QUASICLASSICAL EXPANSION OF YANG–MILLS HEAT KERNELS AND APPROXIMATE CALCULATION OF FUNCTIONAL DETERMINANTS

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We find next-to-leading coefficients in the expansion of Sp exp$(-tK)$ in powers of proper time t, where K is the differential operator for the gluon, ghost or fermion in a background Yang–Mills field. This expansion provides a possibility of estimating functional determinants in arbitrary background fields. For example, for the instanton background our very simple method gives the value of the determinants with an accuracy of a few percent, as compared to the labourious exact calculation.

1. In the quasiclassical approach to quantum field theory one meets the problem of calculating gaussian functional integrals over small quantum oscillations around a given classical background field. In some cases (e.g. for the instanton [1] background field) the functional determinants can be computed exactly [2,3] by finding all the eigenvalues of the appropriate Schrödinger equations. This, however, is a lucky but not a general case. For example, in a monopole field as a background, and also in many other cases of physical interest, the problem of the exact calculation of functional determinants is hopeless. Therefore, in such cases one has to use an approximate method.

In this letter we suggest a very simple method of estimating functional determinants in arbitrary Yang–Mills background fields [‡1]. The method is based on the marriage of a quasiclassical calculation of the high density of eigenvalues in a background field, with the accounting for zero (or approximately zero) eigenvalues.

Let $\bar{A}^a(x, \gamma_1, ..., \gamma_p)$ be a classical background field depending on p parameters γ_i, of which the classical action $\int F^2_{\mu\nu}(\bar{A})\, d^4x$ is, in fact, independent. Using the Faddeev–Popov trick of "introducing a unity" one can write down the statistical weight of a given field configuration $\bar{A}$ as an integral over collective coordinates γ_i and a gaussian functional integral over gluon (B), ghose (χ) and fermion (ψ) fields:

$$Z(\bar{A}) = \exp\left(-\frac{1}{4g^2(M)}\int d^4x\, F^2_{\mu\nu}(\bar{A})\right)\int \mathcal{D}B^a_\mu\,\mathcal{D}\chi^+\mathcal{D}\chi\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi \int\prod^p d\gamma_i\, \det_{(kl)}\left(\int d^4x\,\frac{\partial\bar{A}^a_\mu}{\partial\gamma_k}\psi^{a(l)}_\mu\right)$$

$$\times\prod^p\delta\left(\int d^4x\, B^a_\mu\psi^{a(i)}_\mu\right)\exp\left(-\frac{1}{2g^2(M)}\int d^4x\, B^a_\mu W^{ab}_{\mu\nu}B^b_\nu - \int d^4x\, \chi^{+a}D^2(\bar{A})^{ab}\chi^b - \int d^4x\,\bar{\psi}[i\gamma_\mu\nabla_\mu(\bar{A})-m]\psi\right) \tag{1}$$

Here $\psi^{a(i)}_\mu$ are p functions fixing the zero modes which are not orthogonal to $\partial\bar{A}^a_\mu/\partial\gamma_i$ and which satisfy the background gauge condition [3]: $D^{ab}_\mu(\bar{A})\psi^{b(i)}_\mu = 0$. We use the background Feynman gauge with

$$W^{ab}_{\mu\nu}(\bar{A}) = -D^2(\bar{A})^{ab}\delta_{\mu\nu} - 2F^c_{\mu\nu}(\bar{A})f^{acb}\,,\quad D^{ab}_\mu(\bar{A}) = \partial_\mu\delta^{ab} + f^{acb}\bar{A}^c_\mu\,,\quad \nabla_\mu(\bar{A}) = \partial_\mu - it^a\bar{A}^a_\mu\,. \tag{2}$$

The statistical weight (1) is usually normalized to the free gaussian integral with $\bar{A} = 0$, and also regularized. According to, say, Pauli–Villars regularization scheme, one has to divide the ratio of eq. (1) to the free gaussian inte-

[‡1] A detailed version of this work is to be published [4].

