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YANG-MILLS VACUA IN COULOMB GAUGE

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ABSTRACT

All possible tunnelings exist among the vacua with topological numbers n = 0, + 1/2 and - 1/2 in Coulomb gauge. Thus the gauge symmetry is restored. The transition from the n = 0 vacuum to the n = 1/2 vacuum is also discussed without the restriction of the Coulomb gauge.

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I. INTRODUCTION

The strong interaction is presumably generated by the color SU(3) Yang-Mills theory. The quarks are confined in color singlet states. The problem of confinement is certainly related to restoration of the gauge symmetry. Recently Gribov[1] has discovered a new set of vacua in Coulomb gauge. A crucial question is whether there are tunnelings between the new vacua and the $A_\mu = 0$ vacuum as well as other possible tunnelings. In other words the problem is whether the instanton[2,3] or other field configurations can connect these vacua with finite action in 4-dimensional Euclidean space-time. This is the problem we shall investigate in this paper.

We shall consider SU(2) Yang-Mills theory, since Gribov's vacua are related to the SU(2) structure of the theory. The classical vacua are defined by

$$A_\mu = U^{-1}\partial_\mu U \tag{1.1}$$

where U is a SU(2) unitary matrix. Following Gribov[1] we make the ansatz

$$U = e^{i\alpha(r)\vec{\sigma}\cdot\vec{n}} \tag{1.2}$$

where the σ_i are Pauli matrices, and

$$\vec{n} = \frac{\vec{x}}{r} \quad , \qquad r^2 = \vec{x}^2 \quad .$$

The Coulomb gauge condition is

$$\partial_i A_i = 0 \tag{1.3}$$

which leads to

$$\frac{d^2\alpha}{dr^2} + \frac{2}{r}\frac{d\alpha}{dr} - \frac{\sin 2\alpha}{r^2} = 0 \qquad (1.4)$$

Introducing the variable $\tau = \ell n\, r$, we obtain the equation of motion for a damped pendulum[1,4]

$$\ddot{\alpha} + \dot{\alpha} - \sin 2\alpha = 0 \qquad (1.5)$$

To solve eq. (1.5) we require $\lim_{s \to -\infty} \alpha(\tau) = \text{const.}$ Then the only possibilities are

$$\alpha(\tau)_{\tau\to-\infty} = n\pi \qquad (1.6)$$

or

$$= (n + \tfrac{1}{2})\pi \qquad (1.7)$$

The case (1.7) corresponds to the solution $\alpha(\tau) = (n + \frac{1}{2})\pi$ independent of τ, and the field becomes singular at $r = 0$. We include this case for later discussions. In the case (1.6), when $\tau \to -\infty$ we obtain[1]

$$\alpha(\tau) \to ar + n\pi = ae^{\tau} + n\pi \qquad (1.8)$$

When $\tau \to +\infty$, we obtain

$$\alpha(\tau) \to \pm \frac{\pi}{2} + n\pi \quad . \qquad (1.9)$$

The solutions are completely specified by the value of a as well as r_0, which defines the origin of the coordinate. Thus the solutions of eq. (1.5) may be written as

$$\alpha(r) = G(ar) + n\pi \qquad (1.10)$$

where G(r) is a solution of eq. (1.5) with a = 1. The scale invariant form of the solutions is due to the scale invariance of eq. (1.5). The solution with a = 0 corresponds to $A_\mu = 0$, while the solutions with $a \to \pm\infty$ correspond to $\alpha = \frac{\pi}{2} + n\pi$. Note if we take the limits $a \to \pm\infty$, $\alpha(r)$ becomes a constant. Thus the solutions (1.10) include all the cases. If we introduce a multi-valued function g(r) where

$$g(r) = G(r) + n\pi \qquad (n = 0, \pm 1, \ldots) \qquad (1.11)$$

We have

$$\alpha(r) = g(ar) \quad . \qquad (1.12)$$

Under the ansatz (1.2) which means that the fields are spherically symmetric, there are no other solutions. It is not known whether there are non-spherically symmetric vacua in Coulomb gauge. In this paper we discuss the spherical symmetric ones.

In the next section the topological number of a field configuration will be expressed in terms of the time dependence of the parameter a. All possible tunnelings between the vacua are classified by the behavior of $a(x_4)$. In §3 the ambiguity of a large field is discussed in connection with the instanton. In §4, using the results obtained in §2 and §3, it is shown that the instanton can connect any vacuum. In §5, the transition from the vacuum with topological number n = 0 to the vacuum with n = 1/2 is discussed from a more general viewpoint. Section 6 is devoted to conclusions and discussions. In the Appendix we discuss field configurations which are discontinuous at a certain time and the relation of the distontinuity with the non-uniqueness of large fields.

II. TOPOLOGY OF FIELDS IN COULOMB GAUGE

Since we are interested in tunnelings among various vacua in Coulomb gauge, we require

$$A_\mu \xrightarrow[R\to\infty]{} U^{-1}\partial_\mu U \qquad (R^2 = r^2 + r_4^2) \tag{2.1}$$

in 4-diminsional Euclidean space-time. Thus the problem is to gauge transform the "instanton" into the Coulomb gauge.

