/\rho^4$ in $\Lambda^2(x)$. Then the counterterm action is

$$\Delta A = -\frac{1}{8\pi^2}\int_0^r d^4x\left[\frac{4r^2}{(r^2+x^2)^2}\right]^2 \times\ln\left[\frac{\lambda^2}{\mu^2}\,\frac{4r^2}{(r^2+x^2)^2}\right] = -\frac{1}{8\pi^2}\left(\frac{4\pi^2}{3}\ln\frac{\lambda^2}{\mu^2r^2}+\mathrm{const}\right). \qquad (6.35)$$

This combines with the classical action and determinant associated with this region to give

$$[A]_{\text{instanton}} = 4\pi^2\left(\frac{1}{g_0{}^2}+\frac{1}{24\pi^2}\ln\mu^2r^2\right)+\mathrm{const}\,. \qquad (6.36)$$

The contents of the parentheses are (up to a constant) just $1/g^2(\mu r)$, where g is the effective coupling constant including only the renormalization-group effects of a scalar field. At any event, the chemical potential of a meron core will, as expected, be determined by the effective coupling at the core size.

What about the region $R<x<\infty$? This time, since $R\gg a$, we replace $(x-a)^4/\rho^4$ by x^4/ρ^4, and we find (remember that $a^2=Rr$)

$$\Delta A = -\frac{1}{8\pi^2}\int_R^\infty d^4x\left[\frac{4R^2}{(R^2+x^2)^2}\right]^2 \times\ln\left[\frac{\lambda^2}{\mu^2}\,\frac{x^4}{R^2r^2}\,\frac{4R^2}{(R^2+x^2)^2}\right] = -\frac{1}{8\pi^2}\left(\frac{4\pi^2}{3}\ln\frac{\lambda^2}{\mu^2r^2}+\mathrm{const}\right). \qquad (6.37)$$

Once again, the renormalization scale for the meron core is seen to be r, which is the physical size of the core.

Finally, we must go through the same procedure for the region outside the cores, in the meron region itself. Now we have

$$\Delta A = -\frac{1}{16\pi^2}\int_r^R d^4x\left(\frac{1}{x^2}\right)^2 \times \ln\left[\frac{\lambda^2}{\mu^2 x^2}\,\frac{(x-a)^4}{a^4}\right]. \qquad (6.38)$$

If we keep only the pieces which grow with increasing R,

$$\Delta A = \tfrac{1}{8}\ln\lambda^2 - \ln(\mu^2 dr)\ln\frac{R}{r}, \qquad (6.39)$$

where $d=\sqrt{Rr}$ is the separation distance of the meron pair. Combining this counterterm with the classical action and the regulated determinant associated with the meron region we have

$$[A]_{\text{meron}} = 2\left(\frac{3\pi^2}{g_0^2} + \tfrac{1}{8}\ln\mu^2 dr + 0.52\right)\ln\frac{d}{r}, \qquad (6.40)$$

which is to be interpreted as saying that the meron action has the effective coupling constant evaluated at scale size $\sqrt{dr}$. This is perhaps a bit more complicated than one might like, but not physically unreasonable.

So far we have shown that the determinant serves to identify the effective coupling at which the action of the various components of the smeared meron pair should be evaluated. The computation was carried out for a scalar field, but similar results would have been obtained for higher-spin fields. The details would have been different since the partial-wave decomposition is not so trivial, but these differences are necessary to produce the different renormalization-group behavior of higher-spin fields. The essential aspects of this question will be explained as we discuss the determinant of the gauge fields themselves.

The contribution of the gauge field itself to the functional integral must now be studied, and it is here that the issue of constraints arises since we are attempting to integrate about a gauge field configuration which is not a solution of the equations of motion. We want to specify location and scale size for two meron cores, which is a total of $2(4+1)=10$ constraints. Nine of them are associated with zero modes arising from the symmetries of translation (4 parameters), inversion (4 parameters: arbitrary location of inversion point), and scale invariance. The tenth is the one constraint variable on which the action depends (R/r in the spherical configuration).

We propose to pick out the desired two-meron solution by constraining a small number of partial-wave amplitudes on two spheres, each sphere defined by center coordinates, a_μ and a radius, r. This is a convenient set of parameters from the point of view of conformal invariance: Under inversion, spheres transform into spheres and, furthermore, the obvious volume element, $d^4a\,dr/r^5$, for integrating over constraint parameters is invariant under inversion. Since the whole scheme will be inversion invariant, we are then free to identify a useful choice of constraints in the spherically symmetric configuration.

