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# HYPERFINE SPLITTINGS OF BAG MODEL GLUONIA

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Order $\alpha_s$ contributions to the energy shifts in gluonia are studied in the MIT bag model. The naive result that $M(0^{-+}) > M(0^{++})$ is preserved for $\alpha_s \lesssim 3$, although $M(0^{-+}) < M(2^{++})$ results for $\alpha_s \gtrsim 0.54$. The $0^{-+}$ and $2^{++}$ can thus be identified with $\iota(1440)$ and $\theta(1640)$ for reasonable values of $\alpha_s$ and bag size, in which case the $0^{++}$ mass is predicted to be $\sim 1.0$ GeV.

## 1. Introduction

Recently there has been an upsurge of interest in gluonia – color singlet bound states of two or more gluons – and speculation that low-lying $0^{-+}$ and $2^{++}$ states may already have been manifested at 1440 and 1640 MeV [1–3]. The MIT bag model has been cited as a guide to gluonium masses [4]; it yields degenerate $0^{++}, 2^{++}$ states at $\sim 1$ GeV and $0^{-+}, 2^{-+}$ states at $\sim 1300$ MeV *before* hyperfine splitting effects arising from single gluon exchange and the four-gluon interaction are included.

The recently developed formalism for QCD perturbation theory in a finite spherical cavity [5–10] enables the effects of gluon exchange to be computed directly. To date this has been applied only to quarklei and now for the first time we apply it to gluonia. We find that the $0^{++}, 2^{++}$ degeneracy is lifted, and that it is possible to fit the $0^{-+}$ and $2^{++}$ states with plausible parameter values $\alpha_s = 0.75$, $a^{-1} = 0.22$ GeV. (This radius is essentially the anticipated value from quark bag phenomenology.) With these parameters, the model predicts a scalar glueball at 1.05 GeV.

The paper is organized as follows.

We begin with the QCD hamiltonian, working in the Coulomb gauge, and define the gluon and quark modes in a cavity. Interaction vertices for gluons and/or quarks are then discussed.

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In sect. 3 the derivation of the three-gluon and four-gluon contact and Coulomb vertices is shown in detail, and specific cases are evaluated numerically for lowest mode TE and TM gluons. Application of these results to the second-order energy shift in gg → gg is made in sect. 4. The phenomenological consequences of the mass shifts in $(\mathrm{TE})^2$ and (TE)(TM) gluonia – $0^{++}$, $2^{++}$, and $0^{-+}$ states – are discussed in sect. 5.

Finally, in sect. 6 we briefly summarize our results and conclusions, and we follow this with mathematical appendices on vector spherical harmonics, vector Fierz transforms, cavity Coulomb integrals, and cavity gluon vertices.

## 2. Definitions and preliminaries

### 2.1. HAMILTONIAN

The hamiltonian for quarks and gluons may be written

$$\begin{aligned} H_{\mathrm{QCD}} = \int \mathrm{d}\boldsymbol{x} \colon \Big\{ & H_0(A) + H_0(\psi) + \tfrac{1}{2} g f^{abc} \boldsymbol{H}^a \cdot (\boldsymbol{A}^b \times \boldsymbol{A}^c) \\ & + \tfrac{1}{4} g^2 f^{abc} f^{ade} (\boldsymbol{A}^b \cdot \boldsymbol{A}^d)(\boldsymbol{A}^c \cdot \boldsymbol{A}^e) \\ & - g \bar{\psi} \boldsymbol{\gamma} \cdot \boldsymbol{A}^a \frac{\lambda^a}{2} \psi \Big\} \colon + \frac{1}{8\pi} \int\!\!\int \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{y}\, \rho^a(\boldsymbol{x}) G(\boldsymbol{x}, \boldsymbol{y}) \rho^a(\boldsymbol{y}) , \end{aligned} \tag{2.1}$$

where $H_0(A)$, $H_0(\psi)$ are the "free" gluon and quark hamiltonia

$$\begin{aligned} H_0(A) &= \tfrac{1}{2} |\mathring{\boldsymbol{A}}^a|^2 + \tfrac{1}{2} |\nabla \times \boldsymbol{A}^a|^2 , \\ H_0(\psi) &= i \psi^\dagger \mathring{\psi} , \end{aligned} \tag{2.2}$$

and the charge density to lowest order is

$$\rho^a = \colon g \Big\{ \psi^\dagger \frac{\lambda^a}{2} \psi - f^{abc} \mathring{\boldsymbol{A}}^b \cdot \boldsymbol{A}^c \Big\} \colon \tag{2.3}$$

(Small latin indices are SU(3) $\underline{8}$ color labels.) Higher order corrections arise from ghost loops (covariant gauges) and/or modifications to $G(\boldsymbol{x}, \boldsymbol{y})$ in the Coulomb gauge. Throughout this work we shall employ the Coulomb gauge.

In some earlier bag model applications the empirical magnitude of $g$ has been reduced by using $g' \equiv \frac{1}{2} g$ (thus $g' f^{abc} \boldsymbol{H}^a \cdot \boldsymbol{A}^b \times \boldsymbol{A}^c$ appears in the first line of $H_{\mathrm{QCD}}$). We think that it is better for future applications if we adopt the more conventional normalization forthwith. Consequently some factors of two or four may appear relative to ref. [5–9].

The gluon and quark field operators will be expanded in classical modes as follows:

$$A^a(x,t)=e^a\sum_n\left\{A_n(x)e^{-i\omega_n t}a_n^a+A_n^*(x)e^{i\omega_n t}a_n^{a\dagger}\right\},\tag{2.4}$$

$$\psi^c(x,t)=c\sum_n\left\{\psi_n(x)e^{-i\omega_n t}b_n^c+i\gamma_2\psi_n^*(x)e^{i\omega_n t}d_n^{c\dagger}\right\},\tag{2.5}$$

where $e^a$ and $c$ are unit color 8-vectors and 3-spinors and the sums are over the mode functions $\psi_n$, $A_n$. These fields are normalized so that a one-particle state which is free but confined to the cavity has energy $\omega_n$,

$$H_0=\int dx\!:\!H_0(A)+H_0(\psi)\!:=\sum_{a,n}\omega_n a_n^{a\dagger}a_n^a+\sum_{c,n}\omega_n\left(b_n^{c\dagger}b_n^c+d_n^{c\dagger}d_n^c\right),\tag{2.6}$$

and hence

$$\int dx\, A_{n'}^*\cdot A_n=\frac{1}{2\omega_n}\delta_{nn'},\qquad \int dx\,\psi_{n'}^\dagger\psi_n=\delta_{nn'}.\tag{2.7}$$

### 2.2. GLUON MODES

The normalized classical gluon modes are of TE ("magnetic") and TM ("electric") type:

TE:

$$A_{jm}=\alpha_j^{\mathrm{TE}}j_j(\omega r)Y_{jjm},\tag{2.8}$$

$$A_{jm}^*=(-)^{m+1}A_{j,-m},\tag{2.9}$$

$$H_{jm}=\frac{i\alpha_j^{\mathrm{TE}}}{\sqrt{2j+1}}\left\{\sqrt{j+1}\,j_{j-1}(\omega r)Y_{jj-1m}-\sqrt{j}\,j_{j+1}(\omega r)Y_{jj+1m}\right\},\tag{2.10}$$

$$H_{jm}^*=(-)^{m+1}H_{j,-m};\tag{2.11}$$

TM:

$$A_{jm}=\frac{\alpha_j^{\mathrm{TM}}}{\sqrt{2j+1}}\left\{\sqrt{j+1}\,j_{j-1}(\omega r)Y_{jj-1m}-\sqrt{j}\,j_{j+1}(\omega r)Y_{jj+1m}\right\},\tag{2.12}$$

$$A_{jm}^*=(-)^{m}A_{j,-m},\tag{2.13}$$

$$H_{jm}=i\alpha_j^{\mathrm{TM}}\omega j_j(\omega r)Y_{jjm},\tag{2.14}$$

$$H_{jm}^*=(-)^{m}H_{j,-m},\tag{2.15}$$

with the normalization

$$\alpha_j^{\mathrm{TE}} = \left| \left\{ \chi_j \left( 1 - \frac{j(j+1)}{\chi_j^2} \right) \right\}^{-1/2} j_j(\chi_j)^{-1} a^{-1} \right|, \tag{2.16}$$

$$\alpha_j^{\mathrm{TM}} = | \chi_j^{-1/2} j_{j+1}(\chi_j)^{-1} a^{-1} | . \tag{2.17}$$

The mode numbers $\chi_j \equiv \omega_j a$ and modes are those given by Barnes [11], although his gluon normalizations $\{\alpha_j\}$ are $\sqrt{2}$ too large.

The mode numbers for the first few modes are

TE:

$$j_{j-1}(\chi) = \frac{j}{j+1} j_{j+1}(\chi) \tag{2.18}$$

| $J^P$ | $\chi$ | |
|---|---|---|
| $1^+$ | 2.744 | 6.117 |
| $2^-$ | 3.870 | 7.443 |

(2.19)

TM:

$$j_j(\chi) = 0 \tag{2.20}$$

| $J^P$ | $\chi$ | |
|---|---|---|
| $1^-$ | 4.493 | 7.725 |
| $2^+$ | 5.763 | 9.905 |

(2.21)

As we shall repeatedly consider the lowest TE and TM modes, their properties are of special interest. They are explicitly

$$\boldsymbol{A}_{1m}^{\mathrm{TE}} = \alpha_1 j_1(\omega r) \boldsymbol{Y}_{11m}, \tag{2.22}$$

$$\alpha_1 = 1.819 a^{-1}, \qquad \omega = 2.744 a^{-1}, \qquad \boldsymbol{y}_{11m} = -i\sqrt{\frac{3}{8\pi}}\, \hat{\boldsymbol{r}} \times \boldsymbol{e}_m, \tag{2.23}$$

$$\boldsymbol{A}_{1m}^{\mathrm{TM}} = \alpha_1' \left\{ \sqrt{\tfrac{2}{3}}\, j_0(\omega' r) \boldsymbol{Y}_{10m} - \sqrt{\tfrac{1}{3}}\, j_2(\omega' r) \boldsymbol{Y}_{12m} \right\} \tag{2.24}$$

$$= \frac{\alpha_1'}{\sqrt{24\pi}} \left\{ (2 j_0(\omega' r) - j_2(\omega' r)) \boldsymbol{e}_m + 3 j_2(\omega' r) \hat{\boldsymbol{r}} \cdot \boldsymbol{e}_m \hat{\boldsymbol{r}} \right\}, \tag{2.25}$$

$$\alpha_1' = 2.172 a^{-1}, \qquad \omega' = 4.493 a^{-1}. \tag{2.26}$$

Some properties of the vector spherical harmonics are given in a brief appendix.

