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VORTEX-LINE MODELS FOR DUAL STRINGS

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Abstract: We call attention to the possibility of constructing field theories for the dual string. As an example, we show that a Higgs type of Lagrangian allows for vortex-line solutions, in analogy with the vortex lines in a type II superconductor. These vortex lines can approximately be identified with the Nambu string. In the strong coupling limit we speculate that the vortex lines make up all low energy phenomena. It turns out that this strong coupling limit is "super quantum mechanical" in the sense that the typical action of the theory is very small in comparison with Planck's constant.

1. Introduction and motivation

The many good experimental and theoretical features of the Veneziano model seem to suggest that an underlying string structure [1–4] of hadronic matter is a likely possibility. On the other hand it is not unreasonable to expect that some field theory describes relativistic physics (presumably including strong interactions and certainly electromagnetic interactions). For example, crossing symmetry which is crucial for the Veneziano model, is based on ideas taken from field theory. It is therefore of interest to see if one can cook up a local field theory which gives string structures behaving like dual strings. The spectrum of the Veneziano model would then be brought into contact with a local field theory.

It is the purpose of the present article to point out that it is easy to build up classical field theories allowing for vortex lines (or similar string-like structures) with the property of having equations of motion identical with those of the Nambu dual string [2]. The string description (and thus also the dual string equations of motion) are obtained only in the approximation where the radius of curvatures of the string is much larger than the width of the string. The width can be computed in terms of the parameters of the specific field theory model that one uses.

There are several motivations for being interested in field theories for dual strings:

(i) We have good reasons to believe that both field theory (of a kind which is so far not known) and dual strings (with some yet unknown degrees of freedom) are in fact realized in nature. It is therefore likely that nature has decided to merge some field theory with some dual string structure.

(ii) One may hope that by building some field theory for the dual string one is able to secure that the good properties of the former are inherited by the latter. Thus, for example one might hope that by choosing a field theory with positive definite Hamiltonian one might avoid tachyons in the corresponding dual string model. As we shall see by an example, this hope is apparently not supported when quantum mechanics is taken into account, at least not if this is done in the sloppy way of this paper. But we can still hope that a more careful treatment of quantization can resolve this difficulty.

(iii) One may hope to understand the features of dual strong models better by considering the corresponding field theories, and it would be of great interest to translate the requirement of a critical dimension ($d = 26$ in the conventional model, $d = 10$ in the Neveu-Schwarz model) into a field theoretic language. In particular, the question is as to whether the unphysical amount of dimensions needed in dual models [4] may be related to internal symmetries of the hadrons. Also, the possibility of understanding what happens when a couple of strings collide seems to have a chance in field theory.

(iv) We may use field theory to get ideas for how to modify the existing dual (string) models to get perhaps some day the right model. The trouble with generalizing dual models is that they are so tight, whereas field theory allows many possibilities.

As the main example of a field theory with a dual string structure we consider the theory of an Abelian gauge field coupled to a charged scalar field. This model has been used by Higgs to illustrate the Higgs mechanism. The relevance to dual models of an Abelian gauge field was first pointed out by us in ref. [5]. The Higgs model may also be considered as a relativistic generalization of the Ginzburg-Landau phenomenological field theory of superconductivity (see ref. [6]). In the Ginzburg-Landau case one knows the existence of a vortex-line solution, and it is exactly this fact which allows us to connect the Higgs-type of Lagrangian with the dual string, provided we identify the vortex-line with the dual string.

It turns out (see end of sect. 4) that the limit in which the Higgs Lagrangian gives the dual string solution is a sort of super-quantum mechanical limit. Although this may be bad from the point of view of quantization, it is still interesting that in order to understand hadronic structure in the sense of dual strings one has (in some sense) to consider the quantum of action h to be very large. That is to say, the typical action in the theory is much smaller than the fundamental quantum of action given by Planck's constant h.

In sect. 2 we consider the Higgs Lagrangian in the static case, and it turns out that this Lagrangian becomes identical to the Ginzburg-Landau free energy in the theory of type II superconductors. We recapitulate the most relevant features of this theory, and we discuss how the vortex solution comes out. In sect. 3 we show by a very general argument how the vortex solution leads to a Nambu Lagrangian [1]. This argument is very independent of the specific features of the Lagrangian. In sect. 4 we deal with the problem of getting the width of the vortex line sufficiently small in order that we can get a sufficiently good approximation to the dual string. It is shown that we have to consider the strong coupling limit. In sect. 5 we give a very brief and preliminary discussion of the problem of quantization. In an appendix we deal with the case of a Yang-Mills Lagrangian.

