PHYSICAL REVIEW D VOLUME 22, NUMBER 4 15 AUGUST 1980

Operator ordering and Feynman rules in gauge theories

N. H. Christ and T. D. Lee
Columbia University, New York, New York 10027
(Received 10 April 1980)

The ordering of operators in the Yang-Mills Hamiltonian is determined for the $V_0 = 0$ gauge and for a general noncovariant gauge $\chi(V_i) = 0$, with χ a linear function of the spatial components of the gauge field V_μ. We show that a Cartesian ordering of the $V_0 = 0$ gauge Hamiltonian defines a quantum theory equivalent to that of the usual, covariant-gauge Feynman rules. However, a straightforward change of variables reduces this $V_0 = 0$ gauge Hamiltonian to a $\chi(V_i) = 0$ gauge Hamiltonian with an unconventional operator ordering. The resulting Hamiltonian theory, when translated into Feynman graphs, is shown to imply new nonlocal interactions, even in the familiar Coulomb gauge.

I. INTRODUCTION

Gauge theories have a wide range of applications in physics, from quantum electrodynamics (QED) to quantum chromodynamics (QCD) and the unifying theory of weak and electromagnetic interactions. Because the gauge transformations contain arbitrary functions of time, the usual canonical quantization procedure can only be carried out in a specific gauge. It is natural to inquire whether there are rules to ensure that the quantum theories in different gauges are indeed the same. As we shall see, this question is closely connected with the ordering problem of operators, especially in the non-Abelian Yang-Mills theory because of the intrinsic nonlinear nature of the interaction.

In this paper, we will show that the Yang-Mills theory has a Cartesian realization in the $V_0=0$ gauge; in this gauge the naive ordering of operators in the Hamiltonian is correct. We may then go from the $V_0=0$ gauge to other gauges such as the axial gauge, Coulomb gauge, or covariant gauges via either the operator or the path-integration formalism, thereby resolving whatever ambiguities may arise in these gauges.

The fact that the Euler-Lagrange equations of motion may allow arbitrary functions of time is by no means restricted to relativistic field theories. For convenience of nomenclature, all such theories will be referred to as gauge theories. In the next section, we give one of the simplest examples of such a system.

In Sec. III we review the operator formulation of the Yang-Mills theory in the $V_0=0$ gauge and establish our notation. Next, in Sec. IV, we exploit the residual symmetry of the $V_0=0$ gauge to make a change of coordinates from the Cartesian basis provided by the gauge potentials $V_i(x)$ to new variables $\phi_a(x)$ and $A_i(x)$: $\phi_a(x)$ are pure gauge variables while the gauge potentials $A_i(x)$ obey the constraint $\chi(A_i)=0$. When expressed in terms of these new variables, the gauge-invariant sector of the theory is recognized as identical to the Yang-Mills theory formulated in a noncovariant, $\chi(A_i)=0$ gauge with a specific ordering of operators in the Hamiltonian. This discussion is in precise analogy with the treatment of the simple mechanical system given in Sec. II. As shown in the Appendix, many of our formulas are also identical to those of rigid-body rotation, when one changes from the laboratory frame to the rotating-body frame.

In Sec. V we justify our assertion that the $V_0=0$ gauge provides a Cartesian realization of the quantum Yang-Mills theory: Using functional integration we demonstrate the equivalence of that theory and the usual, covariant-gauge Feynman rules.

The Hamiltonian formulation of the quantum Yang-Mills theory obtained in Sec. IV is not a convenient one for weak-field perturbation theory. In Sec. VI, this Hamiltonian operator theory is translated into the Lagrangian, path-integral language with careful attention paid to the question of operator ordering. This path-integral description implies new nonlocal interactions, called $\mathcal{V}_1+\mathcal{V}_2$ in our paper, that must be added to the usual Feynman rules. Finally, in the conclusion we show explicitly how to equate our Hamiltonian to that obtained by Schwinger for the Coulomb gauge.[1-3] Although the $\mathcal{V}_2$ term is new, the $\mathcal{V}_1$ term was derived by Schwinger in 1962; both have been left out in the conventional treatment[4] of Coulomb-gauge Feynman rules.

II. A SIMPLE MECHANICAL EXAMPLE

Let us consider a point particle in a three-dimensional space at position $\vec{r}$. Its Lagrangian is

$$L=\tfrac{1}{2}(\dot{\vec{r}}-\vec{q}\times\vec{r})^2-V(r)\,, \tag{2.1}$$

where $\vec{q}$ is another coordinate vector, but $\dot{\vec{q}}$ is absent in L. As usual, the dot denotes the time derivative and $r=|\vec{r}|$. From (2.1), one sees immediately that L is invariant under the transformation

$$\vec{r}\to\vec{r}+\vec{\epsilon}\times\vec{r}$$

and (2.2)

$$\vec{q}\to\vec{q}+\dot{\vec{\epsilon}}+\vec{\epsilon}\times\vec{q}$$

where $\vec{\epsilon}=\vec{\epsilon}(t)$ can be an arbitrary infinitesimal vector function of time t. Except for the $\frac{1}{2}(\vec{q}\times\vec{r})^2$ term, this would be the problem of a nonrelativistic charged particle moving in a central potential and under the influence of an external magnetic field.

We may further simplify the problem by imposing the constraint that $\vec{r}$ lies in the (x,y) plane and $\vec{q}=\hat{z}q$, where $\hat{z}$ is the unit vector along the z axis. Equation (2.1) becomes then

$$L=\tfrac{1}{2}(\dot{x}^2+\dot{y}^2)-(x\dot{y}-y\dot{x})q+\tfrac{1}{2}q^2r^2-V(r)\,, \tag{2.3}$$

where x and y are the *Cartesian* coordinates of $\vec{r}$ which is now a two-dimensional vector. In terms of the polar coordinates $x=r\cos\theta$ and $y=r\sin\theta$, (2.3) can be written as

$$L=\tfrac{1}{2}[\dot{r}^2+r^2(\dot{\theta}-q)^2]-V(r) \tag{2.4}$$

and (2.2) is simply the Abelian group of transformations

$$\theta\to\theta+\epsilon(t)$$

and (2.5)

$$q\to q+\dot{\epsilon}(t)\,,$$

where $\epsilon(t)$ can now be any finite function of t.

The invariance group of this simple example shares with the gauge groups of QED or QCD the special feature that its elements contain arbitrary functions of t. Consequently, the canonical procedure from the Lagrangian to the Hamiltonian and to quantization requires a specific choice of the "gauge." The Lagrange equations of motion can, of course, be written down without specifying the gauge. We find in polar coordinates

$$\dot{\theta}-q=0 \tag{2.6}$$

and

$$\ddot{r}+\frac{dV}{dr}=0\,. \tag{2.7}$$

A. $q=0$ gauge

Because of (2.5), any orbit $\vec{r}=\vec{r}(t)$ and $q=q(t)$ can be transformed to one in which $q=0$ at all times. In this gauge, $L=\frac{1}{2}\dot{\vec{r}}^2-V(r)$, the momentum $\vec{p}$ is $\dot{\vec{r}}$ and the Hamiltonian H is $\frac{1}{2}\vec{p}^2+V(r)$. Thus, in quantum mechanics, $\vec{p}=-i\vec{\nabla}$ and

$$H=-\tfrac{1}{2}\nabla^2+V(r)\,. \tag{2.8}$$

The angular momentum operator

$$l=-i\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right) \tag{2.9}$$

commutes with H. To be consistent with the equation of motion (2.6), only eigenstates of H with $l=0$ should be accepted. These eigenstates are all θ independent, and that leads to

$$H=-\frac{1}{2r}\frac{d}{dr}\left(r\frac{d}{dr}\right)+V(r)\,. \tag{2.10}$$

B. $y=0$ gauge

From (2.5), we see that any orbit $\vec{r}=\vec{r}(t)$ and $q=q(t)$ can also be transformed to one with $y=0$ at all times, albeit there are two branches: x can be >0 or <0, with $x=0$ being the point that the Jacobian of the transformation is zero. In the $y=0$ gauge, $r^2=x^2$ and the Lagrangian (2.3) becomes

$$L=\tfrac{1}{2}\dot{x}^2+\tfrac{1}{2}q^2x^2-V(x)\,, \tag{2.11}$$

where, for definiteness, we choose the branch of positive x. Since the above L does not contain $\dot{q}$, we may follow the standard procedure to eliminate q through $\partial L/\partial q=0$, which in the present example is simply $x^2q=0$. Hence, (2.11) becomes

$$L=\tfrac{1}{2}\dot{x}^2-V(x)\,. \tag{2.12}$$

The conjugate momentum p is $\dot{x}$ and the classical Hamiltonian is

$$H=\tfrac{1}{2}p^2+V(x)\,. \tag{2.13}$$

In passing over to quantum mechanics, in order that the spectrum of this Hamiltonian be identical to that of (2.10), it is important not to treat x in the $y=0$ gauge as a Cartesian coordinate; in (2.13) p^2 is the operator

$$-\frac{1}{x}\frac{d}{dx}\left(x\frac{d}{dx}\right)$$

and not $-d^2/dx^2$.

The $q=0$ gauge is the analog of the $V_0=0$ gauge in QED or QCD; the $y=0$ gauge corresponds to either the usual axial gauge or the Coulomb gauge. In this simple example, the choice of which coordinate in what guage is Cartesian can be determined by the one who makes up the problem. In the Yang-Mills theory, however, one is guided by the requirement of relativistic invariance. As we shall see, that leads to the choice of the $V_0=0$ gauge as the starting point.

Before leaving this example, we note that if one wishes, one may choose a more general gauge in which an arbitrary function $\chi(x,y,q)=0$, provided that any point in the (x,y,q) space can be transformed onto the surface $\chi(x,y,q)=0$ through the gauge transformation (2.5). Since this mechanical example is such a simple one, we shall refrain from exhibiting further details, except to remark that in passing over to the path-integration formalism, it is useful to absorb a factor of $r^{1/2}$ into the state vectors so that H, given by (2.10), becomes

$$\bar{H} \equiv r^{1/2} H r^{-1/2} = -\frac{1}{2}\frac{d^2}{dr^2} + V(r) - \frac{1}{8r^2}\,, \qquad (2.14)$$

which changes the volume element from rdr to dr and adds to the potential $V(r)$ a new term $-(8r^2)^{-1}$.

III. $V_0 = 0$ GAUGE

For definiteness, let us consider an SU_N gauge theory consisting of a spin-$\frac{1}{2}$ fermion field ψ which belongs to the N-dimensional representation of the SU_N group and a spin-1 gauge field V_μ^l with $l = 1, 2, \ldots, M$ and $M = N^2 - 1$. The Lagrangian density is

$$\mathcal{L} = -\tfrac{1}{4} V_{\mu\nu}^l V_{\mu\nu}^l - \psi^\dagger \gamma_4 \gamma_\mu D_\mu \psi\,, \qquad (3.1)$$

where all repeated indices are summed over, a dagger denotes Hermitian conjugation,

$$V_{\mu\nu}^l = \frac{\partial}{\partial x_\mu} V_\nu^l - \frac{\partial}{\partial x_\nu} V_\mu^l + g f^{lmn} V_\mu^m V_\nu^n\,,$$
$$D_\mu = \frac{\partial}{\partial x_\mu} - igT^l V_\mu^l\,, \qquad (3.2)$$

$x_\mu = (\vec{r}, it)$, the γ_μ's are the 4×4 Hermitian Dirac matrices, and the T^l's are $N\times N$ matrices that satisfy

$$T^l = T^{l\dagger}\,, \quad \mathrm{Tr}(T^l T^m) \propto \delta^{lm}$$
and (3.3)
$$[T^l, T^m] = if^{lmn} T^n$$

with f^{lmn} the antisymmetric structure constants of the group algebra. For simplicity, the fermion mass has been set equal to zero. The electric and magnetic fields E_i^l and B_i^l are given by the usual expressions

$$E_i^l = iV_{i4}^l = -\dot{V}_i^l - \nabla_i V_0^l + g f^{lmn} V_0^m V_i^n$$
and
$$\epsilon_{ijk} B_k^l = V_{ij}^l = \nabla_i V_j^l - \nabla_j V_i^l + g f^{lmn} V_i^m V_j^n\,,$$

where $V_0^l = -iV_4^l$ and the subscripts i, j, k denote the space indices which vary from 1 to 3. The Lagrangian equations of motion are

$$\gamma_\mu D_\mu \psi = 0 \qquad (3.4)$$
and
$$\frac{\partial}{\partial x_\mu} V_{\mu\nu}^l + g(f^{lmn} V_\mu^m V_{\mu\nu}^n + I_\nu^l) = 0\,, \qquad (3.5)$$

where

$$I_\nu^l = i\psi^\dagger \gamma_4 \gamma_\nu T^l \psi\,.$$

It is convenient to introduce the matrix function

$$V_\mu \equiv T^l V_\mu^l\,. \qquad (3.6)$$

The Lagrangian density (3.1) is invariant under the SU_N transformation

$$V_\mu \to u V_\mu u^\dagger + \frac{i}{g} u \frac{\partial u^\dagger}{\partial x_\mu} \qquad (3.7)$$
and
$$\psi \to u\psi\,, \qquad (3.8)$$

where $u = u(x)$ is any $N\times N$ unitary matrix function of x with $\det u = 1$. From any configuration $V_\mu = F_\mu(\vec{r}, t)$, we may choose $u^\dagger$ to be the following time-ordered function:

$$u^\dagger(\vec{r}, t) = T \exp\left[-i\int_0^t gF_0(\vec{r}, t')\right]\,, \qquad (3.9)$$

where $F_0 = -iF_4$ and T is the time-ordering operator. Hence, $u^\dagger$ satisfies

$$\frac{\partial u^\dagger}{\partial t} = -igF_0 u^\dagger\,. \qquad (3.10)$$

The transformation (3.7) then brings V_μ from the configuration $F_\mu(x)$ to the gauge in which

$$V_0(x) = 0\,. \qquad (3.11)$$

In the $V_0 = 0$ gauge, $E_i^l = -\dot{V}_i^l$ and (3.1) becomes

$$\mathcal{L} = \tfrac{1}{2}(\dot{V}_i^l \dot{V}_i^l - B_i^l B_i^l) - \psi^\dagger \gamma_4 \gamma_\mu D_\mu \psi\,. \qquad (3.12)$$

The conjugate momentum of V_i^l is simply

$$\Pi_i^l = \dot{V}_i^l\,, \qquad (3.13)$$

and the Hamiltonian density is

$$\mathcal{H} = \tfrac{1}{2}(\Pi_i^l \Pi_i^l + B_i^l B_i^l) - g I_i^l V_i^l + \psi^\dagger \gamma_4 \gamma_i \nabla_i \psi\,. \qquad (3.14)$$

The usual canonical quantization procedure leads to

$$[V_i^l(\vec{r}, t), \Pi_j^m(\vec{r}', t)] = i\delta_{ij}\delta^{lm}\delta^3(\vec{r} - \vec{r}')\,, \qquad (3.15)$$
$$\{\psi(\vec{r}, t), \psi^\dagger(\vec{r}', t)\} = \delta^3(\vec{r} - \vec{r}')\,. \qquad (3.16)$$

The equal-time commutators between the V_i's and between the Π_i's are zero; likewise, the equal-time anticommutators between the ψ's and between the $\psi^\dagger$'s are also zero.