## 3. Vertices

### 3.1. QUARK-GLUON VERTEX

In ref. [5] it was shown that the emission vertex for a TE "magnetic" gluon from an S-wave quark mode in a cavity can be written as (with our normalization)

$$H_{\mathrm{I}}=-\int \mathrm{d}x\, \boldsymbol{J}_{nm}\cdot \boldsymbol{A}_k^{*}=ig\boldsymbol{\sigma}\cdot \boldsymbol{e}^{*}\frac{\lambda^a}{2}e^a\left\{\frac{I_{nm}(\chi)}{\sqrt{8\pi\chi}}\right\}a^{-1}, \tag{3.1}$$

where $\boldsymbol{\sigma}$ and $\frac{1}{2}\lambda^a$ give the spatial and color properties of the quark current, and $\boldsymbol{e}$ and $e^a$ are the spatial and color orientations of the gluon. On integrating over the cavity, the spatial dependence of $\boldsymbol{J}\cdot\boldsymbol{A}^*$ yielded the $I_{nm}(\chi)$, which is a function of the modes $n$, $m$ of the initial and final quarks and the gluon mode number $\chi=\omega_{\mathrm{g}}a$.

When the gluon is in the lowest TE mode ($\chi=2.744$) and the S-wave quarks are massless, we find

$$g\left\{\frac{I}{\sqrt{8\pi\chi}}\right\}=0.1389g. \tag{3.2}$$

To fit the hyperfine splittings of quark systems, a value $\alpha_{\mathrm{s}}=2.2$ was required in ref. [12]. From the structure of $H_{\mathrm{I}}$ in (3.1), we see that the effective expansion parameter is $gI/\sqrt{8\pi\chi}$, which is small even though $\alpha_{\mathrm{s}}$ is large. This provides some *a posteriori* justification for the applicability of lowest order perturbation theory in the bag model with large values of $\alpha_{\mathrm{s}}$, although there is reason to believe that the very large $\alpha_{\mathrm{s}}=2.2$ found by the MIT group is an overestimate (see sect. 5) and [13]. As we shall see, gluon bags do not require such a large $\alpha_{\mathrm{s}}$.

### 3.2. THREE-GLUON VERTEX

The three-gluon part of $H_{\mathrm{I}}$ in terms of field operators is★

$$H_{\mathrm{I}}^{(3)}=\tfrac{1}{2}gf^{abc}\int \mathrm{d}x\!:\!\boldsymbol{H}^{A}\cdot(\boldsymbol{A}^{B}\times\boldsymbol{A}^{C})\!: \tag{3.3}$$

Inserting the gluon field expansion in bag modes (2.4), we may separate the part of $H_{\mathrm{I}}$ that creates two gluons ($A$, $B$) from one ($C$):

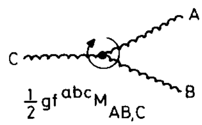
$$H_{\mathrm{I}}^{(3)}(\mathrm{g}\to\mathrm{gg})=\tfrac{1}{2}gf^{abc}M_{AB,C}a_A^{\dagger}a_B^{\dagger}a_C, \tag{3.4}$$

★ Capital letters implicitly are color, mode, and polarization labels, if appropriate. Small letters refer to color only.

where

$$M_{AB,C}=\int \mathrm{d}\boldsymbol{x}\{\boldsymbol{H}^{A*}\cdot(\boldsymbol{A}^{B*}\times \boldsymbol{A}^{C})-\boldsymbol{H}^{B*}\cdot(\boldsymbol{A}^{A*}\times \boldsymbol{A}^{C})+\boldsymbol{H}^{C}\cdot(\boldsymbol{A}^{A}\times \boldsymbol{A}^{B})^{*}\}\,. \tag{3.5}$$

(Note the symmetry $f^{ABC}M_{AB,C}=f^{BAC}M_{BA,C}$.) This (g → gg) operator may be written as a vertex



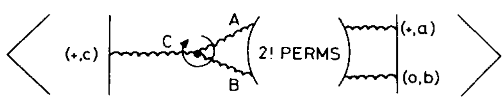

(3.6)

For the special case of all gluons in the lowest TE mode ($\chi=2.744$) and with arbitrary polarizations $\boldsymbol{e}^{A}$, $\boldsymbol{e}^{B}$, $\boldsymbol{e}^{C}$, the overlap integral $M_{AB,C}$ is

$$\begin{aligned}\tfrac{1}{2}gf^{abc}M_{AB,C}^{(\mathrm{TE}^3)}&=-i(\boldsymbol{e}^{A}\times \boldsymbol{e}^{B})^{*}\cdot \boldsymbol{e}^{C}\tfrac{1}{2}gf^{abc}\frac{3\alpha_1^3\omega}{\sqrt{24\pi}}\int_0^a r^2 j_1(\omega r)^2(j_0(\omega r)+j_2(\omega r))\mathrm{d}r\\ &=-i(\boldsymbol{e}^{A}\times \boldsymbol{e}^{B})^{*}\cdot \boldsymbol{e}^{C}\tfrac{1}{2}gf^{abc}\cdot t_1 a^{-1}\,,\end{aligned} \tag{3.7}$$

where $t_1=0.1919$.

As a simple application, consider the amplitude to go from a $(+,c)$ lowest mode TE gluon to a $(+,a)$, $(0,b)$ pair of gluons;




$$\begin{aligned}&=\frac{g}{2}f^{abc}M_{+0,+}^{(\mathrm{TE}^3)}+\frac{g}{2}f^{bac}M_{0+,+}^{(\mathrm{TE}^3)}\\ &=gf^{abc}M_{+0,+}^{(\mathrm{TE}^3)}\\ &=-i(\boldsymbol{e}_{+}\times \boldsymbol{e}_{0})^{*}\cdot \boldsymbol{e}_{+}gf^{abc}t_1 a^{-1}\\ &=-gf^{abc}(0.1919)a^{-1}\,.\end{aligned} \tag{3.8}$$

The polarization vectors are the usual spherical basis vectors $\boldsymbol{e}_{\pm}=\sqrt{\tfrac{1}{2}}\,(\mp\hat{\boldsymbol{x}}-i\hat{\boldsymbol{y}})$ and $\boldsymbol{e}_0=\hat{\boldsymbol{z}}$.

The 3-gluon vertex function for different gluon modes simply changes the numerical coefficient (0.1919) found above for the lowest $(\mathrm{TE})^3$ vertex. Another vertex we shall require is (TM) → (TE, TM), where the TE is again $\chi = 2.744$ and the lowest TM is $\chi' = 4.493$. This vertex is


$$= -i(e^A \times e^B)^* \cdot e_C \tfrac{1}{2} g f^{abc} t_2 a^{-1}, \tag{3.9}$$

where $t_2 = 0.1532$.

### 3.3. FOUR-GLUON VERTEX

The four-gluon part of $H_{\mathrm{I}}$ in terms of field operators is

$$H_{\mathrm{I}}^{(4)} = \tfrac{1}{4} g^2 f^{xab} f^{xcd} \int \mathrm{d}\boldsymbol{x} : (\boldsymbol{A}^A \times \boldsymbol{A}^B) \cdot (\boldsymbol{A}^C \times \boldsymbol{A}^D) : \tag{3.10}$$

As we are primarily interested in gg → gg amplitudes, we again expand the field operators in bag modes (2.4) and keep the part of $H_{\mathrm{I}}^{(4)}$ proportional to $a^{\dagger 2} a^2$:

$$H_{\mathrm{I}}^{(4)}(\mathrm{gg} \to \mathrm{gg}) = \tfrac{1}{2} g^2 \int \mathrm{d}\boldsymbol{x} \big\{ f^{xac} f^{xbd} \big( \boldsymbol{A}^A \times \boldsymbol{A}^{C*} \big) \cdot \big( \boldsymbol{A}^B \times \boldsymbol{A}^{D*} \big)$$
$$+ f^{xab} f^{xcd} \big( \boldsymbol{A}^A \cdot \boldsymbol{A}^{C*} \big) \big( \boldsymbol{A}^B \cdot \boldsymbol{A}^{D*} \big) \big\} a_C^\dagger a_D^\dagger a_A a_B . \tag{3.11}$$

The second term gives zero on color singlets; keeping only the first term, we write it in the form

$$H_{\mathrm{I}}^{(4)}(\mathrm{gg}\mathbf{1} \to \mathrm{gg}\mathbf{1}) = \tfrac{1}{2} g^2 f^{xac} f^{xbd} M_{AB,CD} a_C^\dagger a_D^\dagger a_A a_B . \tag{3.12}$$

The overlap integral is simply

$$M_{AB,CD} = \int \mathrm{d}\boldsymbol{x} \big( \boldsymbol{A}^A \times \boldsymbol{A}^{C*} \big) \cdot \big( \boldsymbol{A}^B \times \boldsymbol{A}^{D*} \big) . \tag{3.13}$$