2. An example of a static vortex solution of the Ginzburg-Landau type

In this section we shall consider a special example of a field theory with a vortex solution. We start from the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}|(\partial_\mu + ieA_\mu)\phi|^2 + c_2|\phi|^2 - c_4|\phi|^4 , \tag{2.1}$$

and the equations of motion are

$$(\partial_\mu + ieA_\mu)^2\phi = -2c_2\phi + 4c_4\phi^2\phi^* , \tag{2.2}$$

$$\partial^\nu F_{\mu\nu} \equiv j_\mu = \tfrac{1}{2}ie(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*) + e^2A_\mu\phi^*\phi . \tag{2.3}$$

We are now looking for a solution of the vortex type‡. In this case the field $F_{\mu\nu}$ has a simple meaning: the field F_{12} measures the number of vortex lines (going in the 3-direction) which pass a unit square in the (12)-plane. This interpretation is identical to the one proposed by us in ref. [5]. We want to identify the vortex line with a dual string, and it is thus necessary that the flux is quantized. To see that this is the case, we use that the flux is given by

$$\Phi = \int F_{\mu\nu}\,\mathrm{d}\sigma^{\mu\nu} = \oint A_\mu(x)\,\mathrm{d}x^\mu , \tag{2.4}$$

where $\mathrm{d}\sigma_{\mu\nu}$ is a two-dimensional surface element in Minkowski-space. Writing

$$\phi = |\phi|\,\mathrm{e}^{i\chi} , \tag{2.5}$$

we get from eq. (2.3)

$$A_\mu = \frac{1}{e^2}\,\frac{j_\mu}{|\phi|^2} - \frac{1}{e}\partial_\mu\chi . \tag{2.6}$$

Next let us perform the integration in (2.4) over a closed loop without any current. Then

$$\Phi = \oint A_\mu(x)\,\mathrm{d}x^\mu = -\frac{1}{e}\oint \partial_\mu\chi(x)\,\mathrm{d}x^\mu . \tag{2.7}$$

‡ Most of the results obtained in this section are known in the theory of type II superconductors (see e.g. ref. [6]). The main new result is the identification of the Ginzburg-Landau theory with the static solution of the Higgs type of Lagrangian (2.1).

The line integral over the gradient of the phase of ϕ does not necessarily vanish. The only general requirement on the phase is that ϕ is single valued, i.e., χ varies by $2\pi n$ (n = integer) when we make a complete turn around a closed loop. Thus,

$$\Phi = n\Phi_0 , \qquad \Phi_0 = -\frac{2\pi}{e} . \tag{2.8}$$

Thus, the flux of vortex lines is quantized, $-2\pi/e$ being the quantum.

We still have to show that the equations of motion (2.2) and (2.3) allow a string-like solution. Let us consider the static case, with a gauge choice $A_0 = 0$. We look for a cylindrically symmetric solution, with axis along the z-direction. We write

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\boldsymbol{r} \times \boldsymbol{e}_z}{r} |A(r)| , \tag{2.9}$$

where $\boldsymbol{e}_z$ is a unit vector along the z-direction. The flux is given by

$$\Phi(r) = 2\pi r |A(r)| , \tag{2.10}$$

so that

$$|H| = \frac{1}{2\pi r} \frac{\mathrm{d}}{\mathrm{d}r} \Phi(r) = \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} (r|A|) . \tag{2.11}$$

With cylindrical symmetry around the z-axis the equations of motion (2.2) and (2.3) give

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}|\phi|\right) + \left[\left(\frac{1}{r} - e|A|\right)^2 - 2c_2 + 4c_4|\phi|^2\right] |\phi| = 0 , \tag{2.12}$$

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r|A|)\right) + |\phi|^2 \left(|A|e^2 - \frac{e}{r}\right) = 0 . \tag{2.13}$$

The exact solution of these two equations has so far not been obtained analytically. We shall be content with a solution of the type where

$$|\phi| \simeq \text{const (for large } r) . \tag{2.14}$$

If we treat $|\phi|$ as a constant, eq. (2.13) can be solved without further approximatio One finds, with c a constant of integration,

$$|A| = \frac{1}{er} + \frac{c}{e} K_1(e|\phi|r) \underset{r\to\infty}{\longrightarrow} \frac{1}{er} + \frac{c}{e}\sqrt{\frac{\pi}{2e|\phi|r}}\, \mathrm{e}^{-e|\phi|r} + \text{lower order terms.} \tag{2.1:}$$

Eq. (2.11) then gives

$$|H| = c|\phi| K_0(e|\phi|r) \underset{r\to\infty}{\longrightarrow} \frac{c}{e}\sqrt{\frac{\pi|\phi|}{2er}}\, \mathrm{e}^{-e|\phi|r} + \text{lower order} . \tag{2.1}$$

Eq. (2.12) is then approximately satisfied if

$$|\phi| \simeq \sqrt{\frac{c_2}{2c_4}}\,, \tag{2.17}$$

with c_2 and c_4 being large so as to take care of deviations of $|A|$ from $1/er$.

We next define the characteristic length λ (called the penetration lenrth in superconductivity),

$$\lambda = \frac{1}{e|\phi|} = \sqrt{\frac{2c_4}{e^2 c_2}}\,. \tag{2.18}$$

λ thus measures (see eq. (2.16)) the region over which the field H is appreciably different from zero.