Remnants of the original gauge transformations (3.7) and (3.8) remain important. In accordance with (3.6), we denote

$$V_i = T^l V_i^l$$
and (3.17)
$$\Pi_i = T^l \Pi_i^l\,.$$

It can be readily verified that the Hamiltonian density $\mathcal{H}$ and the commutation relations are invariant under a time-dependent SU_N transformation

$$V_i \to uV_iu^\dagger - \frac{i}{g}(\nabla_i u)u^\dagger ,$$
$$\Pi_i \to u\Pi_i u^\dagger , \tag{3.18}$$

and

$$\psi \to u\psi ,$$

where $u = u(\vec{r})$ can be any $N\times N$ unitary matrix function of $\vec{r}$ with $\det u = 1$. Since the $u(\vec{r})$'s are time-independent, the invariance group $\{u(\vec{r})\}$ is generated by the $\vec{r}$-dependent operators $\mathcal{G}^l$ which are conserved:

$$\mathcal{G}^l \equiv J^l + \psi^\dagger T^l \psi , \tag{3.19}$$

where

$$J^l \equiv \frac{1}{g} D_i^{lm}\Pi_i^m$$

and (3.20)

$$D_i^{lm} = \delta^{lm}\nabla_i - gf^{lmn}V_i^n .$$

It is straightforward to verify

$$[J^l(\vec{r},t), J^m(\vec{r}\,',t)] = if^{lmn}\delta^3(\vec{r}-\vec{r}\,')J^n(\vec{r},t) , \tag{3.21}$$

$$[\mathcal{G}^l(\vec{r},t), \mathcal{G}^m(\vec{r}\,',t)] = if^{lmn}\delta^3(\vec{r}-\vec{r}\,')\mathcal{G}^n(\vec{r},t) , \tag{3.22}$$

$$[\mathcal{G}^l(\vec{r},t), \psi(\vec{r}\,',t)] = -\delta^3(\vec{r}-\vec{r}\,')T^l\psi(\vec{r},t) , \tag{3.23}$$

$$[\mathcal{G}^l(\vec{r},t), \Pi_i^m(\vec{r}\,',t)] = if^{lmn}\delta^3(\vec{r}-\vec{r}\,')\Pi_i^n(\vec{r},t) , \tag{3.24}$$

and

$$[\mathcal{G}^l(\vec{r},t), V_i^m(\vec{r}\,',t)] = if^{lmn}\delta^3(\vec{r}-\vec{r}\,')V_i^n(\vec{r},t) - \frac{1}{g}\delta^{lm}\nabla_i\delta^3(\vec{r}-\vec{r}\,') , \tag{3.25}$$

where ∇_i is the differential operator with respect to $\vec{r}$. Consequently, $\mathcal{G}^l$ commutes with the Hamiltonian $H = \int \mathcal{H}\, d^3r$ and

$$\dot{\mathcal{G}}^l(\vec{r},t) = i[H, \mathcal{G}^l(\vec{r},t)] = 0 . \tag{3.26}$$

By commuting H with ψ, we obtain the equation of motion (3.4) for ψ. Likewise, by commuting H with V_i^l and Π_i^l, we derive (3.5) for $\nu = i$ which can be 1, 2, or 3 but not 4. We note that $i\mathcal{G}^l/g$ is identical to the left-hand side of (3.5) when $\nu = 4$. Thus, in order to be consistent with all the Lagrangian equations of motion, in the $V_0 = 0$ gauge we require all state vectors $|\ \rangle$ to satisfy

$$\mathcal{G}^l|\ \rangle = 0 . \tag{3.27}$$

In the Schrödinger picture the operators $V_i^l = V_i^l(\vec{r})$ and $\Pi_i^l = \Pi_i^l(\vec{r})$ are all t independent. The state vector in the V_i^l representation is the functional

$$\Psi(V_i) \equiv \langle V_i^l|\ \rangle . \tag{3.28}$$

In this representation, the Hamiltonian H is

$$H = \mathcal{K} + \mathcal{V} , \tag{3.29}$$

where

$$\mathcal{K} = -\tfrac{1}{2}\int \frac{\delta}{\delta V_i^l(\vec{r})}\frac{\delta}{\delta V_i^l(\vec{r})}\, d^3r \tag{3.30}$$

and

$$\mathcal{V} + \int (\tfrac{1}{2}B_i^lB_i^l - gI_i^lV_i^l + \psi^\dagger\gamma_4\gamma_i\nabla_i\psi)d^3r . \tag{3.31}$$

In Sec. IV we shall see how the introduction of curvilinear coordinates can be used to eliminate the constraint (3.27), in complete analogy to the passage from (2.8) to (2.10) in the simple mechanical example discussed in the previous section.

IV. NONCOVARIANT GAUGES

Let us start from the $V_0 = 0$ gauge quantum theory of Sec. III and show how to reach other noncovariant gauges such as the axial or Coulomb gauges. For notational clarity the gauge field in these other gauges will be referred to as $A_\mu = T^lA_\mu^l$. The spatial components of A_μ obey a gauge condition

$$\chi(A_i) = 0 . \tag{4.1}$$

Among possible choices for χ one has

$$\chi(A_i) = \begin{cases} A_3 & \text{in axial gauge ,} \\ \nabla_i A_i & \text{in Coulomb gauge .} \end{cases} \tag{4.2}$$

For simplicity we will treat χ as a linear homogeneous functional of A_i. As in (4.2), we assume $\chi = T^l\chi^l$ to be an $N\times N$ Hermitian matrix with zero trace. Thus at any given space-time point, (4.1) expresses $M = N^2 - 1$ conditions:

$$0 = \chi^l(A_i, \vec{r}) \equiv \int d^3r' \langle \vec{r}, l\,|\,\Gamma_j\,|\,\vec{r}\,', m\rangle A_j^m(\vec{r}\,',t) , \tag{4.3}$$

where the matrix element of Γ_j is real, and as before, the parameters l and m can vary from 1 to M. In addition, we will assume that for every field configuration $V_i(\vec{r},t)$ in the $V_0 = 0$ gauge there exists a unique gauge-transformation matrix $u(\vec{r},t)$ such that[5]

$$V_i = uA_iu^{-1} + \frac{i}{g}u\nabla_i u^{-1} . \tag{4.4}$$

If we view the $N\times N$ matrix u as a function $u(\phi_a)$ of M group parameters ϕ_a, $1 \le a \le M$, then the gauge transformation $u(\vec{r},t)$ in turn specifies M functions $\phi_a(\vec{r},t)$ such that

$$u(\vec{r},t) = u(\phi_a(\vec{r},t)) . \tag{4.5}$$

Equation (4.4) can thus be viewed as expressing the gauge field $V_i(\vec{r},t)$ in terms of the curvilinear coordinates $A_i(\vec{r},t)$, $\phi_a(\vec{r},t)$. For example, if our group is SU_2, then $M=3$ and we might choose the ϕ_a's to be simply the three Euler angles. (See the Appendix for details.)

Because of (3.30), the V_i^l's can be regarded as the Cartesian coordinates. We recall that in any coordinate transformation from a set of Cartesian coordinates $q_1, q_2, \ldots$ to one of curvilinear coordinates $Q_1(q_a)$, $Q_2(q_a), \ldots$, the standard kinetic energy term in the Lagrangian may be written as

$$K \equiv \tfrac{1}{2}\sum_\alpha \dot{q}_\alpha{}^2 = \tfrac{1}{2}\sum_{\alpha,\beta} \dot{Q}_\alpha M_{\alpha\beta}\dot{Q}_\beta \,. \tag{4.6}$$

In the quantum Hamiltonian, the corresponding operator in the coordinate representation is

$$\begin{aligned}\mathcal{K} &\equiv -\tfrac{1}{2}\sum_\alpha \frac{\partial^2}{\partial q_\alpha{}^2}\\ &= -\tfrac{1}{2}\sum_{\alpha,\beta} |M|^{-1/2}\frac{\partial}{\partial Q_\alpha}\left(M^{-1}{}_{\alpha\beta}|M|^{1/2}\frac{\partial}{\partial Q_\beta}\right),\end{aligned} \tag{4.7}$$

where

$$M_{\alpha\beta} = \sum_\gamma \frac{\partial q_\gamma}{\partial Q_\alpha}\frac{\partial q_\gamma}{\partial Q_\beta}, \quad M^{-1}{}_{\alpha\beta} = \sum_\gamma \frac{\partial Q_\alpha}{\partial q_\gamma}\frac{\partial Q_\beta}{\partial q_\gamma}, \tag{4.8}$$

and $|M| = \det M$. Hence,

$$|M|^{-1/2} = \det\left(\frac{\partial Q_\alpha}{\partial q_\beta}\right) \tag{4.9}$$

in which $\partial Q_\alpha/\partial q_\beta$ is the (α,β)th element of the matrix.

Applied to our case, (3.30) becomes

$$\begin{aligned}\mathcal{K} = \tfrac{1}{2}|M|^{-1/2}\int d^3r \int d^3r' [&p_a(\vec{r})|M|^{1/2}M^{-1}(\vec{r},\vec{r}\,')_{ab}p_b(\vec{r}\,') + p_a(\vec{r})|M|^{1/2}M^{-1}(\vec{r},\vec{r}\,')^l_{ai}P^l_i(\vec{r}\,')\\ &+ P^l_i(\vec{r})|M|^{1/2}M^{-1}(\vec{r},\vec{r}\,')^l_{ia}p_a(\vec{r}\,') + P^l_i(\vec{r})|M|^{1/2}M^{-1}(\vec{r},\vec{r}\,')^{lm}_{ij}P^m_j(\vec{r}\,')],\end{aligned} \tag{4.10}$$

where p_a and P_i^l are the momenta conjugate to ϕ_a and A_i^l. In the coordinate representation of the Schrödinger picture, $\phi_a(\vec{r})$ and $A_i^l(\vec{r})$ are all t independent. Clearly,

$$p_a(\vec{r}) = -i\frac{\delta}{\delta\phi_a(\vec{r})}\,. \tag{4.11}$$

However, because of the constraints (4.3) obeyed by $A_i^l(\vec{r})$, its conjugate momentum must be defined more carefully. We can expand $A_i^l(\vec{r})$ in terms of a complete set of real orthonormal functions $f_i^l(\vec{r})_N$, each obeying the same set of constraints

$$\int d^3r' \langle\vec{r},l|\Gamma_j|\vec{r}\,',m\rangle f_j^m(\vec{r}\,')_N = 0 \tag{4.12}$$

for all N, $\vec{r}$, and l. The expansion is

$$A_i^l(\vec{r}) = \sum_N Q_N f_i^l(\vec{r})_N\,. \tag{4.13}$$

The Q_N's may now be viewed as the independent generalized coordinates. The momentum $P_i^l(\vec{r})$ is then given by

$$P_i^l(\vec{r}) = \sum_N f_i^l(\vec{r})_N\left(-i\frac{\partial}{\partial Q_N}\right)\,. \tag{4.14}$$

Thus, $P_i^l(\vec{r})$ also obeys the constraints

$$\chi^l(P_i,\vec{r}) = \int d^3r' \langle\vec{r},l|\Gamma_j|\vec{r}\,',m\rangle P_j^m(\vec{r}\,') = 0. \tag{4.15}$$

The following construction for the functions $f_i^l(\vec{r})_N$ will be useful. We may view $\langle\vec{r},l|\Gamma_i^\dagger|\vec{r}\,',l'\rangle$ as a transformation matrix which maps a vector $|\xi\rangle$ in the functional space $h \equiv \{|\vec{r}\,',l'\rangle\}$ to a vector $|\bar{\xi}\rangle$ in a larger space $\bar{h} \equiv \{|i,\vec{r},l\rangle\}$, where $|\bar{\xi}\rangle$ is given by

$$\langle i,\vec{r},l|\bar{\xi}\rangle \equiv \langle\vec{r},l|\Gamma_i^\dagger|\xi\rangle\,. \tag{4.16}$$

Let $|N\rangle$ be a vector in $\bar{h}$ that is orthogonal to all such $|\bar{\xi}\rangle$'s. If we denote

$$f_i^l(\vec{r})_N \equiv \langle i,\vec{r},l|N\rangle\,, \tag{4.17}$$

then since

$$\langle\bar{\xi}|N\rangle = \int \langle\xi|\Gamma_i|\vec{r},l\rangle f_i^l(\vec{r})_N d^3r = 0 \tag{4.18}$$

for all the $|\xi\rangle$ vectors in h, (4.12) follows.

The inverse of the mass matrix M appearing in (4.10) has been divided into four blocks with

$$\begin{aligned}M^{-1}(\vec{r},\vec{r}\,')_{ab} &= \int d^3r'' \frac{\delta\phi_a(\vec{r})}{\delta V_k^n(\vec{r}\,'')}\frac{\delta\phi_b(\vec{r}\,')}{\delta V_k^n(\vec{r}\,'')}\,,\\ M^{-1}(\vec{r},\vec{r}\,')^l_{ai} &= M^{-1}(\vec{r}\,',\vec{r})^l_{ia}\\ &= \int d^3r'' \frac{\delta\phi_a(\vec{r})}{\delta V_k^n(\vec{r}\,'')}\frac{\delta A_i^l(\vec{r}\,')}{\delta V_k^n(\vec{r}\,'')}\,,\\ M^{-1}(\vec{r},\vec{r}\,')^{lm}_{ij} &= \int d^3r'' \frac{\delta A_i^l(\vec{r})}{\delta V_k^n(\vec{r}\,'')}\frac{\delta A_j^m(\vec{r}\,')}{\delta V_k^n(\vec{r}\,'')}\,.\end{aligned} \tag{4.19}$$

Next, let us determine these functional derivatives in terms of the gauge-fixing function $\chi(A_i,\vec{r})$. Solving Eq. (4.4) for A_i, and considering small variations of A_i, ϕ_a, and V_i we obtain

$$\begin{aligned}\delta A_i &= \delta u^{-1}V_i u + u^{-1}\delta V_i u + u^{-1}V_i\delta u\\ &\quad + \frac{i}{g}\delta u^{-1}\nabla_i u + \frac{i}{g}u^{-1}\nabla_i\delta u\,.\end{aligned} \tag{4.20}$$

By using (3.3), we may convert the above equation into the form

$$\delta A_i^l = U^{ml}\delta V_i^m + \frac{1}{g}\mathfrak{D}_i^{lm}(\lambda_a^m \delta\phi_a), \tag{4.21}$$

where λ_a^l is defined by

$$iu^{-1}\frac{\partial u}{\partial \phi_a} \equiv \lambda_a \equiv T^l \lambda_a^l, \tag{4.22}$$

U^{ml} satisfies

$$u^{-1}T^m u = U^{ml}T^l, \tag{4.23}$$

and $\mathfrak{D}_i^{lm}$ represents the covariant-derivative operator containing the field A_i^n

$$\mathfrak{D}_i^{lm} = \delta^{lm}\nabla_i - gf^{lmn}A_i^n. \tag{4.24}$$

Because of the first two equations in (3.3), the matrix $U = (U^{ml})$ is real and orthogonal, and therefore also unitary.