Consider first the inner instanton core of the configuration. We want to guarantee that it has scale size r and is centered at a. Define

$$I_0 = \int_{r,a_\mu} d\Omega\, A_\mu^a \eta_{a\mu\nu}(x-a)_\nu - \frac{6\pi^2}{g^2}(1-\epsilon),$$
$$I_\nu = \int_{r,a_\mu} d\Omega\, A_\mu^a \eta_{a\mu\nu}\,. \qquad (6.41)$$

The integrals are taken over a sphere of radius r centered at a_μ and ϵ is a small parameter whose significance will shortly emerge. It is easy to show that I_0 and I_ν vanish if A_μ is taken to be the Landau gauge instanton field of scale size $(1+\epsilon)r$ centered at a_μ. I_0 can be thought of as fixing scale size while I_ν fixes location. The reason for the factor of $(1-\epsilon)$ in the definition of I_0 is that if $\epsilon=0$, $I_0=0$ (as well as $I_\nu=0$) for the *meron* field $A_\mu^a=\eta_{a\mu\nu}x_\nu/x^2$. The constraint, by itself, would then not distinguish between the smeared meron and the original singular meron (although a requirement of finite action would).

The Jacobian associated with integrating over a_μ and r about the configuration which satisfies the constraints is easy to compute, and we may establish the identity

$$1 = \frac{1}{g^5}\int \frac{d^4a\,dr}{r^5}(6\pi^2)^5(\tfrac{3}{4})^4\,\delta(I_0(A))_\nu\pi\delta(rI_\nu(A)). \qquad (6.42)$$

The factors of $6\pi^2$ and $1/g$ come from the Jacobian. The factor of $1/g$ arises because the field about which we integrate is $O(1/g)$. To pick out a meron pair, we need two such constraints: one for the inner core and one for the outer core. After inversion, the integration $d^4a\,dr\,d^4a'\,dr'$ can be interpreted as an integration over location and scale size of two meron cores and looks very much like the integration we are used to in the dilute instanton gas. All the expected dimensional factors and nearly all the powers of g are now present.

We now must evaluate the constrained vector-meson functional integral. We set the problem up in the same way as we set up the scalar field functional integral: We work in the spherical configuration, use a dimensionless field variable $a_\mu = xA_\mu$, and use a logarithmic radial coordinate $t = \ln x$. Then the action function is

$$A = \int d\Omega \int_{-\infty}^{\infty} dt[a_{\mu,t}a_{\mu,t} + a_\mu a_\mu + 4a_\mu L^2 a_\mu + 4\sigma x^2 a_\mu T \cdot L_1 a_\mu + (\sigma x^2)^2 a_\mu T^2 a_\mu + \Lambda^2(x) x^2 a_\mu a_\mu + 2a_\mu T_a x^2 f^a_{\mu\nu} a_\mu]. \quad (6.43)$$

$f^a_{\mu\nu}$ is the background field tensor and satisfies

$$\begin{aligned} x^2 f^a_{\mu\nu} &= \frac{4r^2x^2}{(x^2+r^2)^2}\eta_{a\mu\nu}, \quad x<r \\ &= \eta_{a\mu\nu} - \eta_{a\mu\nu\lambda}\hat{x}_\lambda \hat{x}_\nu - \hat{x}_\mu \hat{x}_\lambda \eta_{\alpha\lambda\nu}, \quad r<x<R \\ &= \frac{4R^2x^2}{(x^2+R^2)^2}\eta_{a\mu u}, \quad R<x. \end{aligned} \quad (6.44)$$

The quantities σ and Λ have the same definition as before. The main difference with the scalar case is the presence of the spin coupling term involving $f_{\mu\nu}$. The presence of this term complicates the analysis somewhat since now the partial-wave reduction does not completely diagonalize the action. In fact, since $f^{\text{meron}}_{\mu\nu}$ is neither self-dual nor anti-self-dual, as a general rule four partial waves are coupled. On the other hand, in the meron region the action is still a sum of harmonic oscillators.