To estimate the variation of $|\phi|$ we notice that eq. (2.17) gives the minimum of the potential, i.e.,

$$|\phi| = \phi_0 = \sqrt{\frac{c_2}{2c_4}}\,, \tag{2.19}$$

is the vacuum-value of the field $|\phi|$. Let us write

$$|\phi| = \phi_0 + \rho(x)\,,$$

where $\rho(x)$ give the fluctuations around the vacuum. The first derivatives of the potential

$$\tfrac{1}{2}(-c_2\phi\phi^* + c_4\phi^2\phi^{*2})\,, \tag{2.20}$$

vanish, whereas the second derivative is

$$2c_4|\phi_0|^2 = 2c_2\,, \tag{2.21}$$

which is the mass square of the scalar particle in Higg's mechanism. Then the oscillations in the potential are

$$2c_2\,\rho(x)^2 \tag{2.22}$$

leading to a solution for ρ of the Yukawa-type,

$$\rho(x) \sim e^{-\sqrt{2c_2}\,r}\,. \tag{2.23}$$

We then define a new characteristic length ξ.

$$\xi = \frac{1}{\sqrt{2c_2}}\,. \tag{2.24}$$

Thus, ξ measures the distance that is takes before the field $|\phi|$ reaches it vacuum value.

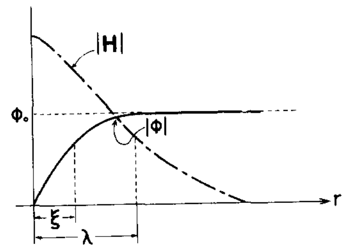

Fig. 1. An example of the behaviour of the fields $|H|$ and $|\phi|$ for a vortex solution.

In fig. 1 we have illustrated the behaviour of the fields $|H|$ and $|\phi|$. It is seen that if ξ and λ are of the same order of magnitude, then we have a well-defined vortex line, or a well defined string. The vacuum state is described by $H = 0$, and $|\phi| = \sqrt{\frac{1}{2}c_2/c_4}$, and the extension of the string is given by $\xi \sim \lambda$. The main point of this section is thus that the Higgs type of Lagrangian (2.1) allows a string-like solution. This is simply due to the fact that the Higgs Lagrangian is a relativistic generalization of the Ginzburg-Landau Lagrangian, which is well known to have vortex solutions.

The constant of integration c introduced in eq. (2.15) is determined by the requirement that the flux $\Phi(r) = 2\pi r|A(r)|$ shall go to zero for $\xi \ll r \ll \lambda$. Now, for $0 < e|\phi|r \ll 1$ we have

$$K_1(e|\phi|r) \approx \frac{1}{e|\phi|r} \tag{2.25}$$

and consequently

$$c = -e|\phi| . \tag{2.26}$$

3. The Nambu Lagrangian from vortex-line structure

In the preceeding section we pointed out that the Higgs-type of Lagrangian

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}|(\partial_\mu + ieA_\mu)\phi|^2 + c_2|\phi|^2 - c_4|\phi|^4 , \tag{3.1}$$

has solutions of the vortex type. By a suitable choice of parameters we can arrange that the width of the vortex-line is much smaller than the radius of curvature of the vortex-line. Of course, in addition to the vortex solution, (3.1) certainly has other solutions. In the next section we shall discuss how to handle some of the other solutions.

In this section we shall concentrate on the vortex contribution to the Lagrangian (3.1), $\mathcal{L}_{\text{vortex}}$ say. We shall assume that the other solutions can be effectively decoupled from the vortex solution, so that it has a meaning to separate out the special vortex-line in the Lagrangian.

In the last section we saw that by choosing the characteristic length λ sufficiently small, the field $F_{\mu\nu}(x)$ is non-vanishing only in a small region, whereas the field $|\phi|$ is nearly always equal to its vacuum value $\sqrt{c_2/2c_4}$, apart from the region in space where $F_{\mu\nu}$ is non-vanishing. Taking $\xi \sim \lambda$, $|\phi|$ practically speaking vanishes when $F_{\mu\nu}$ is non-vanishing, and *vice versa.*

The field $F_{\mu\nu}(x)$ therefore acts as a smeared out δ-function, which is only non-vanishing along the vortex-line. The quantity $-(\frac{1}{4})F^2$ in eq. (3.1) therefore also acts as a smeared out δ-function.

In fig. 2 we have illustrated the vortex-line. For simplicity we shall assume that the end-points are at spatial infinity, or (perhaps better) that the vortex is a closed loop. Since the vortex-contribution to the Lagrangian (3.1) is a smeared out δ-function, which is non-vanishing only along the vortex-line, $\mathcal{L}$ is itself a smeared out δ-function. Of course, $\mathcal{L}$ is also relativistically invariant. It therefore follows that $\mathcal{L}_{\text{vortex}}$ is Lorentz-contracted in the transverse direction, i.e.

$$\mathcal{L}_{\text{vortex}} \propto \sqrt{1 - v_\perp^2}\,, \tag{3.2}$$

due to the motion of the vortex-line in the transverse direction. Let $\mathrm{d}s$ be the element of length along the vortex-line. Since the "transverse length" of the string, $\lambda \sim \xi$, is a constant, it follows that the action is given by (we ignore constants of proportionality)

$$S_{\text{vortex}} = \int \mathrm{d}^4x\ \mathcal{L}_{\text{vortex}} \propto \int \mathrm{d}t\, \mathrm{d}s \sqrt{1 - v_\perp^2}\,. \tag{3.3}$$

