# ZERO MODES OF THE 't HOOFT–POLYAKOV MONOPOLE ☆

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The zero energy eigenvalue problem for quantum fluctuations about the 't Hooft–Polyakov monopole solution is solved explicitly in the Prasad–Sommerfield limit, by exploiting the formal similarity with self-dual euclidean field configurations. The relevance of the results to the spin of the quantized monopole is examined.

A renewed interest in the quantization of the 't Hooft–Polyakov monopole solution [1] has been stimulated by the recent work of Olive and Montonen [2], who advance the intriguing hypothesis that Yang–Mills gauge theories actually possess a greater symmetry on the quantum level than is apparent from the classical lagrangian. This new symmetry would involve an interchange of the roles of electric and magnetic charges. It is proposed that a "dual" formulation of the gauge theory exists, in which the magnetic charge is an explicit Noether symmetry generator on the fundamental fields whereas electric charge is purely topological in character. The field quanta are magnetic monopoles and the vector bosons of the usual formulation appear only in the soliton sector with a topologically conserved electric charge.

In order for this hypothesis to be tenable, the spin of the electrically charged vector boson and the magnetic monopole must be the same, since in the "dual" formulation, the 't Hooft–Polyakov solution, when quantized, *is* the vector boson. Thus, there would seem to be required some subtlety in the quantization of the rotationally symmetric (up to a gauge transformation) monopole solution which endows it with a unique non-zero value of angular momentum, $\hbar$, in its ground state. However, if the quantum ground state of the monopole system is to be rotationally asymmetric (and three-fold degenerate), one would expect to see some evidence of the phenomenon in the excitation spectrum of quantum oscillations about the classical configuration and, in particular, in the zero energy eigenmodes. This work reports on the explicit solution of the zero mode problem in the special limit of the monopole model, considered by Prasad and Sommerfield [3] where the classical fields may be expressed in terms of elementary functions.

Begin with the model lagrangian,

$$\mathcal{L} = -\tfrac{1}{4}F^i_{\mu\nu}F^{i\mu\nu} - \tfrac{1}{2}(D^\mu\Phi)^i(D_\mu\Phi)^i - \tfrac{1}{4}\lambda(\Phi^i\Phi^i - b^2)^2,$$

where

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + e\epsilon^{ijk}A^j_\mu A^k_\nu, \qquad (D_\mu\Phi)^i = \partial_\mu\Phi^i + e\epsilon^{ijk}A^j_\mu\Phi^k \tag{1}$$

and the spherically symmetric static ansatz for the classical fields:

$$A^{ic\ell}_m = \epsilon_{mij}\hat{r}_j A(r)/e, \quad A^{ic\ell}_0 = 0, \quad \Phi^{ic\ell} = \hat{r}_i\phi(r)/e. \tag{2}$$

In the limit $\lambda \to 0$ but $\Phi^i\Phi^i$ is still required to approach $b^2$ as $r \to \infty$, the field equations are satisfied by (2) with:

$$A(r) = \frac{C}{\sinh(Cr)} - \frac{1}{r}, \quad \phi(r) = C\frac{\cosh(Cr)}{\sinh(Cr)} - \frac{1}{r}, \quad C = eb. \tag{3}$$

Now, when $\lambda = 0$, the static Euler–Lagrange equation for the scalar field, $\Phi^i$ is formally identical to that for $A^i_0$ [4]. This formal similarity between $\Phi^i$ and $A^i_0$ persists at the quantum level and allows a solution to

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the zero mode problem by methods primarily developed for the pseudoparticle solutions of the pure Yang–Mills theory [5].

Explicitly, in the $A^i_0 = 0$ gauge, the hamiltonian density for the monopole theory with $\lambda = 0$ is:

$$H = \tfrac{1}{4}F^i_{mn}F^i_{mn} + \tfrac{1}{2}(\partial_0 A^i_m)(\partial_0 A^i_m) + \tfrac{1}{2}(D_m\Phi)^i(D_m\Phi)^i + \tfrac{1}{2}(\partial_0\Phi^i)(\partial_0\Phi^i). \tag{4}$$

The classical solution (2) is time-independent and we are seeking the lowest eigenvalue of the energy about that static configuration. Because of the positive definiteness of each term in (4), the lowest eigenvalue is obtained when the quantum mode functions, $a^i_m$ and $\phi^i$ in the expansion of the fields,

$$A^i_m = A^{i\mathrm{c}\ell}_m + a^i_m, \qquad \Phi^i = \Phi^{i\mathrm{c}\ell} + \phi^i \tag{5}$$

are also time-independent. In that case, the hamiltonian density (4) is identical to the *action*-density of a pure Yang–Mills theory in euclidean space–time, if the former scalar field, $\Phi^i$ is identified with $A^i_0$ of the new description.