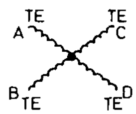
With four gluons in the lowest TE mode and with arbitrary polarizations $\boldsymbol{e}^A, \dots, \boldsymbol{e}^D$, this integral may be evaluated to yield

$$M_{AB,CD}^{(\mathrm{TE}^4)} = (\boldsymbol{e}^A \times \boldsymbol{e}^{C*}) \cdot (\boldsymbol{e}^B \times \boldsymbol{e}^{D*})$$
$$\times \frac{3}{16\pi} \frac{(\alpha_1 a)^4}{\chi^3} \int_0^{\chi} \eta^2 j_1(\eta)^4 \mathrm{d}\eta \cdot a^{-1}$$
$$= (\boldsymbol{e}^A \times \boldsymbol{e}^{C*}) \cdot (\boldsymbol{e}^B \times \boldsymbol{e}^{D*}) (6.177 \cdot 10^{-3}) a^{-1} . \tag{3.14}$$

We may rewrite this as a spin-spin force, as the spin vectors are $\boldsymbol{S}=i\boldsymbol{e}_{\mathrm{i}}\times\boldsymbol{e}_{\mathrm{f}}^{*}$ (see appendix B):

$$M_{AB,CD}^{(\mathrm{TE}^4)}=-(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)(6.177\cdot 10^{-3})a^{-1}. \tag{3.15}$$

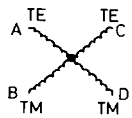
This shows that the 4-gluon contact term is pure spin-spin for (TE)$^2$ glueballs. Attaching the color indices to this matrix element, we have for the (TE)$^4$ vertex and lowest mode gluons the general result




$$= -f^{xac}f^{xbd}f_1(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\alpha_s a^{-1}, \tag{3.16}$$

where $f_1=38.81\cdot 10^{-3}$.

For (TE)(TM) glueballs we have two types of diagrams, a direct one and a crossed one. The direct diagram is pure spin-spin;




$$= -f^{xac}f^{xbd}f_2(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\alpha_s a^{-1}, \tag{3.17}$$

where

$$f_2=\tfrac{1}{24}(\alpha_1 a)^2(\alpha_1' a)^2\int_0^1\eta^2 j_1(\chi\eta)^2[2j_0(\chi'\eta)-j_2(\chi'\eta)]^2\mathrm{d}\eta=11.44\cdot 10^{-3},$$

$$\chi=2.744,\qquad \chi'=4.493. \tag{3.18}$$

The crossed diagram has a rather complicated spin structure. We Fierz transform it into products of bilinears in the first $(\boldsymbol{e}_a,\boldsymbol{e}_c^*)$ and second $(\boldsymbol{e}_b,\boldsymbol{e}_d^*)$ gluon polarization vectors, as discussed in appendix B. The bilinears are

(1) scalar-scalar: (spin-independent)

$$\phi_1\phi_2=(\boldsymbol{e}_A\cdot\boldsymbol{e}_C^*)(\boldsymbol{e}_B\cdot\boldsymbol{e}_D^*); \tag{3.19}$$

(2) spin-spin: (spin-dependent)

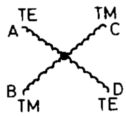
$$\boldsymbol{S}_1\cdot\boldsymbol{S}_2=(i\boldsymbol{e}_A\times\boldsymbol{e}_C^*)\cdot(i\boldsymbol{e}_B\times\boldsymbol{e}_D^*); \tag{3.20}$$

(3) tensor-tensor: (spin-dependent)

$$T_1\cdot T_2=T_1^{ij}T_2^{ij}$$

$$=\left(\frac{e_A{}^i e_C{}^{j*}+e_A{}^j e_C{}^{i*}}{2}-\frac{\delta^{ij}\boldsymbol{e}_A\cdot\boldsymbol{e}_C^*}{3}\right)\begin{pmatrix}A\to B\\ C\to D\end{pmatrix}^{ij}. \tag{3.21}$$

In terms of these invariants and three overlap integrals, the crossed diagram is



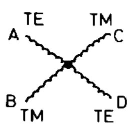

$$= -f^{xac}f^{xbd}\left\{\tfrac{4}{9}I_1(\phi_1\phi_2)+\tfrac{1}{6}(I_1+2I_2+I_3)(\boldsymbol{S}_1\cdot\boldsymbol{S}_2) + \left(\tfrac{1}{3}I_1+\tfrac{2}{5}I_2+\tfrac{1}{5}I_3\right)(\boldsymbol{T}_1\cdot\boldsymbol{T}_2)\right\}\alpha_s a^{-1}. \tag{3.22}$$

The three overlap integrals are

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \tfrac{1}{8}(\alpha_1 a)^2(\alpha_1' a)^2\int_0^1 \eta^2 j_1(\chi\eta)^2 \begin{bmatrix} (2j_0(\chi'\eta)-j_2(\chi'\eta))^2 \\ (2j_0(\chi'\eta)-j_2(\chi'\eta))3j_2(\chi'\eta) \\ 9j_2(\chi'\eta)^2 \end{bmatrix} \mathrm{d}\eta$$

$$= \begin{bmatrix} 34.31 \\ -33.56 \\ 72.32 \end{bmatrix}\cdot 10^{-3}. \tag{3.23}$$

With this result, we may write the crossed $(\mathrm{TE})^2(\mathrm{TM})^2$ vertex in the relatively simple form




$$= -f^{xac}f^{xbd}\left\{f_3(\phi_1\phi_2)+f_4(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)+f_5(\boldsymbol{T}_1\cdot\boldsymbol{T}_2)\right\}\alpha_s a^{-1}, \tag{3.24}$$

where

$$f_3 = 15.25\cdot 10^{-3},$$

$$f_4 = 6.59\cdot 10^{-3},$$

$$f_5 = 12.48\cdot 10^{-3}. \tag{3.25}$$

### 3.4. FOUR-GLUON COULOMB VERTEX

The instantaneous Coulomb interaction between gluons gives a contribution to the hamiltonian of $O(\alpha_s)$. In matrix elements between $|gg\rangle$ states in the bag, we may write this as an effective four-gluon interaction. This interaction is not just a mass shift, as might naively be expected, but has important spin-dependent contributions as well.

The gluon color charge density is

$$\rho^a(\boldsymbol{x}) = -gf^{abc}{:}\mathring{\boldsymbol{A}}_b\cdot\boldsymbol{A}_c{:}$$

$$= -igf^{abc}(\omega_B+\omega_C)\boldsymbol{A}_B^*(\boldsymbol{x})\cdot\boldsymbol{A}_C(\boldsymbol{x})a_B^\dagger a_C. \tag{3.26}$$

The charge density interacts with itself through a Coulomb potential which is a modification of the free space $G_0 = (R)^{-1}$, due to the bag boundary conditions [10]. (This is discussed in appendix C.)

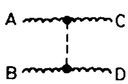
$$H_{\mathrm{Cou}} = \frac{1}{8\pi} \int\int \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\,\rho(\boldsymbol{x})^a G(\boldsymbol{x}, \boldsymbol{y}) \rho(\boldsymbol{y})^a , \tag{3.27a}$$

where $G$ is

$$G(\boldsymbol{x}, \boldsymbol{y}) = \underbrace{\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}}_{G_0(\boldsymbol{x},\,\boldsymbol{y})} + \underbrace{a^{-1}\left\{\xi^{-1} - 2 - \ln\left(\frac{\xi + 1 - \mu\lambda}{2}\right)\right\}}_{C(\boldsymbol{x},\,\boldsymbol{y})}$$

$$\mu = \cos\theta_{xy}, \qquad \lambda = r_x r_y / a^2, \qquad \xi = \sqrt{1 - 2\mu\lambda + \lambda^2}\,. \tag{3.27b}$$

The relevant $a^{\dagger 2} a^2$ part of $H_{\mathrm{Cou}}$ is




$$= \frac{(-ig)^2}{8\pi} f^{xac} f^{xbd} (\omega_A + \omega_C)(\omega_B + \omega_D) I_{\mathrm{Cou}} a_C^\dagger a_D^\dagger a_A a_B , \tag{3.28}$$

$$I_{\mathrm{Cou}} = \iint_{\mathrm{bag}} \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\, G(\boldsymbol{x}, \boldsymbol{y}) [\boldsymbol{A}_A(\boldsymbol{x}) \cdot \boldsymbol{A}_C^*(\boldsymbol{x})][\boldsymbol{A}_B(\boldsymbol{y}) \cdot \boldsymbol{A}_D^*(\boldsymbol{y})] . \tag{3.29}$$

Now consider the Coulomb interaction for $(\mathrm{TE})^2$ glueballs. For a general polarization the $A_m^{\mathrm{TE}}$ field in the lowest mode can be written

$$\boldsymbol{A}_m^{\mathrm{TE}}(\boldsymbol{x}) = -i\sqrt{\frac{3}{8\pi}}\,\alpha_1 j_1(\omega r)(\hat{\boldsymbol{r}} \times \boldsymbol{e}_m) = N_1 j_1(\omega r)(\hat{\boldsymbol{r}} \times \boldsymbol{e}_m) \tag{3.30}$$

where the $\{\boldsymbol{e}_m\}$ are the usual spherical polarization vectors. Inserting these in (3.29), we find the Coulomb integral

$$I_{\mathrm{Cou}}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) = |N_1|^4 \iint_{\mathrm{bag}} \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \left(\frac{j_1(\omega r_x)}{r_x}\right)^2 \left(\frac{j_1(\omega r_y)}{r_y}\right)^2$$

$$\times [(\boldsymbol{x} \times \boldsymbol{a}) \cdot (\boldsymbol{x} \times \boldsymbol{b})][(\boldsymbol{y} \times \boldsymbol{c}) \cdot (\boldsymbol{y} \times \boldsymbol{d})] G(\boldsymbol{x}, \boldsymbol{y}) . \tag{3.31}$$

We have abbreviated $\boldsymbol{e}_a = \boldsymbol{a}, \ldots, \boldsymbol{e}_d^* = \boldsymbol{d}$.

