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VII. Difficulties in Fixing the Gauge in Non-Abelian Gauge Theories

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Abstract:

Some problems arising from the use of the Coulomb gauge in SU(2) Yang–Mills theory are discussed. It is shown that: i) the transversality condition does not fix the gauge uniquely (Gribov ambiguity); ii) there exist physical configurations that cannot be described by a continuous A_μ in the Coulomb gauge.

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1. Introduction

A characteristic feature of any gauge theory is that the number of fields that appear in the Lagrangian is larger than the number of effective degrees of freedom of the theory. Usually one tries to eliminate the redundant variables by imposing a suitable gauge fixing condition. For instance, electrodynamics can be discussed in terms of physical variables (the transverse components of the photon) by choosing the Coulomb gauge:

$$\partial_i A_i = 0. \tag{1.1}$$

Given any configuration $A'_\mu(x)$ one can change it to a purely transverse one, satisfying eq. (1.1), by means of a gauge transformation

$$A_\mu = A'_\mu + \partial_\mu \Lambda. \tag{1.2}$$

In fact, by substituting eq. (1.2) in eq. (1.1) one gets

$$-\Delta\Lambda = \partial_i A'_i, \tag{1.3}$$

which can be inverted, leading to

$$\Lambda = -\frac{1}{\Delta}\partial_i A'_i. \tag{1.4}$$

Let us note, even if it is almost obvious, that the Laplacian $\Delta \equiv \partial_i\partial_i$ is an invertible operator only if one requires that $\Lambda(x)$ is regular everywhere and does not explode for $x \to \infty$. In fact, in order to get one and only one solution of eq. (1.3) one has to impose these boundary conditions in such a way that no solution* of the homogeneous equation

$$\Delta\Lambda = 0 \tag{1.5}$$

exists.

The aim of this lecture is to show that such a simple procedure cannot be directly extended to the non-Abelian case. We will study SU(2) Yang–Mills theory and we will use the following notation:

$$A_\mu = eA^i_\mu \cdot \sigma^i/2\mathrm{i}, \tag{1.6}$$

where e is the coupling constant and σ^i are the Pauli matrices. A gauge transformation on A_μ gives

$$A'_\mu = U^{-1}A_\mu U + U^{-1}\partial_\mu U, \tag{1.7}$$

where the matrices $U(x)$ are SU(2) group elements; $U(x)$ can be parametrized in the quaternionic form:

$$U(x) = U_4 + \mathrm{i}\sigma^i U_i, \tag{1.8}$$

where $U_a(x)$ $(a = 1,2,3,4)$ is a unit four-vector which lies on the unit sphere S_3:

$$U_a U_a = 1. \tag{1.9}$$

* Apart from the trivial one $\Lambda = \text{const.}$ which does not affect A_μ.

If, starting from any $A_\mu(x)$, one tries to gauge transform it by means of eq. (1.7) to get an equivalent potential A'_μ that satisfies the transversality condition

$$\partial_i A'_i = 0, \tag{1.10}$$

one is led to consider the equation

$$\partial_i A_i + [\mathrm{D}_i(A), \partial_i U \cdot U^{-1}] = 0, \tag{1.11}$$

where $\mathrm{D}_i(A)$ is the covariant derivative

$$\mathrm{D}_i(A) = \partial_i + A_i. \tag{1.12}$$

In contrast to its Abelian analogue (1.3), the eq. (1.11) in general does not fix the gauge transformation uniquely. To be more precise, as Gribov [1] was the first to show, for large enough fields the eq. (1.11) admits *several solutions*; later it was shown [2–4] that there exist also configurations such that eq. (1.11) has *no regular solution*. In the next sections we will discuss in detail these different cases, starting with the analysis of the classical vacuum structure.

Throughout this lecture we will limit our considerations to fields that become a pure gauge at very large spatial or temporal distances; that is we always suppose:

$$\lim_{R\to\infty} A_\mu = U^{-1}\partial_\mu U, \tag{1.13}$$

where $R = \sqrt{x^2}$ is the four-dimensional Euclidean distance from the origin. Then regular configurations $A_\mu(x)$ will always have finite Euclidean action and Pontryagin number.

