VACUUM STRUCTURE OF THE SU(3) GAUGE FIELD THEORY IN THE COULOMB GAUGE ☆

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The SU(3) gauge field is studied in the Coulomb gauge. The Gribov ambiguities arising in the Coulomb gauge are analysed. Restricting to a class of spherically symmetric vacua it is shown that there exist non-trivial vacua characterized by a topological number $\eta = 0, \pm\frac{1}{2}$, and ± 2. This must be contrasted with the spherically symmetric SU(2) vacua which are characterized by $\eta = 0, \pm\frac{1}{2}$.

Recently, Gribov [1] has pointed out that for non-abelian gauge theories, the Coulomb gauge is non-unique. In particular, it is shown that there exist in this gauge non-trivial vacua characterized by $A^i \neq 0$ which are gauge equivalent to $A^i = 0$. In an SU(2) gauge theory, Gribov shows that there exist two spherically symmetric non-trivial vacua, besides the trivial one, which are characterized by a topological invariant $\eta = \pm\frac{1}{2}$. Very recently Sciuto [2], Adamello et al. [3] and Abbott and Eguchi [4] have established the role played by the pseudoparticle solutions in bringing about tunnelling between these gauge related vacua. In this brief note we study the non-uniqueness problem in the SU(3) gauge theory, with the hope that it might shed some more light on the problem. We find several interesting features. We find that in the Coulomb gauge there are non-trivial vacua with both integer and half-integer vacua, suggesting that as the gauge group is enlarged the degeneracy of the vacuum increases in the Coulomb gauge.

In order to classify the possible vacua, it is useful to study the quantity [5]

$$\eta = \frac{1}{24\pi^2} \int d^3x\, \epsilon_{ijk}\, \mathrm{Tr}(A^i A^j A^k), \tag{1}$$

where $A^i = (1/2i)\lambda_a A^i_a$ and λ_a ($a = 1, 2, ..., 8$) are the generators of SU(3). In the Coulomb gauge the fields are transverse, $\partial_i A_i = 0$. Gribov [1] has shown that there exists a non-trivial gauge transformation U such that

$$A_i' = U^+ A_i U + U^+ \partial_i U, \quad \partial_i A_i' = 0. \tag{2}$$

Such a U is determined by extremizing the action

$$W = \int d^3x\, \mathrm{Tr}[(\partial_i U^+)(\partial_i U) - 2(\partial_i U) U^+ A_i], \tag{3}$$

subject to $U^+ U = 1$. In particular, in order to study non-trivial Coulomb gauge vacua we set $A_i = 0$ in the integral for W. We can consider two different cases of spherically symmetric solutions to eqs. (2). The first case is the Gribov solution which can always be embedded in an SU(n) gauge theory. These vacua are therefore characterized by the topological number $\eta = \pm\frac{1}{2}$. Besides the SU(2) embedding we may take the SO(3) embedding of SU(3). Thus consider a solution of the form

$$U = \exp(i\alpha(r)\phi)\exp(i\beta(r)\psi), \tag{4}$$

where

$$\phi_{ij} = x_i x_j / r^2 - \tfrac{1}{3}\alpha_{ij}, \quad \psi_{ij} = \epsilon_{ijk} x_k / r.$$

ϕ and ψ are generalized idempotents: $\phi^2 = \frac{1}{3}\phi + \frac{2}{9}$, $\psi^3 = \psi$. This decomposition for U is suggested by the fact that $(\lambda_2, -\lambda_5, \lambda_7)$ generate the SO(3) subgroup of SU(3) and with respect to these the remaining 5 generators transform like a quadrupole moment tensor. Eq. (4) may be written as

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$$U = [(\exp 2\mathrm{i}\alpha/3 - \exp - \mathrm{i}\alpha/3)\phi + \tfrac{1}{3}(\exp 2\mathrm{i}\alpha/3 + 2\exp - 2\mathrm{i}\alpha/3)] \times [1 + \mathrm{i}(\sin\beta)\psi + (\cos\beta - 1)\psi^2].$$

Substituting into eq. (3) one finds for W:

$$W = 8\pi \int \mathrm{d}r\, r^2 \left[\left(\frac{\mathrm{d}\beta}{\mathrm{d}r}\right)^2 + \frac{1}{3}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}r}\right)^2 + \frac{4}{r^2}(1 - \cos\alpha\cos\beta)\right]. \tag{5}$$

Equations for $\alpha(r)$ and $\beta(r)$ are obtained by extremizing W:

$$r(\mathrm{d}^2/\mathrm{d}r^2)(r\beta) - 2\sin\beta\cos\alpha = 0,$$

$$\tfrac{1}{3}r(\mathrm{d}^2/\mathrm{d}r^2)(r\alpha) - 2\sin\alpha\cos\beta = 0.$$

Changing from $r = \exp t$ one finds a pair of coupled pendulum equations:

$$\ddot\beta + \dot\beta - 2\sin\beta\cos\alpha = 0, \quad \ddot\alpha + \dot\alpha - 6\sin\alpha\cos\beta = 0. \tag{6}$$

The dot refers to differentiation with respect to t.