It is convenient to consider a more general integral with the polarization vectors factored out, which we may reduce to two coefficients:

$$I_{ijkl} = \iint_{\text{bag}} \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \left( \frac{j_1(\omega r_x)}{r_x} \right)^2 \left( \frac{j_1(\omega r_y)}{r_y} \right)^2 x_i x_j y_k y_l G(\boldsymbol{x}, \boldsymbol{y}) \tag{3.32}$$

$$= J_1 \delta_{ij} \delta_{kl} + J_2 ( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} ) . \tag{3.33}$$

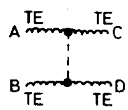
In terms of this the $(\mathrm{TE})^2$ Coulomb integral is

$$I_{\mathrm{Cou}}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) = |N_1|^4 \{ (4J_1 + 2J_2) \boldsymbol{a} \cdot \boldsymbol{c} \boldsymbol{b} \cdot \boldsymbol{d} + J_2 ( \boldsymbol{a} \cdot \boldsymbol{b} \boldsymbol{c} \cdot \boldsymbol{d} + \boldsymbol{a} \cdot \boldsymbol{d} \boldsymbol{b} \cdot \boldsymbol{c} ) \} . \tag{3.34}$$

To classify the interaction according to its spin dependence, we must Fierz transform this expression into a product of bilinears in the first $(\boldsymbol{e}_a, \boldsymbol{e}_c^*)$ gluon and second $(\boldsymbol{e}_b, \boldsymbol{e}_d^*)$ gluon polarization vectors, as we did previously for the $(\mathrm{TE})^2(\mathrm{TM})^2$ four-gluon vertex. We find that the Coulomb integral is a mixture of scalar-scalar and tensor-tensor:

$$I_{\mathrm{Cou}}(abcd) = |N_1^4| \{ (4J_1 + \tfrac{8}{3} J_2)( \phi_1 \phi_2 ) + 2J_2 (T_1 \cdot T_2) \} . \tag{3.35}$$

Replacing $J_1$ and $J_2$ by $n_1 a^5$ and $n_2 a^5$ ($n_1$ and $n_2$ are dimensionless), the general $(\mathrm{TE})^4$ vertex in the lowest mode $(\omega a = 2.744)$ is




$$= -f^{xac} f^{xbd} \underbrace{\left( \frac{9 \chi^2 \alpha_1^4 a^4}{16 \pi^2} \right)}_{(4.698)} \{ (2n_1 + \tfrac{4}{3} n_2)( \phi_1 \phi_2 ) + n_2 (T_1 \cdot T_2) \} \alpha_s a^{-1}$$

$$= -f^{xac} f^{xbd} \{ c_1 ( \phi_1 \phi_2 ) + c_2 (T_1 \cdot T_2) \} \alpha_s a^{-1} . \tag{3.36}$$

The numbers $n_1$ and $n_2$ are given by the following integrals over unit spheres:

$$\iint_0^1 \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} a\, G(a\boldsymbol{x}, a\boldsymbol{y}) j_1(\chi r_x)^2 j_1(\chi r_y)^2 \begin{bmatrix} 1 \\ \cos^2\theta_x \cos^2\theta_y \end{bmatrix} = \begin{bmatrix} 9n_1 + 6n_2 \\ n_1 + 2n_2 \end{bmatrix} , \tag{3.37}$$

$$(\chi = \omega a = 2.744) .$$

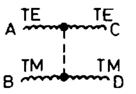
Numerical evaluation gives the results

$$c_1 = (90 \pm 1) \cdot 10^{-3} , \qquad c_2 = (40.5 \pm 0.2) \cdot 10^{-3} . \tag{3.38}$$

The Coulomb interaction for (TE)(TM) glueballs may be treated similarly. The required (TM) vector potential is

$$A_m^{\mathrm{TM}} = N_1'\{(2j_0(\omega' r) - j_2(\omega' r))\boldsymbol{e}_m + 3j_2(\omega' r)(\hat{\boldsymbol{r}}\cdot\boldsymbol{e}_m)\hat{\boldsymbol{r}}\} \tag{3.39}$$

and we may similarly write the overlap integral (3.29) in the hamiltonian (3.27) in terms of integrals times bilinear products of polarization tensors. The Coulomb interaction is scalar-scalar and tensor-tensor for the direct (TE)(TM) → (TE)(TM) graph, and pure spin-spin for the crossed graph (TE)(TM) → (TM)(TE);




$$= -f^{xac}f^{xbd}\underbrace{\left(\frac{\chi\chi'\alpha_1^2\alpha_1'^2a^4}{32\pi^2}\right)}_{(0.6093)}\{n_3(\phi_1\phi_2) + n_4(T_1\cdot T_2)\}\alpha_s a^{-1} \tag{3.40}$$

$$= -f^{xac}f^{xbd}\{c_3(\phi_1\phi_2) + c_4(T_1\cdot T_2)\}\alpha_s a^{-1}. \tag{3.41}$$

As before, we may write the numbers $n_3$ and $n_4$ in terms of double integrals over unit spheres:

$$n_3 = 2m_0 + 2m_1 + \tfrac{4}{3}m_2, \qquad n_4 = -2m_2, \tag{3.42}$$

$$3m_0 = \iint_0^1 \mathrm{d}x\,\mathrm{d}y\, aG(ax, ay)j_1(\chi r_x)^2[2j_0(\chi' r_y) - j_2(\chi' r_y)]^2, \tag{3.43}$$

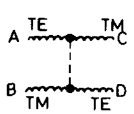
$$\begin{bmatrix} 3m_1 + 2m_2 \\ \tfrac{1}{3}m_1 + \tfrac{2}{3}m_2 \end{bmatrix} = \iint_0^1 \mathrm{d}x\,\mathrm{d}y\, aG(ax, ay)j_1(\chi r_x)^2[4j_0(\chi' r_y) + j_2(\chi' r_y)]$$

$$\times j_2(\chi' r_y)\begin{bmatrix} 1 \\ \cos^2\theta_x \cos^2\theta_y \end{bmatrix}, \tag{3.44}$$

which gives the numerical values (with $\chi = 2.744$, $\chi' = 4.493$) of

$$c_3 = (122 \pm 3)\cdot 10^{-3}, \qquad c_4 = (1.5 \pm 0.1)\cdot 10^{-3}. \tag{3.45}$$

Finally, the crossed interaction is pure spin-spin:




$$= -f^{xac}f^{xbd}\underbrace{\left(\frac{(\chi+\chi')^2\alpha_1^2\alpha_1'^2a^4}{128\pi^2}\right)}_{(0.6471)} n_5(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\alpha_s a^{-1} \tag{3.46}$$

$$= -f^{xac}f^{xbd}c_5(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\alpha_s a^{-1}, \tag{3.47}$$

$$n_5 = \iint_0^1 \mathrm{d}x\,\mathrm{d}y\, aG(ax, ay)\{j_1(\chi r_x)[2j_0(\chi' r_x) - j_2(\chi' r_x)]\}\{r_x \to r_y\}\cos\theta_x \cos\theta_y\,, \tag{3.48}$$

$$c_5 = (72.4 \pm 0.2)\cdot 10^{-3}\,. \tag{3.49}$$

This completes the set of Coulomb interaction effective four-gluon vertices we require to calculate the $O(\alpha_s)$ energy shift of low-lying $(TE)^2$ and (TE)(TM) glueballs. This calculation is described in the following section.

## 4. gg → gg second-order energy shift

If $|\phi\rangle = |g_0 g_0\rangle$ is a state of two bag-model gluons in a definite mode and $|\psi\rangle$ is the physical glueball state including interactions, then to $O(\alpha_s)$

$$\begin{aligned}\langle\psi|H|\psi\rangle &= \langle\phi|\left(H_0 + H_{\mathrm{Cou}} + H^{(4)}\right)|\phi\rangle \\ &\quad + \sum_{x=\phi g} \frac{\langle\phi|H^{(3)}|x\rangle\langle x|H^{(3)}|\phi\rangle}{(E_\phi - E_{\phi g}) \equiv -\omega_g} \\ &= E_0(\text{glue}) + \delta E(\text{glue}) = (A_0 + A_1\alpha_s)a^{-1}\,,\end{aligned} \tag{4.1}$$

where $H_0$ is the free hamiltonian and $H_{\mathrm{cou}}$, $H^{(3)}$ and $H^{(4)}$ are the Coulomb interaction, the three-gluon interaction, and the four-gluon interaction, respectively.

In a fixed cavity of radius $a$, the energy of the two-gluon system plus the bag is, for both gluons in the lowest TE mode ($\chi = 2.744$),

$$E_{\mathrm{glue}} + E_{\mathrm{bag}} = \underbrace{Aa^{-1}} + \underbrace{\tfrac{4}{3}\pi B_0 a^3}_{\text{bag energy}}\,, \tag{4.2a}$$

where $A = A_0 + A_1\alpha_s + \cdots$, and $A_0 = 2\chi = 5.488$. Minimizing this energy with respect to $a$, we find that at the optimum radius $a_0$ the bag plus glue energy is just $\frac{4}{3}$ times the glue energy $Aa^{-1}$ alone:

$$\left(E_{\mathrm{glue}} + E_{\mathrm{bag}}\right)|_{a_0^4 = A/4\pi B_0} = Aa_0^{-1} + \tfrac{4}{3}\pi B_0 a_0^3 = \underbrace{\left(\tfrac{4}{3}\right)\cdot\left(Aa_0^{-1}\right)}_{\text{glue + bag energy}} \tag{4.2b}$$

For simplicity we assume that this radius $a_0$ is the same for all glueballs. In a strict spherical bag calculation, the radii vary both with $\alpha_s$ and from state to state. We find that these model-dependent variations are typically $\sim 10\%$ effects, and that they do not significantly alter our conclusions.

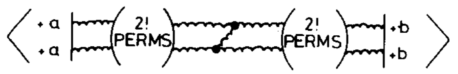
We neglect other $a^{-1}$ effects such as gluon self-energy diagrams ($\sim \alpha_s a^{-1}$) and the Casimir contributions ($\sim a^{-1}$), which are not yet well understood [13, 14]*, though we show elsewhere that they cannot be large on phenomenological grounds [23].