2. The vacuum structure

In this section we will work at a *fixed time* x_4 and we will discuss the classical vacuum degeneracy in the Coulomb gauge. For pure gauge fields

$$A_\mu = U^{-1}\partial_\mu U, \qquad F_{\mu\nu} = 0, \tag{2.1}$$

the gauge fixing condition (1.11) reduces to the simple form

$$\partial_i(U^{-1}\partial_i U) = 0, \tag{2.2}$$

which is the non-Abelian analogue of the homogeneous equation (1.5). In order to study eq. (2.2), we must first carefully make the boundary conditions precise. We will consider two kinds of boundary conditions.

2.1. Strong boundary conditions (SBC)

One can impose that the limit of the group element $U(x)$ for large distance exists and *does not depend on the direction*:

$$\lim_{r\to\infty} U(r, \theta, \varphi) = \text{const.}; \tag{2.3}$$

in eq. (2.3) we have used spherical coordinates: $r = \sqrt{x^2}$, θ is the colatitude and φ is the azimuth. In terms of the potential A_μ, given by eq. (2.1), the condition (2.3) means that A_i vanishes faster than $1/r$ for $r \to \infty$

$$\lim_{r\to\infty} rA_i(x) = 0. \tag{2.4}$$

So the condition (2.3) is actually very strong and it excludes relevant physical configurations; for instance in presence of a magnetic monopole the potential A_i has just a $1/r$ behaviour. However the strong boundary condition (2.3) has the advantage that it compactifies the R_3 space to a sphere S_3 (by identifying all the points at infinity of R_3). Then the mapping $x \to U(x)$ is a mapping $S_3 \to S_3$ and it is characterized by an homotopy class in $\pi_3(S_3)$. Therefore one can define a topological number

$$\Phi = \frac{1}{24\pi^2}\int d^3x\, \varepsilon_{ijk}\, \mathrm{Tr}\,(A_iA_jA_k) = -\frac{1}{2\pi^2}\int_{I(R_3)} d\alpha\, d\beta\, d\gamma\, \sin^2\alpha \sin\beta. \tag{2.5}$$

In the last paragraph we have used the expression (2.1) for A_i, the parametrizations (1.8) and

$$U_a = \begin{pmatrix} \sin\alpha\sin\beta\cos\gamma \\ \sin\alpha\sin\beta\sin\gamma \\ \sin\alpha\cos\beta \\ \cos\alpha \end{pmatrix}. \tag{2.6}$$

It is clear by eq. (2.5) that Φ must be an integer [for continuous mapping satisfying eq. (2.3)], namely, the number of times the group manifold S_3 is covered by the image $I(R_3)$ of the space R_3.

2.2. *Weak boundary conditions* (*WBC*)

Another possibility is that the limit for large r of the group element $U(x)$ exists but *does depend on the direction*:

$$\lim_{r\to\infty} U(r,\theta,\varphi) = U(\theta,\varphi). \tag{2.7}$$

In such a case, which allows for magnetic monopoles, the potential $A_i(x)$ has the following asymptotic behaviour:

$$A_i(x) \underset{r\to\infty}{\sim} O(1/r). \tag{2.8}$$

If the weak boundary conditions (2.7) are imposed, the space R_3 is no longer compactified to S_3 but rather to the ball B_3 in 3-dimensional space. Hence the quantity Φ defined in eq. (2.5) loses any topological meaning* and a priori can be any real number; the image $I(R_3)$ can cover the group manifold S_3 an incomplete number of times, as the image of the boundary ∂B_3 does not have to shrink to a point.

Let us note that SBC or WBC can be imposed not only on the vacuum states but on any configuration satisfying eq. (1.13). It is important to note that one can consistently choose either weak

* A topological meaning of Φ can be recovered if one imposes some further conditions on $U(\theta,\phi) \equiv U(x/r)$ defined in (2.7). For instance by requiring

$$U(-x/r) = \pm U(x/r) \tag{2.9}$$

one gets that ϕ must be integer or half integer respectively [5].

or strong boundary conditions; in fact one can prove [3] by quite elegant semiclassical arguments, both in the temporal and in the Coulomb gauge, that no tunnelling between configurations with different boundary conditions can occur. In most of the rest of this lecture we shall use the strong boundary conditions (2.3); only in section 5 we shall briefly discuss the weak boundary conditions (2.7).