Virtually all of the analysis of the scalar field determinant carries through here. The only difference is that since partial waves are coupled, the quantities m and μ are 4×4 matrices instead of numbers. The algebra of combining the determinants of the several regions and dividing out the instanton determinant goes through with the understanding that functions of μ and m, now matrices, must be interpreted by taking the determinant. The constraints affect only a few $l=0$ and $l=1$ partial waves and therefore have no effect on the divergences or their renormalization.

To analyze the renormalization of the $\ln(R/r)$ term in the action we must once again compute the regulated sum of meron eigenvalues. Despite the more complicated meron mass matrix (compared to the scalar case), the eigenvalues may still be found explicitly and the regulated sum computed. The result is to replace the classical action, $(6\pi^2/g_0^2)\ln(d/r)$, by a renormalized action, $S_m(d, r, \mu)$, which grows with d and has the simple property that[33]

$$\frac{\partial}{\partial \ln d} S_m = \frac{3}{4}\,\frac{8\pi^2}{g_n^2(\bar{\mu}d)} - C_n. \quad (6.45)$$

In this expression C_n is a constant [found to have the value 2.89 for SU(2) and 6.55 for SU(3)] and g_n is the same Pauli-Villars definition of the coupling constant adopted in Sec. III:

$$\frac{8\pi^2}{g_2^2(\bar{\mu}d)} = \frac{8\pi^2}{g_0^2} - \tfrac{22}{3}\mu d \simeq \tfrac{22}{3}\ln\frac{1}{\bar{\mu}d}, \quad (6.46)$$

$$\frac{8\pi^2}{g_3^2(\bar{\mu}d)} = \frac{8\pi^2}{g_0^2} - 11\mu d \simeq 11\ln\frac{1}{\bar{\mu}d}. \quad (6.47)$$

The essential feature of the determinant $[\exp(-S_m)]$ is the power of d with which it decreases: If it falls too rapidly, the merons never get far enough apart to be effective as separate entities. In the next section we will see that the critical condition is just

$$\frac{\partial}{\partial \ln d} S_m = 8, \quad (6.48)$$

or

$$\frac{8\pi^2}{g^2(\bar{\mu}d)} = \tfrac{32}{3} + C_n = \begin{cases} 13.6, & \text{SU(2)} \\ 17.2, & \text{SU(3)}. \end{cases} \quad (6.49)$$

Thus there is a critical coupling and separation at which meron dominated vacuum physics will set in. By comparison with the arguments of Sec. III we see that for SU(2) merons come in at a scale where instanton (and presumably also meron) density is low, while in SU(3) merons come in at larger, but still small densities.

The dependence of S_m on the core radii is also simple. Taking two cores of independent radii r and r' and keeping d fixed, it can be shown that

$$\begin{aligned} \frac{\partial}{\partial \ln r} S_m &= -\frac{3}{8}\left(\frac{8\pi^2}{g_n^2(\bar{\mu}r)} - C_n\right), \\ \frac{\partial}{\partial \ln r'} S_m &= -\frac{3}{8}\left(\frac{8\pi^2}{g_n^2(\bar{\mu}r')} - C_n\right). \end{aligned} \quad (6.50)$$

Thus the derivatives of S_m with respect to $\ln d$, $\ln r$, or $\ln r'$ are the same as those of the bare action with g_0 replaced by an appropriate coupling renormalized at d, r, or r'.

Finally, we should comment on an instability which is present in the lowest partial wave. Since the meron pair action increases without limit as we increase the separation, there is a point where, as we increase the separation, it becomes energetically favorable to create a meron-antimeron pair out of the vacuum and convert the original two-meron configuration into a configuration consisting of an instanton and an anti-instanton. This is reflected in the occurrence of a negative s-wave eigenvalue as we increase R/r. The calculation we have described is strictly valid in the region $1 \le R/r < \exp(\pi/2\sqrt{2})$ where there are *no* negative eigenvalues at all. In fact, meron pairs do not occur in isolation and we expect the density of pairs of a given scale size to be comparable to the density of instantons of comparable scale size. In SU(3) at least, when the meron effective coupling is large enough for the meron degree of freedom to be significant, the meron density is comparable to one. Then a meron is never far enough from a neighbor for the above-mentioned instability to be relevant. The converse of the existence of this instability is that when an instanton and anti-instanton get too close to each other, it will be energetically favorable for them to convert to a meron-antimeron pair. The critical separation is related to the stability requirement $1 \leq R/r \leq \exp(\pi/2\sqrt{2})$ and corresponds to a density of a few percent. That is, when the integrated instanton density exceeds a few percent, neighboring instanton–anti-instanton pairs will collapse to meron-antimeron configurations. Consequently, it may be that merons dominate all the semiclassical vacuum fluctuations—not just at the confinement scale. This point will be investigated further.