The three- and four-gluon interactions will generate spin-dependent energy shifts proportional to $\alpha_s a^{-1}$. Finally, the Coulomb interaction gives rise to comparable spin-dependent shifts as well as relatively small spin-independent ones.

Now we shall derive the $O(\alpha_s)$ corrections to the zeroth-order bag model $(TE)^2$ gg state energies, using the diagrammatic techniques discussed in sect. 3. We proceed by calculating the $A_1\alpha_s a^{-1}$ gluon energy shift in (4.2a); the total gluon plus bag energy shift with our approximations is always $\frac{4}{3}$ times this.

### 4.1. $2^{++}$

First consider the lowest-lying $2^{++}$ gg state, and the effect of one transverse gluon exchange. This energy shift is diagrammatically

$\frac{1}{\sqrt{16}}$  $\frac{1}{\sqrt{16}}$ (4.3)





$$= \left(-if^{cab}t_1(\boldsymbol{e}_C\times\boldsymbol{e}_A)^*\cdot\boldsymbol{e}_B\tfrac{1}{2}ga^{-1}\right)^* \underbrace{(2!)}_{\substack{AC\\ \text{perms}}}\ \underbrace{(2!)}_{\substack{EF\\ \text{perms}}}\ \delta_{BE}\frac{1}{(-\omega_x)}$$

$$\times\left(-if^{efd}t_1(\boldsymbol{e}_E\times\boldsymbol{e}_F)^*\cdot\boldsymbol{e}_D\tfrac{1}{2}ga^{-1}\right) \tag{4.4}$$

$$= f^{xfd}f^{xab}\left(\frac{4\pi t_1^2}{\chi}\right)(\boldsymbol{e}_A\times\boldsymbol{e}_B^*)\cdot\boldsymbol{e}_X(\boldsymbol{e}_D\times\boldsymbol{e}_F^*)\cdot\boldsymbol{e}_X^*\alpha_s a^{-1}. \tag{4.5}$$

This gives a transverse gluon exchange energy shift for the $2^{++}$ state of

$$\delta E^{\mathrm{T}}_{\mathrm{glue}}(2^{++}) = \tfrac{1}{16}(2!)^2\left(-f^{xba}f^{xab}\frac{4\pi t_1^2}{\chi}\boldsymbol{S}_1\cdot\boldsymbol{S}_2\alpha_s a^{-1}\right) = \frac{24\pi t_1^2}{\chi}\alpha_s a^{-1}. \tag{4.6}$$

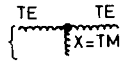
Recalling the contribution of exchanging only the lowest TE mode from appendix D (D.1),

$$t_1 = 0.1919\,, \tag{4.7}$$

we find

$$\delta E^{\mathrm{T}}_{\mathrm{glue}}(2^{++}) = 1.012\alpha_s a^{-1}. \tag{4.8}$$

* Baake [15] notes that the physical vacuum outside the perturbative bag will also contribute a $Za^{-1}$ term to the zero-point energy, and that this exterior contribution is usually arbitrarily neglected.

Exchange of higher modes gives zero $\{$ 

$\}=0$ or much smaller additional contributions $\{$ 

$\}$ ; the next highest (6.117; $1^+$) TE exchange increases $\delta E^{\mathrm{T}}$ $(2^{++})$ to

$$\delta E^{\mathrm{T}}_{\mathrm{glue}}(2^{++}) = (1.012 + 0.008)\alpha_s a^{-1}. \tag{4.9}$$

Exchange of angular excitations also gives very small contributions, due to small angular overlap integrals.

The four-gluon contact term may be treated similarly, to give

$$\delta E^{(4)}_{\mathrm{glue}}(2^{++}) = \frac{1}{\sqrt{16}} \Big\langle \begin{smallmatrix}\cdot a\\ \cdot a\end{smallmatrix}$$


$$\begin{smallmatrix}\cdot b\\ \cdot b\end{smallmatrix} \Big\rangle \frac{1}{\sqrt{16}} \tag{4.10}$$

$$= \tfrac{1}{16}(2!)^2\left(-f^{xac}f^{xac}f_1 \boldsymbol{S}_1 \cdot \boldsymbol{S}_2 \alpha_s a^{-1}\right) \tag{4.11}$$

$$= -6 f_1 \alpha_s a^{-1}. \tag{4.12}$$

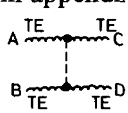
Recalling the four-gluon overlap integral for (2.744) mode from appendix D (D.3), we have

$$f_1 = 38.81 \cdot 10^{-3}, \tag{4.13}$$

so

$$\delta E^{(4)}_{\mathrm{glue}}(2^{++}) = -0.2329 \alpha_s a^{-1}. \tag{4.14}$$

The final contribution to the $2^{++}$ energy shift is the Coulomb energy. We recall the form of the effective four-gluon vertex due to Coulomb exchange for $(\mathrm{TE})^2$ glueballs from appendix D (D.6).



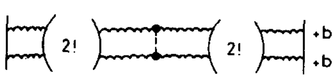

$$= -f^{xac}f^{xbd}\left(c_1(\phi_1\phi_2) + c_2(T_1 \cdot T_2)\right)\alpha_s a^{-1}. \tag{4.15}$$

For $j = 2$, the tensor and scalar bilinear products are

$$\langle 2^{++}|\phi_1\phi_2|2^{++}\rangle = 1, \qquad \langle 2^{++}|T_1 \cdot T_2|2^{++}\rangle = \tfrac{1}{6}. \tag{4.16}$$

We now take the matrix element of (4.15) just as with the four-point interaction:

$$\delta E^{\mathrm{Cou}}_{\mathrm{glue}}(2^{++}) = \frac{1}{\sqrt{16}} \Big\langle \begin{smallmatrix}\cdot a\\ \cdot a\end{smallmatrix}$$




$$\begin{smallmatrix}\cdot b\\ \cdot b\end{smallmatrix} \Big\rangle \frac{1}{\sqrt{16}}$$

$$= \tfrac{1}{16}(2!)^2\left[\left(-f^{xab}f^{xab}\right)\left(c_1 + \tfrac{1}{6}c_2\right)\alpha_s a^{-1}\right] = -6\left(c_1 + \tfrac{1}{6}c_2\right)\alpha_s a^{-1},$$

$$\delta E^{\mathrm{Cou}}_{\mathrm{glue}}(2^{++}) = -(0.581 \pm 0.006)\alpha_s a^{-1}. \tag{4.17}$$

As a check, we note that using the (incorrect) free space $G_0 = (R_{xy})^{-1}$ gives $\delta E = -3.56\alpha_s a^{-1}$, which compares quite favorably with the naive estimate for a flat charge distribution in free space:

$$\delta E_{\mathrm{Cou}}(\mathrm{flat}) = \underbrace{\left(\tfrac{3}{5}\alpha_s a^{-1}\right)}_{\text{charged sphere}} \underbrace{(-3)}_{\mathbf{8}\cdot\mathbf{8}\text{ color factor}} \underbrace{(2)}_{\text{two gluons}} = -3.6\alpha_s a^{-1}. \tag{4.18}$$

Adding the three contributions – transverse, four-gluon, and Coulomb – we find a $2^{++}$ shift and total energy of

$$\delta E_{\mathrm{glue}}(2^{++}) = \Bigg( \underbrace{1.012}_{\text{transverse gluon exchange}} \quad \underbrace{-0.233}_{\text{four-gluon interaction}} \quad \underbrace{-0.581}_{\text{color Coulomb}} \Bigg) \alpha_s a^{-1}, \tag{4.19}$$

$$E_{\mathrm{tot}}(2^{++}) = \underbrace{\tfrac{4}{3}}_{\text{glue + bag energy}} (5.488 + 0.198\alpha_s)a^{-1} = (7.32 + 0.26\alpha_s)a^{-1}. \tag{4.20}$$

### 4.2. $0^{++}$

The $0^{++}$ energy may be derived similarly, using the matrix elements of $\boldsymbol{S}_1 \cdot \boldsymbol{S}_2$ and $T_1 \cdot T_2$ given in the appendix. Here we simply quote the result. One transverse gluon exchange and the four-gluon interaction are pure spin-spin, and the Coulomb interaction is mixed scalar-scalar (spin-independent) and tensor-tensor. With the change to $j = 0$, the three contributions become

$$\delta E_{\mathrm{glue}}(0^{++}) = \Big( \underbrace{-2.024}_{\text{transverse}} \ \underbrace{+0.466}_{\text{4-gluon}} \ \underbrace{-0.945}_{\text{Coulomb}} \Big) \alpha_s a^{-1}, \tag{4.21}$$

so

$$E_{\mathrm{tot}}(0^{++}) = \tfrac{4}{3}\left(2\chi a^{-1} + \delta E_{\mathrm{glue}}(0^{++})\right) = (7.32 - 3.34\alpha_s)a^{-1}. \tag{4.22}$$

Finally, we consider the odd-parity glueballs with $J^{PC} = 0^{-+}, 2^{-+}$ made of (TE)(TM) gluons in the lowest mode. This problem will be discussed only schematically, as the techniques are the same as those used above in treating even parity $(\mathrm{TE})^2$ glueballs.

### 4.3. $2^{-+}$ AND $0^{-+}$

The $2^{-+}$ and $0^{-+}$ (TE)(TM) glueball states are

$$|2^{-+}\rangle, |0^{-+}\rangle = \sum_{a=1}^{8} \sqrt{\frac{1}{8}}\, |\mathrm{TE}^a\rangle |\mathrm{TM}^a\rangle \otimes \mathrm{C.G.\ coeffs.} \tag{4.23}$$

As for the $0^{++}$ and $2^{++}$ glueballs, there are contributions to the $O(\alpha_s)$ energy shift due to (1) transverse gluon exchange, (2) the four-gluon interaction, and (3) the color Coulomb interaction. Here, we shall first discuss (1) for the $2^{-+}$ glueball, following which we give a unified discussion of the terms (2) and (3) for both the $0^{-+}$ and $2^{-+}$ states.