Furthermore, the pure Yang–Mills solution:

$$A^{i\mathrm{c}\ell}_m = \epsilon_{mij}\hat{r}_j A(r)/e, \qquad A^{i\mathrm{c}\ell}_0 = \hat{r}_i\phi(r)/e, \tag{6}$$

is euclidean self-dual, $F^i_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^i_{\rho\sigma}$ (with $\epsilon_{0123} = +1$) so that the zero energy eigenmode problem for the actual monopole solution (2) is identical to the problem of finding the zero-action modes about the self-dual euclidean solution (6) with $A^i_0$ identified with $\Phi^i$.

The zero action eigenmodes about a euclidean self-dual Yang–Mills field have been found by Brown et al. [5]. The zero mode eigenfunctions can be constructed from spinor solutions of the equation,

$$-\mathrm{i}\sigma_\mu D^{ij}_\mu \chi^j = [-\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\,\delta^{ij} - \mathrm{i}\epsilon^{ijk}A(r)(\boldsymbol{\sigma}\times\hat{r})^k + \epsilon^{ijk}\phi(r)\hat{r}^k]\,\chi^j = 0, \tag{7}$$

where $\sigma_0 = \begin{pmatrix}-\mathrm{i} & 0\\ 0 & -\mathrm{i}\end{pmatrix}$ and $\boldsymbol{\sigma}$ are the Pauli matrices. $\chi^i = \begin{pmatrix}\alpha^i\\ \beta^i\end{pmatrix}$ is a two-component Lorentz spinor and a vector in the SU(2) group space. Expanding the gauge fields about the self-dual solution (6),

$$A^i_\mu = A^{i\mathrm{c}\ell}_\mu + a^i_\mu \tag{8}$$

the eigenmodes of zero action, in terms of the solutions to equation (7) are:

$$-\mathrm{i}\sigma^\dagger_\mu a^i_\mu = \begin{pmatrix}\alpha^i & -\beta^{i*}\\ \beta^i & \alpha^{i*}\end{pmatrix}, \tag{9a}$$

$$-\mathrm{i}\sigma^\dagger_\mu a^i_\mu = \begin{pmatrix}\mathrm{i}\alpha^i & \mathrm{i}\beta^{i*}\\ \mathrm{i}\beta^i & -\mathrm{i}\alpha^{i*}\end{pmatrix}, \tag{9b}$$

which satisfy a background gauge condition:

$$(D_\mu a_\mu)^i = \partial_\mu a^i_\mu + e\epsilon^{ijk}A^{j\mathrm{c}\ell}_\mu a^k_\mu = 0. \tag{10}$$

Thus, each time-independent spinor solution to (7) yields *two* zero-energy gauge field modes in the background gauge.

Now, precisely the spinor equation (7) has been thoroughly analyzed by Jackiw and Rebbi [6] in the somewhat different context of fermion-monopole scattering. The results of their analysis are summarized here.

There are two distinct normalizable solutions to (7), given by:

$$\chi^i = N[f_1(r)\hat{r}^i\boldsymbol{\sigma}\cdot\hat{r} + f_2(r)(\sigma^i - \hat{r}^i\boldsymbol{\sigma}\cdot\hat{r}]\,\chi_{\uparrow,\downarrow}$$

where

$$\chi_\uparrow = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \chi_\downarrow = \begin{pmatrix}0\\1\end{pmatrix}. \tag{11}$$

The functions $f_1(r)$ and $f_2(r)$ may be expressed in terms of the functions,

$$\rho(r) = \frac{1}{r} + A(r) = \frac{C}{\sinh(Cr)},$$

$$F(r) = \frac{1}{2}\left[\phi(r) - \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{1}{r} = \phi(r)\right] \tag{12}$$

and the regular solution to the differential equation,

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2} - \left(F^2 + \frac{\mathrm{d}F}{\mathrm{d}r} + 2\rho^2\right)u = 0 \tag{13}$$

through

$$f_1(r) = \frac{1}{r^2}\,u(r)\exp\left[-\int_0^r \mathrm{d}r'\,F(r')\right],$$

$$f_2(r) = \frac{1}{2r^2\rho(r)}\frac{\mathrm{d}}{\mathrm{d}r}(r^2 f_1(r)). \tag{14}$$

The factor $N$ in (11) is a finite normalization constant.