The requirement that $\chi(A_i + \delta A_i, \vec{r})$ should also vanish then relates δV_i and $\delta\phi_a$:

$$0 = \int d^3r' l\langle \vec{r}, l|\Gamma_i|\vec{r}', m\rangle \left[U^{nm}(\vec{r}')\delta V_i^n(\vec{r}') + \frac{1}{g}\mathfrak{D}_i^{mn}(\lambda_a^n(\vec{r}')\delta\phi_a(\vec{r}'))\right], \tag{4.25}$$

which can be solved for $\delta\phi_a(\vec{r})$, thereby determining

$$\frac{\delta\phi_a(\vec{r})}{\delta V_i^l(\vec{r}')} = -g\lambda^{-1}(\vec{r})_a^m \langle \vec{r}, m|(\Gamma_j\mathfrak{D}_j)^{-1}\Gamma_i|\vec{r}, n\rangle U^{ln}(\vec{r}'), \tag{4.26}$$

where $(\Gamma_j\mathfrak{D}_j)^{-1}$ is the inverse of the matrix $\Gamma_j\mathfrak{D}_j$ whose matrix elements are

$$\langle \vec{r}, l|\Gamma_j\mathfrak{D}_j|\vec{r}', l'\rangle \equiv \int d^3r'' \langle \vec{r}, l|\Gamma_j|\vec{r}'', m\rangle \times \langle \vec{r}'', m|\mathfrak{D}_j|\vec{r}', l'\rangle, \tag{4.27}$$

with $\mathfrak{D}_j$ as the antisymmetric matrix, defined by

$$\langle \vec{r}, l|\mathfrak{D}_j|\vec{r}', m\rangle = \mathfrak{D}_j^{lm}(\vec{r})\delta^3(\vec{r} - \vec{r}'). \tag{4.28}$$

The matrix $\lambda^{-1}(\vec{r})$ is the inverse of the $M\times M$ matrix $\lambda(\vec{r}) \equiv (\lambda_a^n(\vec{r}))$, with $(\lambda^{-1})_a^m \lambda_a^n = \delta^{mn}$. If (4.26) is used in (4.21), we can also obtain

$$\frac{\delta A_i^m(\vec{r})}{\delta V_j^l(\vec{r}')} = U^{lm}(\vec{r})\delta_{ij}\delta^3(\vec{r} - \vec{r}') - \langle \vec{r}, m|\mathfrak{D}_i(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j|\vec{r}', n\rangle U^{ln}(\vec{r}'). \tag{4.29}$$

Note that (4.29) automatically obeys

$$\int d^3r \langle \vec{r}'', l''|\Gamma_i|\vec{r}, m\rangle \frac{\delta A_i^m(\vec{r})}{\delta V_j^l(\vec{r}')} = 0. \tag{4.30}$$

The determinant $|M|$ in (4.10) can now be determined explicitly. From (4.9), we see that $|M|$ is invariant under an orthogonal transformation among the Cartesian coordinates q_α. By using the notation of (4.13) and (4.16), we may consider the variations

$$\delta A_i^l(\vec{r}) = \sum_N f_i^l(\vec{r})_N \delta Q_N \tag{4.31}$$

and

$$\delta V_i^l(\vec{r}') = U^{lm}[\delta \bar{V}_i^m(\vec{r}')_I + \delta \bar{V}_i^m(\vec{r}')_{II}], \tag{4.32}$$

with

$$\delta \bar{V}_i^l(\vec{r}')_I \equiv \sum_\xi \langle \vec{r}', l|\Gamma_i^\dagger(\Gamma_j\Gamma_j^\dagger)^{-1/2}|\xi\rangle \delta q_\xi \tag{4.33}$$

and

$$\delta \bar{V}_i^l(\vec{r}')_{II} = \sum_N f_i^l(\vec{r}')_N \delta q_N, \tag{4.34}$$

where ξ runs over any complete set of orthonormal basis vectors $|\xi\rangle$ in h. Hence, the matrix $(\partial Q_\alpha/\partial q_\beta)$ in (4.9) now takes a 2×2 block form. We find

$$|M|^{-1/2} = \det \begin{bmatrix} -g\lambda^{-1}(\Gamma_i\mathfrak{D}_i)^{-1}(\Gamma_j\Gamma_j^\dagger)^{1/2} & 0 \\ X & 1 \end{bmatrix} \tag{4.35}$$

where X is a rectangular matrix whose matrix elements are

$$X_{N\xi} = \frac{\partial Q_N}{\partial q_\xi} = -\int f_i^l(\vec{r})_N \langle \vec{r}, l|\mathfrak{D}_i(\Gamma_k\mathfrak{D}_k)^{-1}(\Gamma_j\Gamma_j^\dagger)^{1/2}|\xi\rangle d^3r.$$

From (4.35), we see that

$$|M|^{1/2} = \det(-g^{-1}(\Gamma_j\Gamma_j^\dagger)^{-1/2}\Gamma_i\mathfrak{D}_i\lambda) = \text{const}\, \mathcal{J}|\lambda|, \tag{4.36}$$

where $|\lambda| = \prod_{\vec{r}} \det\lambda(\vec{r})$, $\mathcal{J}$ is the Faddeev-Popov determinant

$$\mathcal{J} = \det|\Gamma_i\mathfrak{D}_i|, \tag{4.37}$$

and const $= \det|-g^{-1}(\Gamma_j\Gamma_j^\dagger)^{-1/2}|$ which is independent of both $A_i(\vec{r})$ and $\phi_a(\vec{r})$.

Our partial derivative formulas (4.26) and (4.29) may also be substituted into the expression (4.10) for $\mathcal{K}$ yielding

$$\mathcal{K} = \tfrac{1}{2}\mathcal{J}^{-1}\int d^3r P_i^l(\vec{r})\mathcal{J}P_i^l(\vec{r}) + \tfrac{1}{2}\mathcal{J}^{-1}|\lambda|^{-1}\int\int d^3r\, d^3r' [-P_i^l(\vec{r})\mathfrak{D}_i - gp_a(\vec{r})(\lambda^{-1})_a^l]\mathcal{J} \times (\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j\Gamma_j^\dagger(\mathfrak{D}_{k'}^\dagger\Gamma_{k'}^\dagger)^{-1}|\lambda|[\mathfrak{D}_{i'}P_{i'}^{l'}(\vec{r}') - g(\lambda^{-1})_{a'}^{l'}p_{a'}(\vec{r}')], \tag{4.38}$$

where in the double integral the matrix element $\langle\vec{r}, l|O|\vec{r}', l'\rangle$ between any pair of momenta is simply written as O. The above formula appears quite similar to the familiar Coulomb-gauge formula, $P_i^l(\vec{r})$ replacing the transverse momenta, the quantity in square brackets the charge density, and the central matrix element the Coulomb Green's function. However, the analogy is not yet completely precise because of the presence of the angle variables $\phi_a(\vec{r})$, their conjugate momenta $p_a(\vec{r})$, and the matrix λ.

In order to eliminate the angular dependence, it is useful to first construct the group generators in terms of the p_a's. Let us introduce the following parallel expression to (4.22):

$$i\frac{\partial u}{\partial \phi_a}u^{-1} \equiv \Lambda_a \equiv T^l \Lambda_a^l . \tag{4.39}$$

Because of (4.23), Λ_a^l and λ_a^m are related by

$$\Lambda_a^l = U^{lm}\lambda_a^m , \tag{4.40}$$

which implies

$$\det\Lambda = \det\lambda . \tag{4.41}$$

Since u is unitary and T^l Hermitian, both Λ_a^l and λ_a^l are real. We define two sets of operators $\{j^l(\vec{r})\}$ and $\{J^l(\vec{r})\}$:

$$p_a \equiv \lambda_a^l j^l \equiv \Lambda_a^l J^l . \tag{4.42}$$

Hence,

$$\begin{aligned} &j^l = (\lambda^{-1})_a^l p_a, \quad J^l = (\Lambda^{-1})_a^l p_a , \\ &j^l = U^{ml}J^m \text{ and } J^l = U^{lm}j^m . \end{aligned} \tag{4.43}$$

On account of (4.5) and (4.11), the $\vec{r}$ dependence of j^l, J^l, λ_a^l, and Λ_a^l is entirely through their dependence on $\phi_a(\vec{r})$ and $p_a(\vec{r})$. By using (3.3) and by differentiating (4.22) and (4.39) with respect to ϕ_b, we can verify that

$$[j^l(\vec{r}), j^m(\vec{r}')] = -if^{lmn}j^n(\vec{r})\delta^3(\vec{r}-\vec{r}') \tag{4.44}$$

and

$$[J^l(\vec{r}), J^m(\vec{r}')] = if^{lmn}J^n(\vec{r})\delta^3(\vec{r}-\vec{r}') . \tag{4.45}$$

Similarly, through differentiation of (4.23), it follows that

$$\begin{aligned} &[J^l(\vec{r}), U^{mn}(\vec{r}')] = if^{lms}U^{sn}(\vec{r})\delta^3(\vec{r}-\vec{r}') , \\ &[j^l(\vec{r}), U^{mn}(\vec{r}')] = -if^{lns}U^{ms}(\vec{r})\delta^3(\vec{r}-\vec{r}') , \end{aligned} \tag{4.46}$$

which together with (4.43)–(4.45) lead to

$$[J^l(\vec{r}), j^m(\vec{r}')] = 0 . \tag{4.47}$$

Furthermore, by using

$$|\lambda|^{-1}\frac{\partial}{\partial\phi_a}|\lambda| = (\lambda^{-1})_b^m\frac{\partial}{\partial\phi_a}\lambda_b^m$$

and by differentiating (4.22), we can derive, after some manipulation, a useful commutation relation

$$[p_a, (\lambda^{-1})_a^l|\lambda|] = 0 . \tag{4.48}$$

Thus, for arbitrary functions $f(\phi_a)$ and $g(\phi_a)$, j^l satisfies the Hermiticity condition

$$\int f^* j^l g\, d\tau_\phi = \left(\int g^* j^l f\, d\tau_\phi\right)^* , \tag{4.49}$$

where $d\tau_\phi = |\lambda|\Pi d\phi_a$. Likewise, J^l is also Hermitian. That there should exist two sets of operators $\{J_a^l(\vec{r})\}$ and $\{-j_a^l(\vec{r})\}$, both satisfying the same group algebra and mutually commuting, has a simple geometrical meaning. In the case of the SU_2 group, this situation is identical to the familiar problem of rigid-body rotation, with J^l as the angular momentum operator in the laboratory frame and j^l that in the body frame. The details will be given in the Appendix.

We shall now show that the operator J^l defined by (4.43) is equal to $g^{-1}D_i^{lm}\Pi_i^m$, given by (3.20). From (4.4), (4.23), and (4.39) we find

$$V_i^l = U^{lm}A_i^m - \frac{1}{g}\Lambda_a^l\nabla_i\phi_a , \tag{4.50}$$

which together with (3.20) and (4.24) leads to

$$D_i^{lm}U^{mn} = U^{lm}\mathfrak{D}_i^{mn} . \tag{4.51}$$

The operator $D_i^{lm}\Pi_i^m$ can be written as a linear combination of the momenta p_a and P_i^l:

$$\begin{aligned} D_i^{lm}(\vec{r})\Pi_i^m(\vec{r}) &= -iD_i^{lm}(\vec{r})\frac{\delta}{\delta V_i^m(\vec{r})} \\ &= \int d^3r'\left[D_i^{lm}(\vec{r})\frac{\delta\phi_a(\vec{r}')}{\delta V_i^m(\vec{r})}p_a(\vec{r}') + D_i^{lm}(\vec{r})\frac{\delta A_j^n(\vec{r}')}{\delta V_i^m(\vec{r})}P_j^n(\vec{r}')\right]. \end{aligned} \tag{4.52}$$

The substitution of Eqs. (4.26) and (4.29) simplifies this considerably. For example, by using the transpose of (4.51), we find

$$\begin{aligned} D_i^{lm}(\vec{r})\frac{\delta A_j^n(\vec{r}')}{\delta V_i^m(\vec{r})} &= \langle\vec{r}', n|(U^\dagger\delta_{ij} - \mathfrak{D}_j(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_i U^\dagger)D_i|\vec{r}, l\rangle \\ &= -\langle\vec{r}', n|(\delta_{ij} - \mathfrak{D}_j(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_i)\mathfrak{D}_i U^\dagger|\vec{r}, l\rangle \\ &= 0 \end{aligned} \tag{4.53}$$

in which the matrix $\langle\vec{r}, l|U|\vec{r}', m\rangle = U^{lm}(\vec{r})\delta^3(\vec{r}-\vec{r}')$ is real, and $\langle\vec{r}, l|D_i|\vec{r}', m\rangle = D_i^{lm}(\vec{r})\delta^3(\vec{r}-\vec{r}')$ is antisymmetric, like $\mathfrak{D}_i$. Likewise,

$$D_i^{lm}(\vec{r})\frac{\delta\phi_a(\vec{r}')}{\delta V_i^m(\vec{r})} = g\lambda^{-1}(\vec{r})_a^m U^{lm}(\vec{r})\delta^3(\vec{r}-\vec{r}') . \tag{4.54}$$

Consequently, (4.52) becomes

$$D_i^{lm}\Pi_i^m = g(\Lambda^{-1})_a^l p_a = gJ^l , \tag{4.55}$$

which establishes the identity between the two definitions of J^l, (3.20) and (4.43).

Let us finally examine the requirement (3.27) that the physical subspace of the $V_0=0$ gauge Hilbert space be annihilated by the operator $\mathcal{G}^l=J^l+\psi^\dagger T^l\psi$. Under the transformation (4.4), the state vector (3.28) becomes a functional of A_i and ϕ_a. Equation (3.27) can be written as

$$\left[\frac{\delta}{\delta\phi_a(r)}+i\Lambda_a^l\psi^\dagger T^l\psi\right]\Psi[A_i,\phi_a]=0\,, \tag{4.56}$$

an equation which is easily solved:

$$\Psi[A_i,\phi_a]=\mathfrak{U}(\phi_a)\tilde{\Psi}(A_i)\,. \tag{4.57}$$

Here $\tilde{\Psi}(A_i)$ is any state vector depending on the vector potential A_i and the fermionic degrees of freedom, but independent of ϕ_a. $\mathfrak{U}(\phi_a)$ is the unitary transformation acting on the fermionic degrees of freedom which represents the group element $u(\phi_a)$. Since the generators of $\mathfrak{U}(\phi_a)$ are simply $i\psi^\dagger T^l\psi$, $1\le l\le N$, the equation

$$\frac{\partial\mathfrak{U}}{\partial\phi_a}\mathfrak{U}^{-1}=-i\Lambda_a^l\psi^\dagger T^l\psi \tag{4.58}$$

is the Hilbert-space analog of (4.39) and this equation implies directly that the state $\Psi[A_i,\phi_a]$ defined in Eq. (4.57) does indeed satisfy Gauss' law, Eq. (4.56). [In the SU_2 case, if we represent u in terms of the Euler angles a, b, c as in (A1) of the Appendix

$$u=e^{-i\tau_z b/2}e^{-i\tau_y a/2}e^{-i\tau_z c/2}\,,$$

then the corresponding $\mathfrak{U}$ is given by

$$\mathfrak{U}=e^{-i\mathcal{T}_z b/2}e^{-i\mathcal{T}_y a/2}e^{-i\mathcal{T}_z c/2}\,,$$

where $\mathcal{T}_i=\int d^3r\,\psi^\dagger\frac{1}{2}\tau_i\psi$].

If the Hamiltonian H is applied to this gauge-invariant subspace of the Hilbert space, it is convenient to work directly with the ϕ-independent state vectors $\tilde{\Psi}(A)$ and to absorb the unitary operator $\mathfrak{U}(\phi_a)$ into H. The resulting Hamiltonian,

$$\tilde{H}=\mathfrak{U}(\phi)^{-1}H\mathfrak{U}(\phi)\,, \tag{4.59}$$

is then extremely simple. For example, (4.58) and (4.40) imply that under conjugation by $\mathfrak{U}$, the combination $(\lambda^{-1})_a^m p_a$ appearing in $\mathcal{K}$ becomes

$$\begin{aligned}(\lambda^{-1})_a^m\mathfrak{U}^{-1}p_a\mathfrak{U}&=-(\lambda^{-1})_a^m\Lambda_a^l\mathfrak{U}^{-1}\psi^\dagger T^l\psi\mathfrak{U}\\&=-(\lambda^{-1})_a^m\Lambda_a^l(U^{-1})^{l'l}\psi^\dagger T^{l'}\psi\\&=-\psi^\dagger T^m\psi\,.\end{aligned} \tag{4.60}$$

Likewise,

$$\mathfrak{U}^{-1}\psi^\dagger\alpha_j(-i\nabla_j-gV_j^lT^l)\psi\mathfrak{U}=\psi^\dagger\alpha_j(-i\nabla_j-gA_j^lT^l)\psi\,. \tag{4.61}$$

Thus, from (3.29), (4.38), and (4.59) we find

$$\begin{aligned}\tilde{H}(P,A)=\;&\tfrac{1}{2}\mathcal{J}^{-1}\int d^3r\,P_i^l(\vec{r})\mathcal{J}P_i^l(\vec{r})\\&+\tfrac{1}{2}\mathcal{J}^{-1}\int d^3r\int d^3r'\,\{[-P_i^l(\vec{r})\mathfrak{D}_i+g\psi^\dagger T^l\psi]\mathcal{J}\times(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j\Gamma_j^\dagger(\mathfrak{D}_{k'}^\dagger\Gamma_{k'}^\dagger)^{-1}[\mathfrak{D}_{i'}P_{i'}^{l'}(\vec{r}')+g\psi^\dagger T^{l'}\psi]\}\\&+\int d^3r[\tfrac{1}{2}B_j^lB_j^l+\psi^\dagger\alpha_j(-i\nabla_j-gA_j^lT^l)\psi]\,,\end{aligned} \tag{4.62}$$

where, as in (4.37), $\mathcal{J}=\det|\Gamma_k\mathfrak{D}_k|$. The angle variables have now been completely eliminated from the problem. The Yang-Mills quantum theory, when restricted to the states $\tilde{\Psi}(A)$, becomes very close to the canonical theory that would have been naively proposed for the gauge $\chi[A]$. However, the precise ordering of the operators in (4.62), in particular, the appearance of the Jacobian $\mathcal{J}$, is not the conventional one. In fact, as will be shown in Sec. VI, the operator ordering in (4.62) yields additional vertices in the Feynman rules. Although for the case of the Coulomb gauge, the operator ordering implied by (4.62) can be shown to be identical to that proposed by Schwinger[1] as will be discussed in Sec. VII, the derivation presented above appears to be particularly simple and clear, the kinetic energy in (4.62) being essentially the familiar formula (4.7) for the Laplacian in curvilinear coordinates.