There are contributions to the $2^{-+}$ energy shift due to both (TE) and (TM) gluon exchange, as the $(\mathrm{TM})^2(\mathrm{TE})$ vertex is non-zero as well as the $(\mathrm{TE})^3$ vertex. Recalling these vertices (D.1), (D.2), we proceed as in the $|2^{++}(\mathrm{TE})^2\rangle$ energy calculation:

$$\delta E^{\mathrm{T}}_{\mathrm{glue}}(2^{-+}) = \frac{1}{\sqrt{8}}$$


$$\frac{1}{\sqrt{8}} \tag{4.24}$$

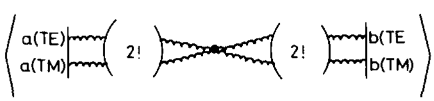
$$= 24\pi \left\{ \underbrace{\frac{t_2^2}{\chi'}}_{x=\mathrm{TM}} + \underbrace{\frac{t_1 t_2}{\chi}}_{x=\mathrm{TE}} \right\} \alpha_s a^{-1} = 1.202\alpha_s a^{-1}. \tag{4.25}$$

In contrast to the usual result that exchange of the lowest gluon modes gives the total energy shift of the lowest glueballs to a very good accuracy, here we find that a second excited mode gives an important contribution. Exchange of the second TE mode ($\chi = 6.117$) gives an additional $0.388\alpha_s a^{-1}$, which gives a combined shift

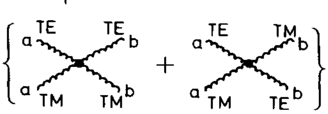
$$\delta E^{\mathrm{T}}_{\mathrm{glue}}(2^{-+}) = 1.590\alpha_s a^{-1}. \tag{4.26}$$

Repeating this calculation for general external polarizations, we find that this interaction is pure spin-spin, as it was for $(\mathrm{TE})^2$ glueballs. This means the pseudoscalar matrix element is $\delta E(0^{-+}) = -3.180\alpha_s a^{-1}$.

The four gluon contact term produces both direct and crossed diagrams (D.4), (D.5):

$$\frac{1}{\sqrt{8}}$$


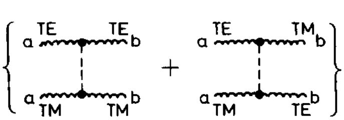
$$\frac{1}{\sqrt{8}} \tag{4.27}$$

$$\delta E^{(4)}_{\mathrm{glue}} = \tfrac{1}{4}$$


$$\tag{4.28}$$

$$= -6(f_2(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2) + f_3(\phi_1\phi_2) + f_4(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2) + f_5(T_1 \cdot T_2))\alpha_s a^{-1}, \tag{4.29}$$

and the Coulomb diagram also gives both direct and crossed contributions (D.7), (D.8):

$$\delta E^{\mathrm{Cou}}_{\mathrm{glue}} = \tfrac{1}{4}$$


$$\tag{4.30}$$

$$= -6(c_3(\phi_1\phi_2) + c_4(T_1 \cdot T_2) + c_5(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2))\alpha_s a^{-1}. \tag{4.31}$$

Specializing these general matrix elements to $j=0$ and $j=2$, we find for the complete $O(\alpha_s a^{-1})$ $2^{-+}$ and $0^{-+}$ glueball mass shifts

$$\delta E_{\text{tot}}(2^{-+}) = (\tfrac{4}{3}) \qquad (1.590 \qquad -0.212 \qquad -1.17)\alpha_s a^{-1}, \tag{4.32}$$

$$\delta E_{\text{tot}}(0^{-+}) = \underbrace{(\tfrac{4}{3})}_{\substack{\text{bag}\\+\\\text{glue}\\\text{energy}}} \left( \underbrace{-3.180}_{\substack{\text{transverse}\\\text{gluon}\\\text{exchange}}} + \underbrace{0.}_{\substack{\text{four-gluon}\\\text{term}}} + \underbrace{0.12}_{\substack{\text{color}\\\text{Coulomb}}} \right) \alpha_s a^{-1}, \tag{4.33}$$

$$E_{\text{tot}}(2^{-+}) = (9.649 + 0.28\alpha_s)a^{-1}, \tag{4.34}$$

$$E_{\text{tot}}(0^{-+}) = (9.649 - 4.08\alpha_s)a^{-1}. \tag{4.35}$$

## 5. Glueball phenomenology

There are at present two experimental glueball candidates, a pseudoscalar $\iota(1440)$ and a tensor $\theta(1640)$ [3]. These are not in agreement with the zeroth-order bag model predictions due to Jaffe and Johnson [4]:

$$E(0^{-+}) = 1.29 \text{ GeV}, \tag{5.1}$$

$$E(2^{++}) = 0.96 \text{ GeV}. \tag{5.2}$$

To see the effect of the $O(\alpha_s)$ radiative corrections, recall the results we found in (4.20), (4.35) for the tensor and pseudoscalar masses,

$$E(2^{++}) = [7.32 + 0.26\alpha_s]a^{-1}, \tag{5.3}$$

$$E(0^{-+}) = [9.65 - 4.08\alpha_s]a^{-1}. \tag{5.4}$$

First we shall make contact with the earlier results of the MIT group. Taking $\alpha_s = 0$ and $B_0^{1/4} = 0.146$ GeV, and determining the bag radius by minimizing the bag + glue + zero-point energy $-Z_0 a^{-1} = -1.84a^{-1}$ [12] (which has since been disclaimed by Johnson [13]), we find essentially Jaffe and Johnson's result:

$$E(2^{++}) = E(0^{++}) = 0.97 \text{ GeV}|_{a^{-1}=0.199 \text{ GeV}}, \tag{5.5}$$

$$E(2^{-+}) = E(0^{-+}) = 1.30 \text{ GeV}|_{a^{-1}=0.180 \text{ GeV}}. \tag{5.6}$$

Thorn has calculated some of the spin-dependent corrections to these masses in an unpublished calculation [16]. He kept the old MIT parameters and considered transverse gluon exchange and the four-gluon contact term and assumed that self-energy diagrams cancel the Coulomb interaction. (See appendix C.) We also find that he determined the bag radius after adding these terms to the energy. This procedure gives

$$E\left\{\begin{matrix}2^{++}\\0^{++}\end{matrix}\right\}=\left[\left(2\chi-Z_0+\left\{\begin{matrix}0.78\\-1.56\end{matrix}\right\}\alpha_s\right)a^{-1}+\tfrac{4}{3}\pi B_0 a^3\right]\Bigg|_{\substack{a^{-1}=\left\{\begin{smallmatrix}0.181\\0.403\end{smallmatrix}\right\}\text{ GeV}\\ \alpha_s=2.2\\ B_0^{1/4}=0.146\text{ GeV}\\ Z_0=1.84}}$$

$$=\left\{\begin{matrix}1.29\text{ GeV}\\0.12\text{ GeV}\end{matrix}\right\} \tag{5.7,8}$$

This result has led to a postulated $2^{++}$ glueball degenerate with the f(1270) [17]. The missing $0^{++}$ is explained away by saying that it mixes with the vacuum, and that an orthogonal linear combination is elevated to $\sim 1$ GeV.

Now we return to our result for these masses with $\alpha_s$ and $a^{-1}$ arbitrary, including the four-gluon and cavity Coulomb interactions, but leaving out the zero-point energy. As there are candidate pseudoscalar and tensor glueball states recently seen at SLAC [3], we shall fit our two parameters to these states. Thus,

$$E(2^{++})=[7.32+0.26\alpha_s]a^{-1}=1.64\text{ GeV}, \tag{5.9}$$

$$E(0^{-+})=[9.65-4.08\alpha_s]a^{-1}=1.44\text{ GeV}. \tag{5.10}$$

This requires the parameters

$$a^{-1}=0.218\text{ GeV}, \tag{5.11}$$

$$\alpha_s=0.748. \tag{5.12}$$

The cavity radius we find, $a^{-1}=0.22$ GeV, is essentially the same as the old MIT radius $a^{-1}\sim 0.19$ GeV. This radius is not really a free parameter if the bag model is correct – the quark bag strength $B_0^{1/4}=0.146$ GeV determines the MIT gg radius, and this is indeed close to our independent fit.

The strong $\alpha_s=0.75$ compares favorably with the typical values of $\alpha_s=0.6$–$0.8$ found in potential model studies of the light meson spectrum [18], and a potential model of transverse gluon bound states gives $M(2^{++})/M(0^{-+})$ correctly with the similar value of $\alpha_s=0.6$ [19].

An important prediction of the model is the mass of the light scalar glueball. Recall that we found

$$E(0^{++})=[7.32-3.34\alpha_s]a^{-1}. \tag{5.13}$$

With the fitted parameters (5.11), (5.12), we predict

$$E(0^{++})=1.05\text{ GeV}. \tag{5.14}$$

This relatively low scalar glueball mass is the clearest test of the MIT bag model that we have found relative to the expectation of Shifman [20] of $M(0^{++}) \sim 1.4$ GeV and the models of Barnes [19] and Suura [21], both of which give the equivalent result $M(0^{++}) = M(0^{-+})$. It is interesting that a recent SU(2) lattice calculation [24] found $M(0^{++})/M(0^{-+}) = 0.61 \pm 0.12$, which is consistent with the bag result $M(0^{++}) < M(0^{-+})$.

Historically, the MIT group fitted the light S-wave mesons and baryons with a value of $\alpha_s = 2.2$ [12]. Obviously, this is a disaster for the gg spectrum, as it predicts $0^{++}$ and $0^{-+}$ glueballs with masses of O(100 MeV), far below the 1.5 GeV region.

It is also disturbing that the MIT value $\alpha_s = 2.2$ is far from the potential model results, which typically give $\alpha_s(b\bar{b}) \sim 0.25$, $\alpha_s(c\bar{c}) \sim 0.4$, and $\alpha_s(q\bar{q}) \sim 0.7$. What is wrong with the MIT bag model?