To summarize, the contribution of a meron pair to the functional integral may be written as

$$C\int \frac{d^4x\,dr}{r^5}\,\frac{d^4x'\,dr'}{r'^5}\times\exp\left[-\frac{4\pi^2}{g^2(r\mu)}-\frac{4\pi^2}{g^2(r'\mu)}-S_m(d,r,r')\right], \quad (6.51)$$

where $d=|x-x'|$, the meron separation, and C is a number containing some powers of g_0^{-1} from the constraints. At the one-loop level C is independent of r, r', and d (assuming of course that $d \gg r, r'$) and we have isolated the complete dependence on the interesting parameters. Each core has a chemical potential $4\pi^2/g^2(r)$ and an entropy of position measured in units of the core size. The integration over core sizes is convergent at $r=0$ by asymptotic freedom. To go to large r we would have to determine the way in which the factors of g_0^{-1} in C depend on r, r' and d. Unlike the case of an instanton the renormalization group does not suffice here and a two-loop calculation or some further physical insight is needed.

D. The dynamical effects of merons

In the examples reviewed at the beginning of this section confinement was achieved by reducing the Euclidean functional integral to the partition function of a plasma and using our understanding of plasma dynamics (more specifically, charge screening) to establish the confining behavior of the quark-antiquark potential. Turning to four-dimensional gauge theories, we found a class of field configurations (merons) which have the essential properties of the lower dimensional plasma pseudoparticles (most importantly $A_\mu \sim r^{-1}$ for large r) and conjectured that a plasma of such pseudoparticles would account for confinement in realistic gauge theories. The merons, of course, are rather strange objects and the preceding lengthy discussion was aimed at convincing the reader that they make a perfectly sensible contribution to the functional integral and have entropy of position, etc., just as do the instantons.

There are significant differences, of course. Most importantly, the meron cannot be discussed in isolation, even when we smear its core, because of the logarithmic divergence of its action at infinity. This can be understood, by the way, in vacuum tunneling language: The meron can readily be shown to describe the transition between an n vacuum and an "$n+\frac{1}{2}$" vacuum, and the general argument of Sec. II shows that such an event cannot have finite action. Pairs of merons have finite action, but this action grows logarithmically as they are pulled apart.

Two-dimensional experience, summarized in our paper on the effects of massless fermions on two-dimensional instantons,[18] has shown that a logarithmic potential between pseudoparticles, though growing without limit, does not prevent the passage to a pseudoparticle plasma phase: Entropy can beat energy provided the temperature (coupling constant) is high enough (the essential elements of this mechanism work in four dimensions as well as two). Our study of the meron determinant showed that the coupling constant that goes with the logarithmic interaction energy depends on meron separation and scale size and, because of asymptotic freedom, increases (decreases) as these quantities increase (decrease). We will shortly discuss ways of estimating the critical coupling constant at which the merons are "liberated" and will find that it is gratifyingly small. The general picture then is that small merons are closely bound in pairs and not much different in their effect on vacuum quantities from instantons, while sufficiently large scale-size merons are not, in spite of their divergent action when considered separately. One might therefore expect many of the vacuum fluctuation effects associated with "half vacuum tunneling" to survive, at least in a qualitative fashion.

There are some technical problems which stand in the way of a quantitative study of these questions. Most serious is our lack of a suitable trial function to describe the meron plasma. We know what a meron *pair* looks like when it is recognizably a deformation of an instanton and can of course describe a sufficiently dilute gas of such pairs. In a pair, the group orientation of the individual merons is rigidly locked together, while that is presumably not the case in the more general configuration appropriate to the plasma phase. At the moment, the best we can do is to examine how the meron degree of freedom affects vacuum properties on the other side of the "phase transition" where we have a gas of meron pairs and look for singular behavior signaling the onset of a transition. The idea is the following: We have already seen that instantons in the Yang-Mills vacuum lead to a paramagnetic susceptibility which increases with increasing scale size. This susceptibility arises from the response of the instanton color dipole to the applied field. The two-meron configuration can be regarded as a deformation of an instanton and could arise in response to an applied field. We will now include these deformations in our calculation of the susceptibility and will find that it is enhanced. More to the point, we will find that at some scale size the susceptibility begins to diverge, signaling an imminent phase transition. This occurs, classically, at the rather small coupling $g^2/8\pi^2=\frac{3}{32}$, which is in fact the value of the coupling at which the free energy of an isolated meron vanishes.