The Pontryagin index is given by

$$q = -\frac{1}{16\pi^2}\int d^4x \ \ Tr(F_{\mu\nu}\tilde{F}_{\mu\nu}) \quad , \tag{2.2}$$

which may also be written in the following form.

$$q = -\frac{1}{24\pi^2}\ \epsilon_{\mu\nu\lambda\rho}\int Tr(A_\nu A_\lambda A_\rho)\ d^3\sigma_\mu \quad . \tag{2.3}$$

Consider a very large sphere S^3 in the 4-dimensional space-time; then the vector fields in eq. (2.3) may be written as pure gauges, as in eq. (2.1). From the analysis in §1, at fixed x_4 the pure gauge is characterized by a ($\vec{r}_o(x_4)$ is fixed for all x_4 unless it is stated otherwise). The function $\alpha(r, x_4)$ can be written in the form

$$\alpha(r,x_4) = g(a(x_4)r) \quad . \tag{2.4}$$

Thus the Pontryagin index is specified completely by the function $a(x_4)$.

To connect q with $a(x_4)$, we begin with investigation of g(x) defined (1.11). The x dependence of g(x) is shown in Figure 1. Except for the detailed behavior, it is similar to that of $\tan^{-1}x$.

We assume that $a(x_4)$ is finite as $x_4\to\pm\infty$. [We will discuss later the case where $a(x_4) \to$ infinity at $x_4\to\pm\infty$. In this case the fields of the vacuum are singular at r = 0 when $x_4\to\pm\infty$]. Then we have

$$\lim_{x_4\to+\infty}\ \lim_{r\to 0}\ \alpha(x_4 r) = n(+\infty)\pi \tag{2.5a}$$

and

$$\lim_{x_4\to-\infty}\ \lim_{r\to 0}\ \alpha(x_4 r) = n(-\infty)\pi \quad . \tag{2.5b}$$

From eqs. (2.3) and (2.5) we obtain

$$q = n(+\infty) - n(-\infty) \quad . \tag{2.6}$$

We also obtain eq. (2.6) from the definition that q is the winding number for the mapping of $S^3 \to SU(2)$, by writing

$$U = \Pi \cdot u_4 + i\sigma_i u_i \tag{2.7}$$

where

$$u_i^2 + u_4^2 = 1 \quad . \tag{2.8}$$

It should be noted that $\alpha(x_4, r)$ is a continuous function on large sphere S^3.

Equation (2.6) implies that in the case of the instantons (with $q \geq 1$), $\alpha(r, x_4)$ takes different branches of the function g at $x_4 \to +\infty$, $r \to 0$ and $x_4 \to -\infty$, $r \to 0$. From Fig. 1 and eq. (1.12) we conclude that $a(x_4)$ must be divergent at some x_4. We assign the topological number $n(x_4)$ to each singular point x_4 as follows: If $a(x_4)$ takes $+\infty \to -\infty$ as $x_4 = \bar{x}_4 - \varepsilon \to \bar{x}_4 + \varepsilon$ $(\varepsilon > 0)$, $n(x_4) = +1$ for $x_4 = \bar{x}_4$; if it takes $-\infty \to +\infty$, then $n(x_4) = -1$; if it takes either $-\infty \to -\infty$ or $+\infty \to +\infty$, $n(x_4) = 0$. Then we obtain

$$q = \sum n(x_4) \tag{2.9}$$

where the summation is over singular points x_4.

From the above analysis the behavior of $a(x_4)$ for the case of the instanton with q = 1 should be as shown in Fig. 2, if the instanton can in fact be gauge transformed into the Coulomb gauge. The functions $a(x_4)$ are differentiable, at least, up to the first

order. There are nine types of tunnelings depending on the values of $a(x_4)$ when $x_4 \to \pm\infty$ $(a \gtreqless 0)$.

If $\alpha(r, x_4)$ takes the form of $\tan^{-1} a(x_4)r$ instead of $g(a(x_4)r)$, the same conclusions as above are obtained: The function $a(x_4)$ should be divergent at some x_4 for the instanton with q = 1. The original form of the instanton obtained by Belavin et al.[2] has exactly this form and $a(x_4)$ is given by $1/x_4$ which is singular at $x_4 = 0$.

It may be worthwhile to note that the singularities of a, $a = \pm\infty$ do not imply a singularity for the field at r = 0, since the form of the field (2.1) can only be used for $r \to \infty$ with x_4 finite. The field is also continuous at x_4 such that $a(x_4) = 0$, although the asymptotic values for g change as $r \to \infty$ when x_4 passes this value of x_4 as follows:

$$g(a(x_4)r) \longrightarrow \begin{cases} \pi/2 + (n\pi) & \text{for } a > 0 \\ 0 + (n\pi) & \text{for } a = 0 \\ -\pi/2 + (n\pi) & \text{for } a < 0 \end{cases} \tag{2.10}$$

The behavior of $g(a(x_4)r)$ at x_4 such that $a(x_4) = 0$ is similar to that of $\tan^{-1} x_4 r$ at $x_4 = 0$.

Now we define the topological number of a vacuum field by the following equation

$$n = \frac{-1}{24\pi^2} \int d^3x \; \varepsilon_{ijk} \, \mathrm{Tr} \, \{A_i A_j A_k\}$$

$$= \frac{-1}{24\pi^2} \int d^3x \; \varepsilon_{ijk} \, \mathrm{Tr} \, \{U^{-1}\partial_i U \, U^{-1}\partial_j U \, U^{-1}\partial_k U\} \tag{2.11}$$

Then we have

$$n = \begin{cases} +1/2 & \text{for } a > 0 \\ 0 & \text{for } a = 0 \\ -1/2 & \text{for } a < 0 \end{cases} \tag{2.12}$$

Thus we have three kinds of vacua which have different topological numbers. It means that we can classify tunnelings between these vacua into 9 types, each of which corresponds to one of the figures in Fig. 2.