The vacuum structure in the Coulomb gauge is completely clarified by the following *theorem* (ref. [3], see also ref. [6]): the transversality condition (1.1), with the strong boundary conditions (2.3), fixes the vacuum uniquely, i.e. the only solution of eq. (2.2) is $U(\boldsymbol{x}) = \text{const.}$ which implies $A_i(\boldsymbol{x}) \equiv 0$.

The proof [3] consists of two steps.

i) The SBC imply that, for large r, $U(\boldsymbol{x})$ has the following asymptotic form:

$$U(\boldsymbol{x}) = M + N(\boldsymbol{x}), \tag{2.10}$$

where $M = \text{const.}$ and $N(\boldsymbol{x})$ vanishes for $r \to \infty$. By substituting eq. (2.10) in eq. (2.2) one gets that the asymptotic behaviour of $N(\boldsymbol{x})$ is the following:

$$N(\boldsymbol{x}) \underset{r\to\infty}{=} \mathrm{O}(1/r). \tag{2.11}$$

ii) Eq. (2.2), that in terms of the four-vector $U_a(\boldsymbol{x})$ takes the form:

$$\Delta U_a = -(\partial_i U_b \cdot \partial_i U_b) U_a, \tag{2.12}$$

can be imagined to stem from a variational principle applied to the "action" of the non-linear σ model in three dimensions:

$$W = \int \mathrm{d}^3 x\, \partial_i U_a\, \partial_i U_a; \qquad U_a U_a = 1. \tag{2.13}$$

It is easy to prove, by scaling arguments, that $U_a(\boldsymbol{x}) = \text{const.}$ is the only solution of eq. (2.12) with *finite* "action" W. However from (2.10) and (2.12) we see that SBC lead to a finite W [if $U_a(\boldsymbol{x})$ is regular everywhere, as we always assume]; then SBC imply that the only solution of (2.12) is $U_a(\boldsymbol{x}) = \text{const.}$ – Q.E.D.

We anticipate, as we shall see in section 5, that, on the contrary, if one uses WBC the Coulomb condition (2.1) is not able even to fix the vacuum uniquely. In the next section we shall show that the SBC, which are able to remove any ambiguity of the Coulomb gauge for the vacuum states, also allow the existence of several different solutions of eq. (1.11), if the field A_μ is large enough.

3. The Gribov ambiguity [1]

Still assuming SBC, let us discuss the Coulomb gauge eq. (1.11) for non-vanishing $F_{\mu\nu}$. Let us consider a simple example*, starting from a field which is already transverse:

$$A_i = \mathrm{i}\varepsilon_{ijk} \frac{x_j \sigma_k}{r^2} f(r) \tag{3.1}$$

* We do not write the time dependence explicitly as we shall work at fixed time throughout this section.

where $f(r)$ is any smooth function with the boundary conditions:

$$\lim_{r\to 0} f(r) = O(r); \qquad \lim_{r\to\infty} f(r) = 0; \tag{3.2}$$

the first condition is necessary to avoid singularities at $x = 0$ and the second one is the transcription of the SBC [eq. (2.4)]. We will consider the subclass of gauge transformations which preserve the spherical symmetry of A_i:

$$U = \exp(\mathrm{i}\alpha(r)\boldsymbol{\sigma}\cdot\boldsymbol{x}/r); \tag{3.3}$$

regularity of A_i at the origin and SBC (2.3) fix the asymptotic behaviour of $\alpha(r)$:

$$\begin{aligned} &\text{a)} \quad \alpha(r) \xrightarrow[r\to 0]{} n\pi + \gamma r \\ &\text{b)} \quad \alpha(r) \xrightarrow[r\to\infty]{} m\pi, \end{aligned} \tag{3.4}$$

where m and n are integers and γ is an arbitrary real constant. By inserting eq. (3.1) and eq. (3.3) in eq. (1.11) one gets the following form of the Coulomb gauge condition [1]:

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\alpha}{\mathrm{d}r} - \frac{\sin 2\alpha}{r^2}(1 + 2f) = 0. \tag{3.5}$$

By the change of variables

$$s = \ln r \tag{3.6}$$

eq. (3.5) becomes the equation of a damped pendulum with an external force (fig. 1):

$$\ddot{\alpha} + \dot{\alpha} - \sin 2\alpha\cdot(1 + 2f) = 0, \tag{3.7}$$

where

$$\dot{\alpha} = \mathrm{d}\alpha/\mathrm{d}s, \qquad \ddot{\alpha} = \mathrm{d}^2\alpha/\mathrm{d}s^2.$$

The boundary conditions (3.4) become

$$\begin{aligned} &\text{a)} \quad \alpha(s) \xrightarrow[s\to -\infty]{} n\pi + \gamma e^s \qquad \text{implying } \dot{\alpha}(s) \xrightarrow[s\to -\infty]{} \gamma e^s \\ &\text{b)} \quad \alpha(s) \xrightarrow[s\to +\infty]{} m\pi. \end{aligned} \tag{3.8}$$

Hence the pendulum starts at "time" $s = -\infty$ from the unstable equilibrium position $\alpha(s) = n\pi$ with vanishing velocity. Then, if $f(s) \geqslant -\frac{1}{2}$, only two possibilities are allowed:

i) The pendulum does not move from the unstable equilibrium position [this corresponds to choose $\gamma = 0$ in (3.8a)].

ii) After some oscillations the pendulum falls down in the stable equilibrium point

$$\alpha(s) \to (n \pm \tfrac{1}{2})\pi \qquad \text{as} \qquad s \to +\infty. \tag{3.9}$$

The case (i) corresponds to the trivial solution $U(x) = \pm 1$; the case (ii) is forbidden by the strong boundary conditions (3.8b). Therefore in this particular case we obtain again that the vacuum is not degenerate (with SBC), as we have proved in general in section 2. However, if for a sufficiently long period of "time" the external force $f(s)$ is negative enough (fig. 2), $\alpha(s)$ can start from 0 (or $n\pi$), move away ($\gamma \neq 0$), come back to 0 (or $m\pi$) under the effect of the external force $f(s)$, and finally remain there.

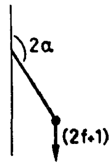

Fig. 1.

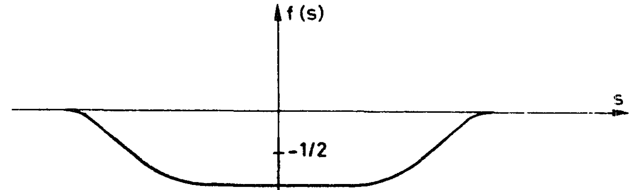

Fig. 2.

Such a kind of solution of (3.7) [and thus of (1.11)] is not trivial and satisfies the SBC (2.3); hence we have shown, following Gribov, that several different solutions of the gauge fixing conditions (1.10) and (1.11) can exist, even if SBC are imposed. Then the transversality condition (1.11) does not actually fix the gauge completely but leaves a certain amount of ambiguity.

This phenomenon can be better understood if one looks for *infinitesimal* gauge transformations which preserve transversality; in other words, we wonder if transverse potentials A_i ($\partial_i A_i = 0$) exist, such that one can find an infinitesimal transformation,

$$U(\boldsymbol{x}) = I + \mathrm{i}\varepsilon(\boldsymbol{x}),$$
$$\varepsilon(\boldsymbol{x}) = \varepsilon_i(\boldsymbol{x})\cdot\sigma_i, \qquad \varepsilon_i \ll 1, \tag{3.10}$$

that sends A_i into an A'_i which is still transverse ($\partial_i A'_i = 0$). Under these hypotheses eq. (1.11) becomes

$$0 = \partial_i[\partial_i + A_i, \varepsilon] \stackrel{\text{def.}}{=} \Delta(A)\varepsilon. \tag{3.11}$$