Let us conclude this section by specializing our resulting Hamiltonian operator in (4.62) to the Coulomb gauge for the group SU_2. In that case the matrix Γ_j becomes

$$\langle\vec{r},l\,|\,\Gamma_j\,|\,\vec{r}',m\rangle=\nabla_j\,\delta^3(\vec{r}-\vec{r}')\delta^{lm}\,. \tag{4.63}$$

The operator $\Gamma_k\mathfrak{D}_k$ becomes

$$\Gamma_k\mathfrak{D}_k=\nabla_k\mathfrak{D}_k=\nabla_k(\nabla_k\delta^{lm}-g\epsilon^{lmn}A_k^n) \tag{4.64}$$

so that the Jacobian $\mathcal{J}$ of (4.37) can be recognized as the familiar, Coulomb-gauge, Faddeev-Popov determinant. Equation (4.62) becomes

$$\tilde{H}=\int d^3r\,[\tfrac{1}{2}\mathcal{G}^{-1}\vec{\Pi}_i^T\circ\mathcal{G}\vec{\Pi}_i^T+\tfrac{1}{2}\vec{B}_i^2+\psi^\dagger\alpha_i(-i\nabla_i)-\tfrac{1}{2}g\vec{\tau}\cdot\vec{A}_i\psi]$$
$$+\tfrac{1}{2}g^2\int d^3r\,d^3r'\mathcal{G}^{-1}\{[-\vec{\Pi}_i^T(\vec{r})\times\vec{A}_i(\vec{r})+\tfrac{1}{2}\psi^\dagger(\vec{r})\vec{\tau}\psi(\vec{r})]^m$$
$$\cdot\langle\vec{r},m|(\nabla_j\mathcal{D}_j)^{-1}(-\nabla^2)(\nabla_{j'}\mathcal{D}_{j'})^{-1}|\vec{r}',m'\rangle\mathcal{G}\cdot[\vec{A}_k(\vec{r}')\times\vec{\Pi}_k(\vec{r}')+\tfrac{1}{2}\psi^\dagger(\vec{r}')\vec{\tau}\psi(\vec{r}')]^{m'}\},\quad(4.65)$$

where $T^l=\frac{1}{2}\tau^l$, all SU_2 vectors are indicated by arrows, and P_i^l is replaced by the more familiar notation $(\vec{\Pi}_i^T)^l$. Both $\vec{A}_i$ and $\vec{\Pi}_i^T$ are transverse. Note that the matrix $\langle\vec{r},m|(\Gamma_k\mathcal{D}_k)^{-1}(\Gamma_j\Gamma_j^\dagger)(\mathcal{D}_k^\dagger\Gamma_k^\dagger)^{-1}|\vec{r}',m'\rangle$ has reduced to precisely the usual non-Abelian, Coulomb, Green's function.

V. COVARIANT GAUGES

We will now show that the Cartesian operator ordering of the $V_0=0$ gauge Hamiltonian (3.29)–(3.31) defines a quantum theory identical to the theory determined by the usual covariant-gauge Feynman rules. The transformation of the $V_0=0$ gauge quantum theory of Sec. III into a covariant gauge is most easily done using the Feynman path-integral formalism.[6]

For the case of a single coordinate q and conjugate momentum p the path integral for the Schrödinger Green's function applied to a state $|\ \rangle$ is obtained by writing

$$\langle q'|e^{-iH(t'-t)}|\ \rangle=\lim_{\mathfrak{N}\to\infty}\prod_{n=1}^{\mathfrak{N}}\int dq(n)\langle q(n+1)|[1-i\epsilon H(p,q)]|q(n)\rangle\langle q(1)|\ \rangle,\quad(5.1)$$

where $q(n)$ denotes the coordinate q at time $t_n=t+(n-1)\epsilon$ with $\epsilon=(t'-t)/\mathfrak{N}$ and $q(\mathfrak{N}+1)=q'$. Each matrix element in the product on the right-hand side of (5.1) is represented by

$$\langle q(n+1)|1-i\epsilon H(p,q)|q(n)\rangle=\int\frac{dp(n)}{2\pi}e^{ip(n)[q(n+1)-q(n)]}\left[1-i\epsilon H\left(p(n),\frac{q(n)+q(n+1)}{2}\right)\right],\quad(5.2)$$

so that (5.1) becomes

$$\langle q'|e^{-iH(t'-t)}|\ \rangle=\lim_{\mathfrak{N}\to\infty}\prod_{n=1}^{\mathfrak{N}}\int\frac{dq(n)dp(n)}{2\pi}\exp\left\{i\epsilon\left[\frac{p(n)[q(n+1)-q(n)]}{\epsilon}-H\left(p(n),\frac{q(n+1)+q(n)}{2}\right)\right]\right\}\langle q(1)|\ \rangle,\quad(5.3)$$

the familiar Hamiltonian path integral for a one-dimensional problem. Our substitution of the variable $\frac{1}{2}[q(n+1)+q(n)]$ for the operator q in $H(p,q)$ of (5.1) corresponds to the Weyl-ordered form of the Hamiltonian. As can be readily verified, for a classical Hamiltonian $p^2f(q)+pu(q)+v(q)$, the Weyl form of the quantum-mechanical Hamiltonian is

$$H=\tfrac{1}{4}[p^2f(q)+2pf(q)p+f(q)p^2]+\tfrac{1}{2}[pu(q)+u(q)p]+v(q),\quad(5.4)$$

where q and p are operators. By substituting this expression into the left-hand side of (5.2), one sees that it leads precisely to the right-hand side.

Equation (5.3) can be immediately generalized to represent the $V_0=0$ gauge Schrödinger Green's function by a path integral for the gauge theory without fermions:

$$\langle V'|e^{-iH(t'-t)}|\rangle=\lim_{\mathfrak{N}\to\infty}\int\prod_{n=1}^{\mathfrak{N}}d[V(n)]d[\Pi(n)]$$
$$\times\left\{\exp\left[\sum_{n=1}^{\mathfrak{N}}i\epsilon\int d^3r\,\mathrm{tr}[\Pi_i(n)[V_i(n+1)-V_i(n)]\epsilon^{-1}\right.\right.$$
$$\left.\left.-\tfrac{1}{2}\Pi_i(n)\Pi_i(n)-\tfrac{1}{2}B_i(n)B_i(n)]\right]\right\}\Psi[V_i(1)],\quad(5.5)$$

where $\Psi[V_i]$ is introduced by (3.28), $V_i(n)$ and $\Pi_i(n)$ refer to the matrix form (3.17) of the fields $V_i(\vec{r},t_n)$ and $\Pi_i(\vec{r},t_n)$, with $t_n=t+(n-1)\epsilon$. For simplicity, the position dependence is suppressed, $V_i(\mathfrak{N}+1)=V_i'$, and the differentials $d[V(n)]$ and $d[\Pi(n)]$ stand, respectively, for the products

$\prod_{i,l,\vec{r}} dV_i^l(\vec{r},t_n)$ and $\prod_{i,l,\vec{r}} d\Pi_i^l(\vec{r},t_n)$.

To include fermions in the path-integral description, it is necessary to introduce a representation of the fermion Hilbert space as a space of polynomials of generators $\hat{q}_1, \hat{q}_2, \ldots$ of a Grassmann algebra[7]

$$\{\hat{q}_\alpha, \hat{q}_\beta\} = 0$$

for all α and $\beta = 1, 2, \ldots$. Their differentials $d\hat{q}_\alpha$ and derivative operators $\partial/\partial\hat{q}_\alpha$ satisfy

$$\left\{\frac{\partial}{\partial\hat{q}_\alpha}, \hat{q}_\beta\right\} = \frac{\partial\hat{q}_\beta}{\partial\hat{q}_\alpha} = \delta_{\alpha\beta} \tag{5.6}$$

and

$$\{d\hat{q}_\alpha, \hat{q}_\beta\} = \{d\hat{q}_\alpha, d\hat{q}_\beta\} = 0 . \tag{5.7}$$

In addition, there are the usual integration rules

$$\int d\hat{q}_\alpha = 0 \text{ and } \int \hat{q}_\alpha d\hat{q}_\beta = \delta_{\alpha\beta} . \tag{5.8}$$

Let

$$\{\psi_\alpha(\vec{r})\} \tag{5.9}$$

be a complete orthonormal set of c-number single-particle spinor functions. By introducing for each $\psi_\alpha(\vec{r})$ a Grassmann generator $\hat{q}_\alpha$, we can represent a multiparticle state vector $|\ \rangle$ with the probability amplitude $C_r(\alpha_1, \ldots, \alpha_r)$ for the states $\psi_{\alpha_1}, \ldots, \psi_{\alpha_r}$ to be occupied by the polynomial

$$\langle q|\rangle \equiv \sum_{r=0}^{\infty} \sum_{\alpha_1<\alpha_2<\cdots<\alpha_r} C_r(\alpha_1, \ldots, \alpha_r) \times \prod_{\beta\neq\alpha_i} \hat{q}_\beta \tag{5.10}$$

in which, for definiteness, the product of the $\hat{q}_\beta$'s is arranged in the order of increasing β. Equation (5.10) is the $\hat{q}$ representation of the bra vector $|\ \rangle$; its ket vector $\langle\ |$ in the $\hat{q}$ representation then assumes the form

$$\langle\ |\hat{q}\rangle \equiv \sum_{r=0}^{\infty} \sum_{\alpha_1<\alpha_2<\cdots<\alpha_r} \pm C_r^*(\alpha_1, \ldots, \alpha_r)\hat{q}_{\alpha_1}\hat{q}_{\alpha_2}\cdots\hat{q}_{\alpha_r} , \tag{5.11}$$

where the $\pm$ sign is determined by the normalization condition

$$\langle\ |\ \rangle = \int \langle\ |q\rangle \prod_\alpha d\hat{q}_\alpha \langle q|\ \rangle = \sum_{r=0}^{\infty} \sum_{\alpha_1<\alpha_2<\cdots<\alpha_r} |C_r(\alpha_1, \ldots, \alpha_r)|^2 . \tag{5.12}$$

With the above definition, a multiplication $\hat{q}_\alpha$ onto $\langle\hat{q}|\ \rangle$ becomes the annihilation operator for the αth state, while the differentiation $\partial/\partial\hat{q}_\alpha$ is the corresponding creation operator.

The "Fourier transform" of any polynomial $\psi(\hat{q})$ is given by

$$\phi(\overline{q}) \equiv \int \exp\left(\sum_\alpha \overline{q}_\alpha \hat{q}_\alpha\right)\psi(\hat{q})\prod_\alpha d\hat{q}_\alpha , \tag{5.13}$$

where the $\overline{q}_\alpha$'s anticommute with each other as well as with the $\hat{q}_\alpha$'s. By using (5.6) and (5.8), we see that

$$\overline{q}_\beta \phi(\overline{q}) = \int \exp\left(\sum_\alpha \overline{q}_\alpha\hat{q}_\alpha\right)\frac{\partial\psi(\hat{q})}{\partial\hat{q}_\beta}\prod_\alpha d\hat{q}_\alpha , \tag{5.14}$$

and therefore the usual partial-integration rule holds. The "δ function" is defined by

$$\begin{aligned}\langle\hat{q}|\hat{q}'\rangle &\equiv \delta(\hat{q}-\hat{q}') \\ &\equiv \int \prod_\alpha d\overline{q}_\alpha \exp\sum_\alpha \overline{q}_\alpha(\hat{q}_\alpha - \hat{q}'_\alpha) \\ &= \prod_\alpha(\hat{q}'_\alpha - \hat{q}_\alpha)\end{aligned} \tag{5.15}$$

since, for arbitrary $\psi(\hat{q})$,

$$\psi(\hat{q}) = \int \delta(\hat{q}-\hat{q}')\tilde{\prod_\alpha} d\hat{q}'_\alpha \psi(\hat{q}') , \tag{5.16}$$

where the order of product $\tilde{\prod}_\alpha$ is the transpose of that in $\prod_\alpha$; i.e., if in (5.15) $\prod_\alpha d\overline{q}_\alpha = d\overline{q}_1 d\overline{q}_2\cdots$, then $\delta(\hat{q}-\hat{q}') = (\hat{q}'_1 - \hat{q}_1)(\hat{q}'_2 - \hat{q}_2)\cdots$ in the same order, but in (5.16) $\tilde{\prod}_\alpha d\hat{q}'_\alpha = \cdots d\hat{q}'_2 d\hat{q}'_1$ in the transposed order. Hence, the inverse transform of (5.13) is

$$\psi(\hat{q}) = \int \exp\left(\sum_\alpha \hat{q}_\alpha\overline{q}_\alpha\right)\phi(\overline{q})\tilde{\prod_\alpha} d\overline{q}_\alpha . \tag{5.17}$$

As an illustration, we first examine the case of a single mode with only one generator $\hat{q}$. The corresponding basis vectors of the Hilbert space can be chosen to be $|l\rangle$, where l = occupation number = 0 or 1. In accordance with (5.10)–(5.12), we may write

$$\begin{aligned}&\langle\hat{q}|0\rangle = \hat{q}, \quad \langle\hat{q}|1\rangle = 1 , \\ &\langle 0|\hat{q}\rangle = -1, \quad \langle 1|\hat{q}\rangle = \hat{q}.\end{aligned} \tag{5.18}$$

Therefore, we have the usual orthonormality relation

$$\langle l|l'\rangle = \int \langle l|\hat{q}\rangle d\hat{q}\langle\hat{q}|l'\rangle = \delta_{ll'}$$

and the completeness condition

$$\sum_{l=0,1} \langle\hat{q}'|l\rangle\langle l|\hat{q}\rangle = -\hat{q}' + \hat{q} = \delta(\hat{q}'-\hat{q}) . \tag{5.19}$$

If a Hamiltonian H has $|\,l\rangle$ as its eigenvector with eigenvalue $E_l = lE$, then the matrix representation of e^{-iHt} is

$$\langle \hat{q}'|\, e^{-iHt}\,|\,\hat{q}\rangle = \sum_{l=0,1} \langle \hat{q}'|\, l\rangle\, e^{-iE_l t}\langle l\,|\,\hat{q}\rangle = -\hat{q}' + e^{-iEt}\hat{q}\,. \tag{5.20}$$