Equation (2.3) can be written in the form

$$q = n_+ - n_- + n_s \tag{2.13}$$

where

$$n = \frac{-1}{24\pi^2} \int d^3x\, \varepsilon_{ijk}\, \mathrm{Tr}\, \{A_i A_j A_k\} \Big|_{x_4 = \pm\infty} \tag{2.14}$$

and

$$n_s = \frac{-1}{24\pi^2} \int dx_4 dS^i \varepsilon_{i\alpha\beta\gamma} \mathrm{Tr}\, \{A_\alpha A_\beta A_\gamma\} \Big|_{r=\infty} \tag{2.15}$$

Here we take the limit $r \to \infty$ first. Equation (2.13) states that the Pontryagin index equals the difference of the topological numbers of vacua at $x_4 = +\infty$ and at $x_4 = -\infty$, plus the contribution from the spacial boundary.

Since the Pontryagin index is gauge invariant, it equals unity in any type of tunneling. For example, in the case of

tunneling from the $n = 0$ vacuum to the $n = 1/2$ vacuum n_s should be $+ 1/2$ since $n_+ = 1/2$ and $n_- = 0$. If the function $a(x_4)$ behaves as is shown in Fig. 2(f), this condition, i.e., $n_s = 1/2$ is automatically satisfied.

It should be noted that the 9 types of tunnelings seem quite different from the viewpoint of eq. (2.13), however, they are not so different from the viewpoint of Fig. 2.

III. AMBIGUITIES OF FIELDS

Gribov[1] pointed out not only the non-uniqueness of the vacuum but also the ambiguities of other field configurations in Coulomb gauge. Let us discuss the ambiguities in connection with the instanton. The field configuration of the instanton[2] at $x_4 = 0$ is given by

$$\begin{aligned} A_i^a &= 2\, \frac{\varepsilon_{aib} x_b}{r^2 + \rho^2} \\ A_o^a &= -2\, \frac{x_a \sigma_a}{r^2 + \rho^2} \end{aligned} \tag{3.1}$$

This field automatically satisfies the Coulomb gauge condition and is exactly the case discussed by Gribov.

Let us review the work of Gribov. Consider first a field $A_i = A_i^a\, \sigma^a/2i$ which satisfies the Coulomb gauge condition and makes a gauge transformation

$$A_i' = U^{-1} A_i U + U^{-1} \partial_i U \tag{3.2}$$

Requiring

$$\partial_i A_i' = 0 \tag{3.3}$$

we obtain

$$\partial_i (U^{-1} A_i U + U^{-1} \partial_i U) = 0 \tag{3.4}$$

If there are nontrivial solutions for U, it means that the gauge condition (1.3) does not specify uniquely the vector field A_μ.

Take a special case where A_i takes a simple form

$$A_i = \varepsilon_{aib} \frac{x_b}{r^2} f(r) \frac{\sigma_a}{2i} , \tag{3.5}$$

$$f(r) \to 0 \quad \text{as} \quad r \to 0$$

and assume the form of U as

$$U = e^{i\beta(r)\vec{\sigma}\vec{n}} \tag{3.6}$$

Then eq. (3.4) reduces to the equation of motion for a damped pendulum

$$\ddot{\beta} + \dot{\beta} - \sin 2\beta \, (1 - f(\tau)) = 0 \tag{3.7}$$

where $\tau = \ln r$. In this case the potential depends on time.

Now return to eq. (3.1). In this case $f(\tau)$ is given by

$$f(\tau) = \frac{2e^2 \tau}{e^{2\tau} + \rho^2} \tag{3.8}$$

The potential behaves as

$$-\sin^2\beta \qquad \text{for } \tau \to -\infty \tag{3.9a}$$

and

$$+\sin^2\beta \qquad \text{for } \tau \to +\infty \tag{3.9b}$$

As the time (τ) increases the coefficient of $\sin^2\beta$ increases monotonously from -1 to +1.

The asymptotic behavior of $\beta(\tau)$ as $\tau \to -\infty$ is the same as that of the solutions $\alpha(\tau)$ of eq. (1.5). $\beta(\tau)$ approaches to $n\pi$, except for the constant solution $\beta(\tau) = (n \pm 1/2)$. We choose $n = 0$ without losing any generality. The function $\beta(\tau)$ is completely determined by the coefficient f'of e^τ as $\tau \to -\infty$. [This b corresponds to a of $\alpha(\tau)$ in §2]. Thus the asymptotic value of $\beta(\tau)$ as $\tau \to +\infty$ is also determined by the value of b.

When $b = 0$, $\beta = 0$. As long as the value of b is small enough, $\beta(+\infty) = 0$. If we gradually increase the value of b, $\beta(\tau)$ reaches another stable point π. At the particular value of b, b_o, where the transition from $\beta(+\infty) = 0$ to $\beta(+\infty) = \pi$ occurs, the asymptotic valur of $\beta(\tau)$ as $\tau \to +\infty$ becomes $\pi/2$, which is a position of unstable equilibrium. If the value of b decreases from zero, the same thing as above happens except for a change of sign. Thus $\beta(+\infty) = 0$ for $-b_o < b \leq 0$, and $\beta(+\infty) = -\pi/2$ for $b = -b_o$.

Let us now discuss the function $\alpha(\tau)$ which characterizes the field A', where A' is defined by eq. (3.2). The relation between $\alpha(\tau)$ and $\beta(\tau)$ is given by

$$\lim_{r\to\infty} \left(\tan^{-1} \frac{r}{x_4} + \beta(r) \right) = \lim_{r\to\infty} \alpha(r) \tag{3.10}$$

or

$$\lim_{r\to\infty}\left(\beta(r)+\frac{\pi}{2}\right)=\lim_{r\to\infty}\alpha(r)$$

Thus $\alpha(+\infty) = 0$ for $b = -b_o$, $\alpha(+\infty) = \pi/2$ for $-b_o < b < b$ and $\alpha(+\infty) = \pi$ for $b = b_o$. From this we conclude that the value of a changes according to the value of b as follows: $a = 0$ for $b = -b_o$, a increases as b increases up to zero, $a = +\infty$ at $b = -\varepsilon$, $a = -\infty$ at $b = +\varepsilon$, then a decreases as b increases from zero and $a = 0$ for $b = b_o$. Thus as b changes from $-b_o$ to $+b_o$, a takes all the possible values.