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gral by the same ratio with fields B, χ and ψ having a regulator mass M. Hence, the normalized and regularized statistical weight takes the form

$$Z(\bar{A})_{\rm reg} = \exp\left(-\frac{1}{4g^2(M)}\int d^4x\, F^2_{\mu\nu}(\bar{A})\right)\int \prod^{p} d\gamma_i \det{}_{(kl)}\left(\int d^4x\, \frac{\partial \bar{A}^a_\mu}{\partial\gamma_k}\psi^{a(l)}_\mu\right)\det{}^{-1/2}_{(kl)}\left(\int d^4x\, \psi^{a(k)}_\mu\psi^{a(l)}_\mu\right)$$

$$\times\left(\frac{M}{g(M)\sqrt{2\pi}}\right)^p\left(\frac{\det' W^{ab}_{\mu\nu}}{\det(-\partial^2\delta^{ab}\delta_{\mu\nu})}\right)^{-1/2}_{\rm reg}\left(\frac{\det(-D^{2ab})}{\det(-\partial^2\delta^{ab})}\right)_{\rm reg}\left(\frac{\det(i\not{\nabla}-m)}{\det(i\not\partial-m)}\right)_{\rm reg}. \tag{3}$$

Here $\det' W$ means that in calculating this determinant one is not to take into account the p zero eigenvalues of W.

2. To calculate the three functional determinants in eq. (3) it is convenient to use the Schwinger proper time representation combined with the ζ regularization [5] (equivalent to the Pauli–Villars scheme). One has

$$\left(\frac{\det K}{\det K_0}\right)_{\rm reg} = \exp\left(-\lim_{s\to 0}\frac{d}{ds}\frac{M^{2s}}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\,{\rm Sp}(e^{-tK}-e^{-tK_0})\right), \tag{4}$$

where Sp is understood in the functional sense. For small t (corresponding to large eigenvalues of K) there exists a quasiclassical expansion in powers of t:

$${\rm Sp}\exp(-tW^{ab}_{\mu\nu}) = (1/16\pi^2)\,4d(G)V^{(4)}/t^2 + \tfrac{10}{3}C_2(G)F_2 - tC_2(G)(\tfrac{2}{45}F_3 + \tfrac{6}{5}I_3) + O(t^2), \tag{5}$$

$${\rm Sp}\exp(tD^{2ab}) = (1/16\pi^2)\,d(G)V^{(4)}/t^2 - \tfrac{1}{6}C_2(G)F_2 + tC_2(G)(-\tfrac{1}{90}F_3 + \tfrac{1}{30}I_3) + O(t^2), \tag{6}$$

$${\rm Sp}\{\exp[-t(i\not{\nabla}-m)^2]\} = (1/16\pi^2)\,4d(F)V^{(4)}/t^2 + \tfrac{4}{3}r(F)F_2 - {\rm tr}(F)(\tfrac{2}{45}F_3 + \tfrac{8}{15}I_3) + O(t^2) + O(m^2). \tag{7}$$

Here $d(G)$ and $d(F)$ are dimensions of the gluon and fermion representations, $C_2(G)$ and $C_2(F)$ are the corresponding values of the quadratic Casimir operators, $r(F) = C_2(F)d(F)/d(G)$.

By $F_{2,3}$ and I_3 we denote the gauge-invariant combinations built of the background field:

$$F_2 = (1/32\pi^2)\int d^4x\, F^a_{\mu\nu}(\bar{A})F^a_{\mu\nu}(\bar{A}),\quad F_3 = (1/32\pi^2)\int d^4x\, F^a_{\lambda\mu}F^b_{\mu\nu}F^c_{\nu\lambda}f^{abc},$$

$$I_3 = (1/32\pi^2)\int d^4x\,(D^{ab}_\alpha F^b_{\alpha\beta})(D^{ac}_\gamma F^c_{\gamma\beta}). \tag{8}$$

The latter invariant is zero for a background field with no sources, owing to the equation of motion. The first terms in eqs. (5)–(7) are cancelled by the free operators in eq. (4). Note the absence of a $O(t^{-1})$ term, which is due to the gauge invariance. The $O(t^0)$ terms leading to the charge renormalization were calculated in ref. [6]. The $O(t)$ terms were calculated by us for this work ‡2.