Now

$$v_\perp = \partial \boldsymbol{x}/\partial t - \partial \boldsymbol{x}/\partial s(\partial \boldsymbol{x}/\partial t \cdot \partial \boldsymbol{x}/\partial s) \tag{3.4}$$

and hence

$$S_{\text{vortex}} \propto \int \mathrm{d}t\, \mathrm{d}s \sqrt{1 - \left(\frac{\partial \boldsymbol{x}}{\partial t}\right)^2 + \left(\frac{\partial \boldsymbol{x}}{\partial t}\ \frac{\partial \boldsymbol{x}}{\partial s}\right)^2}\,. \tag{3.5}$$

Introducing a different parametrization of the vortex, $\boldsymbol{x}(s, t) = \boldsymbol{x}(\sigma, t)$ this leads to the Nambu action [2] (discussed in detail by Goddard, Goldstone, Rebi and Thorn [4])

$$S_{\text{vortex}} \propto \int_{-\infty}^{+\infty} \mathrm{d}t \int_{\sigma_1}^{\sigma_2} \mathrm{d}\sigma \frac{\mathrm{d}s}{\mathrm{d}\sigma} \sqrt{1 - v_\perp^2}\,, \tag{3.6}$$

provided one chooses a frame [4] where the parameter τ is identified with the time t.

It is rather clear that the arguments presented above do not depend on the details of the Lagrangian (3.1). All we need is that a vortex solution exists.

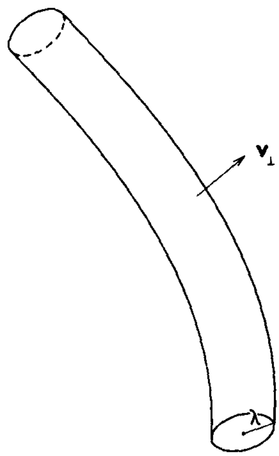

Fig. 2. Lorentz contraction of the vortex in the transverse direction.

Thus we have obtained the following rather remarkable result: the action of the vortex is proportional to the area of the surface swept out by the vortex in space and time.

This result forms the basis of our identification of the vortex solution of the Higgs-type Lagrangian (which is, of course, only a special example) with the dual string, described by the Nambu Lagrangian. It must be emphasized that our considerations are entirely classical, and so far no quantum effects are included.

4. The strong coupling limit

Our example, namely the field theory of the Landau-Ginzburg type, is the one used by Higgs to illustrate the Higgs mechanism. He showed that the theory by an appropriate gauge choice is revealed to be a theory of a massive scalar and a massive vector meson. In fact he chooses a gauge where the phase of the charged scalar field vanishes. (This would be a very inappropriate gauge choice for a vortex state). Choosing this gauge and putting

$$\phi(x) = \phi_0 + \rho(x)\,, \tag{4.1}$$

where ϕ_0 is given by

$$\phi_0 = \sqrt{\frac{c_2}{2c_4}}\,, \tag{4.2}$$

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}(\partial_\mu\rho)^2 + \tfrac{1}{2}A_\mu^2(\phi_0+\rho)^2 + \frac{c_2^2}{4c_4} - 2c_2\rho^2 - 2\sqrt{2c_2c_4}\rho^3 - c_4\rho^4 \,. \tag{4.3}$$

This form is easily seen to describe a vector meson with mass

$$m_V = e\phi_0 = e\sqrt{\frac{c_2}{2c_4}}\,, \tag{4.4}$$

and a scalar meson with mass

$$m_S = \sqrt{2c_2} \tag{4.5}$$

interacting with each other.

The compton wave lengths of these two mesons

$$\lambda = m_V^{-1}\,, \tag{4.6}$$

$$\xi = m_S^{-1} \tag{4.7}$$

give the width of the string. In the case $\lambda \gg \xi$ discussed above, ξ is the radius of an inner core in the string where $|\phi|$ deviates appreciably from its vacuum ϕ_0, while λ is the radius of the string determined by the width of the flux bundle (compare with fig. 1). In order that the strings are really strings, that is to say are thin‡, we must have the penetration depth λ and the coherence length ξ small compared to the characteristic length, which for a dual string model is $\sqrt{\alpha'}$. Here α' is the universal slope for the state of the string with two ends. According to the paper by Goldstone, Goddard, Rebbi, and Thorn [4], the energy density along the dual string is

$$\text{energy density} = \frac{\gamma_\perp}{2\pi\alpha'} = \frac{1}{2\pi\alpha'\sqrt{1-v_\perp^2}}\,, \tag{4.8}$$

where $v_\perp$ is the transverse velocity of the string.

We can thus make a classical estimate of the connection of the universal slope α' to the three parameters, c_2, c_4 and e of the Landau-Ginzburg-like Lagrangian (2.1) by calculating the energy-density at rest for the vortex solution given in sect. 2.

The magnetic energy-density along the vortex string is

$$\frac{1}{2}\int_0^\infty |H|^2\, 2\pi r\,\mathrm{d}r = \tfrac{1}{2}\phi_0^4 e^2 \int_0^\infty K_0(e\phi_0 r)^2\, 2\pi r\,\mathrm{d}r = \pi\phi_0^2 \int_0^\infty K_0(y)^2 y\,\mathrm{d}y$$

$$= \frac{c_2}{c_4}\frac{\pi}{2}\int_0^\infty K_0(y)^2 y\,\mathrm{d}y\,. \tag{4.9}$$

The integral $\int_0^\infty K_0(y)^2 y\,\mathrm{d}y$ converges because $K_0(y)$ behaves no worse than a logarithm for $y \to 0$ and decays exponentially for $y \to \infty$. The integral is thus of order unity.