Up to now, we have analyzed the arbitrariness of the field of the instanton at $x_4 = 0$. At $x_4 \neq 0$, the original form[2] does not satisfy the Coulomb gauge condition. We make a gauge transformation (3.2) where U is given by (3.6). Then the Coulomb gauge condition (3.3) implies, as shown by Wadia and Yoneya[4],

$$\frac{\partial^2\beta}{\partial r^2}+\frac{2}{r}\frac{\partial\beta}{\partial}-\frac{x_4^2-r^2+\rho^2}{r^2(R^2+\rho^2)}\sin 2\beta-\frac{2x_4 r}{r^2(R^2+\rho^2)}\cos 2\beta$$
$$-\frac{2x_4(x_4^2+\rho^2)}{f(R^2+\rho^2)^2} = 0 \qquad (3.11)$$

This equation is a little complicated. However, if $x_4 << \rho$, the potential is given by $-\sin^2\beta$ in the limit $\tau\to-\infty$, while it is given by $+\sin^2\beta$ in the limit $\tau\to+\infty$. Other terms are small perturbations at finite time. The qualitative behavior of the solution is the same as in the case of $x_4 = 0$.

Thus as long as $x_4 << \rho$, the field of the instanton can be gauge transformed to the field with arbitrary a.

IV. INSTANTONS IN COULOMB GAUGE

With the preparation of the preceding two sections (§2 and §3) we are ready to make gauge transormations which take the instanton (q = 1) into Coulomb gauge. Specifying the gauge transformation at x_4 by $a(x_4)$, the problem is whether we can draw figures as in Fig. 2. First choose $\lim_{x_4\to-\infty} a(x_4) = a(-\infty)$, and then make continuous gauge transformation of the instanton up to $x_4^{(1)}$ where $x_4^{(1)} < 0$ and $|x_4^{(1)}| << \rho$. Next choose $\lim_{x_4\to+\infty} a(x_4) = a(+\infty)$ and do the same thing up to $x_4^{(2)}$ where $x_4^{(2)} > 0$ and $x_4^{(2)} << \rho$. The problem reduces to whether we can smoothly connect $a(x_4)$ between $x_4^{(1)}$ and $x_4^{(2)}$. We know that if $|x_4| << \rho$ an arbitrary value of a can be realized by suitably choosing b. Thus by choosing a continuous function $b(x_4)$ for $|x_4| << \rho$, we have a continuous $a(x_4)$ as shown in Fig. 2. [In the cases where a crosses zero as in Fig. 2 e ~ i, the range of b should be widened from the region $-b_o < b < b_o$ to get a continuous $b(x_4)$. This point will be discussed in more detail in the Appendix.]

In conclusion all the nine types of tunnelings can be realized in Coulomb gauge. Since the values of a at $x_4 = \pm\infty$ can be chosen arbitrarily, there are infinitely many ways of describing tunneling in Coulomb gauge. All of them have the same field strength configurations. This is due to ambiguities[1] in defining the field in Coulomb gauge.

In the cases of instanton with $q \geq 2$ the functions $a(x_4)$ is roughly those obtained by patching two or more of the solutions for $q = 1$, displayed in Fig. 2. In general $\vec{r}_o(x_4)$ depend on x_4. No special difficulties exist to construct instantons with $q \geq 2$.

One can construct θ-vacua[3] by introducing surface variables as dynamical variables as in Ref. 4.

V. TUNNELING FROM THE n = 0 VACUUM TO THE n = 1/2 VACUUM

It has been shown that there is tunneling from the vacuum with $n = 0$ to the vacuum with $n = 1/2$. Here we would like to elucidate the situation further. In $A_o = 0$ gauge[2], if the topological number of the vacuum at $x_4 = -\infty$ is an integer the instanton only connects it with a vacuum with integer topological number. This is trivial. Since $n_s = 0$ in (2.13), n_+ should be an integer if n_- is an integer. However, if we take other gauges, this is not in general true.

In Coulomb gauge the $n = 1/2$ vacuum field cannot be expressed in terms of elementary functions, nor can the instantons which are determined from the solutions of eq. (3.11). Thus it may be instructive to lift the restriction of the Coulomb gauge in order to show explicitly how to transform the instanton to the form which connects the $n = 0$ vacuum with the $n = 1/2$ vacuum. It may also help to understand how this kind of tunneling can occur by looking at the specific form of the solutions.

If the form of the field is restricted to the following form:

$$A_i^a = \varepsilon_{aib} \frac{x_b}{r^2} (\phi(r, x_4) + 1)$$
$$A_0^a = 0 \qquad , \tag{5.1}$$

the action is given by

$$A = \frac{8\pi}{g^2} \int_{-\infty}^{\infty} dx_4 \int_0^{\infty} dr \left[\frac{1}{2} \left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{1}{2} \left(\frac{\partial\phi}{\partial x_4}\right)^2 + \frac{1}{4r^2} (1 - \phi^2)^2 \right] \tag{5.2}$$

$\phi = -1$ corresponds to the $n = 0$ vacuum and $\phi = 1$ to the $n = 1/2$ vacuum with $a = \infty$. As discussed by Gribov[1], due to the singularity at $r = 0$ of the last term of eq. (5.2), it seems there are no tunnelings between them.