One immediately realizes that the determinant of the operator $-\Delta(A)$ is the Faddeev–Popov determinant for the Coulomb gauge [1]. Hence, infinitesimal transformations preserving transversality exist only for those configurations A_i which correspond to a vanishing eigenvalue of $\Delta(A)$ and thus to vanishing Faddeev–Popov determinant. The eigenvalue equation of $\Delta(A)$ is

$$-\Delta(A)\varepsilon = \lambda\varepsilon \tag{3.12}$$

or, explicitly

$$-\Delta\varepsilon - \partial_i[A_i, \varepsilon] = \lambda\varepsilon. \tag{3.12'}$$

One realizes by (3.12') that, for A_i vanishing or small, $-\Delta(A)$ has only positive eigenvalues; however if A_i increases, an eigenvalue λ_1 can cross zero and change sign; if A_i still increases, another positive eigenvalue λ_2 can change sign, and so on. Then one can divide the space of transverse configurations $A_i(\partial_i A_i = 0)$ into regions with different sign of the Faddeev–Popov determinant, separated by boundaries, where the Faddeev–Popov determinant vanishes (fig. 3) [1]. The situation is well illustrated by the previous example; if one considers infinitesimal transformations ($\alpha \ll 1$), eq. (3.5) becomes

$$0 = \Delta\alpha(r) - 2\frac{1 + 2f(r)}{r^2}\alpha(r) \stackrel{\text{def.}}{=} \Delta(A)\alpha, \tag{3.13}$$

where Δ is the Laplacian in spherical coordinates, and $-\Delta(A)$ is the Faddeev–Popov operator. The eigenvalue equation for $\Delta(A)$ is

$$-\Delta(A)\alpha = \lambda\alpha, \tag{3.14}$$

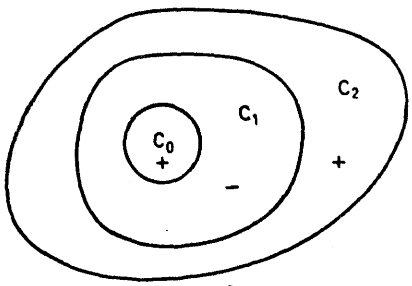

Fig. 3.

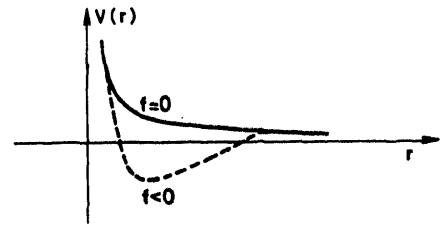

Fig. 4.

that is

$$-\Delta\alpha + 2\frac{1 + 2f(r)}{r^2}\alpha = \lambda\alpha. \tag{3.14'}$$

Equation (3.14′) looks like the radial part of a Schrödinger equation; it has no bound states for $f = 0$, but it has an increasing number of negative eigenvalues as $f(r)$ becomes more and more negative (fig. 4); hence the picture of fig. 3 is confirmed. In his second paper Gribov [1] has found another interesting result; for any transverse configuration close to a boundary (fig. 3), there exists another gauge equivalent transverse configuration again close to the boundary but on the other side of it. The situation in the configuration space of A_i can then be depicted as in fig. 5, where the horizontal line represents the hypersurface corresponding to the transversality condition $\partial_i A_i = 0$. The numbered lines represent the orbits generated by gauge transformations; two lines labelled by different numbers represent different physical situations, while the orbits labelled by the same number but with a different number of primes represent the same physical situation, expressed in gauges which are not continuously deformable to each other (we will come back to this point later). The points where the Faddeev–Popov determinant vanishes (the boundaries in fig. 3) are represented in fig. 5 by •.

We shall conclude this section by noting that the Coulomb gauge is unambiguous (each orbit is crossed once and only once by the hypersurface $\partial_i A_i = 0$) only in a *sub*region of C_0, close to the potential $A_i \equiv 0$ [in fig. 5 the orbit 3″ crosses the hypersurface $\partial_i A_i = 0$ once with $\det(-\Delta(A_i)) > 0$ – region C_0 – and once with $\det(-\Delta(A_i)) < 0$ – region C_1; hence there exist fields in C_0 which admit Gribov copies].