Returning to the general case, with the help of these equations we can easily write down the fermionic analog of (5.2):

$$\langle \hat{q}(n+1)|\, 1 - i\epsilon H\,|\,\hat{q}(n)\rangle = \int \prod_\alpha d\bar{q}_\alpha(n) \exp\Big[\sum_\beta \bar{q}_\beta(n)(\hat{q}_\beta(n+1) - \hat{q}_\beta(n))\Big] \times \Big[1 + i\epsilon \sum_{\gamma,\gamma'} \bar{q}_\gamma(n) H_{\gamma\gamma'}\hat{q}_{\gamma'}(n)\Big]\,, \tag{5.21}$$

where in our case, because of (3.1),

$$H_{\gamma\gamma'} = -i\int d^3r\, \psi_\gamma^\dagger(\vec{r})\alpha_j D_j \psi_{\gamma'}(\vec{r}) \tag{5.22}$$

with $\psi_\gamma(\vec{r})$ as a member of the c-number spinor set (5.9). Thus, if we define the anticommuting functions

$$\psi(n) \equiv \psi(\vec{r}, t_n) \equiv \sum_\alpha \psi_\alpha(\vec{r})\hat{q}_\alpha(n)\,,$$
$$\bar{\psi}(n) \equiv \bar{\psi}(\vec{r}, t_n) \equiv -\sum_\alpha \bar{\psi}_\alpha(\vec{r})\bar{q}_\alpha(n)\,, \tag{5.23}$$

where $\bar{\psi}_\alpha = \psi_\alpha^\dagger\gamma_4$, $t_n = t + (n-1)\epsilon$ as before, and write $\int\prod_\alpha dq_\alpha(n)$ and $\int\prod_\alpha d\bar{q}_\alpha(n)$ as $\int d[\psi(n)]$ and $\int d[\bar{\psi}(n)]$, respectively, the fermionic Green's function becomes

$$\langle \psi'|e^{-iH(t'-t)}|\,\rangle = \lim_{\mathfrak{N}\to\infty}\int \prod_{n=1}^{\mathfrak{N}} d[\bar{\psi}(n)]\,d[\psi(n)] \times \Big\{\exp\Big[\sum_{n=1}^{\mathfrak{N}}\int d^3r\,[-\bar{\psi}(n)\gamma_4(\psi(n+1) - \psi(n)) - i\epsilon\bar{\psi}(n)\gamma_j D_j\psi(n)]\Big]\Big\}\Psi[\psi(1)]\,, \tag{5.24}$$

where $\Psi[\psi(1)] = \langle \psi(1)|\,\rangle$. Equations (5.5) and (5.24) can be combined to give a path-integral expression for the Yang-Mills and fermion Green's function. By using the Cartesian form $\Pi_i(n)\,\Pi_i(n)$ for the kinetic energy in (5.5), with no additional terms we have explicitly incorporated the operator ordering of the $V_0 = 0$ gauge Hamiltonian (3.29)–(3.31).

We can now transform the $V_0 = 0$ gauge quantum theory defined by (5.5) and (5.24) into a noncovariant gauge, $\chi(A_i) = 0$, by a simple change of variables. Of course, we must be careful when transforming $V(n+1) - V(n)$ and $\psi(n+1) - \psi(n)$ to keep all terms in the exponent through order ϵ if we intend the transformed theory to be equivalent to the original one. Such a procedure is completely equivalent to the operator manipulations of Sec. IV and leads to the same result.[8]

In this section we wish to use the path-integral formulation to transform to a covariant gauge. This can be done most economically if we introduce into the integrand of (5.5) the factor

$$1 = \int \prod_{n=1}^{\mathfrak{N}} d[\phi_a(n)]\,\mathfrak{J}\left(\exp\left\{\sum_{n=1}^{\mathfrak{N}}\int d^3r\,\frac{-i\epsilon}{2\alpha}\,\mathrm{tr}\Big[\nabla_i A_i(\vec{r}, t_n) + \frac{1}{\epsilon}(A_0(\vec{r}, t_{n+1}) - A_0(\vec{r}, t_n))\Big]^2\right\}\right), \tag{5.25}$$

where $A_i(n) = A_i(\vec{r}, t_n)$ and $A_0(n) = A_0(\vec{r}, t_n)$ are Hermitian matrices, defined by

$$A_i(n) = u(n)^{-1}V_i(n)u(n) + \frac{i}{g}u(n)^{-1}\nabla_i u(n) \tag{5.26}$$

and

$$A_0(n) = -\frac{i}{g\epsilon}\ln[u(n)^{-1}u(n+1)] = -\frac{i}{g\epsilon}u(n)^{-1}(u(n+1) - u(n)) + \frac{i}{2g\epsilon}[u(n)^{-1}(u(n+1) - u(n))]^2 + \cdots\,, \tag{5.27}$$

with

$$u(n) = u(\phi_a(n)) \tag{5.28}$$

the $N\times N$ unitary matrix function of the M group parameters $\phi_a(n)=\phi_a(\vec{r},t_n)$ in (4.5), and the differential $d[\phi_a(n)]=\Pi_{a,\vec{r}}\, d\phi_a(\vec{r},t_n)$. Except for a numerical factor, the Jacobian $|\mathfrak{J}|$ is given by

$$|\mathfrak{J}|=\det\langle\vec{r},t_n|\mathfrak{J}|\,\vec{r}',t_n\rangle$$
$$=\det\frac{\delta}{\delta\phi_a(n)}\left[\nabla_i A_i(n')+\frac{(A_0(n'+1)-A_0(n'))}{\epsilon}\right]. \quad (5.29)$$

In order to match the final configuration of (5.5), we set $A_i(\mathfrak{N}+1)=V_i(\mathfrak{N}+1)=V_i'$, i.e.,

$$u(\mathfrak{N}+1)=1. \quad (5.30)$$

Furthermore, since the integrand of (5.25) is invariant under $A_0(n)\to A_0(n)+\text{constant}$ for all $n=1,2,\ldots,\mathfrak{N}+1$, we may choose $A_0(\mathfrak{N}+1)=0$.

We now examine the region where the summand in the exponent of (5.25) is of order 1. Because in the summand the coefficient of $[A_0(n+1)-A_0(n)]^2$ is proportional to ϵ^{-1}, we expect only those configurations with $A_0(n+1)-A_0(n)$ of order $\epsilon^{1/2}$ to contribute to the integral over $\phi_a(n)$. Thus since $A_0(\mathfrak{N}+1)=0$, $A_0(n)$ should be of order $(\mathfrak{N}-n)^{1/2}\epsilon^{1/2}=O(1)$ and, from (5.27), $u(n+1)-u(n)$ is only of order ϵ.

If we perform $\int d[\Pi(n)]$ in (5.5) we find, up to a numerical factor,

$$\int d[\Pi(n)]\exp\left(i\epsilon\sum_n\int d^3r\,\mathrm{tr}\{\Pi_i(n)[V_i(n+1)-V_i(n)]\epsilon^{-1}-\tfrac{1}{2}\Pi_i(n)\Pi_i(n)\}\right)$$
$$=\exp\left(i\tfrac{1}{2}\sum_n\int d^3r\,\mathrm{tr}[V_i(n+1)-V_i(n)]^2\epsilon^{-1}\right). \quad (5.31)$$

Next we can use (5.26) to express this exponent in terms of A_i and u. A particularly symmetrical form is obtained if the quantity in square brackets in (5.31) is conjugated with the matrix $[u(n)^{-1}u(n+1)]^{1/2}u(n+1)^{-1}$:

$$\begin{aligned}\epsilon^{-1}\mathrm{tr}[V_i(n+1)-V_i(n)]^2=\epsilon^{-1}\mathrm{tr}\Big\{&[u(n)^{-1}u(n+1)]^{1/2}A_i(n+1)[u(n)^{-1}u(n+1)]^{-1/2}\\&+\frac{i}{g}[u(n)^{-1}u(n+1)]^{1/2}\nabla_i[u(n)^{-1}u(n+1)]^{-1/2}\\&-[u(n)^{-1}u(n+1)]^{-1/2}A_i(n)[u(n)^{-1}u(n+1)]^{1/2}\\&-\frac{i}{g}[u(n)^{-1}u(n+1)]^{-1/2}\nabla_i[u(n)^{-1}u(n+1)]^{1/2}\Big\}^2.\end{aligned} \quad (5.32)$$

Using (5.27) to replace $u(n)^{-1}u(n+1)$ by a function of $A_0(n)$,

$$u(n)^{-1}u(n+1)=1+ig\epsilon A_0(n)+O(\epsilon^2),$$

we can expand[9] the right-hand side of Eq. (5.32) through order ϵ:

$$\epsilon^{-1}\mathrm{tr}[V_i(n+1)-V_i(n)]^2=\epsilon\,\mathrm{tr}\{[A_i(n+1)-A_i(n)]\epsilon^{-1}+ig[A_0(n),A_i(n)]+\nabla_iA_0(n)\}^2+0(\epsilon^{3/2}) \quad (5.33)$$

where we treat $A_i(n+1)-A_i(n)$ as of order $\epsilon^{1/2}$. Thus, if we change integration variables from $V_i(n)$, $\psi(n)$, and $\overline{\psi}(n)$ to the unitarily equivalent set $A_i(n)$,

$$\psi'(n)=u(n)^{-1}\psi(n)$$
and (5.34)
$$\overline{\psi}'(n)=\overline{\psi}(n)u(n),$$

the exponentials in (5.5), (5.24), and (5.25) combine to give the usual covariant-gauge action up to terms vanishing with ϵ:

$$\begin{aligned}&i\epsilon\sum_n\int d^3r\,\mathrm{tr}\Big\{\tfrac{1}{2}[(A_i(n+1)-A_i(n))\epsilon^{-1}+ig[A_0(n),A_i(n)]+\nabla_iA_0(n)]^2\\&\qquad-\tfrac{1}{2}[B_i(n)]^2-\frac{1}{2a}[\nabla_iA_i(n)+(A_0(n+1)-A_0(n))\epsilon^{-1}]^2\Big\},\\&-i\epsilon\sum_n\int d^3r\Big\{\overline{\psi}'(n)\Big[\frac{\gamma_4(\psi'(n+1)-\psi'(n))}{i\epsilon}+gA_0(n)\gamma_4\psi'(n+1)+\gamma_i(\nabla_i-igA_i(n))\psi'(n)\Big]\Big\}.\end{aligned} \quad (5.35)$$

Finally, we must use Eq. (5.27) to change the integration variable $\phi_a(n)$ to $A_0^m(n)$ where $T^m A_0^m(n) = A_0(n)$. The resulting Jacobian $\det[\delta\phi_{a'}(n')/\delta A_0^m(n)]|\mathcal{G}|$ can then be worked out explicitly. By using (5.27), we see that the matrix $\delta A_0^m(n)/\delta\phi_a(n')$ can be resolved into two terms:

$$\frac{\delta A_0^m(n)}{\delta\phi_a(n')}=\frac{1}{g\epsilon}(\delta_{nn'}\langle a|G(n)|m\rangle + \delta_{n+1n'}\langle a|G'(n)|m\rangle), \tag{5.36}$$

where, on account of (4.22),

$$\langle a|G(n)|m\rangle = \lambda_a^m(n) - \tfrac{1}{2}g\epsilon f^{klm}\lambda_a^k(n)A_0^l(n) + O(\epsilon^2). \tag{5.37}$$

Because of the boundary condition (5.30), $A_0^m(\mathfrak{N})$ depends only on $u(\mathfrak{N})$. Hence, we have

$$\langle a|G'(\mathfrak{N})|m\rangle = 0,$$

and therefore

$$\det\frac{\delta A_0^m(n)}{\delta\phi_a(n')} = \prod_{n=1}^{\mathfrak{N}} \det\frac{\langle a|G(n)|m\rangle}{g\epsilon}. \tag{5.38}$$

From (5.37), it follows that

$$\begin{aligned}\det\langle a|G(n)|m\rangle &= [1-\tfrac{1}{2}g\epsilon f^{klm}\lambda_a^k(n)A_0^l(n)(\lambda^{-1}(n))_a^m] \\ &\quad\times \det\lambda(n) + O(\epsilon^2) \\ &= \det\lambda(n) + O(\epsilon^2),\end{aligned} \tag{5.39}$$

where $\lambda(n)$ stands for the $M\times M$ matrix $\lambda_a^m(n)$. Since $\mathfrak{N}=O(\epsilon^{-1})$, when $\epsilon\to 0$ we can drop the $O(\epsilon^2)$ term in (5.39) and derive

$$\det\frac{\delta\phi_a(n')}{\delta A_0^m(n)} = \text{const}\times\left[\prod_{n=1}^{\mathfrak{N}}\det\lambda(n)\right]^{-1}. \tag{5.40}$$

By following steps very similar to those that led to (4.36), we may express $|\mathcal{G}|$ of (5.29) as

$$|\mathcal{G}| = \text{const}\times\mathcal{G}_c\times\prod_n \det\lambda(n), \tag{5.41}$$

which when multiplied by (5.40) gives, in the limit $\epsilon\to 0$, a constant times the usual, covariant-gauge Faddeev-Popov determinant

$$\mathcal{G}_c = \det\{\partial_\mu\mathfrak{D}_\mu\}. \tag{5.42}$$

Consequently, apart from a constant multiplicative factor, the original $V_0=0$ gauge Schrödinger Green's function has been transformed precisely into the usual covariant-gauge expression

$$\langle V'|e^{-iH(t'-t)}|\rangle = \int d[A_\mu]d[\psi']d[\bar\psi']\mathcal{G}_c \exp\left\{i\epsilon\int d^4x\,\mathrm{tr}\left[-\tfrac{1}{4}F_{\mu\nu}{}^2 - \bar\psi'\gamma_\mu\mathfrak{D}_\mu\psi' - \frac{1}{2\alpha}(\partial_\mu A_\mu)^2\right]\right\}, \tag{5.43}$$

where

$$d[A_\mu]d[\psi']d[\bar\psi'] = \lim_{n\to\infty}\prod_{n=1}^{\mathfrak{N}} d[A_\mu(n)]d[\psi'(n)]d[\bar\psi'(n)].$$

VI. NONCOVARIANT-GAUGE FEYNMAN RULES

The quantum Yang-Mills theory in the gauge $\chi(A_i)=0$ is completely specified by the Hamiltonian operator (4.62) derived in Sec. IV. However, if this Hamiltonian is divided into free and interacting pieces,

$$H_{\text{free}} + H_{\text{int}}, \tag{6.1}$$

the resulting Dyson-Wick perturbation theory in H_{int} will be complicated because of the quadratic dependence of H_{int} on the canonical momenta P_i^l. In the interaction representation the P_i^l become $\dot A_i^l$, the Wick contraction of $\dot A_i^l(x)$ and $\dot A_j^l(y)$ contains noncovariant $\delta(x_0-y_0)$ terms, these δ-function terms can be systematically resummed and the result represented as new, nonpolynomial additions[10] to H_{int}. The vertices implied by this modified H'_{int}, when joined with "naive" Feynman propagators, then provide the correct Feynman perturbation series for the original quantum theory.

In this section we will derive these simplified Feynman rules for the noncovariant-gauge Hamiltonian (4.62). In practice, these Feynman rules are most easily obtained using functional integration.[6] We first represent the Schrödinger Green's function by an integral over classical trajectories in phase space as in (5.1). Instead of immediately evaluating the Gaussian integral over P_i^l, we next replace the term in H_{int} quadratic in P_i^l by a Gaussian integral with respect to a new variable A_0^l coupling linearly with P_i^l. Finally, the integration over P_i^l is carried out leaving a Lagrangian functional integral over A_i^l and A_0^l which, if the original Hamiltonian was Weyl ordered, directly specifies the proper Feynman rules.