‡ A string is by definition a thin practically one dimensional structure.

A crude estimate of the energy per unit length of the string due to the core where $|\phi|$ no longer has its vacuum value also gives something of the order of magnitude

$$\xi^2 c_2 \phi_0^2 \approx \phi_0^2 \,, \tag{4.10}$$

since $c_2\phi_0^2$ is the energy density. Thus we can conclude that the energy-density along the string to be identified with $1/2\pi\alpha'$ in the dual model is

$$\frac{1}{2\pi\alpha'} \sim \phi_0^2 \sim \frac{c_2}{c_4} \,. \tag{4.11}$$

The exact ratio $\alpha' c_2/c_4$ can be computed numerically by solving the differential equations. What is important, however, is just that it is of order unity.

The order of magnitude of the characteristic length for the hadrons, being the lower quantum mechanical levels of the dual string, is thus

$$\sqrt{\alpha'} \approx \sqrt{\frac{c_4}{c_2}} \,. \tag{4.12}$$

In order that thin strings are a good approximation, we thus need to have

$$\sqrt{\alpha'} \gg \lambda, \xi \,, \tag{4.13}$$

which again implies

$$\frac{\sqrt{\alpha'}}{\lambda} \approx e \gg 1 \tag{4.14}$$

and

$$\frac{\sqrt{\alpha'}}{\xi} \approx \sqrt{c_4} \gg 1 \,. \tag{4.15}$$

These requirements might also be written

$$m_V, m_S \gg \frac{1}{\sqrt{\alpha'}} \,, \tag{4.16}$$

which means that the particles corresponding directly to the local fields have masses m_V and m_S much larger than the typical hadron masses. Thus in this limit low energy phenomena (low energy meaning energies of the order of $1/\sqrt{\alpha'}$) should be dominated by hadrons, i.e. dual strings, and not by the fundamental vector and scalar particles in the theory revealed by Higgs.

We may hope that a third kind of excitation is not going to be important at low energies, so that we may have a pure dual string theory in the low energy range of this strong coupling limit of the Landau-Ginzburg-like theory, in which the two coupling constants e and c_4 are infinitely large.

We thus would like to suggest that such a strong coupling limit is the one in which a dual string theory emerges, and so we would like to postulate that if nature were to be described by the Landau-Ginzburg-like model of ours, it would have chosen fairly large values of e and $\sqrt{c_4}$, so that we could get pure dual string properties of the theory at low energies.

The limit e, $\sqrt{c_4}$ large, make however, classical field considerations very doubtful, because it is in a certain sense a super quantum-mechanical limit, i.e. something like $\hbar \to \infty$. This is at first unclear in our formulation, because we used the quantum of action as a unit $\hbar = 1$.

To see that our limit $e, \sqrt{c_4} \to \infty$ is a super quantum mechanical one, we may just remark that the masses m_V and m_S of the fundamental particles of the field theory are typical harmonic oscillator frequencies, the classical solution of the equations revealing the particles as solutions $\phi(x) \approx \epsilon\, e^{-ipx}$, $A_\mu \approx \epsilon_\mu\, e^{-ipx}$, where the coefficients ϵ and ϵ_μ are small quantities. But the typical energy of the field theory is rather the energy of say a vortex line, with a length of the order of magnitude of its width, and that is of the order of magnitude λ/α'.

So in our limit $e, \sqrt{c_4} \to \infty$ the typical energy is much smaller than the typical frequency, since from (4.14) and (4.16)

$$\frac{\lambda}{\alpha'} \ll m_S, m_V \tag{4.17}$$

in the strong coupling limit. That means that the typical action of the theory is, using eq. (4.6) and (4.14),

$$\frac{\lambda}{\alpha' m_V} = \frac{1}{\alpha' m_V^2} \approx \frac{1}{e^2} \ll 1 , \tag{4.18}$$

so that from the point of view of strong coupling theory in this limit the quantum of action $\hbar = 1$ is tremendously big compared to the typical amount of action e^{-2}. So the theory is very very far from being classical, since in a classical theory the typical action, say 1 erg. sec, is very large compared to $\hbar$.

The extreme quantum mechanical nature of the theory in the strong coupling limit ($e, \sqrt{c_4} \to \infty$) has the implication that if we want to justify our classical solution of the fields around and in a vortex line in sect. 2 by estimating the fluctuations in a coherent state simulating our classical solution, we may be in a very bad shape.