The four zero modes obtained from (11) in conjunction with (9) may be written in the concise covariant form,

$$a^i_\mu(\lambda) = N\hat{r}_i\hat{r}_j\eta_{j\mu\lambda}(f_1 - f_2) + N\eta_{i\mu\lambda}f_2, \tag{15}$$

where $\lambda = 0, 1, 2, 3$ labels the four modes and $\eta_{i\mu\nu}$ is the self-dual tensor defined by:

$$\eta_{i\mu\nu} = \begin{cases} \epsilon_{i\mu\nu} & \mu, \nu = 1, 2, 3 \\ \delta_{i\nu} & \mu = 0; \nu = 1, 2, 3 \\ -\delta_{i\mu} & \nu = 0; \mu = 1, 2, 3 \\ 0 & \mu = \nu = 0 \end{cases} \tag{16}$$

Furthermore, it is readily verified that the differential equation (13) is solved by:

$$u(r) = r^2 \frac{\mathrm{d}\phi}{\mathrm{d}r} \exp\left\{\int_0^r \mathrm{d}r' \phi(r')\right\} \tag{17}$$

so that (14) gives $f_1$ and $f_2$ explicitly:

$$f_1(r) = \frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{1}{r^2} - \frac{C^2}{\sinh^2(Cr)},$$
$$f_2(r) = \rho\phi = \frac{C^2\cosh(Cr)}{\sinh^2(Cr)} - \frac{C}{r\sinh(Cr)}. \tag{18}$$

Each of the zero modes possesses a simple interpretation in terms of the classical solution (6). For $\lambda = l = 1, 2, 3$ we obtain three translational zero energy modes in the background gauge (10). In fact, performing one of the three infinitesimal gauge transformations given by:

$$\Lambda^i(l) = N\epsilon_{lij}\hat{r}_j A(r) \tag{19}$$

brings $a^i_\mu(l)$ into the form,

$$a^i_\mu(l)' = a^i_\mu(l) + (D_\mu\Lambda(l))^i = N\partial_l A^{i\,\mathrm{c}\ell}_\mu, \tag{20}$$

which explicitly is the form of translational zero modes.

The fourth mode, $\lambda = 0$ is a pure gauge mode, present because (10) does not completely eliminate gauge degrees of freedom. Explicitly,

$$a^i_\mu(\lambda = 0) = (D_\mu\Lambda)^i, \quad \Lambda^i = -N\hat{r}_i\phi(r). \tag{21}$$

Thus, returning to the original 't Hooft–Polyakov solution (2), we have obtained all of the zero energy eigenmodes for the gauge fields $(a^i_m)$ and the scalar field $(\phi^i)$. There are the anticipated three translational modes,

$$a^i_m(l) = N\hat{r}_i\hat{r}_j\epsilon_{jml}(f_1 - f_2) + N\epsilon_{iml}f_2, \tag{22}$$

$$\phi^i(l) = N\hat{r}_i\hat{r}_l(f_1 - f_2) - N\delta_{il}f_2, \tag{22}$$

satisfying the condition,

$$\partial_m a^i_m + e\epsilon^{ijk}A^{j\,\mathrm{c}\ell}_m a^k_m + e\epsilon^{ijk}\Phi^{j\,\mathrm{c}\ell}\phi^k = 0 \tag{23}$$

for each $l$ and there is a pure gauge mode permitted by the gauge condition, eq. (23). This condition, which is imposed only on the time-independent zero energy modes restricts the time-independent gauge transformations left free by the $A^i_0 = 0$ condition alone. Of course, the $A^i_0$ gauge choice completely fixes the time dependent gauge freedom and specifies $a^i_0(l) = 0$ to complete (22).

These are exactly the results to be expected if the monopole has zero spin. With the zero energy spectrum completely determined, the system can be treated by any of the established canonical techniques for the quantization of soliton solutions [7], once a suitably well-defined choice of gauge has been made. We simply have collective position coordinates, corresponding to the three zero energy translational modes (22), and a spectrum of positive energy excitations, corresponding to monopole-boson scattering or bound states. Any asymmetry or degeneracy in the ground state would have to be artificially inserted. Canonical quantization appears to preclude any subtle one loop effect which would realize the spin one conjecture for the 't Hooft–Polyakov monopole, necessary for the proposed electric-magnetic symmetry in the Yang–Mills gauge theory.

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