The first step in this procedure is the most difficult. We must rearrange the operators in the Hamiltonian (4.62) into Weyl order. The Weyl ordering of the Laplacian (4.7) in curvilinear coordinates,

$$\mathcal{K} = -\tfrac{1}{2}\sum_{\alpha,\beta}|M|^{-1/2}\frac{\partial}{\partial Q_\alpha}M^{-1}{}_{\alpha\beta}|M|^{1/2}\frac{\partial}{\partial Q_\beta}, \tag{4.7}$$

is quite straightforward. The result is particularly simple if we extract a factor of $|M|^{-1/4}$ from the state vectors:

$$|M|^{1/4}\mathcal{K}|M|^{-1/4}=-\frac{1}{2}\left\{\frac{1}{4}\frac{\partial}{\partial Q_\alpha}\frac{\partial}{\partial Q_\beta}M^{-1}{}_{\alpha\beta}+\frac{1}{2}\frac{\partial}{\partial Q_\alpha}M^{-1}{}_{\alpha\beta}\frac{\partial}{\partial Q_\beta}+\frac{1}{4}M^{-1}{}_{\alpha\beta}\frac{\partial}{\partial Q_\alpha}\frac{\partial}{\partial Q_\beta}\right\}+\frac{1}{8}\left(\frac{\partial}{\partial Q_\alpha}\frac{\partial Q_\beta}{\partial q_\gamma}\right)\left(\frac{\partial}{\partial Q_\beta}\frac{\partial Q_\alpha}{\partial q_\gamma}\right), \quad (6.2)$$

in which the differentiations appearing inside the curly brackets are arranged in Weyl order, as in (5.4).

However, complications arise because what we are interested in is the Weyl ordering of the Hamiltonian (4.62) which, having been simplified by restriction to gauge-invariant states, (4.57), no longer has the form (4.7). In order to apply this simple formula to the case at hand, we must return to the form (4.38) for the gauge-theory Hamiltonian in which the angle variables ϕ_a and their conjugate momenta p_a appear. Equation (6.2) then gives a Weyl-ordered form for H, from which ϕ_a and p_a must again be eliminated as follows: The terms in curly brackets in (6.2) which contain derivatives with respect to ϕ_a are

$$\begin{aligned}-\tfrac{1}{2}\int d^3r\int d^3r'\Big\{&\langle\vec{r},l\,|(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j\Gamma_j^\dagger(\mathfrak{D}_{k'}^\dagger\Gamma_{k'}^\dagger)^{-1}|\,\vec{r},l'\rangle g^2\\&\times\left[\frac{1}{4}\frac{\delta}{\delta\phi_a(\vec{r})}\frac{\delta}{\delta\phi_{a'}(\vec{r}\,')}\lambda^{-1}(\vec{r})_a^l\lambda^{-1}(\vec{r}\,')_{a'}^{l'}+\frac{1}{2}\frac{\delta}{\delta\phi_a(\vec{r})}\lambda^{-1}(\vec{r})_a^l\lambda^{-1}(\vec{r}\,')_{a'}^{l'}\frac{\delta}{\delta\phi_{a'}(\vec{r}\,')}\right.\\&\left.+\tfrac{1}{4}\lambda^{-1}(\vec{r})_a^l\lambda^{-1}(\vec{r}\,')_{a'}^{l'}\frac{\delta}{\delta\phi_a(\vec{r})}\frac{\delta}{\delta\phi_{a'}(\vec{r}\,')}\right]\\&+ig[P_i^l(\vec{r})\langle\vec{r},l\,|\mathfrak{D}_i(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j\Gamma_j^\dagger(\mathfrak{D}_{k'}^\dagger\Gamma_{k'}^\dagger)^{-1}|\,\vec{r}\,',l'\rangle+\langle\vec{r},l\,|\mathfrak{D}_i(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j\Gamma_j^\dagger(\mathfrak{D}_{k'}^\dagger\Gamma_{k'}^\dagger)^{-1}|\,\vec{r}\,',l'\rangle P_i^l(\vec{r})]\\&\times\left[\frac{1}{2}\frac{\delta}{\delta\phi_{a'}(\vec{r}\,')}\lambda^{-1}(\vec{r}\,')_{a'}^{l'}+\tfrac{1}{2}\lambda^{-1}(\vec{r}\,')_{a'}^{l'}\frac{\delta}{\delta\phi_{a'}(\vec{r}\,')}\right]\Big\}. \end{aligned}\quad (6.3)$$

If this operator is applied to the state

$$|\lambda|^{1/2}\mathfrak{U}(\phi_a)\tilde{\Psi}(A_i), \quad (6.4)$$

the result is

$$\begin{aligned}-\tfrac{1}{2}|\lambda|^{1/2}\mathfrak{U}(\phi_a)\int d^3r\int d^3r'\Big\{&\langle\vec{r},l\,|(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j\Gamma_j^\dagger(\mathfrak{D}_{k'}^\dagger\Gamma_{k'}^\dagger)^{-1}|\,\vec{r}\,',l\rangle g^2\\&\times\left[-\psi^\dagger(\vec{r}\,')T^{l'}\psi(\vec{r}\,')\psi^\dagger(\vec{r})T^l\psi(\vec{r})+\frac{1}{4}\left(\frac{\delta}{\delta\phi_a(\vec{r})}\lambda^{-1}(\vec{r}\,')_{a'}^{l'}\right)\left(\frac{\delta}{\delta\phi_{a'}(\vec{r}\,')}\lambda^{-1}(\vec{r})_a^l\right)\right]\\&+g[P_i^l(\vec{r})\langle\vec{r},l\,|\mathfrak{D}_i(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j\Gamma_j^\dagger(\mathfrak{D}_{k'}^\dagger\Gamma_{k'}^\dagger)^{-1}|\,\vec{r}\,',l\rangle\\&+\langle\vec{r},l\,|\mathfrak{D}_i(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j\Gamma_j^\dagger(\mathfrak{D}_{k'}^\dagger\Gamma_{k'}^\dagger)^{-1}|\,\vec{r}\,',l'\rangle P_i^l(\vec{r}\,')]\psi^\dagger(\vec{r}\,')T^{l'}\psi(\vec{r}\,')\Big\}\tilde{\Psi}(A_i). \end{aligned}\quad (6.5)$$

In obtaining this expression we have made repeated use of the identities (4.48) and (4.60). The state (6.4) above is simply (4.57) multiplied by $|\lambda|^{1/2}$, the ϕ_a-dependent part of the factor $|M|^{1/4}$ introduced in (6.2).

Next let us examine the remainder term in (6.2):

$$\begin{aligned}\frac{1}{8}&\left(\frac{\partial}{\partial Q_\alpha}\frac{\partial Q_\beta}{\partial q_\gamma}\right)\left(\frac{\partial}{\partial Q_\beta}\frac{\partial Q_\alpha}{\partial q_\gamma}\right)\\&=\tfrac{1}{8}\int d^3r\int d^3r'\int d^3r''\Big\{-[P_i^l(\vec{r}),\langle\vec{r}\,',l'|\mathfrak{D}_{i'}(\Gamma_{k'}\mathfrak{D}_{k'})^{-1}\Gamma_j\,|\,\vec{r}\,'',l''\rangle][P_{i'}^{l'}(\vec{r}\,'),\langle\vec{r},l\,|\mathfrak{D}_i(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j\,|\,\vec{r}\,'',l''\rangle]\\&\qquad-2ig[P_i^l(\vec{r}),\lambda^{-1}(\vec{r}\,')_a^{l'}\langle\vec{r}\,',l'\,|(\Gamma_{k'}\mathfrak{D}_{k'})^{-1}\Gamma_j\,|\,\vec{r}\,'',n'\rangle U^{l''n'}(\vec{r}\,'')]\\&\qquad\times\left(\frac{\delta}{\delta\phi_{a'}(\vec{r}\,')}\langle\vec{r},l\,|\,\delta_{ij}-\mathfrak{D}_i(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j\,|\,\vec{r}\,'',n\rangle U^{l''n}(\vec{r}\,'')\right)\\&\qquad+g^2\left(\frac{\delta}{\delta\phi_a(\vec{r})}\lambda^{-1}(\vec{r}\,')_a^{l'}\langle\vec{r}\,',l'\,|(\Gamma_{k'}\mathfrak{D}_{k'})^{-1}\Gamma_j\,|\,\vec{r}\,'',n'\rangle U^{l''n'}(\vec{r}\,'')\right)\\&\qquad\times\left(\frac{\delta}{\delta\phi_{a'}(\vec{r}\,')}\lambda^{-1}(\vec{r})_a^l\langle\vec{r},l\,|(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j\,|\,\vec{r}\,'',n\rangle U^{l''n}(\vec{r}\,'')\right)\Big\}. \end{aligned}\quad (6.6)$$

The last term in (6.6) can be evaluated quite explicitly if we use the commutation relation (4.44) in the form

$$\lambda^{-1}(\vec{r})^l_a \frac{\delta\lambda^{-1}(\vec{r}\,')^{l'}_b}{\delta\phi_a(\vec{r})} - \lambda^{-1}(\vec{r}\,')^{l'}_{a'} \frac{\delta\lambda^{-1}(\vec{r})^l_b}{\delta\phi_{a'}(\vec{r}\,')} = f^{ll'n}\lambda^{-1}(\vec{r})^n_b\delta^3(\vec{r}-\vec{r}\,') \tag{6.7}$$

with the result

$$\tfrac{1}{8}g^2\int d^3r\int d^3r'\Big\{\langle\vec{r},l\,|(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j\Gamma_j^\dagger(\mathfrak{D}_{k'}^\dagger\Gamma_{k'}^\dagger)^{-1}|\vec{r}\,',l'\rangle\Big(\frac{\delta}{\delta\phi_a(\vec{r})}\lambda^{-1}(\vec{r}\,')^{l'}_{a'}\Big)\Big(\frac{\delta}{\delta\phi_{a'}(\vec{r}\,')}\lambda^{-1}(\vec{r})^l_a\Big)$$
$$-\langle\vec{r},l'\,|(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j t^l - t^l(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j\,|\vec{r},n\rangle\langle\vec{r},l\,|(\Gamma_{k'}\mathfrak{D}_{k'})^{-1}\Gamma_j t^{l'}\,|\vec{r},n\rangle\delta^3(\vec{r}-\vec{r}\,')\Big\}, \tag{6.8}$$

where $(t^l)_{mn} = -if^{lmn}$ is the adjoint representation analog of the generator T^l. The ϕ-dependent term in this result cancels the ϕ dependence of (6.5) so the combination of (6.5) and (6.8) will not depend on ϕ.

The second term in (6.6), linear in $P_i^l(\vec{r})$, is also not hard to simplify. Because the vector $|\bar{\xi}\rangle$ defined by

$$\langle i,\vec{r},l\,|\bar{\xi}\rangle \equiv \langle\vec{r},l\,|[\delta_{ij} - \mathfrak{D}_i(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j)]\,|\vec{r}\,'',n\rangle \tag{6.9}$$

is automatically orthogonal to $|\xi\rangle$,

$$\langle i,\vec{r},l\,|\xi\rangle = \langle\vec{r},l\,|\Gamma_i^\dagger|\,\xi\rangle$$

given by (4.16), we can replace

$$P_i^l(\vec{r}) = \sum_N f_i^l(\vec{r})_N\Big(-i\frac{\partial}{\partial Q_N}\Big)$$

by $-i\delta/\delta A_i^l(\vec{r})$ so that the second term becomes

$$\tfrac{1}{4}g^2\int d^3r\int d^3r'\langle\vec{r}\,',l'\,|(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_i t^l\,|\vec{r},m\rangle\langle\vec{r},m\,|(\Gamma_{k'}\mathfrak{D}_{k'})^{-1}\Gamma_j\,|\vec{r}\,',n\rangle\langle\vec{r},l\,|(\delta_{ij} - \mathfrak{D}_i(\Gamma_{k''}\mathfrak{D}_{k''})^{-1}\Gamma_j)t^{l'}\,|\vec{r}\,',n\rangle. \tag{6.10}$$

This result can be combined with the second, ϕ-independent term in (6.8) to yield

$$\mathcal{V}_1(A) = \tfrac{1}{8}g^2\int d^3r\langle\vec{r},l'\,|(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j\,|\vec{r},l\rangle\langle\vec{r},m\,|(\Gamma_{k'}\mathfrak{D}_{k'})^{-1}\Gamma_j(t^{l'}t^l + t^l t^{l'})\,|\vec{r},m\rangle$$
$$+\tfrac{1}{4}g^2\int d^3r\int d^3r'\langle\vec{r}\,',l'\,|(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_i t^l\,|\vec{r},m\rangle\langle\vec{r},m\,|(\Gamma_{k'}\mathfrak{D}_{k'})^{-1}\Gamma_j t^{l'}\,|\vec{r}\,',n\rangle\langle\vec{r},l\,|\mathfrak{D}_i(\Gamma_{k''}\mathfrak{D}_{k''})^{-1}\Gamma_j\,|\vec{r}\,',n\rangle,$$

which, on account of the identity, valid for arbitrary states $|X\rangle$, $|Y\rangle$, and $|Z\rangle$,

$$\int d^3r f^{abc}\{\langle\vec{r},a\,|\mathfrak{D}_i\,|X\rangle\langle\vec{r},b\,|Y\rangle\langle\vec{r},c\,|Z\rangle + \langle\vec{r},a\,|X\rangle\langle\vec{r},b\,|\mathfrak{D}_i\,|Y\rangle\langle\vec{r},c\,|Z\rangle + \langle\vec{r},a\,|X\rangle\langle\vec{r},b\,|Y\rangle\langle\vec{r},c\,|\mathfrak{D}_i\,|Z\rangle\} = 0 \tag{6.11}$$

can be further reduced to

$$\mathcal{V}_1(A) = \tfrac{1}{8}g^2\int d^3r\langle\vec{r},l'\,|(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j\,|\vec{r},l\rangle \times\langle\vec{r},m\,|(\Gamma_{k'}\mathfrak{D}_{k'})^{-1}\Gamma_j t^{l'}t^l\,|\vec{r},m\rangle. \tag{6.12}$$

In order to simplify the first term in (6.6), we observe that

$$-i\frac{\delta}{\delta A_{i'}^{l'}(\vec{r}\,')}\langle\vec{r},l\,|\mathfrak{D}_i(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j\,|\vec{r}\,'',l''\rangle = g\langle\vec{r},l\,|[-\delta_{ii'} + \mathfrak{D}_i(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_{i'}]t^l\,|\vec{r}\,',n\rangle \times\langle\vec{r}\,',n\,|(\Gamma_{k'}\mathfrak{D}_{k'})^{-1}\Gamma_j\,|\vec{r}\,'',l''\rangle$$

and therefore

$$[P_{i'}^{l'}(\vec{r}\,'),\langle\vec{r},l\,|\mathfrak{D}_i(\Gamma_k\Gamma_k^\dagger)^{-1}\Gamma_j\,|\vec{r}\,'',l''\rangle], \tag{6.13}$$

regarded as vectors in the functional space $\{|i,\vec{r},l\rangle\}$, are both orthogonal to $\langle i,\vec{r},l\,|\xi\rangle$ given by (4.16). Since in the operator identity

$$P_i^l(\vec{r}) = -i\frac{\delta}{\delta A_i^l(\vec{r})} + i\int d^3\bar{r}\langle\vec{r},l\,|\Gamma_i^\dagger(\Gamma_{\bar{k}}\Gamma_{\bar{k}}^\dagger)^{-1}\Gamma_{\bar{i}}\,|\vec{r},\bar{l}\rangle\frac{\delta}{\delta A_{\bar{i}}^{\bar{l}}(\vec{r})}$$

the second term is parallel to $\langle i,\vec{r},l\,|\xi\rangle$ and consequently orthogonal to (6.13), we can simplify the first term in (6.6) by replacing $P_i^l(\vec{r})$ by $-i\delta/\delta A_i^l(\vec{r})$, and $P_{i'}^{l'}(\vec{r}\,')$ by $-i\delta/\delta A_{i'}^{l'}(\vec{r}\,')$. Thus, the first term in (6.6) becomes

$$\mathcal{V}_2(A) = \tfrac{1}{8}g^2\int d^3r\int d^3r'\{\langle\vec{r}\,',l'\,|\delta_{i'i} - \mathfrak{D}_{i'}(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_i\,|\vec{r},n\rangle\langle\vec{r},l\,|\delta_{ii'} - \mathfrak{D}_i(\Gamma_{k'}\mathfrak{D}_{k'})^{-1}\Gamma_{i'}\,|\vec{r}\,',n'\rangle \times\langle\vec{r},n\,|t^l(\Gamma_{\bar{k}}\mathfrak{D}_{\bar{k}})^{-1}\Gamma_j\Gamma_j^\dagger(\mathfrak{D}_{k'}^\dagger\Gamma_{k'}^\dagger)^{-1}t^{l'}\,|\vec{r}\,',n'\rangle\}. \tag{6.14}$$