5. The problem of quantization

We now mention a few words concerning the problem of quantizing the above string scheme. As mentioned near the end of sect. 4, if one calculates the characteristic action of the present theory, then it turns out to be much larger than Planck's constant. Thus we have an extreme quantum mechanical problem, where one might expect important fluctuations in a coherent state approximating our classical vortex solution. Thus we may be in serious trouble in going from the classical theory to the quantized theory. We have no real solution to offer to this problem. The only thing we can say at present is the following: suppose that one could show that in the strong coupling limit the classical theory has only one stable solution, namely the vortex solution. The arguments in sect. 3 then shows that the Nambu Lagrangian comes out from the field theory Lagrangian, i.e.,

$$\mathcal{L}\begin{pmatrix}\text{classical}\\ \text{field theory}\end{pmatrix} \to \mathcal{L}\begin{pmatrix}\text{classical}\\ \text{Nambu}\end{pmatrix} \text{ (strong coupling limit)}\,. \tag{5.18}$$

This limiting behaviour would of course not be correct if we had other stable solutions in addition to the vortex solution, and in spite of various attempts we have not succeeded in convincing ourselves that only the vortex solution is stable (classically). However, suppose that the limit (5.18) is correct. Then we could quantize the theory just by quantizing the Nambu Lagrangian, in which case we should obtain the usual quantized dual string.

Actually the somewhat optimistic remarks above are dubious. The point is that we have formulated our theory in four dimensions, and hence we should run into trouble with respect to quantization‡, and we should also obtain a tachyon. However, the classical Hamiltonian is certainly positive definite, and it appears as a mystery how it can generate a non-definite spectrum in the quantized version. However, it may be that the solution to this apparent paradox is that the positive definite character of a classical Hamiltonian may not carry over into quantum field theory. For example, if the theory has to be renormalized, it is not at all guaranteed that the signs of the renormalized couplings are the same as the signs of the bare couplings, and hence positive definiteness may impose different conditions on the couplings in a classical theory (where the bare couplings enter) and in a qunatized theory (where the renormalized coupling enter). A recent discussion of this possibility has been given by 't Hooft [7]. We do, however, not yet know to what extent the strong coupling limit (the vortex solution) should be renormalized.

6. Conclusions

We have seen that it is possible to make field theories that (classically) have solutions corresponding to vortex-lines that are one dimensional structures moving around e.g. according to the equation of motion of the Nambu dual string. In order that the length of the string in one of the lowest mass eigenstates (i.e. one of the lowest hadron states) should be much larger than the width of it, we had to choose a strong coupling limit $e, c_4 \to \infty$ in the Ginzburg-Landau model used as the example.

It should be stressed that the Ginzburg-Landau model is only an example. Many models could easily be proposed, that would all lead to the simple Nambu string with no extra degrees of freedom in the strong coupling limit at low energies. For

‡ Notice that in order to formulate the theory in an arbitrary number of dimensions, we have to generalize the Higgs mechanism to an arbitrary number of dimensions. This has not yet been done.

instance it is easy to add some extra fields giving particles with high masses of the order of magnitude of m_V and m_S to the Ginzburg-Landau theory discussed above, without giving further vortex-lines so that the vortex-lines would still behave like Nambu dual strings. Also, in the appendix we discuss the case of Yang-Mills fields, which does not lead to additional degrees of freedom.

One is also not restricted to work in 3+1 dimensions. In fact, probably the simplest non-trivial vortex-line model is the so-called Sine-Gordon theory in 2+1 dimensions having the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \varphi(x))^2 + c \cos(d\varphi(x)) ,$$

giving the equation of motion

$$\partial_\mu \partial^\mu \varphi(x) + cd \sin(d\varphi(x)) = 0 ,$$

i.e., the Sine-Gordon-equation. It is readily seen that this theory classically allows for a static solution

$$\varphi(x) = \frac{4}{d} \arctan \exp(\sqrt{c}\, d x_2)$$

describing a vortex line along the x_1-axis. The field $\varphi(x)$ changes by $2\pi/d$ across the vortex line, which has a width of the order of magnitude $1/\sqrt{c}d$. Analogous to the case of the Ginzburg-Landau theory, this width is equal to the Compton wavelength for the particle of the theory, as is seen by considering the Klein-Gordon equation with mass square $m^2 = cd^2$ obtained as the weak field limit of the Sine-Gordon-equation. The energy density along the vortex line is

$$\frac{1}{2\pi\alpha'} = \int_{-\infty}^{+\infty} \left[\left(\frac{4}{d} \frac{\partial}{\partial x} \arctan \exp(\sqrt{c} d x) \right)^2 + 2c \sin^2(2 \arctan \exp(\sqrt{c} d x) \right] \mathrm{d}x \approx \sqrt{c}/d ,$$

so that strings that are narrow (vortex lines) compared to the hadronic length $\sqrt{\alpha'} \approx d^{\frac{1}{2}} c^{-\frac{1}{4}}$ are obtained in the strong coupling (and super quantum mechanical) limit $m\sqrt{\alpha'} \approx c^{-\frac{1}{4}}$ are obtained in the strong coupling (and super quantum mechanical) vortex lines move as Nambu dual strings.

We have shown the equivalence of string models with a certain set of solutions of some field theories in a classical approximation, namely the vortex lines. We believe, but we have not proven that in the strong coupling limit at low energies all states of the Ginzburg-Landau field theory are states that can be described as states of some system of strings. This hope cannot be taken to be true for all theories having vortex lines, since theories can be cooked up, which have e.g. zero-dimensional structures in excess of the one dimensional one. That is to say one could make a field-theory model which also had kink like type of singularity similar to the solution in the 1+1 dimensional Sine-Gordon theory. Such theories could be built in higher dimensions too.

That we have to take a super quantum mechanical limit necessitates that, (i) the field theories to be used should be quantized and (ii) the equivalence of the field theory and the corresponding string theory be proven on a quantum mechanical level (that can only be done, if at all, in a low energy and strong coupling limit).