However the form of the field (5.1) is too stringent to discuss the tunneling between them. Even if the field asymptotically approaches the form given by (5.1) as $x_4 \to \pm\infty$, it may have a more general form at finite x_4. The solutions for tunnelings in Coulomb gauge obtained in the previous sections and the solutions obtained below have a more general form.

Following Witten[5], we write

$$A_i^a = \frac{\phi_2+1}{r^2} \varepsilon_{iab} x_b + \frac{\phi_1}{r^2} (\delta_{ia}r^2 - x_i x_a) + A_1 \frac{x_i x_a}{r^2}$$
$$A_o^a = A_o \frac{x^a}{r} \tag{5.3}$$

The action is then given by

$$A = \frac{8\pi}{g^2} \int_{-\infty}^{\infty} dx_4 \int_0^{\infty} dr \left[\frac{1}{2}(D_\mu\phi_i)^2 + \frac{1}{8} r^2 F_{\mu\nu}{}^2 + \frac{1}{4r^2} (1 - \phi_1^2 - \phi_i^2)^2 \right] \tag{5.4}$$

where $\mu = \nu = 0, 1$ (subscript o corresponds to x_4 and subscript 1 to r), $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $D_\mu \phi_i = \partial_\mu \phi_i + \varepsilon_{ij} A_\mu \phi_j$.

The general form of the instanton (q = 1) under the assumption of spherical symmetry is given by

$$\phi_1 = \sin 2f + \frac{2r^2}{R^3+\rho^2} \left(\frac{x_4}{r} \cos 2f - \sin 2f \right) \quad , \tag{5.5}$$

$$\phi_2 = -\cos 2f + \frac{2r^2}{R^2+\rho^2} \left(\cos 2f + \frac{x_4}{r} \sin 2f \right)$$

$$A_1 = 2 \frac{\partial f}{\partial r} + \frac{2x_4}{R^2+\rho^2}$$

and

$$A_o = 2 \frac{\partial f}{\partial x_4} - \frac{2r}{R^2+\rho^2}$$

where f is an arbitrary function of r and x_4, $f(r, x_4)$, and $R^2 = f^2 + {}_4^2$. Note again that we are not restricted to Coulomb gauge in this section.

The field (5.5) may be obtained from the solution A_μ^B of Belavin et al.[2] by the following gauge transformation

$$A_\mu = U^{-1} A_\mu{}^B U + U^{-1} \partial_\mu U \tag{5.6}$$

where

$$U = \exp\,(if(r, x_4)\, \vec{\sigma}\vec{n}) \quad . \tag{5.7}$$

The action is independent of the function $f(r, x_4)$ and is $8\pi^2/g^2$, as it should be.

In order that the field (5.5) connect the n = 0 vacuum to the n = 1/2 vacuum, the function $f(r, x_4)$ must be constrained so that the topological number of U (5.7) [defined by eq. (2.11)] at $x_4 = -\infty$ is + 1/2 and at $x_4 = +\infty$ is zero. Otherwise the function $f(r, x_4)$ is arbitrary. We can see that with this constraint the field (5.5) satisfies these requirements by direct calculation, or by noting that the original form A_μ^B connects the n = - 1/2 vacuum with the n = + 1/2 vacuum. There are a lot of such functions. Simple examples are

$$f(r, x_4) = \frac{r}{\sqrt{r^2+c^2}} \; h(x_4) \tag{5.8}$$

where c is a constant and $h(x_4)$ is an arbitrary function satisfying **the following condition:**

$$\lim_{x_4 \to -\infty} h(x_4) \longrightarrow \frac{\pi}{2} \tag{5.9}$$

and

$$\lim_{x_4 \to +\infty} h(x_4) \longrightarrow 0 \quad .$$

For example

$$h(x_4) = \frac{1}{2} \left(-\tan^{-1} \frac{x_4}{d} + \frac{\pi}{2} \right) \tag{5.10}$$

where d is a constant. Another example is

$$f(r,x_4) = -\tan^{-1} \frac{r}{x_4} + \frac{1}{2} \left(-\tan^{-1} \frac{x_4}{d} + \frac{\pi}{2} \right) \quad . \tag{5.11}$$

In this case the initial vacuum and the final vacuum are those given by eq. (5.1) with $\phi = \pm 1$. The field is singular at $r = 0$ and the Pontryag in index q comes also for the surface integral surrounding the $r = 0$ axis. In Coulomb gauge a similar thing happens in the transition from $\phi = -1$ to $\phi = +1$.

The action (5.4) is divergent at $r = 0$ unless $1 - \phi_1^2 - \phi_2^2 = 0$ at $r = 0$. The functions ϕ_1 and ϕ_2 at $r = 0$ are given by

$$\phi_1 = \sin 2f(r, x_4) + 0(r^2)$$

$$\phi_2 = -\cos 2f(r, x_4) + 0(r^2) \tag{5.12}$$

Thus the action is finite as it should be. If we restrict ourselves to one Higgs meson (ϕ_2) in 2-dimensional space time, there is no tunneling from $\phi_2 = -1$ to $\phi_2 = +1$. Since we have two Higgs mesons, we can rotate around the circle $\phi_1^2 + \phi_2^2 = 1$ from $\phi_2 = -1$ to $\phi_2 = +1$.

In $A_o = 0$ gauge, the transition from the $n = 0$ vacuum to the $n = 1/2$ vacuum is forbidden. One topological reason has been given above. Another reason is that the term

$$\int T_r \, (\dot{A}_i)^2 \; d^4x \tag{5.13}$$

of the action diverges[6,7] if we try to make such configurations. However in the above examples this divergence does not occur because the combination

$$\int T_r(\dot{A}_i + \partial_i A_o + [A_i, A_o])^2 \, d^4x \tag{5.14}$$

appears. This integral is finite, since the integrand decreases as $1/(R^2)^2$ as $R^2 \to \infty$.