3. Let us first estimate $\det -D^2$. The operator is positive definite, and for background fields which are smooth enough, one can hope that the density of eigenvalues does not differ drastically from that of the free operator $-\partial^2$. Therefore, the difference ${\rm Sp}(\exp tD^2 - \exp t\partial^2)$ should be a rapidly decreasing function of t, whose expansion at small t is given by eq. (6). Note that actually we have gone as far as the fourth term in the quasiclassical expansion, so that one can expect that eq. (6) represents adequately the density of eigenvalues all the way down to

‡2 Actually, these terms can be extracted from the results of ref. [7] where the quasiclassical expansion up to the $O(t)$ terms were computed for a differential operator on a general riemannian manifold. We are grateful to the referee for drawing our attention to ref. [7].

rather low states. (Our experiment with the instanton background confirms this expectation, see below). All the information about a concrete external field $\bar{A}$ is accumulated in the values of $F_{2,3}$ and I_3.

To estimate the integral in eq. (4), let us cut the integration range from above at some δ and then find the best value of δ (for the given number of terms in the quasiclassical expansion) from a stability requirement. Substituting eq. (6) in eq. (4) and differentiating in respect to δ we find the best value of δ

$$\delta_{\mathrm{gh}} = 15F_2/(-F_3 + 3I_3)\,, \tag{9}$$

and hence ($\gamma_{\mathrm{E}} = 0.5772$ ...)

$$\left(\frac{\det -D^2}{\det -\partial^2}\right)_{\mathrm{reg}} \approx \exp\left[\tfrac{1}{6}N_{\mathrm{c}}F_2\,(\ln M^2\delta_{\mathrm{gh}} + \gamma_{\mathrm{E}} - 1)\right]\,. \tag{10}$$

Naturally, this estimation has a sense only if $\delta > 0$.

4. Turning to $\det' W$, one should recall that the gluon operator W is assumed to have p zero modes. Therefore, in order to calculate $\det' W$ according to eq. (4), one should subtract the number of zero modes p in the integrand. After the subtraction one can assume that the quantity

$$\mathrm{Sp}\left[\exp(-tW^{ab}_{\mu\nu}) - \exp(t\partial^2\delta^{ab}\delta_{\mu\nu})\right] - p$$

is again a rapidly decreasing function of t whose expansion is given by eq. (5). Repeating the same manoeuver as in the case of $\det -D^2$, one finds the stability point for the cut-off

$$\delta_{\mathrm{gl}} = \left[\tfrac{10}{3}C_2(G)F_2 - p\right]/C_2(G)\left(\tfrac{2}{45}F_3 + \tfrac{6}{5}I_3\right)\,, \tag{11}$$

and, hence, one obtains an estimate

$$\left(\frac{\det' W^{ab}_{\mu\nu}}{\det -\partial^2\delta^{ab}\delta_{\mu\nu}}\right)^{-1/2}_{\mathrm{reg}} \approx \exp\left[\tfrac{1}{2}\left[\tfrac{10}{3}C_2(G)F_2 - p\right](\ln M^2\delta_{\mathrm{gl}} + \gamma_{\mathrm{E}} - 1)\right]\,. \tag{12}$$

5. The value of the fermion determinant depends significantly on the fermion mass m. We shall here consider a case of small m. We shall also assume that the external field $\bar{A}$ produces q zero fermion modes when m is set to zero.