We proceed as in the computation of the instanton gas susceptibility. To evaluate the interaction of a meron pair with a weak constant external field, $F_{\mu\nu}^{\rm ext}$, we only need to know the meron pair field at large distances. Let the separation between the merons be Δ_μ and let x (large compared to Δ) be the position at which $F_{\mu\nu}^{\rm meron}$ is evaluated. In the gauge described earlier in this section it is easy to see that (for $x\gg\Delta$)

$$F_{\mu\nu}^a(x)\sim\frac{\Delta^2}{x^4}M_{\mu\mu'}(\hat{x})M_{\nu\nu'}(\hat{x})\mathfrak{F}^a_{\mu'\nu'}(\hat{\Delta})\,,$$
$$\mathfrak{F}^a_{\mu\nu}=C_{aa'}(-\eta_{a'\mu\nu}+\hat{\Delta}_\mu\hat{\Delta}_\lambda\eta_{a'\lambda\nu}+\eta_{a'\mu\lambda}\hat{\Delta}_\lambda\hat{\Delta}_\nu)\,, \quad (6.52)$$

where $M_{\mu\nu}(x)=(g_{\mu\nu}-2x_\mu x_\nu/x^2)$ is the conformal inversion matrix and $C_{aa'}$ is a group matrix describing the group orientation of the meron pair. Since the meron pair is basically a deformation of an instanton it is perhaps not surprising that $F_{\mu\nu}$ falls as x^{-4} just as does the field of an instanton. What plays the role of scale size is the separation, Δ, of the meron pair so that for fixed x, the field (and the interaction with an external field) grows like Δ^2 with increasing separation. Unlike the instanton case, the field is neither self-dual nor anti-self-dual.

By the arguments of Sec. III the interaction energy of the pair with a weak external field is

$$\delta S_{\rm int}=2\int d\Omega\,\hat{x}_\mu(-\tfrac{1}{2}F_{\nu\rho}^{a\,{\rm ext}}x_\rho)F_{\mu\nu}^{\rm meron\ pair}\,, \quad (6.53)$$

where the integral is taken over a large sphere of radius x. One easily finds that

$$\delta S=-\frac{\pi^2\Delta^2}{2g^2}F_{\nu\rho}^{a\,{\rm ext}}\mathfrak{F}^a_{\nu\rho}(\hat{\Delta})\,. \quad (6.54)$$

As before, $\langle\delta S\rangle=0$ when we average over gauge orientations. The effective interaction for $F_{\mu\nu}^{\rm ext}$ is thus $\frac{1}{2}\delta S^2$ averaged over gauge orientations and meron coordinates while weighted with the appropriate meron pair density functions. For the purposes of this crude analysis we shall ignore the integration over meron scale sizes and imagine them to be held fixed at some optimal value. This does not affect the qualitative point we want to make. The meron pair has two position coordinates which we must integrate over. Nothing, except possibly $F_{\mu\nu}^{\rm ext}$, depends on the center-of-mass position coordinate and we leave that until last. For the moment we only concern ourselves with the integral over Δ, the meron pair separation coordinate. It must be integrated $d^4\Delta$ with the appropriate meron pair weight function. This weight contains the logarithmically growing meron pair interaction energy, which cuts off the Δ integration and makes the meron pair picture sensible.

The result of all this is an effective interaction

$$\delta S_{\rm eff}=\frac{1}{n}\frac{\pi^4C}{2g^4}\int d^4x\,(F_{\mu\nu}^aF_{\mu\nu}^a)^{\rm ext}\times\int d^4\Delta\,e^{-S_m(\Delta)}\Delta^4\,, \quad (6.55)$$

where $n=3$ (8) if the gauge group is SU(2) [SU(3)], C is the normalization factor of the meron coordinate integration arising from the meron determinant, and $g(\Delta)$ is the effective coupling that goes with the logarithmic interaction energy [symbols have the meaning of Eq. (6.51)]. This is the same sort of expression which led us to conclude that the instanton gas behaves like a medium with magnetic susceptibility. In the case at hand

$$\chi=+\int d^4\Delta\,C\frac{1}{n}\frac{8\pi^2}{g^2(\Delta)}\frac{\Delta^4}{4}e^{-S_m(\Delta)}\,. \quad (6.56)$$

This expression is superficially like the instanton contribution to the susceptibility, with the scale-size parameter replaced by Δ. The integral begins to diverge when $(\partial/\partial\ln\Delta)S_m=8$.