Now let us compare the Coulomb gauge instanton solution which tunnels from the $n = 0$ vacuum to the $n = 1/2$ vacuum with the meron solution[7,8]. A meron solution may be written in Coulomb gauge as follows[7,11]

$$A_2^a = (1 + \frac{x_4}{R}) \; \frac{\varepsilon_{iab} x_b}{r^2} \tag{5.15}$$

$$A_o^a = 0$$

This configuration connects the two vacua discussed above ($\phi = \pm 1$). However the action is infinite. Our solution has finite action. Our solution has non-zero A_o at some finite x_4, while the meron solution has $A_o = 0$ for all x_4. If we try to gauge transform our solution to $A_o = 0$ gauge, we obviously obtain the instanton in $A_o = 0$ which connect the $n = 0$ vacuum with the $n = 1$ vacuum. Thus in Coulomb gauge there are tunnelings between the $n = 0$ vacuum and the $n = 1/2$ vacuum, while in $A_o = 0$ there are no such tunnelings.

VI. CONCLUSION AND DISCUSSION

As pointed out by Gribov, the classical vacuum is not unique in SU(2) Yang-Mills theory even after imposing the Coulomb gauge condition. There are three kinds of vacua which have topological number $n = 0$, $+ 1/2$, and $- 1/2$. The vacuum with $n = 0$ is $A_\mu = 0$.

The vacua with $n = + 1/2$ and $n = - 1/2$ are specified by four parameters, $\vec{r}_o$ and a, and the field A_i^a decreases as $1/r$ as $r \to \infty$.

It is shown that there are all possible tunnelings (9 types) between these vacua. In other words the instanton is not unique in Coulomb gauge and can be gauge thrnsformed so as to produce any transition between these (infinitely many) vacua. This is due to the ambiguity at defining large fields in Coulomb gauge. This ambiguity is also pointed out by Gribov. In $A_o = 0$ gauge the transition from the $n = 0$ vacuum to the $n = 1/2$ vacuum is forbidden. This is a special feature inherent to $A_o = 0$ gauge. A detailed account concerning this point has been given.

Even in QED the Coulomb gauge condition does not uniquely specify the vacuum. The field $A_i = c_i$ (c_i's are constants) is a pure gauge and satisfies the Coulomb gauge condition. However, there is no tunneling between this vacuum and the $A_\mu = 0$ vacuum. We may choose one of them as a vacuum. [We usually require that the field decrease as $r \to \infty$.] Thus the gauge symmetry is spontaneously broken. The photon can be interpreted[9] as a Nambu-Goldstone particle for this spontaneous symmetry breakdown.

Since it is shown that there are tunnelings between the $n = 0$ vacuum and the $n = 1/2$ vacua in the SU(2) Yang-Mills theory we cannot simply require that the field decreases faster than $1/r$ as $r \to \infty$. Furthermore even the tunneling from the $n = 0$ vacuum to the $n = 0$ vacuum occurs through a field configuration which decreases as $1/r$. [If we restrict ourselves to fields which decrease faster than $1/r$, the action becomes too singular and diverges, as shown in the Appendix.] The fact that the gauge

symmetry is restored by tunnelings suggests also the non-existence of massless gluons.

The fundamental problem remaining is how to quantize the field canonically, if there is a way at all, when the gauge condition does not uniquely specify the field. This problem is connected with the question of how to quantize the theory by the path integral formalism - does one choose one of equivalent fields or average them? The Faddeev-Popov determinant will be modified. What is shown in this paper should be taken into account in attempting to solve this problem and it may also help to find the solution. We hope we can discuss it elsewhere.

There are several papers[10,11,12,13] which discuss the same problem as this paper. However, they overlook the tunneling between the $n = 0$ vacuum and the $n = 1/2$ vacua and some conclusions in them contradict our results.

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APPENDIX

Discontinuous Fields and Ambiguities of Fields

If we restrict $a(x_4)$ to be zero for all time x_4 as Jackiw, Muzinich and Rebbi[12] assumed, then there are no continuous field configurations for the instanton which connect the n = 0 vacuum to the n = 0 vacuum. Let us review their results. The field is discontinuous at a certain time $x_4 = \bar{x}_4$, that is,

$$\lim_{x_4 \to \bar{x}_4 - \varepsilon} A_\mu(x, x_4) = A_\mu^{(1)} \quad ,$$

$$\lim_{x_4 \to \bar{x}_4 + \varepsilon} A_\mu(x, x_4) = A_\mu^{(2)} \tag{A.1}$$

and

$$A_\mu^{(2)} = U^{-1} A_\mu^{(1)} U + U^{-1} \partial_\mu U \quad . \tag{A.2}$$

The Pontryagin index is given by

$$q = n_+ - n_- + n_s + n(U) \tag{A.3}$$

where n(U) is the same as defined in eq. (2.11). Here $n_+ = n_- = n_s = 0$ and $n(U) = 1$.