Also we have to understand how it can happen that tachyons appear in a model having at first a positive definite Hamiltonian density, as is easily seen to be the case for the Higgs-Ginzburg-Landau model. Further, we should like to know what the sifnificance of the critical dimension ($d = 26$ in the conventional model) is in terms of field-theory models like the ones discussed, but we have not even made the 26-dimensional classical Ginzburg-Landau model yet.

If some day one understands the quantum properties better, the possibility of constructing field-theory models for strings like the ones we discussed, might be an easier way to come across a good (possibly unitary) Veneziano-model than to make a string model directly.

First of all we want to thank Don Weingarten for pointing out how to make vortex-line Sine-Gordon models in any dimensions, and C.H. Tze for finding literature. We also thank B. Zumino for calling our attention to the fact that the equation we discussed is well known as the Ginzburg-Landau equation. Secondly we want to thank our colleagues at the Niels Bohr Institute and CERN for helpful discussions.

Appendix. Discussion of the Yang-Mills type of model

It is natural to ask whether it is possible to produce strings from Lagrangian having internal degrees of freedom, e.g. isospin or SU(3). If we can manage to keep the vortex solutions for such Lagrangians, this would indeed be very nice from the point of view of the dual string, since the string would then carry internal degrees of freedom, and would therefore perhaps lead to a more realistic spectrum of hadrons (and perhaps to a more realistic dual amplitude, provided we could solve the problem of colliding strings). This, unfortunately, does not seem to be the case.

An example of a Lagrangian with internal degrees of freedom which immediately comes to the mind, is the Yang-Mills type of Lagrangian. Here we introduce a field $B_{\mu\nu}$,

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu - g B_\mu \times B_\nu \,. \tag{A.1}$$

Defining the dual field

$$B^*_{\mu\nu} = \tfrac{1}{2} \epsilon_{\mu\nu\alpha\beta} B^{\alpha\beta} \,, \tag{A.2}$$

it is easily seen that

$$(\partial^\mu + g B^\mu \times) B^*_{\mu\nu} = 0, \tag{A.3}$$

which is the analogue of the Maxwell equation

$$\partial^\mu F^*_{\mu\nu} = 0 \,, \tag{A.4}$$

which states that magnetic monopoles do not exist.

The form of eq. (A.3) poses a problem with respect to the interpretation of (magnetic) flux lines. From eq. (A.4) one can derive that

$$0 = \int_{V_\mu} \partial^\mu F^*_{\mu\nu} \mathrm{d}V^\nu = \int_{S_1} F_{\mu\nu} \mathrm{d}\sigma^{\mu\nu} - \int_{S_2} F_{\mu\nu} \mathrm{d}\sigma^{\mu\nu} \,, \tag{A.5}$$

where V_μ is a volume bounded by the two surfaces S_1 and S_2. Eq. (A.5) tells us that no magnetic flux lines can start inside the volume. As far as eq. (A.3) is concerned, a similar procedure leads to

$$-g \int_{V_\mu} B^\mu \times B^*_{\mu\nu} \, \mathrm{d}V^\nu = \int_{S_1} B_{\mu\nu} \, \mathrm{d}\sigma^{\mu\nu} - \int_{S_2} B_{\mu\nu} \, \mathrm{d}\sigma^{\mu\nu} \,. \tag{A.6}$$

Thus, at least in general flux lines can originate inside the volume. If, however, we concentrate on the static situation, we can take $B_o = 0$, and $B_{ok} = 0$, and hence we have the interpretation that the flux is given by

$$\mathbf{\Phi} = \int B_{\mu\nu} \, \mathrm{d}\sigma^{\mu\nu} \text{ (static case)} \,. \tag{A.7}$$

Next let us consider a specific Yang-Mills Lagrangian, namely

$$\begin{aligned} \mathcal{L} = &-\tfrac{1}{4} B_{\mu\nu} B^{\mu\nu} + \tfrac{1}{2} [(\partial_\mu + g B_\mu \times) \boldsymbol{\phi}]^2 + \tfrac{1}{2} [(\partial_\mu + g B_\mu \times) \boldsymbol{\psi}]^2 \\ &+ c_2 \boldsymbol{\phi}^2 - c_4 (\boldsymbol{\phi}^2)^2 + d_2 \boldsymbol{\psi}^2 - d_4 (\boldsymbol{\psi}^2)^2 + e_2 \boldsymbol{\phi}\boldsymbol{\psi} - e_4 (\boldsymbol{\phi}\boldsymbol{\psi})^2 \,, \end{aligned} \tag{A.8}$$

where the fields $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are isovector fields. The reader may wonder why we introduce two fields $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ and not just one field. The reason for this will turn out later, where we shall see that in order to have a vortex solution at least two isovector fields are needed.