Thus when Δ crosses the threshold defined by $8\pi^2/g^2(\Delta)=\frac{32}{3}+C_n$ the integration over Δ begins to diverge with larger contributions coming from larger separations. This is a familiar enough situation in mean field theory calculations and typically signals the onset of a new phase in which the constituents described by the parameter Δ are liberated. The new feature is that the "temperature," or effective coupling constant, cannot be affected from the outside but has its own internal dynamics.

The critical effective coupling is rather small giving us a reasonable hope that quantum corrections will not substantially modify this qualitative picture. It is also not too much different from the effective coupling at which the instantons themselves begin to have large effects on coupling renormalization, etc. In fact, the meron degree of freedom places a natural cutoff on instanton scale sizes themselves: Beyond $8\pi^2/g^2 = \frac{32}{3} + C_n$, the instantons do not really exist, but dissociate into merons. Since the meron plasma should provide the ultimate confining potential, this is another piece of evidence that the confinement scale size is defined by $8\pi^2/g^2 \sim 12$ [for SU(2)] or 17 [for SU(3)].

To lend some weight to our conjecture that the meron plasma confines we will now look at the effect of the meron pair degree of freedom on the static quark-antiquark potential. In contrast to the instanton case, we will find a crossover to a potential which rises with distance at essentially the critical coupling of the preceding paragraph. We have discussed most of the mechanics of such a calculation in Sec. IV. Everything said there (apart from the specific choice of instanton field) is valid here so long as we are dealing with a dilute gas of meron pairs. We use Eq. (4.6) for the contribution of a specific configuration to the energy

$$\delta\epsilon = \frac{-1}{\mathrm{tr}\,(1)}[\mathrm{tr}\,(U^{(+)}U^{(-)} - 1)], \tag{6.57}$$

where $U^{(\pm)}$ are the loop integrals taken over the quark (anti-quark) trajectories. Then, holding the time coordinate of *one* meron fixed, we must integrate over the remaining seven meron position coordinates, including the proper weight function, to compute E (once more, we neglect the meron scale-size integrals, which do not affect the growth rate of the potential).

The meron pair field is taken to be

$$A_\mu^a = \eta_{a\mu\nu}\left[\frac{(x - x^{(1)})_\nu}{(x - x^{(1)})^2} + \frac{(x - x^{(2)})_\nu}{(x - x^{(2)})^2}\right], \tag{6.58}$$

with $x^{(1)}$ and $x^{(2)}$ denoting the four-vector locations of the two merons. The two legs of the quark loop have fixed spatial coordinates $\vec{R}^{\pm}$ and the integral for U has the form

$$U = P\exp\left\{i\frac{\vec{\tau}}{2}\cdot\int_{-\infty}^{\infty} dx_0\left[\frac{\vec{R} - \vec{x}^{(1)}}{(x_0 - x_0^{(1)})^2 + (\vec{R} - \vec{x}^{(1)})^2} + \frac{\vec{R} - \vec{x}^{(2)}}{(x_0 - x_0^{(2)})^2 + (\vec{R} - \vec{x}^{(2)})^2}\right]\right\} \tag{6.59}$$

where $\vec{R} = \vec{R}^{(+)}$ for $U^{(+)}$ and $U^{(-)}$ is $U^{(\dagger)}$ with $\vec{R} = \vec{R}_0^{(-)}$. The path-ordering instruction, P, is not superfluous and it does not seem possible to evaluate this integral analytically except in special cases.