The results (6.5), (6.8), (6.12), and (6.14) can now be combined, giving a Weyl-ordered form for the $\chi(A_i)=0$ gauge Hamiltonian

$$\overline{H}(P,A)\equiv \mathcal{J}^{1/2}\tilde{H}(P,a)\mathcal{J}^{-1/2}$$
$$=\frac{1}{2}\int d^3r\, P_i^l(\vec r)P_i^l(\vec r)+\mathcal{V}_1(A)+\mathcal{V}_2(A)$$
$$+\frac{1}{2}\int d^3r\int d^3r'\{[-P_i^l(\vec r)\mathfrak{D}_i+g\psi^\dagger T^l\psi](\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j\Gamma_j^\dagger(\mathfrak{D}_{k'}^\dagger\Gamma_{k'}^\dagger)^{-1}[\mathfrak{D}_{i'}P_{i'}^{l'}(\vec r\,')+g\psi^\dagger T^{l'}\psi]\}_W$$
$$+\int d^3r[\tfrac{1}{2}B_j^lB_j^l+\psi^\dagger\alpha_j(-i\nabla_j-gA_j^lT^l)\psi]\,, \tag{6.15}$$

where $\mathcal{J}$ and $\tilde{H}(P,A)$ are given by (4.37) and (4.62), respectively, and $\{\ \}_W$ indicates that the enclosed operators are to be Weyl-ordered, i.e., arranged in the same order as those inside the curly brackets in (6.2). Thus, with the substitution

$$\tilde{\Psi}(A)=\mathcal{J}^{-1/2}\overline{\Psi}(A)\,, \tag{6.16}$$

the time-dependent Schrödinger equation $i\dot{\tilde{\Psi}}(A)=\tilde{H}(P,A)\tilde{\Psi}(A)$ becomes

$$i\dot{\overline{\Psi}}(A)=\overline{H}(P,A)\overline{\Psi}(A)\,. \tag{6.17}$$

As was discussed in Sec. V, the Schrödinger Green's function, applied to the state $\overline{\Psi}(A)$, can be written as

$$\langle A'|e^{-i\overline{H}(t'-t)}|\rangle=\lim_{n\to\infty}\int\prod_{n=1}^{n}d[A(n)]\,d[P(n)]\,d[\overline{\psi}(n)]\,d[\psi(n)]$$
$$\times\left(\exp\left\{i\epsilon\sum_{n=1}^{n}\int d^3r\left[P_i^l(n)(A_i^l(n+1)-A_i^l(n))\epsilon^{-1}\right.\right.\right.$$
$$\left.\left.\left.-\overline{H}\left(P(n),\frac{A(n+1)+A(n)}{2}\right)\right]\right\}\right)\overline{\Psi}[A_i(1)]\,, \tag{6.18}$$

where all notations are the same as those in (5.5) and (5.24), except for the replacements of $V(n)$ and $\Pi(n)$ by $A(n)$ and $P(n)$; hence, $A_i(\mathfrak{N}+1)=A_i'$, $d[A(n)]=\Pi_{i,l,\vec r}\, dA_i^l(\vec r,t_n)$, $A_i^l(n)=A_i^l(\vec r,t_n)$, $\overline{\psi}(n)=\psi^\dagger(\vec r,t_n)\gamma_4$, ... The quadratic interaction of the momenta $P_i^l(\vec r)$ can be removed by introducing a Gaussian integral over a new variable $A_0^l(n)=A_0^l(\vec r,t_n)$:

$$\exp\left\{-i\epsilon\tfrac{1}{2}\sum_{n=1}\int d^3r\int d^3r'[-P_i^l(\vec r)\mathfrak{D}_i+g\psi^\dagger T^l\psi](\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j\Gamma_j^\dagger(\mathfrak{D}_{k'}^\dagger\Gamma_{k'}^\dagger)^{-1}(\mathfrak{D}_{i'}P_{i'}^{l'}(\vec r\,')+g\psi^\dagger T^{l'}\psi]\right\}$$
$$=\int\prod_{n=1}^{n}d[A_0(n)]\,\mathcal{J}\exp\left\{i\epsilon\sum_{n=1}\left[\int d^3r[P_i^l(n)\mathfrak{D}_iA_0^l(n)-g\psi^\dagger(n)T^l\psi(n)A_0^l(n)]\right.\right.$$
$$\left.\left.+\tfrac{1}{2}\int d^3r\int d^3r'A_0^l(\vec r,n)\langle l,\vec r|\mathfrak{D}_k^\dagger\Gamma_k^\dagger(\Gamma_j\Gamma_j^\dagger)^{-1}\Gamma_{k'}\mathfrak{D}_{k'}|l',\vec r\,'\rangle A_0^{l'}(\vec r\,',n)\right]\right\}, \tag{6.19}$$

where, apart from a constant factor, $\mathcal{J}=\det|\Gamma_k\mathfrak{D}_k|$ as in (4.37), which differs from

$$[\det(\mathfrak{D}_k^\dagger\Gamma_k^\dagger(\Gamma_j\Gamma_j^\dagger)^{-1}\Gamma_{k'}\mathfrak{D}_{k'})]^{1/2}$$

only by another A_i^l-independent factor. One should note that just as $\overline{H}$ in (6.18) is evaluated at the symmetric point $\frac{1}{2}[A(n+1)+A(n)]$, the same rule must also apply to the operator $\mathfrak{D}_i$ wherever it appears in (6.19). Finally, we can perform the P_i^l integrations. Because P_i obeys the constraint (4.15), $\Gamma_jP_j=0$, we have

$$\int d[P(n)]\exp\left(i\epsilon\int d^3r\left\{P_i^l(n)[(A_i^l(n+1)-A_i^l(n))\epsilon^{-1}+\mathfrak{D}_iA_0^l(n)]-\tfrac{1}{2}P_i^l(n)P_i^l(n)\right\}\right)$$
$$=\exp\left\{i\epsilon\left[\int d^3r\,\tfrac{1}{2}[(A_i^l(n+1)-A_i^l(n))\epsilon^{-1}+\mathfrak{D}_iA_0^l(n)]^2\right.\right.$$
$$\left.\left.-\tfrac{1}{2}\int d^3r\,d^3r'A_0^l(\vec r)\langle l,\vec r|\mathfrak{D}_k^\dagger\Gamma_k^\dagger(\Gamma_j\Gamma_j^\dagger)^{-1}\Gamma_{k'}\mathfrak{D}_{k'}|l',\vec r\,'\rangle A_0^{l'}(\vec r\,')\right]\right\}. \tag{6.20}$$

Substituting (6.19) into (6.18), applying (6.20) to the P_i^l integrations and taking the limit $\epsilon\to 0$, we find

$$\langle A'|e^{-iH(t'-t)}|\rangle = \int d[A_\mu]\mathcal{J}d[\bar\psi(n)]d[\psi(n)]\exp\left(i\int_t^{t'}dt_1\left\{-\mathcal{V}_1-\mathcal{V}_2+\int d^3r\left[-\tfrac{1}{4}F^l_{\mu\nu}F^l_{\mu\nu}-\bar\psi\gamma^\mu D_\mu\psi\right]\right\}\right). \tag{6.21}$$

Perturbation expansion of (6.21) gives the usual Feynman rules for the gauge $\chi(A_i)=0$ with the addition of the new term $\mathcal{V}_1+\mathcal{V}_2$.

In order to justify this statement, we must recall the connection between occurrence of the symmetric combination $\frac{1}{2}[A(n+1)+A(n)]$ in our definition of (6.18) and (6.19) and the usual Feynman rules. This connection can be most easily illustrated by considering the case of a simple one-dimensional harmonic oscillator with frequency ω. Let the "contraction" between any two functions $F(t)$ and $G(t)$ of the coordinate $q(t)$ and the velocity dq/dt be given by the standard expression

$$F(t)^{\cdot}G(t')^{\cdot} \equiv \frac{\int d[q]F(t)G(t')\exp(-i\int L(q,dq/dt)dt)}{\int d[q]\exp(-i\int L(q,dq/dt)dt)}. \tag{6.22}$$

With our convention of replacing $q(t)$ by the symmetric form $\frac{1}{2}[q(n+1)+q(n)]$ and dq/dt by the antisymmetric form $[q(n+1)-q(n)]\epsilon^{-1}$, and then taking the limit $\epsilon\to0$, it is straightforward to verify that

$$q(t)^{\cdot}q(t')^{\cdot}=D(t-t')=\frac{i}{2\pi}\int\frac{e^{-ik_0(t-t')}}{k_0^{\,2}-\omega^2}dk_0, \tag{6.23}$$

$$\left[\frac{dq(t)}{dt}\right]^{\cdot}q(t')^{\cdot}=\frac{d}{dt}D(t-t'), \tag{6.24}$$

and

$$\left[\frac{dq(t)}{dt}\right]^{\cdot}\left[\frac{dq(t')}{dt'}\right]^{\cdot}=-\frac{d^2}{dt^2}D(t-t')$$
$$=\omega^2D(t-t')+i\delta(t-t'). \tag{6.25}$$

These equations correspond to the usual "covariant" Feynman propagators.

We note that in (6.22) when $t=t'$, the product $F(t)G(t)$ for $F=dq/dt$ and $G=q$ is, because of our choice,

$$\frac{1}{2\epsilon}[q(n+1)-q(n)][q(n+1)+q(n)]$$
$$=\frac{1}{2\epsilon}[q(n+1)^2-q(n)^2],$$

which, after the $dq(n)$ and $dq(n+1)$ integrations, clearly gives zero, in accordance with the right-hand side of (6.24),

$$\frac{d}{dt}D(0)=\frac{1}{2\pi}\int\frac{k_0}{k_0^{\,2}-\omega^2}dk_0=0.$$

For $F=G=dq/dt$, the factor $F(t)G(t)$ is

$$\frac{1}{\epsilon^2}[q(n+1)-q(n)]^2$$

whose integrated value is $\frac{1}{2}\omega+i\epsilon^{-1}$, which in the limit $\epsilon\to0$ agrees with the right-hand side of (6.25). This is to be contrasted with the usual Wick definition: the vacuum expectation value of $t([dq(t)/dt][dq(t')/dt'])$, or $T(p(t)p(t'))$, which is finite when $t=t'$ and differs from (6.25) by $i\delta(t-t')$. It is this difference that, when summed directly in the Dyson-Wick perturbation formalism, leads to the Jacobian[10] or Faddeev-Popov ghost[11] term.

Our result (6.21) implies that for the gauge $\chi(A_i)=0$, in addition to the Jacobian $e^{\ln\mathcal{J}}$, the usual Feynman rules must be augmented by the potential-like term $\mathcal{V}_1+\mathcal{V}_2$ given in (6.12) and (6.14). Although the derivation is somewhat lengthy, the physical origin of both terms is the same as that of $r^{1/2}$ and $-(8r^2)^{1/2}$ in (2.14), for the simple mechanical example.

The Jacobian is usually converted into an additional term $-i\delta(0)\ln\mathcal{J}$ in the Lagrangian. In contrast, the new $\mathcal{V}_1$ and $\mathcal{V}_2$ terms have very different characteristics; they are both real and without the $\delta(0)$ factor. When expanded in power series of A^l_i, each can be written as

$$\mathcal{V}_\alpha=O(g^4A^2)+O(g^5A^3)+O(g^6A^4)+\cdots, \tag{6.26}$$

where $\alpha=1$ or 2. [There is an $O(g^2)$ constant term in $\mathcal{V}_2$, which can be dropped.] For example, in the Coulomb gauge and for SU_2, to order g^4 we have

$$\mathcal{V}_1(A)=-\frac{g^4}{4}\int d^3r\,d^3r'\,d^3r''$$
$$\times K_{ij}(\vec r-\vec r\,')K_{ik}(\vec r-\vec r\,'')\vec A_j(\vec r\,')\cdot\vec A_k(\vec r\,''), \tag{6.27}$$

where

$$K_{ij}(\vec\rho)=\left[\tfrac{1}{3}\delta_{ij}\delta^3(\vec\rho)-\frac{1}{4\pi\rho^5}(3\rho_i\rho_j-\rho^2\delta_{ij})\right]\frac{1}{4\pi\rho}, \tag{6.28}$$

where $\rho=|\vec\rho|$ and the components of $\vec A_j$ are A^l_j. The corresponding expression for $\mathcal{V}_2(A)$ is similar, but more complicated. In (6.28), the singular term $\propto\delta^3(\vec\rho)/\rho$ gives rise to infinites which are presumably relevant for the cancellation of divergences from the usual two-loop g^4-Feynman graphs; the remainder is finite when $\rho\neq0$ and leads to new nonlocal interactions between the gauge fields which could be of physical importance.

VII. CONCLUSION

The $\chi(A_i)=0$ gauge generating function $Z(J_\mu)$ implied by the discussion of Sec. VI is

$$Z(J_\mu)=\int d[A_\mu]\,\delta(\chi(A_l)) \times \exp\Big(i\int d^4x\,\{-\tfrac{1}{4}F^l_{\mu\nu}F^l_{\mu\nu}+J^l_\mu A^l_\mu - i\,\delta(0)\langle x,l|\ln(\Gamma_k\mathfrak{D}_k)|x,l\rangle - \mathcal{V}_1(A)-\mathcal{V}_2(A)\}\Big)\,. \tag{7.1}$$

The first two terms in the exponent make up the usual classical action which, if a factor of g is absorbed into A_μ, is of order g^{-2}. The third Faddeev-Popov term is of order g^0 and although peculiar to the functional-integral description, makes an essential contribution to the Feynman rules. The final two terms, of order g^2, have been the object of the discussion in this paper, and are given explicity by (6.12) and (6.14). If Planck's constant is distinguished from one, then these three classes of terms are of order $\hbar^{-1}$, $\hbar^0$, and $\hbar$, respectively.