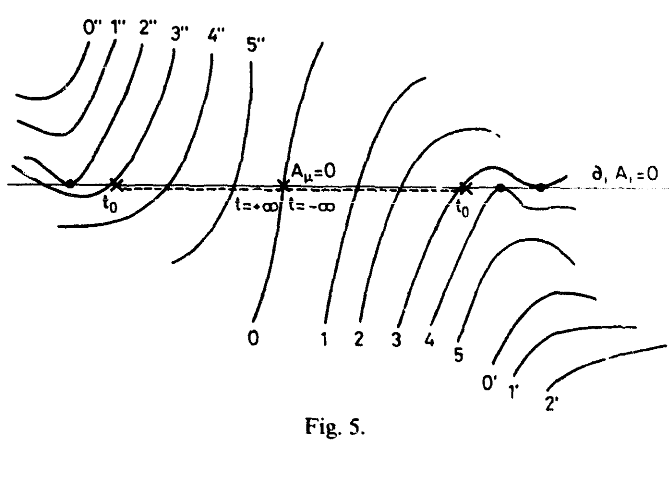

Fig. 5.

4. Configurations not attainable continuously from the Coulomb gauge

In this section we are going to show [2–4] that there are physical time-dependent configurations $A_\mu(x)$ that cannot be gauge transformed, in a smooth way, to satisfy the transversality condition (1.10) together with the SBC [(1.13) and (2.3)]. In fact let us consider the Pontryagin number*

$$q = -\frac{1}{32\pi^2}\int d^4x\, \varepsilon_{\mu\nu\alpha\beta}\, \mathrm{Tr}\,(F_{\mu\nu}F_{\alpha\beta}). \tag{4.1}$$

If A_μ and $F_{\mu\nu}$ are regular everywhere Gauss's theorem allows us to transform the right-hand side of (4.1) into an integral over a very large close surface S_3' homotopic to S_3; by using (1.13), (1.8) and (2.6) and by a suitable choice of S_3' one can write

$$q = \Phi_+ - \Phi_- + \Phi_L, \tag{4.2}$$

where $\Phi_\pm$ are the quantities defined in eq. (2.5), calculated at time $x_4 = \pm\infty$, and the lateral flux Φ_L is given by

$$\Phi_L = \frac{1}{8\pi^2}\int_{-\infty}^{+\infty} dx_4 \lim_{r\to\infty}\int d^2S_i\, \varepsilon_{ijk}\, \mathrm{Tr}\,(A_jA_kA_4). \tag{4.3}$$

Now one immediately realizes that SBC (2.4), together with the requirement that A_4 is smooth everywhere** (also for $r \to \infty$), force Φ_L to vanish. Moreover by the theorem proved in section 2 we know that SBC prevent vacuum degeneracy and then imply $\Phi_+ = \Phi_- = 0$.

Therefore we have proved that only configurations with trivial topology ($q = 0$) can be obtained in the Coulomb gauge, with SBC, in a continuous way. However, one has to note that Jackiw, Muzinich and Rebbi [4] were able to prove, by studying spherically symmetrical configurations, that the single instanton [8] can be written in the Coulomb gauge with SBC if a *time discontinuity* is allowed. A possible time evolution of this kind is drawn in fig. 5 by a dashed line. One starts at $t = -\infty$ from the vacuum $A_\mu \equiv 0$ and then the field increases (moving towards right in fig. 5). One cannot however overcome the point • where the Faddeev–Popov determinant vanishes, because in such a case one could not reach orbits like 5 or 0′. The orbit 0′ represents pure gauge fields of the form

$$A_\mu = U^{-1}\partial_\mu U, \tag{4.4}$$

with

$$U(x) \sim \exp\,[i\alpha(r)\boldsymbol{\sigma}\cdot \boldsymbol{x}/r], \qquad \alpha(0) = 0, \qquad \alpha(\infty) = \pi, \tag{4.5}$$

where $\sim$ means homotopic. The mapping $\boldsymbol{x} \to U(\boldsymbol{x})$ given in (4.5) is not topologically trivial and cannot be continuously deformed to the trivial one; for such an $U(\boldsymbol{x})$ the topological number Φ defined in (2.5) is 1. Therefore it is clear from the theorem of section 2 that the orbit 0′ cannot cross the hypersurface $\partial_i A_i = 0$. However if at a certain time t_0 one performs a non-trivial gauge transformation U^{-1}, with U given by (4.5), one jumps to the left of fig. 5 and can continue to move

* For simplicity we work in Euclidean space-time.