There is some simplification if $|\vec{R} - \vec{x}^{(2)}| \gg |\vec{R} - \vec{x}^{(1)}|$. In that case we can show that

$$U \sim M_2\left[\frac{\pi}{4} - \tfrac{1}{2}\tan^{-1}\left(\frac{x_0^{(1)} - x_0^{(2)}}{|\vec{R} - \vec{x}^{(2)}|}\right)\right] M_1\left(\frac{\pi}{2}\right) M_2\left[\frac{\pi}{4} + \tfrac{1}{2}\tan^{-1}\left(\frac{x_0^{(1)} - x_0^{(2)}}{|\vec{R} - \vec{x}^{(2)}|}\right)\right], \tag{6.60}$$

where

$$M_i(\theta) = \exp\left[i\left(\theta\vec{\tau}\cdot\frac{\vec{R} - \vec{x}^{(i)}}{|\vec{R} - \vec{x}^{(i)}|}\right)\right], \tag{6.61}$$

so that, in particular, $M_i(\pi/2) = i\vec{\tau}\cdot(\vec{R} - \vec{x}^{(i)})/|\vec{R} - \vec{x}^{(i)}|$.

Now if $\vec{x}^{(2)}$ is far from both the quark and antiquark lines, so that $(\vec{R}^{(+)} - \vec{x}^{(2)})/|\vec{R}^{(+)} - \vec{x}^{(2)}| \sim (\vec{R}^{(-)} - \vec{x}^{(2)})/|\vec{R}^{(-)} - \vec{x}^{(2)}|$, the M_2 matrices drop out when we form $\mathrm{tr}(U^{(+)}U^{(-)})$. Then

$$\mathrm{tr}(U^{(+)}U^{(-)}) \sim \mathrm{tr}\left[M_1^{(+)}\left(\frac{\pi}{2}\right)M_1^{(-)}\left(-\frac{\pi}{2}\right)\right] = \frac{(\vec{R}^{(+)} - \vec{x}^{(1)})\cdot(\vec{R}^{(-)} - \vec{x}^{(1)})}{|\vec{R}^{(+)} - \vec{x}^{(1)}||\vec{R}^{(-)} - \vec{x}^{(1)}|}\ . \tag{6.62}$$

δE is proportional to $\mathrm{tr}(U^{(+)}U^{(-)} - 1)$ which vanishes, according to the above result, whenever $|\vec{R}^{(+)} - \vec{R}^{(-)}|$, $|\vec{R}^{(-)} - \vec{x}^{(1)}|$ become large compared to $r = |\vec{R}^{(+)} - \vec{R}^{(-)}|$. Roughly speaking we get a contribution of order 1 to $\mathrm{tr}(U^{(+)}U^{(-)} - 1)$ whenever meron 1 is within a sphere of radius $\sim r$ centered about the mean position of the quark-antiquark pair and meron 2 is outside that sphere. The quark-antiquark interaction energy is obtained by integrating over meron positions and scale sizes with the weight function obtained from the meron determinant. We will not worry about the scale-size integration (which would be important if we wanted the value of the potential and not just its r dependence) and discuss the integration over position only. The determinant gives a factor of $\exp[-S_m(\Delta)]$, where Δ is the four-dimensional distance between the merons. Roughly speaking, given the geometry of the integrations and the fact that g^2 is relatively slowly varying, this should behave roughly like $r^{-6\pi^2/g^2(r)+3C_n/4}$. We have *seven* position coordinates to integrate over (the time coordinate of meron 1 is held fixed in this calculation). The $r^{-6\pi^2/g^2}$ renders the integral convergent and the geometry of the integral is such that the seven dimensional coordinate integrals must behave like r^7. In summary (and very approximately)

$$\delta E \propto r^{7-6\pi^2/g^2(r)+3C_n/4} \; . \qquad (6.63)$$

This gives an alternate estimate of the critical coupling. The quark-antiquark potential starts to increase with separation, signaling the onset of confinement, when $6\pi^2/g^2(r) = 7+3C_n/4$ or $8\pi^2/g^2(r) = \frac{28}{3}+C_n$. This is close enough to the value that emerges from the susceptibility calculation to reassure us that the underlying physics is the same. To see the actual phase transition at $\frac{32}{3}+C_n$ requires a different calculation. Cutting off the integration over Δ so that $8\pi^2/g_n^2(\Delta) - C_n \geq \frac{32}{3}$ and taking r very large yields, as we saw for instantons at the end of Sec. IV, a (finite) mass renormalization for each quark and a renormalized Coulomb interaction. The coefficient of r^{-1} (static coupling) blows up when $8\pi^2/g_n^2(\Delta)$ approaches $\frac{32}{3}+C_n$, as it should because this is where the magnetic susceptibility diverges.