As we have seen, the extra terms $\mathcal{V}_1$ and $\mathcal{V}_2$ are precisely those which arise when the term in the quantum Hamiltonian $\overline{H}$, quadratic in the conjugate momenta $P^l_i(\vec{r})$, is Weyl ordered. In the pure Yang-Mills case, Eq. (6.15) can be written as

$$\overline{H}(P,A)=\tfrac{1}{2}\int d^3r\{[E^l_i(\vec{r})]^2+[B^l_i(\vec{r})]^2\}_W + \mathcal{V}_1+\mathcal{V}_2\,, \tag{7.2}$$

where

$$E^l_i(\vec{r})=P^l_i(\vec{r})+\int d^3r'\langle\vec{r},l|\Gamma^\dagger_i(\mathfrak{D}^\dagger_k\Gamma^\dagger_k)^{-1}\mathfrak{D}_j|\vec{r}\,',l'\rangle \times P^{l'}_j(\vec{r}\,')\,, \tag{7.3}$$

and, as before, the subscript W denotes Weyl ordering. For the case of the Coulomb gauge, our result (7.2) can be directly compared with that of Schwinger.[1,2] In our notation, Schwinger's Hamiltonian takes the form

$$\tfrac{1}{2}\int d^3r\{[E^l_i(\vec{r})_W]^2+[B^l_i(\vec{r})]^2\}+\mathcal{V}_1\,, \tag{7.4}$$

where the operator $E^l_i(\vec{r})$ defined by (7.3) is first to be Weyl ordered and then squared. The equality of the expression (7.4) and $\overline{H}$ in (7.2) follows from the identity

$$-\frac{1}{8}\left[\frac{\partial}{\partial Q_N}F^\gamma_N+F^\gamma_N\frac{\partial}{\partial Q_N}\right]^2=-\frac{1}{8}\left[\frac{\partial}{\partial Q_N}\frac{\partial}{\partial Q_N}F^\gamma_N F^\gamma_{N'}+2\frac{\partial}{\partial Q_N}F^\gamma_N F^\gamma_{N'}\frac{\partial}{\partial Q_{N'}}+F^\gamma_N F^\gamma_{N'}\frac{\partial}{\partial Q_N}\frac{\partial}{\partial Q_{N'}}\right]+\frac{1}{8}\frac{\partial F^\gamma_N}{\partial Q_{N'}}\frac{\partial F^\gamma_{N'}}{\partial Q_N}\,, \tag{7.5}$$

in which we may let Q_N be that given by (4.13) and F^γ_N be

$$\frac{\delta Q_N}{\delta V^l_j(\vec{r})}=\int d^3r' f^{l'}_i(\vec{r}\,')_N\langle\vec{r}\,',l|\delta_{ij}-\mathfrak{D}_i(\Gamma_k\mathfrak{D}_k)^{-1}\Gamma_j|\vec{r},l\rangle\,. \tag{7.6}$$

The last term on the right-hand side of (7.5) can then be written as

$$\int d^3r\frac{\partial}{\partial Q_N}\left[\frac{\delta Q_{N'}}{\delta V^l_j(\vec{r})}\right]\frac{\partial}{\partial Q_{N'}}\left[\frac{\delta Q_N}{\delta V^l_j(\vec{r})}\right]\,, \tag{7.7}$$

which is precisely the first term inside the curly brackets in (6.6), the quantity defined as $\mathcal{V}_2$. Thus, $[E^l_i(\vec{r})_W]^2$ equals $[E^l_i(\vec{r})^2]_W+\mathcal{V}_2$ and the Hamiltonians (7.2) and (7.4) are identical.

Although the detailed arguments used in this paper to deduce the Hamiltonian operator $\overline{H}$ differ significantly from the method used by Schwinger, there is a close relationship between the initial physical assumptions. While Schwinger[1] determines $\overline{H}$ so that the Lorentz group generators obey the proper commutation relations, we show that $\overline{H}$ is equivalent to a $V_0=0$ gauge Hamiltonian with Cartesian operator ordering. Of course, the Lorentz group generators are gauge invariant and can be easily seen to obey the proper commutation relations when expressed in the $V_0=0$ gauge. Hence, these two methods lead to the same Hamiltonian operator. However, it is the Weyl-ordered $\overline{H}$ of (7.2) given in this paper that must be used to deduce the Feynman rules.

Note added in proof. We wish to thank J.-L. Gervais, I. Muzinich, and T. N. Tudron for informing us of the following papers which have also discussed the transformation from the $V_0=0$ gauge to the Coulomb gauge along lines similar to those in Sec. IV of our paper: V. N. Gribov, lecture at the 12th Winter School of the Leningrad Nuclear Physics Institute, 1977 (unpublished); J.-L. Gervais and B. Sakita, Phys. Rev. D 18, 453 (1978); M. Creutz, I. Muzinich, and T. N. Tudron, *ibid.* 19, 531 (1979); T. N. Tudron, Syracuse University Report No. SU-4217-156 (unpublished). Finally, we are indebted to R. Marmelius for bringing the work of R. Utiyama and J. Sakamoto [Prog. Theor. Phys. 55, 1631 (1976)] to our attention. These authors discuss the Coulomb-gauge operator-ordering problem from a viewpoint quite similar to ours. However, their technique for solving the Gauss constraint is significantly different from ours and the result seems more complex.

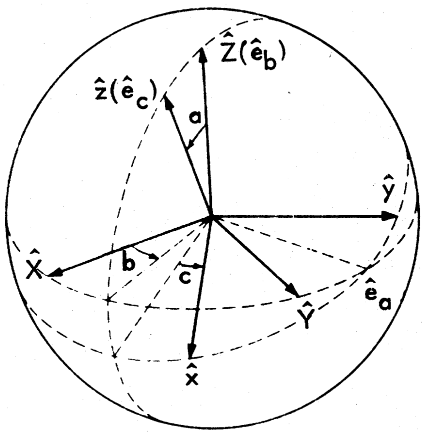

FIG. 1. The basis vectors of Σ_{lab} are $\hat{X}$, $\hat{Y}$, $\hat{Z}$ and those of Σ_{body} are $\hat{x}$, $\hat{y}$, $\hat{z}$. The rotation sequence "*cab*" takes Σ_{lab} to Σ_{body}: first an angle b around $\hat{Z}(\hat{e}_b)$, then an angle a around $\hat{e}_a$, and finally an angle c around $\hat{z}(\hat{e}_c)$.

ACKNOWLEDGMENTS

We thank Edward Yao and Charles Sommerfield for bringing the work of J. Schwinger and R. P. Treat to our attention. This research was supported in part by the U. S. Department of Energy.

APPENDIX

In this appendix, we consider the special case of SU_2 and give a simple physical interpretation of the operators j^l, J^l and their commutation relations (4.44)–(4.47). There are three group parameters ϕ_a, which may be chosen to be the Eulerian angles a, b, and c. Thus, the u matrix of (4.5) can be written as

$$u = e^{-i\tau_z b/2} e^{-i\tau_y a/2} e^{-i\tau_z c/2}, \tag{A1}$$

where τ_x, τ_y, τ_z are the usual Pauli matrices.

We recall that the origin of Eulerian angles a, b, c lies in the description of a rigid-body rotation. As shown in Fig. 1, there are two reference systems, the laboratory frame Σ_{lab} and the body frame Σ_{body}. Each frame is defined by a basis of three orthogonal unit vectors: $\hat{X}$, $\hat{Y}$, $\hat{Z}$ for Σ_{lab} and $\hat{x}$, $\hat{y}$, $\hat{z}$ for Σ_{body}. To go from Σ_{lab} to Σ_{body}, we first rotate an angle b along

$$\hat{Z} \equiv \hat{e}_b,$$

which moves the y axis from $\hat{Y}$ to $\hat{e}_a$; then an angle a along $\hat{e}_a$ which rotates the z axis from $\hat{Z}$ to $\hat{z}$; finally an angle c along

$$\hat{z} \equiv \hat{e}_c .$$

Consider now a point P in space, whose coordinates in Σ_{lab} and Σ_{body} are, respectively, X, Y, Z and x, y, z. Let us define

$$R \equiv \tfrac{1}{2}(\tau_x X + \tau_y Y + \tau_z Z)$$

and

$$r \equiv \tfrac{1}{2}(\tau_x x + \tau_y y + \tau_z z) .$$

From the definition of the Eulerian angles, it follows that

$$R = uru^\dagger ,$$

where u is given by (A1). (Here, R plays the role of V_i in the gauge-field case, and r the role of A_i.)

We keep Σ_{lab} fixed, and consider the rotation of Σ_{body} by changing a, b, and c. The angular velocity vector is

$$\dot{a}\hat{e}_a + \dot{b}\hat{e}_b + \dot{c}\hat{e}_c , \tag{A2}$$

where the dot denotes a time derivative. Let us refer to the components of (A2) in Σ_{body} and Σ_{lab} as ω^l and Ω^l, respectively. It is useful to define

$$iu^\dagger \dot{u} \equiv \omega = \tfrac{1}{2}\vec{\tau} \cdot \vec{\omega} \tag{A3}$$

and

$$i\dot{u}u^\dagger \equiv \Omega = \tfrac{1}{2}\vec{\tau} \cdot \Omega . \tag{A4}$$

Then, the matrices ω and Ω are related by

$$\Omega = u\omega u^\dagger , \tag{A5}$$

and the components of $\vec{\omega}$ and $\vec{\Omega}$ are the aforementioned ω^l and Ω^l, related by

$$\Omega^l = U^{lm}\omega^m \tag{A6}$$

with U^{lm} given by (4.23). The quantities λ_a^l and Λ_a^l, defined by (4.22) and (4.39), are related to ω^l and Ω^l by

$$\omega^l = \lambda_a^l \dot{\phi}_a \quad \text{and} \quad \Omega^l = \Lambda_a^l \dot{\phi}_a . \tag{A7}$$

The Lagrangian L of a rigid body with no external forces is a function only of ω^1, ω^2, and ω^3:

$$L = L(\omega^l) .$$

Through (A2), L is also a function of a, b, c and $\dot{a}, \dot{b}, \dot{c}$. The conjugate momenta of a, b, and c are given, respectively, by

$$p_a = \frac{\partial \omega^l}{\partial \dot{a}} \frac{\partial L}{\partial \omega^l}, \quad p_b = \frac{\partial \omega^l}{\partial \dot{b}} \frac{\partial L}{\partial \omega^l} \quad \text{and} \; p_c = \frac{\partial \omega^l}{\partial \dot{c}} \frac{\partial L}{\partial \omega^l} . \tag{A8}$$

A comparison between (A8) and (4.43) gives

$$j^l = \frac{\partial L}{\partial \omega^l} , \tag{A9}$$

which is simply the component of the angular momentum vector in Σ_{body}. The same vector viewed in Σ_{lab} carries the component J^l:

$$\hat{x}j^1 + \hat{y}j^2 + \hat{z}j^3 = \hat{X}J^1 + \hat{Y}J^2 + \hat{Z}J^3 , \tag{A10}$$

or $J^l = U^{lm} j^m$, as in (4.43).

In the quantum theory,

$$\vec{J} \equiv \hat{X}J^1 + \hat{Y}J^2 + \hat{Z}J^3 \tag{A11}$$

is the rotational operator. Its components in $\Sigma_{\rm lab}$ satisfy the usual commutation relation

$$[J^l, J^m] = i\epsilon^{lmn}J^n\,. \tag{A12}$$

On the other hand, its components the j^l's in $\Sigma_{\rm body}$ do not. From (A10), we have $j^1 = \hat{x}\cdot\vec{J}$ and similar expressions for J^2 and J^3. Since $\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors fixed in the body frame, under a rotation they transform like $\vec{J}$; all rotate like vectors. Hence, the scalar products $\hat{x}\cdot\vec{J}$, $\hat{y}\cdot\vec{J}$, and $\hat{z}\cdot\vec{J}$ are invariants, and that gives

$$[J^l, j^m] = 0\,. \tag{A13}$$

Likewise, we can show that

$$[j^l, j^m] = -i\epsilon^{lmn}j^n\,. \tag{A14}$$

Equations (4.44)–(4.47) are simply generalizations of these standard commutation rules, but for the gauge field.

It is important to note that in the generalized Laplacian, $\mathcal{K}$ of (4.38), the angular momentum (i.e., charge) operator that appears naturally is j^l. Since $j^l = U^{ml}J^m$, the Gauss theorem (3.27) takes on the form

$$j^l|\rangle = -U^{ml}\psi^\dagger T^l\psi|\rangle\,, \tag{A15}$$

which is the alternative expression for (4.60).

In terms of a, b, and c the components of $\vec{\omega}$ and $\vec{\Omega}$ are given by

$$\begin{aligned}\omega^1 &= \dot{a}\sin c - \dot{b}\sin a\cos c\,,\\ \omega^2 &= \dot{a}\cos c + \dot{b}\sin a\sin c\,,\\ \omega^3 &= \dot{b}\cos a + \dot{c}\end{aligned} \tag{A16}$$

and

$$\begin{aligned}\Omega^1 &= -\dot{a}\sin b + \dot{c}\sin a\cos b\,,\\ \Omega^2 &= \dot{a}\cos b + \dot{c}\sin a\sin b\,,\\ \Omega^3 &= \dot{b} + \dot{c}\cos a\,.\end{aligned} \tag{A17}$$

The matrices λ and Λ can be obtained by differentiating these expressions with respect to $\dot{a}$, $\dot{b}$, and $\dot{c}$. Likewise, the components j^l and J^l can be expressed in terms of $p_a = -i\partial/\partial a$, $p_b = -i\partial/\partial b$, and $p_c = -i\partial/\partial c$:

$$\begin{aligned}j^1 &= \sin c\, p_a - \frac{\cos c}{\sin a}p_b + \cos c\,\frac{\cos a}{\sin a}p_c\,,\\ j^2 &= \cos c\, p_a + \frac{\sin c}{\sin a}p_b - \sin c\,\frac{\cos a}{\sin a}p_c\,,\\ j^3 &= p_c\,,\end{aligned} \tag{A18}$$

and

$$\begin{aligned}J^1 &= -\sin b\, p_a - \cos b\,\frac{\cos a}{\sin a}p_b = \frac{\cos b}{\sin a}p_c\,,\\ J^2 &= \cos b\, p_a - \sin b\,\frac{\cos a}{\sin a}p_b + \frac{\sin b}{\sin a}p_c\,,\\ J^3 &= p_b\,.\end{aligned} \tag{A19}$$

By using (A18) and (A19), one can also verify the commutation relations (A12)–(A14) directly.

[1]J. Schwinger, Phys. Rev. 127, 324 (1962).

[2]J. Schwinger, Phys. Rev. 130, 406 (1963).

[3]R. P. Treat, Phys. Rev. D 12, 3145 (1975).

[4]These terms do not appear, for example, in the Coulomb gauge, Hamiltonian quantum mechanics of Eq. (13.24) in E. S. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973).

[5]With appropriate boundary conditions, the existence of such a unique gauge transformation can be proven for the axial gauge. However, for the Coulomb gauge, $U(r,t)$ may either fail to exist if we require continuity in time [N. H. Christ, A. H. Guth, and E. J. Weinberg, Nucl. Phys. B114, 61 (1976)] or fail to be unique [V. N. Gribov, lecture at the 12th Winter School of the Leningrad Nuclear Physics Institute, 1977 (unpublished)]. Of course, the Coulomb gauge may well provide a satisfactory description of processes which do not involve these nonperturbative field configurations.

[6]Application of the path-integral formalism to gauge theories appears in E. S. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973), and references therein. The influence of Hamiltonian operator ordering on the path-integral formalism has been discussed by B. S. De-Witt, Rev. Mod. Phys. 29, 377 (1957); S. F. Edwards and Y. V. Gulyaev, Proc. R. Soc. London A279, 229 (1964); D. W. McLaughlin and L. S. Schulman, J. Math. Phys. 12, 2520 (1971); F. A. Berezin, Theor. Math. Phys. (USSR) 6, 141 (1971); K. S. Chang, J. Math. Phys. 13, 1723 (1972); M. Mizrahi, *ibid.* 16, 2201 (1975); J.-L. Gervais and A. Jevicki, Nucl. Phys. B110, 93 (1976).

[7]F. A. Berezin, *The Method of Second Quantization* (Academic, New York, 1966).

[8]This point has been stressed in the context of soliton quantization by J.-L. Gervais and A. Jevicki, Nucl. Phys. B110, 93 (1976).

[9]It is in this step that a similar, path-integral treatment of the noncovariant $\chi(A_i)=0$ gauge generates additional terms. In that case, the gauge-fixing term in curly brackets in (5.25) is replaced by $\delta[\chi(A_i)]$ so that $[A_0(n+1)-A_0(n)]^2/\epsilon$ no longer appears in the exponent and $A_0(n)$ becomes of order $\epsilon^{-1/2}$, being constrained only by its appearance in (5.33). Consequently, in contrast to the present covariant case, there will be additional $A_0{}^2$ and $A_0{}^3$ terms that can enter (5.33) in order ϵ. These terms then lead to the nonlocal potentials $\mathcal{V}_1$ and $\mathcal{V}_2$ of Sec. VI.

[10]T. D. Lee and C. N. Yang, Phys. Rev. 128, 885 (1962).

[11]V. N. Popov and L. D. Faddeev, Kiev ITP report (unpublished); L. D. Faddeev and V. N. Popov, Phys. Lett. 25B, 29 (1967); R. P. Feynman, Acta. Phys. Pol. 24, 697 (1963).