From eqs. (1) and (4) one can compute the topological number η. After a lengthy calculation one obtains the result

$$\eta = -\frac{1}{6\pi}\int_0^\infty \mathrm{d}r\, r^2 \epsilon_{ijk}\,\mathrm{Tr}(A_i A_j A_k) \tag{7}$$
$$= (2/\pi)[\beta(r) - \cos\alpha(r)\sin\beta(r)]_{r=0}^{r=\infty}.$$

In order to explore the possible allowed values of η we consider the solutions to eq. (6).

Case I. If $\beta(r) \equiv 0$ then

$$A_i = \exp(-\mathrm{i}\alpha\phi)\partial_i \exp(\mathrm{i}\alpha\phi). \tag{8}$$

Such an A_i has vanishing η. In this case, it is clear that the equation for α is $\ddot\alpha + \dot\alpha - 6\sin\alpha = 0$, which is qualitatively similar to the SU(2) case. Hence a solution with $\alpha(0) = 0$ and $\alpha(\infty) = \pm\pi/2$ exists and hence one has non-trivial vacuum A_i which is characterized by $\eta = 0$.

Case II. If $\alpha = 0$, β satisfies $\ddot\beta + \dot\beta - 2\sin\beta = 0$. This equation is identical to the Gribov pendulum equation after identifying β with 2α in Gribov's equation. Hence there exist solutions to this equation with $\beta(-\infty) = 0$ and $\beta(\infty) = \pm\pi$, giving rise to $\eta = \pm 2$.

Although this result is rather unusual, it is to be noted that there exist pseudoparticle solutions to the SU(3) gauge theory with $q = \pm 1$ and $q = \pm 4$ [6]. It is clear that the pseudoparticles with $q = \pm 4$ connect two vacua with topological quantum number $\eta = \pm 2$, while the ones with $q = \pm 1$ connect two vacua with $\eta = \pm 1/2$ as discussed in refs. [2,4]. This further demonstrates that the pseudoparticle solutions are closely related to the non-uniqueness of the Coulomb vacuum of non-abelian gauge theories.

Besides these two cases, other solutions to eq. (6) may exist as illustrated by the following analysis. The equations of motion (6) may be thought of as representing a two dimensional pendulum with friction in a potential

$$V(\alpha, \beta) = 2[\cos\sqrt{3}\,\alpha\cos\beta - 1]. \tag{9}$$

A particle starting at $\alpha(t = -\infty) = 0$, $\beta(t = -\infty) = 0$, will roll down the potential surface and will come to rest as $t \to \infty$ at the minimum points $\alpha = \pm\pi/\sqrt{3}$, $\beta = 0$ and $\beta = \pm\pi$, $\alpha = 0$. These two cases give solutions that generalize those found in case I and II, respectively. The potential (9) has, in addition, saddle points at $\alpha = \pm\pi/2, \pm 3\pi/2$, and $\beta = \pm\pi/2, \pm 3\pi/2$. Near the saddle points equations of motion can be linearized and one finds that three out of four eigenvalues represent solutions along which the particle converges to the saddle point. Thus it is possible that there exist solutions in which the particle starting from the origin comes to rest at the saddle points. In the case where $\alpha(r = 0) = \beta(r = 0) = 0$ and $\alpha(r \to \infty) = \beta(r \to \infty) = \pm\pi/2$ the topological number is ± 1. While in the other case it is ± 3.

Since we are dealing with a potential in two dimensions, it is conceivable that the pendulum may circumvent the maxima by going around them and thus it is possible to envisage solutions where $\alpha(r)$ and $\beta(r)$ approach minima beyond $\beta = \pi$ as $r \to \infty$. We are investigating these possibilities at present.

To summarize, we have shown that in an SU(3) gauge theory there exist in the Coulomb gauge besides the trivial vacuum:

(i) a non-trivial vacuum $A_1 \neq 0$ with $\eta = 0$,
(ii) vacua with topological charge $\eta = \pm\frac{1}{2}$,
(iii) non-trivial vacua with $\eta = \pm 2$.

The second case arises from the SU(2) embedding of SU(3) and case (iii) from the SO(3) embedding. Finally we have indicated that vacua characterized by $\eta = \pm 1$ and ± 3 are also possible.

References

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