Now we are looking for a solution which quantizes the flux (A.7) in a way similar to the Higgs (Ginzburg-Landau) Lagrangian discussed in sect. 2. Defining the current to be

$$-\partial^\mu B_{\mu\nu} + g B_{\mu\nu} \times B^\mu \equiv j_\nu \,, \tag{A.9}$$

i.e., $j_\nu = 0$ for a free Yang-Mills field, we obtain from the Lagrangian (A.8) by varying B_μ

$$j_\nu = g(\boldsymbol{\phi} \times \partial_\nu \boldsymbol{\phi}) + g(\boldsymbol{\psi} \times \partial_\nu \boldsymbol{\psi}) + g^2 (B_\nu \times \boldsymbol{\phi}) \times \boldsymbol{\phi} + g^2 (B_\nu \times \boldsymbol{\psi}) \times \boldsymbol{\psi} \,. \tag{A.10}$$

Considering now the static solution with cylindrical symmetry we see that the term $B_\mu \times B_\nu$ in eq. (A.1) is smaller than the term $\partial_\mu B_\nu - \partial_\nu B_\mu$ in $B_{\mu\nu}$ for large distances from the axis of symmetry. Thus we can write the flux (A.7) as

$$\Phi = \oint B_\mu(x)\,\mathrm{d}x^\mu \quad \text{(static case, large distance)}\,, \tag{A.11}$$

provided we integrate over a circle with large radius. For such large distances j_ν vanishes, and eq. (A.10) leads to

$$(B_\nu \times \phi)\times\phi + (B_\nu \times \psi)\times\psi = (B_\nu\phi)\phi + (B_\nu\psi)\psi - (\phi^2+\psi^2)B_\nu$$
$$= \frac{1}{g}(\phi\times\partial_\nu\phi + \psi\times\partial_\nu\psi)\,. \tag{A.12}$$

Now the ground state (the vacuum) of the Lagrangian (A.8) corresponds to

$$\phi^2 = \frac{c_2}{2c_4}\,, \qquad \psi^2 = \frac{d_2}{2d_4}\,, \qquad \phi\psi = \frac{e_2}{2e_4}\,, \tag{A.13}$$

where we assume that ϕ and ψ are not in the same (or opposite) direction, i.e.

$$\frac{c_2 d_2}{c_4 d_4} > \left(\frac{e_2}{e_4}\right)^2 . \tag{A.14}$$

Thus, in the vacuum the lenghts of the isovectors ϕ and ψ are fixed, and in addition the projection of one vector on the other is fixed. This ensures that in any frame of reference in isospace at least three components of ϕ and ψ are non-vanishing (e.g., ϕ_1, ψ_1 and ψ_2).

Now let us go to a frame of reference where $\phi_3 = \psi_3 = 0$. This corresponds to selecting the flux lines in the 3-direction. It is easily seen that the condition (A.12) for the current to vanish at large distances imposes the condition on B_ν that

$$B^1_\nu = B^2_\nu = 0\,, \qquad B^3_\nu = -\frac{1}{g}\partial_\nu\chi\,, \tag{A.15}$$

where χ is the phase of $\phi_1 + i\phi_2$. Notice that due to the last condition (A.13) the phase of $\psi_1 + i\psi_2$ is the same as χ + constant. Inserting eq. (A.15) in eq. (A.11) and using the fact that the phase is only unique modulus $2\pi n$ we get that the flux is given by

$$|\Phi| = 2\pi n/g\,, \tag{A.16}$$

i.e., the flux is quantized‡. This result only follows if we have at least two fields ϕ and ψ.

Having obtained the flux quantization we then go to the strong coupling limit, in order that the width of the vortex line is made sufficiently small. The width is given

‡ We have not convinced ourselves that there are no other quantas than $2\pi/g$.

by the order of magnitude of the Compton wave lengths of the Higgs particles. As before, we have scalar particles with masses

$$m_S = \sqrt{2c_2}\,, \qquad m'_S = \sqrt{2d_2}\,. \tag{A.17}$$

The Higgs vector particle is obtained from the seagull terms in the Lagrangian (A.8), i.e. from

$$\tfrac{1}{2}g^2(B_\mu \times \phi)^2 + \tfrac{1}{2}g^2(B_\mu \times \psi)^2 = \tfrac{1}{2}g^2\,[B_\mu^2(\phi^2 + \psi^2) - (B_\mu\phi)^2 - (B_\mu\psi)^2]\,. \tag{A.18}$$

Inserting the vacuum values (A.13) it is easily seen that all vector mesons B_μ acquire a mass,

$$m_V = g\sqrt{\frac{c_2}{2c_4} + \frac{d_2}{2d_4}}\,. \tag{A.19}$$

Proceeding as in sect. 4 we can now go to the strong coupling limit $g \to \infty$, $c_4 \to \infty$, $d_4 \to \infty$ (or $m_S, m'_S, m_V \gg 1/\sqrt{\alpha'}$), which then gives us the string solution. From the very general argument in sect. 3, we know that this solution corresponds to the classical Nambu Lagrangian. However, one can easily see that due to gauge invariance no new degrees of freedom are introduced in the string Lagrangian.

Note added in proof

In addition to the term (4.9) the magnetic energy-density also contains a term coming from the seagull term in the Lagrangian. The latter term gives rise to an energy-density of the order of magnitude

$$(c_2/c_4)\log(\lambda/\xi).$$

As long as $\log(\lambda/\xi)$ is not too large, this term does not change any of the conclusions in sect. 4. See ref. [8] for example.

A preprint of L.J. Fassie [9] with a similar idea has appeared.

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