Once we have complete control over the meron determinant we must redo this calculation with a view to establishing the detailed r dependence and numerical magnitude of the potential in the preconfinement region where such a computation can be trusted. Once we have a handle on the meron plasma we must also verify that the onset of a growing potential is indeed followed ultimately by a linearly rising potential. This is a harder task, but the indication of confinement seen here is suggestive and encouraging to say the least.

VII. CONCLUSIONS

In this paper we have initiated a systematic exploration of the relevant degrees of freedom and the dynamics of quantum chromodynamics. We have discovered that to a large extent the dynamical properties of QCD are a consequence of the structure of the vacuum arising from the tunneling between degenerate, classically stable vacuums, and that the relevant degrees of freedom can be taken to be the Euclidean path histories that can be used to calculate the tunneling amplitudes in the semiclassical approximation. We have seen that the nonperturbative structure of the vacuum that arises due to this tunneling explains the major features of QCD, i.e., the dimensional transmutation which determines the size of the hadrons and the strong-interaction coupling constant, the source of dynamical chiral-symmetry breaking, and the mechanism responsible for confinement.

Let us summarize the picture of QCD that has emerged from our investigation. The general structure of the θ vacuum as a superposition of n vacuums and the solution of the U(1) problem follow from general considerations. The detailed dynamics of QCD, however, depend crucially on the scale of phenomena under investigation. At very short distance asymptotic freedom guarantees that the effective coupling is small enough so that the theory is essentially free. The density of instantons of small size vanishes rapidly and at short distances the "dilute-gas approximation" applies. Since the up and down quark masses are very light, tunneling is suppressed at short distances or, alternatively, instantons of small size are tightly bound to anti-instantons. We have shown that at distances, ρ_A, corresponding to couplings of order $g^2/8\pi^2 \sim \frac{1}{20}$ chiral-symmetry breaking occurs via a Nambu–Jona-Lasinio–type mechanism generated by the effective determinantal interactions provided by instantons. At this distance therefore the quark dynamical mass will become substantial and tunneling will be restored. Alternatively, instantons of size greater than ρ_A will exist in isolation. At this distance scale the instanton gas is dilute and can be treated by mean field theory techniques. Their dynamical effects are large and significant. For example, we have seen how they substantially modify the Coulomb potential between massive quarks. They are certainly much greater in importance than the standard perturbative corrections. Physically the reason that such tunneling effects are so large for small coupling is the existence of many distinct tunneling paths (degrees of freedom for instantons). The interactions in the dilute instanton gas, which behaves as a paramagnetic medium of magnetic dipoles, cause a renormalization of the effective coupling resulting in a rapid increase in its value at a sharply defined distance $\rho_C \sim 0.2\mu^{-1}$. At this distance, where the effective coupling is of order $g^2/8\pi^2 \sim \frac{1}{10}$ instantons dissociate into merons. The latter configurations are such as to provide a mechanism for confinement of quarks. In the confinement phase we conjecture that the vacuum is dominated by configurations that can be described as a plasma of merons, leading to a picture of electric confinement in ordinary space similar to that in a magnetic superconductor. We associate ρ_C with the confinement scale or the size of a hadron and $g^2/8\pi^2 \sim \frac{1}{10}$ with the hadronic coupling constant, thereby concluding that in QCD the coupling is always relatively small and that many aspects of the strong interactions can be treated by semiclassical methods.

Much work remains to be done. The quantitative details of the chiral phase transition require further investigation. A formalism for generating systematic improvements of the dilute-gas approximation should be developed. Most important we must gain control over the meron configurations and the confinement phase where our understanding is still at a qualitative and conjectural stage. Finally we must proceed with the construction of hadronic states, and the calculation of their masses, couplings, and scattering amplitudes. This is a difficult problem in a theory such as QCD where it is not even clear what is the best way to begin the attack. Perhaps the most reasonable strategy is to construct a phenomenological model of hadrons which would play a role similar to that of the Landau-Ginzburg theory in superconductivity. Given our picture of confinement as described above, the most likely candidate for such a model is a string model. This would be a string of confined electric flux joining massive quarks. The size of the string would be given by the Debye length of the meron plasma. The dynamics of the string as well as the amplitude for the breaking of the string could be determined by Euclidean functional techniques. Such a model would be useful for calculating hadronic masses, as well as offering the possibility of constructing a model of hadrons that could be used in Minkowski space to calculate scattering amplitudes.

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