First we would like to show in this Appendix that if we admit this kind of discontinuity of fields, the tunneling from the n = 0 vacuum to the n = 1/2 vacuum is trivial. Consider two field configurations: the above one (a) which connect the n = 0 vacuum with the n = 0 vacuum and which has a discontinuity at $x_4 = \bar{x}_4$,

and the other one (b) which connects the n = -1/2 vacuum with the n = +1/2 vacuum and which is known to be continuous. Then use the field (a) for $x_4 < \bar{x}_4$ and the field (b) for $x_4 > \bar{x}_4$. The field is discontinuous at $x_4 = \bar{x}_4$, that is,

$$\lim_{x_4 \to \bar{x}_4 - \varepsilon} A_\mu(x, x_4) = A_\mu^{(1)}$$

$$\lim_{x_4 \to \bar{x}_4 + \varepsilon} A_\mu(x, x_4) = A_\mu^{(3)} \tag{A.4}$$

and

$$A_\mu^{(3)} = U'^{-1} A_\mu^{(1)} U' + U'^{-1} \partial_\mu U' \tag{A.5}$$

In this case $n_+ = 1/2$ and $n(U') = 1/2$, thus $q = 1$. This is the instanton which connects the n = 0 vacuum with n = 1/2 vacuum and which is discontinuous $x_4 = \bar{x}_4$. We can obtain in this way any of the nine possible types of tunnelings.

Next we would like to show that this type of discontinuity leads to an infinite action. The Pontryagin index can be expressed as a surface integral (eq. 2.15) surrounding the singularities of the field, and therefore it is well defined under any singular gauge transformation. However, the action becomes ambiguous or divergent at some singularities. Formally we can prove the following: by the gauge transformation

$$A_\mu' = U^{-1} A_\mu U + U^{-1} \partial_\mu U \tag{A.6}$$

the field strength transforms covariantly

$$F'_{\mu\nu} = U^{-1} F_{\mu\nu} U \quad . \tag{A.7}$$

Thus we have

$$T_r(F'^{\,2}_{\mu\nu}) = T_r(F_{\mu\nu}^{\,2}) \tag{A.8}$$

This proves the invariance of the action under the gauge transformation.

However if U contains $\Theta(x_4)$ as, e.g.,

$$U = V(x_i)\ \theta(x_4) + 1\cdot\Theta(-x_4) \tag{A.9}$$

the formal proof breaks down. The relations, e.g.,

$$\begin{aligned}(U^{-1}\partial_4 U)\cdot(U^{-1}\partial_i U) \\ = (-\partial_4 U^{-1}U)\cdot(U^{-1}\partial_i U) \\ = -\partial_4 U^{-1}(UU^{-1})\partial_i U \\ = -\partial_4 U^{-1}\cdot\ \partial_i U\end{aligned} \tag{A.10}$$

do not hold. Actually, explicit calculation gives us

$$F'_{i4} = U^{-1}F_{i4}U + \delta(x_4)D \tag{A.11}$$

where D has some complicated terms and is not zero.

We have

$$\int T_r(F'^{\,2}_{\mu\nu})\ d^4x \sim \delta(0) \tag{A.11}$$

Thus the action is infinite. This means that there is no tunneling by this kind of field configuration.

However, we have shown in the text that there are continuous field configurations for the instanton. The discontinuity discussed above implies non-uniqueness of the field[12]. Because of this non-uniqueness, the singularity can be smoothed out. Although this point has been already discussed in the text, we would like to discuss it here from a different point of view.

As we have done in §4, choose $\lim_{x_4\to-\infty} a(x_4) = a(-\infty)$, and then make a continuous gauge transformation of the instanton up to $\bar{x}_4$ where $|x_4| \ll \rho$. Next choose $\lim_{x_4\to+\infty} = a(+\infty)$ and do the same thing up to $\bar{x}_4$. In general the field becomes discontinuous at $\bar{x}_4$ as in eqs. (A.1) and (A.2)

$$\begin{aligned}\lim_{x_4\to\bar{x}_4-\varepsilon} A_\mu(x_i, x_4) &= A_\mu^{(1)} \\ \lim_{x_4\to\bar{x}_4+\varepsilon} A_\mu(x_i, x_4) &= A_\mu^{(2)}\end{aligned} \tag{A.1}$$

and

$$A_\mu^{(2)} = U^{-1}A_\mu^{(1)}U + U^{-1}\partial_\mu U \tag{A.2}$$

Equations (A.1) and (A.2) mean that at $x_4 = \bar{x}_4$ there are two equivalent fields $A^{(1)}$ and $A^{(2)}$. If the fields $A^{(1)}$ and $A^{(2)}$ are specified by $a^{(1)}$ and $a^{(2)}$ (defined by eq. (1.8)), a can take any value between $a^{(1)}$ and $a^{(2)}$, from the argument in §3. Suppose the discontinuous fields are specified by $a(x_4)$ shown in Fig. 3. $a(x_4)$ is discontinuous at $\bar{x}_4$. The value of a can take any value between $a^{(1)}$ and $a^{(2)}$ at $x_4 = \bar{x}_4$. A similar thing can be proven around $\bar{x}_4$ by changing the discontinuous point back and forth. This fact is enough to smoothly connect the line of a around $\bar{x}_4$.

To get a smooth curve in the cases of Fig. 2 e ~ i, $\beta(+\infty)$ [defined in §3] should reach $3/2\pi$ at some particular b, b_1. This can be concluded by analyzing eq. (3.7), but by the above argument it is concluded more easily.

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FIGURE CAPTIONS

Fig. 1 x dependence of the function $g(x)$.

Fig. 2 Nine types of tunnelings and the behavior of $a(x_4)$ for each case.

Fig. 3 Discontinuous function $a(x_4)$ at $x_4 = \bar{x}_4$.