Calculating the determinant according to eq. (4), let us divide the integration range into two parts: from 0 to δ and from δ to ∞. In the first range we shall use the expansion (7), and in the second we shall take into account only the would-be zero eigenvalues, which are shifted to m^2. Thus,

$$\left(\frac{\det(\mathrm{i}\bar{\nabla\!\!\!/} - m)}{\det(\mathrm{i}\partial\!\!\!/ - m)}\right)_{\mathrm{reg}} \approx \exp\left[-\tfrac{1}{2}\lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{M^{2s}}{\Gamma(s)}\int_0^\delta \mathrm{d}t\; t^{s-1}\left(\tfrac{4}{3}F_2 - \tfrac{2}{45}tF_3 - \tfrac{8}{15}tI_3\right)r(F) + q\int_\delta^\infty \mathrm{d}t\; t^{s-1}\,\mathrm{e}^{-tm^2}\right)\right]\,. \tag{13}$$

Differentiating in respect to δ we find its best value for this approximation:

$$\delta_{\mathrm{f}} = \left[r(F)\tfrac{4}{3}F_2 - q\right]/r(F)\left(\tfrac{2}{45}F_3 + \tfrac{8}{15}I_3\right) + O(m^2)\,. \tag{14}$$

Substituting this value in eq. (13) we get

$$(\det(\mathrm{i}\bar{\nabla\!\!\!/} - m)/\det(\mathrm{i}\partial\!\!\!/ - m))_{\mathrm{reg}} \approx (m/M)^q \exp\left[-\tfrac{1}{2}\left[\tfrac{4}{3}F_2 r(F) - q\right](\ln M^2\delta_{\mathrm{f}} + \gamma_{\mathrm{E}} - 1)\right]\,. \tag{15}$$

6. Let us now check, how this crude method works in the case of an instanton background field for which exact calculations of the determinants are available [2,3]. In the instanton case one has (ρ is the instanton size)

$$F_2 = 1\,,\quad F_3 = -\tfrac{12}{5}\rho^{-2}\,,\quad I_3 = 0\,.$$

The number of zero modes are: $p = 4N_c$, $q = 1$ (for fermions in the fundamental representation of SU(N_c) $r(F) = 1/2$). From eqs. (9), (11) and (14) we find

$$\delta_{gh} = \delta_{gl} = \delta_f = \tfrac{25}{4}\rho^2 .$$

Note that this is an anomalously large value. It means that the quasiclassical expansion [eqs. (5)–(7)] indeed controls an anomalously wide range of the operators' spectra, and one looks forward to a good accuracy in estimating the determinants. Indeed, we obtain from eqs. (10), (12) and (15):

$$(\det -D^2)_{reg} = (M\rho)^{N_c/3}\, e^{0.235 N_c} , \quad \text{exact: } (M\rho)^{N_c/3}\, 1.15^{-1}\, e^{0.292 N_c};$$

$$(\det{}' W_{\mu\nu})_{reg}^{-1/2} = (M\rho)^{-2N_c/3}\, e^{0.235(-2N_c)}, \quad \text{exact: } (M\rho)^{-2N_c/3} 1.15^2\, e^{0.292(-2N_c)};$$

$$[\det(i\not{\nabla} - m)]_{reg} = (m\rho)(M\rho)^{-2/3}\, e^{0.235} , \quad \text{exact: } (m\rho)(M\rho)^{-2/3}\, e^{0.292} .$$

The fermion determinant is computed to an accuracy of 5.5%. In the case of the ghost and gluon determinants the accuracy is better than 3% for SU(2) and SU(3) groups. For larger groups it is natural to speak of the accuracy in the logarithmic scale, which is very good.

To complete the job, one has to calculate the zero-modes determinant in eq. (3), which can always be done analytically. For the sake of completeness we quote the result of this calculation for the instantons of the SU(N_c) group [3]:

$$\int d^4x \int \frac{d\rho}{\rho^5} \left(\frac{8\pi^2}{g^2(M)}\right)^{2N_c} (M\rho)^{4N_c} \frac{\pi^{-2}\, 2^{2-2N_c}}{(N_c - 1)!\,(N_c - 2)!} .$$

We note finally that an approximate method of evaluating multi-instanton determinants was proposed in ref. [8]

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