The answer is simply that the bag model quark wave functions are uncorrelated, which means that there is no short-distance spike in $\psi(r)$ in the S-wave states due to the strong attraction of the Coulomb potential. As hyperfine shifts in S-wave states (e.g. spin-spin) are sensitive to short distance parts of $\psi(r)$, $(\langle \delta(\boldsymbol{r}) \rangle, \langle r^{-3} \rangle, \dots)$, they are badly underestimated by the smooth $j_0^2 + j_1^2$ bag model distributions. To compensate for the small bag model hyperfine matrix elements in fitting the experimental S-wave hadron spectrum, one must increase $\alpha_s$ to an unreasonably large value. This problem with quark bags can presumably be avoided by allowing mixing of radially excited quark modes through the Coulomb interaction, leading to a Coulomb spike in $\psi_{\text{bag}}$.

This shortcoming of bag wave functions does not occur for gg states because they all have non-trivial angular dependence of the wave function in the relative coordinate, and always see an $L_{\text{eff}} \geqslant 1$; $\psi_{gg}(0)$ must be zero. The bag model gluon wave functions are a reasonable first approximation to this situation.

## 6. Conclusions

We conclude that the bag model gg states have important $O(\alpha_s)$ corrections to their masses, and that the recently discovered glueball candidates $\iota(1440)$ and $\theta(1640)$ can be accounted for with plausible parameter values. We find a strong coupling strength $\alpha_s = 0.75$ which is much smaller than the earlier MIT value $\alpha_s = 2.2$, and we discuss reasons for this discrepancy. Our results are shown to reduce to the earlier results of Jaffe and Johnson in the limit $\alpha_s \to 0$ and to those of Thorn as a special case.

With the approximations discussed in the text, we find that the MIT bag model predicts a scalar glueball at 1.05 GeV, well below the pseudoscalar glueball mass.

We should like to thank J. Baake, L. Castillejo, M. Chanowitz and C.B. Thorn for interesting and useful comments.

## Appendix A

VECTOR SPHERICAL HARMONICS

The vector spherical harmonics $\{\boldsymbol{Y}_{jlm}(\hat{\Omega})\}$ are very useful functions for describing vector fields confined to a cavity. They are defined as the Clebsch-Gordon product of ordinary spherical harmonics $\{Y_{lm}\}$ and the spin one spherical unit vectors $\{\boldsymbol{e}_m\}$:

$$\boldsymbol{Y}_{jlm}(\hat{\Omega}) = \sum_{\mu} \langle l\mu, m-\mu | jm \rangle Y_{l\mu}(\hat{\Omega}) \boldsymbol{e}_{m-\mu}. \tag{A.1}$$

They are eigenfunctions of $\boldsymbol{J}^2$, $\boldsymbol{L}^2$, and $\boldsymbol{S}^2$ with eigenvalues $j(j+1)$, $l(l+1)$ and 2, respectively.

They are most useful in manipulations involving $\nabla$ and in doing angular integrals, due to the properties [22]

$$\nabla\left(f(r)Y_{jm}\right) = \left\{\sqrt{\frac{j}{2j+1}}\left(\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{j+1}{r}f\right)\boldsymbol{Y}_{jj-1m} - \sqrt{\frac{j+1}{2j+1}}\left(\frac{\mathrm{d}f}{\mathrm{d}r} - \frac{j}{r}f\right)\boldsymbol{Y}_{jj+1m}\right\}; \tag{A.2}$$

$$\nabla\cdot\left(f(r)\boldsymbol{Y}_{jj+1m}\right) = -\sqrt{\frac{j+1}{2j+1}}\left(\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{j+2}{r}f\right)Y_{jm}, \tag{A.3}$$

$$\nabla\cdot\left(f(r)\boldsymbol{Y}_{jjm}\right) = 0, \tag{A.4}$$

$$\nabla\cdot\left(f(r)\boldsymbol{Y}_{jj-1m}\right) = \sqrt{\frac{j}{2j+1}}\left(\frac{\mathrm{d}f}{\mathrm{d}r} - \frac{j-1}{r}f\right)Y_{jm}, \tag{A.5}$$

$$\nabla\times\left(f(r)\boldsymbol{Y}_{jj+1m}\right) = i\sqrt{\frac{j}{2j+1}}\left(\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{j+2}{r}f\right)\boldsymbol{Y}_{jjm}, \tag{A.6}$$

$$\nabla\times\left(f(r)\boldsymbol{Y}_{jjm}\right) = \frac{i}{\sqrt{2j+1}}\left\{\sqrt{j}\left(\frac{\mathrm{d}f}{\mathrm{d}r} - \frac{j}{r}f\right)\boldsymbol{Y}_{jj+1m} + \sqrt{j+1}\left(\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{j+1}{r}f\right)\boldsymbol{Y}_{jj-1m}\right\}, \tag{A.7}$$

$$\nabla\times\left(f(r)\boldsymbol{Y}_{jj-1m}\right) = i\sqrt{\frac{j+1}{2j+1}}\left(\frac{\mathrm{d}f}{\mathrm{d}r} - \frac{j-1}{r}f\right)\boldsymbol{Y}_{jjm}, \tag{A.8}$$

$$\boldsymbol{Y}^*_{jlm} = (-)^{j+l+m+1}\boldsymbol{Y}_{jl-m}, \tag{A.9}$$

$$\int \boldsymbol{Y}^*_{jlm}\cdot\boldsymbol{Y}_{j'l'm'}\mathrm{d}\Omega = \delta_{jj'}\delta_{ll'}\delta_{mm'}. \tag{A.10}$$

A few specific $\{Y_{jlm}\text{'s}\}$ for small $j$ and $l$ are

$$Y_{10m} = \frac{1}{\sqrt{4\pi}} e_m , \tag{A.11}$$

$$Y_{11m} = -i\sqrt{\frac{3}{8\pi}}\, \hat{r} \times e_m , \tag{A.12}$$

$$Y_{12m} = \frac{1}{\sqrt{8\pi}} \{ e_m - 3(\hat{r}\cdot e_m)\hat{r} \} . \tag{A.13}$$

A final useful result is the cross product of two vector spherical harmonics, which is implicit in the relation

$$\int \mathrm{d}\Omega Y_{j,l,m} \cdot \left( Y_{j_2 l_2 m_2} \times Y_{j_3 l_3 m_3} \right) = i\sqrt{\frac{3}{2\pi}} \left[ \prod_{i=1}^{3} \sqrt{(2j_i+1)(2l_i+1)}\,(-)^{j_i+1} \right]$$
$$\times \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \\ 1 & 1 & 1 \end{Bmatrix} , \tag{A.14}$$

where the curly bracket is a $9j$ symbol and the $3j$ symbols are related to CG coefficients by

$$\langle j_1 m_1, j_2 m_2 | j_3 m_3 \rangle = (-)^{j_1 - j_2 - m_3} \sqrt{2 j_3 + 1} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} . \tag{A.15}$$

## Appendix B

### VECTOR FIERZ TRANSFORMATION

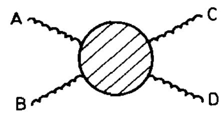
The $\mathrm{gg} \to \mathrm{gg}$ matrix elements we have derived are necessarily proportional to the four external gluon polarization vectors $e_A, e_B, e_C^*, e_D^*$. It is convenient to write the matrix elements in terms of $t$-channel invariants analogous to the spin-spin operator familiar from $\mathrm{e^+e^-}$ and $\mathrm{q\bar{q}}$ hamiltonians, as these operators are diagonal on states of definite $J^{PC}$. The three SO(3) tensors we may construct from $e_A$ and $e_C^*$, for example, are

$$\phi_1(\text{scalar}) = (\boldsymbol{e}_A \cdot \boldsymbol{e}_C^*), \tag{B.1}$$

$$\boldsymbol{S}_1(\text{spin vector}) = (i\boldsymbol{e}_A \times \boldsymbol{e}_C^*), \tag{B.2}$$

$$T_1^{ij}(\text{tensor}) = \left(\tfrac{1}{2}\left(e_A{}^i e_C^{j*} + e_A{}^j e_C^{i*}\right) - \tfrac{1}{3}\delta^{ij}\boldsymbol{e}_A \cdot \boldsymbol{e}_C^*\right). \tag{B.3}$$

The most general gg → gg effective hamiltonian may be written as a sum of three such invariant products:




$$H_{\text{eff}} = h_\phi(\phi_1\phi_2) + h_S(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2) + h_T(T_1 \cdot T_2). \tag{B.4}$$

Reducing invariant products of tensors in the $s$- or $u$-channel [for example the $u$-channel spin-spin term $(i\boldsymbol{e}_A \times \boldsymbol{e}_D^*)\cdot(i\boldsymbol{e}_B \times \boldsymbol{e}_C^*)$] to this standard form requires an SO(3) Fierz transformation. The general result is given below

$$I^{(m)}(\boldsymbol{a}, \boldsymbol{b}) \cdot I^{(m)}(\boldsymbol{a}', \boldsymbol{b}') = \sum_{n=1}^{3} \underbrace{c^{mn}}_{\substack{\text{Fierz}\\ \text{coeffs.}}} I^{(n)}(\boldsymbol{a}, \boldsymbol{a}') \cdot I^{(n)}(\boldsymbol{b}, \boldsymbol{b}'),$$

$$I^{(1)} = \phi, \qquad I^{(2)} = S^i, \qquad I^{(3)} = T^{ij}, \tag{B.5}$$

$\{c^{mn}\}$

| $m$; $n$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | $\frac{1}{3}$ | $-\frac{1}{2}$ | 1 |
| 2 | $-\frac{2}{3}$ | $\frac{1}{2}$ | 1 |
| 3 | $\frac{5}{9}$ | $\frac{5}{12}$ | $\frac{1}{6}$ |

(B.6)

In the example quoted previously, the $u$-channel spin-spin force is the following sum of $t$-channel invariants:

$$(i\boldsymbol{e}_A \times \boldsymbol{e}_D^*) \cdot (i\boldsymbol{e}_B \times \boldsymbol{e}_C^*) = \tfrac{2}{3}\phi_1(\boldsymbol{e}_A, \boldsymbol{e}_C^*)\,\phi_2(\boldsymbol{e}_B, \boldsymbol{e}_D^*) + \tfrac{1}{2}\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 - T_1 \cdot T_2. \tag{B.7}$$

The antisymmetry of $\boldsymbol{S}$ and symmetry of $\phi$ and $T$ has been used to obtain this result.