** Of course if one admits that A_4 can become singular as $r \to \infty$, one can obtain any Φ_L, and also tunnel from SBC to WBC (see refs. [4] and [7]).

to the right until reaching the vacuum again at time $x_4 = +\infty$. Of course if one considers a multi-instanton built by instantons well separated in time, one can repeat this procedure as many times as it is necessary.

5. The case of weak boundary conditions

If one uses WBC [eq. (2.7)] instead of SBC the situation becomes much more complicated. One immediately realizes [1] that now even the vacuum is degenerate; in fact the theorem of section 2 does not apply anymore. Looking at the spherical situation (3.1), (3.3) with $f(r) \equiv 0$, one sees that now eq. (3.7) has non-trivial solutions, as the behaviour (3.9) is no longer excluded. Hence the vacuum has at least a degeneracy depending on four continuous parameters: the three coordinates of the origin and the size γ.

A complete analysis of the vacuum structure in the general case, with WBC, is not available at present; however a classification of some a priori possible configurations is given in ref. [3].

Of course when the field increases one still has all the ambiguity which is present in the SBC case, but moreover, one has extra-ambiguities of the same kind as the vacuum degeneracy.

One could hope that, having paid such a price in terms of ambiguity, at least one would be free from discontinuities. In fact the vacua with the boundary conditions (3.9) have Φ [defined in (2.5)] equal to $\mp\frac{1}{2}$, respectively; then, as is shown in ref. [2], it is possible to describe the single instanton [8] configuration by a continuous $A_\mu(x)$. However, even if a rigorous proof does not exist, it is almost certain that multi-instanton configurations cannot be written in a continuous way in the Coulomb gauge even if WBC are used [3].

6. Final remarks

The Coulomb gauge situation can be summarized in the following way. There are two possible Hilbert spaces, corresponding to the choice of SBC or of WBC. These two spaces do not communicate with each other (theorem proved in ref. [3]) at least at the semiclassical level. The space with SBC has a single classical vacuum which is the usual one $A_\mu(x) \equiv 0$; on the other hand, the space with WBC has two classes of degenerate vacua (the Gribov ones, with $\phi = \pm\frac{1}{2}$) which can tunnell into each other via the single instanton. In both spaces there are ambiguities, and configurations with topological charge $|q| \geqslant 1$ (SBC) or $|q| \geqslant 2$ (WBC) cannot be described by a smooth $A_\mu(x)$.

We can conclude our lecture by wondering if the pathologies we have described so far are properties of the Coulomb gauge and disappear by using other gauges. As a matter of fact, Gribov [1] has shown that ambiguities are present also in the covariant gauge $\partial_\mu A_\mu = 0$, and Montonen [9] has shown that pathologies quite similar to those we have described affect the unitary gauge of spontaneously broken gauge theories. More generally, a theorem due to Singer [10] states that, if the gauge field is defined *on the manifold* S_4, it is impossible to find a continuous and unambiguous gauge fixing condition. The Coulomb gauge with SBC can be seen as a particular case of this theorem, as it forces all the points at infinity of the space time R_4 to have the same image under the mapping $x \to U(x)$. However, there exist other gauges which do not satisfy the hypotheses of the Singer theorem, like the temporal or the axial gauge, where some direction at infinity is selected out. In these gauges, or in their improved versions [11, 12] no ambiguities or discontinuities arise. It seems then safer to work in these gauges and to drop the Coulomb one. However, one has to quote that Gribov [1] and Bender, Eguchi and Pagels [13] have proposed using the peculiar features of the Coulomb gauge to obtain quark confinement; however, their results seem to be gauge dependent [14, 15] and do not yet give a final answer to this challenging problem of QCD.

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