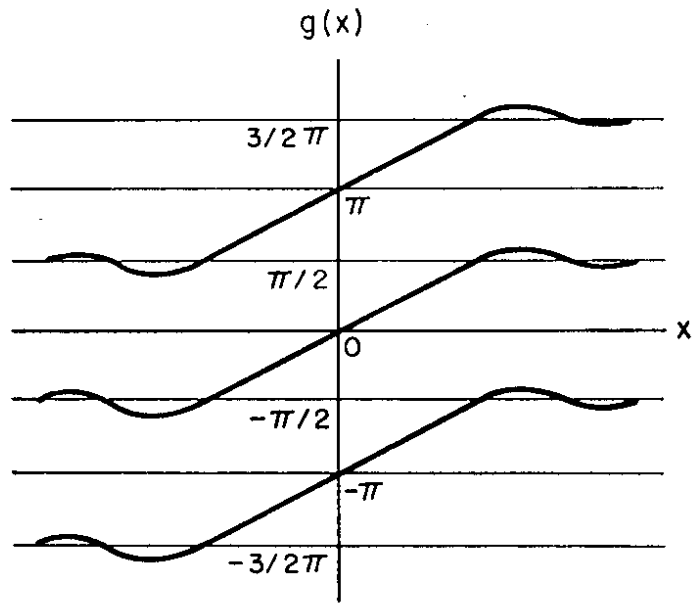

FIG. 1

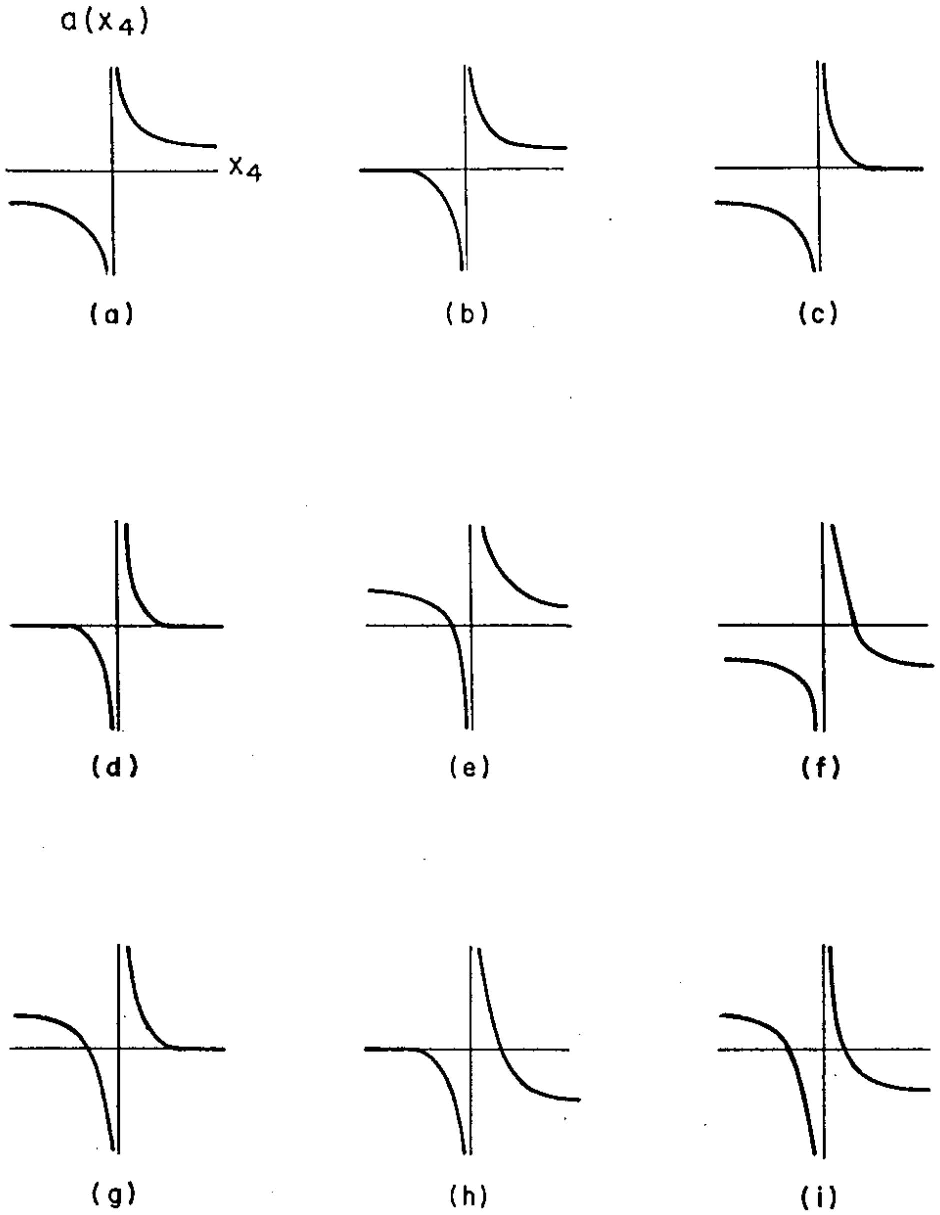

FIG. 2

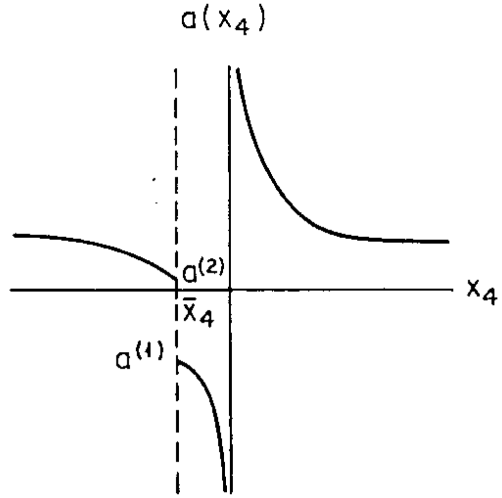

FIG. 3

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