This transformation is useful because the $t$-channel invariants have simple eigenvalues on the $J^{PC}$ gg states considered in the text of the paper:

$$(\phi_1\phi_2)|J^{PC}\rangle = |J^{PC}\rangle, \tag{B.8}$$

$$(S_1\cdot S_2)|J^{PC}\rangle = [\tfrac{1}{2}J(J+1)-2]\,|J^{PC}\rangle, \tag{B.9}$$

$$(T_1\cdot T_2)|J^{PC}\rangle = [\tfrac{1}{6}\delta_{J2}-\tfrac{5}{6}\delta_{J1}+\tfrac{5}{3}\delta_{J0}]\,|J^{PC}\rangle. \tag{B.10}$$

## Appendix C

### CAVITY COULOMB INTEGRALS

In free space, the Coulomb Green function is given by $(R_{xy})^{-1}$, and the energy of an assemblage of charges is just an overlap integral of the charge density self-interacting through this Green function.

If instead we assemble the same charges in a cavity and impose the bag boundary conditions $\hat{\boldsymbol{n}}\cdot \boldsymbol{E}^a|_S=0$, we find that the Green function must be modified to insure that the boundary condition be satisfied for an arbitrary charge distribution. As the Green function is modified, so the energy needed to assemble the charges is different than if they were brought together in free space.

This modified cavity Green function for a cavity of radius $a$ was derived by Lee [10]:

$$G(\boldsymbol{x},\boldsymbol{y}) = \underbrace{R^{-1}_{x,y}}_{\text{free space}} + \underbrace{a^{-1}\left\{\left[\sum_{l=1}^{\infty}\frac{l+1}{l}\left(\frac{r_x r_y}{a^2}\right)^l P_l(\cos\theta_{xy})\right]-1\right\}}_{\text{cavity modification}}, \tag{C.1}$$

$$\begin{aligned} G(\boldsymbol{x},\boldsymbol{y}) &= R^{-1}_{x,y} + a^{-1}\left\{\left[\xi^{-1}-1-\ln\left(\frac{\xi+1-\mu\lambda}{2}\right)\right]-1\right\} \\ &= G_0(\boldsymbol{x},\boldsymbol{y}) + C(\boldsymbol{x},\boldsymbol{y}), \end{aligned} \tag{C.2}$$

where

$$\mu=\cos\theta_{xy}, \qquad \lambda = r_x r_y/a^2, \qquad \xi=\sqrt{1-2\mu\lambda+\lambda^2}\,. \tag{C.3}$$

The final $-1$ in the cavity modification $C(\boldsymbol{x},\boldsymbol{y})$ deserves comment. It is imposed by requiring that the integrals $(1/8\pi)\int\int_{\text{bag}}\rho_x G_{xy}\rho_y\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}$ and $\frac{1}{2}\int_{\text{bag}}|\mathbf{E}|^2\mathrm{d}\boldsymbol{x}$ inside the bag give the same energy; we may check this $-1$ by noting that a flat abelian charge distribution $\rho_0=Q/V$ gives an interaction energy of $0.1\alpha_s a^{-1}$, calculated either using $E=(1/8\pi)\int\int_{\text{bag}}\rho_x G_{xy}\rho_y\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}$ or $\frac{1}{2}\int_{\text{bag}}|\boldsymbol{E}|^2\mathrm{d}\boldsymbol{x}$. Lee states that the constant

$(-1)$ in $G_{xy}$ has no effect on the energy of color singlets, as it is $\sim :Q^a::Q^a:$. This is only true if we include self-energy diagrams without renormalization, in addition to the Coulomb gluon exchange. We have considered only gluon exchange and contact diagrams, as the self-energy diagrams are presumably cancelled by counterterms which define the gluon propagator at some renormalization point. The constant in $G_{xy}$ does indeed shift the color singlet energy if we neglect the self-energy diagrams, as this makes $G_{\text{constant}} \sim :Q^aQ^a:$ rather than $:Q^a::Q^a:$. $:Q^aQ^a:$ is not zero on color singlets in general. We believe this procedure is correct, as it is what one would do in treating positronium to this order. There, Coulomb photon exchange is certainly not cancelled by self-energy diagrams.

The effect of replacing $G_0(\boldsymbol{x}, \boldsymbol{y})$ by the cavity $G(\boldsymbol{x}, \boldsymbol{y})$ (C.2) is just to change the numerical values of the five coefficients $(c_1, c_2, c_3, c_4, c_5)$ which we evaluated previously in deriving the effective Coulomb vertices with a cavity Coulomb propagator (3.36), (3.41), (3.47). These coefficients with a cavity Green function, together with the free space $(G_0)$ coefficients for comparison, are found to be

| | $G_0 = R_{xy}^{-1}$ | $G$ = cavity Green function |
|---|---|---|
| $c_1$(TE$^2$ scalar) | $(590 \pm 1) \cdot 10^{-3}$ | $(90 \pm 1) \cdot 10^{-3}$ |
| $c_2$(TE$^2$ tensor) | $(23.1 \pm 0.2) \cdot 10^{-3}$ | $(40.5 \pm 0.2) \cdot 10^{-3}$ |
| $c_3$(TE TM scalar) | $(622 \pm 3) \cdot 10^{-3}$ | $(122 \pm 3) \cdot 10^{-3}$ |
| $c_4$(TE TM tensor) | $(-2.0 \pm 0.1) \cdot 10^{-3}$ | $(1.5 \pm 0.1) \cdot 10^{-3}$ |
| $c_5$(TE TM spin-spin) | $(29.2 \pm 0.2) \cdot 10^{-3}$ | $(72.4 \pm 0.2) \cdot 10^{-3}$ |

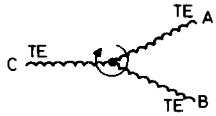
Two important effects are evident; first, the large spin-independent Coulomb shifts $c_1, c_3$ are essentially eliminated when the charges are assembled in a cavity. Hence, the $\sim -3\alpha_s a^{-1}$ Coulomb shift in free space is missing in cavity glueballs. Second, the spin-spin crossed diagram given by $c_5$ is approximately double the free space result; this term in the cavity cancels 40% of the transverse gluon exchange spin-spin term instead of the 20% found with $G_0$. It is more important than the four-gluon interaction spin-spin term! Finally, we note that the tensor-tensor spin-dependent Coulomb terms are always rather small.

## Appendix D

VERTICES

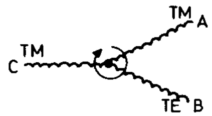
In this appendix we collect all the three-gluon (t), four-gluon (f) and Coulomb (c) vertices we have evaluated in the text. TE and TM refer to the $\omega a = 2.744$ and 4.493

gluon modes, respectively:



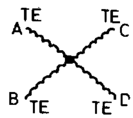

$$= -if^{abc}t_1(e^A \times e^B)^* \cdot e^C \tfrac{1}{2} g a^{-1},$$

$$t_1 = 0.1919; \tag{D.1}$$



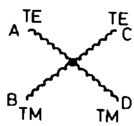

$$= -if^{abc}t_2(e^A \times e^B)^* \cdot e^C \tfrac{1}{2} g a^{-1},$$

$$t_2 = 0.1532; \tag{D.2}$$



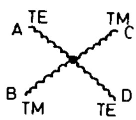

$$= -f^{xac}f^{xbd}f_1(S_1 \cdot S_2)\alpha_s a^{-1},$$

$$f_1 = 38.81 \cdot 10^{-3}; \tag{D.3}$$




$$= -f^{xac}f^{xbd}f_2(S_1 \cdot S_2)\alpha_s a^{-1},$$

$$f_2 = 11.44 \cdot 10^{-3}; \tag{D.4}$$

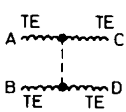



$$= -f^{xac}f^{xbd}\{f_3(\phi_1\phi_2) + f_4(S_1 \cdot S_2) + f_5(T_1 \cdot T_2)\}\alpha_s a^{-1},$$

$$f_3 = 15.25 \cdot 10^{-3},$$

$$f_4 = 6.59 \cdot 10^{-3},$$

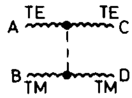
$$f_5 = 12.48 \cdot 10^{-3}; \tag{D.5}$$




$$= -f^{xac}f^{xbd}\{c_1(\phi_1\phi_2) + c_2(T_1 \cdot T_2)\}\alpha_s a^{-1},$$

$$c_1 = (90 \pm 1) \cdot 10^{-3},$$

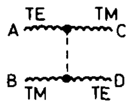
$$c_2 = (40.5 \pm 0.2) \cdot 10^{-3}; \tag{D.6}$$




$$= -f^{xac}f^{xbd}\{c_3(\phi_1\phi_2)+c_4(T_1\cdot T_2)\}\alpha_s a^{-1},$$

$$c_3=(122\pm 3)\cdot 10^{-3},$$

$$c_4=(1.5\pm 0.1)\cdot 10^{-3}; \tag{D.7}$$




$$= -f^{xac}f^{xbd}c_5(\boldsymbol{S}_1\cdot \boldsymbol{S}_2)\alpha_s a^{-1},$$

$$c_5=(72.4\pm 0.2)\cdot 10^{-3}. \tag{D.8}$$

We have neglected a term in the four-gluon vertices which gives zero on $s$-channel color singlets (3.11).

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