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SU(2) Yang-Mills Coulomb Green's functions in the presence of a generalized Wu-Yang configuration

R. L. Stuller
Institut für Theoretische Physik, Freie Universität Berlin, Germany
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The Coulomb propagator in a generalized Wu-Yang configuration A is considered and its dependence on zero modes is investigated. It is found that for (i) A equal to a pure Wu-Yang monopole there are no zero modes and (ii) A equal to a pure gauge one zero mode appears. Pure-gauge boundary conditions at the origin lead to an ambiguity which must be resolved by introducing a new parameter θ ($0<\theta<\pi$). For those cases without zero modes an exact multipole expansion is constructed.

INTRODUCTION

It was Gribov[1] who first pointed out that the SU(2) Coulomb-gauge Yang-Mills Green's function D_A^{bc} defined by[2]

$$-[\nabla^k_{xab}+g\epsilon_{adb}A_d^k(x)]\nabla_x^k D_A^{bc}(\vec{x},\vec{x}')=\delta^{ac}\delta^3(\vec{x}-\vec{x}') \tag{1}$$

does not in general exist for arbitrary configurations $A_a^k(x,t)\,[\nabla^k A_a^k(x,t)=0]$ since the differential operator appearing in Eq. (1) can have zero modes. This fact renders the quantization procedure of Ref. 2 incomplete since its formal solution involves a functional integral over all configurations A_a^k (subject to $\nabla^k A_a^k=0$), including those for which D_A is ill defined. Subsequent analyses[3]) have exposed additional technical features of the Coulomb gauge and there have been various proposals on how to solve the "Gribov zero-mode problem" in the fundamental formulation of the theory.[4,5] In this note we examine solutions to Eq. (1) for the configurations

$$A_a^k(\vec{x})=-\frac{i}{g}\epsilon^{kai}\frac{x^i}{r^2}\alpha(r),\quad r=|\vec{x}|\ , \tag{2}$$

with $\alpha(r)$ a real-valued function. This choice is motivated by the growing suspicion, based on general arguments[6] and from recent developments in lattice gauge theories[7] that the confinement phenomenon is to be associated with a kind of magnetic degeneracy of the ground state. For were we to calculate the magnetic energy $\mathcal{H}_M$ associated with A_a^k we would find[8]

$$\mathcal{H}_M=\frac{1}{2g^2}\int d^3x\left\{\frac{[\alpha(r)^2-2\alpha(r)]^2}{r^4}+\frac{2}{r^2}\left[\frac{d\alpha(r)}{dr}\right]^2\right\}. \tag{3}$$

Choosing $\alpha(r)$ so that the integral exists implies "almost degeneracy" at least for large coupling. Choosing $\alpha(r)=2$ (A_a^k a pure gauge[8]) implies "complete degeneracy," since $\mathcal{H}_M=0$ *for all values of the coupling*.

Our first objective will be to investigate D_A for the case $\alpha=$ constant and to display the zero-mode phenomenon as an explicit function of α. It will turn out that for the case $\alpha=1$ (infinite-energy Wu-Yang monopole) D_A is well defined, while for the case $\alpha=2$ (zero-energy pure gauge) D_A develops a single zero mode.

Our second objective is a brief discussion of zero modes for the $\alpha(r)$'s having $\mathcal{H}_M<\infty$.

THE MULTIPOLE EXPANSION

Substituting Eq. (2) in Eq. (1) we find

$$-\nabla^2\delta_{ab}+\frac{\alpha(r)}{r^2}L^iT^i_{ab}\ D_A^{bc}(\vec{x},\vec{x}')=\delta^{ac}\delta^3(\vec{x}-\vec{x}')\ , \tag{4}$$

with the orbital angular momentum $L^i=-i\epsilon^{ijk}x^j\nabla^k$ and T^i_{aib}, the spin-1 representation of SU(2). Our strategy now becomes clear. The differential operator in (4) may be thought of as the Hamiltonian of a nonrelativistic particle of spin $T=1$ moving in the spin-orbit potential

$$V=\frac{\alpha(r)}{r^2}\vec{\mathrm{L}}\cdot\vec{\mathrm{T}}\ . \tag{5}$$

Define

$$H\equiv-\nabla^2 1+\frac{\alpha(r)}{r^2}\vec{\mathrm{L}}\cdot\vec{\mathrm{T}}\ , \tag{6}$$

where the matrices 1 and $\vec{\mathrm{T}}$ act on the intrinsic spin indices. Because of the spin-orbit coupling neither L nor T are constants of the motion, but the total angular momentum $J_i=L_i+T_i$ is, i.e.,

$$[H,J^i]=0\ . \tag{7}$$

Recalling that in spherical coordinates

$$-\nabla^2=-\left(\frac{i}{r}\frac{\partial}{\partial r}r\right)^2+\frac{L^2}{r^2}, \tag{8}$$

H can be written

$$H=-\left(\frac{1}{r}\frac{\partial}{\partial r}r\right)^2+\frac{1}{r^2}\left[L^2+\frac{\alpha(r)}{2}(J^2-L^2-T^2)\right]. \tag{9}$$

The operators J^2, L^2, T^2 commute among themselves and with H, so maximum use will be made of the symmetry of the problem if we expand D_A in eigenfunctions of J^2, L^2, and T^2. To this end, we define vector spherical harmonics[9] as

$$Y^a_{JLM}(\Omega)=\sum_{m=-L}^{+L}\sum_{n=-1}^{+1}Y_{Lm}(\Omega)e^a_n\langle Lm1n|JLM\rangle \qquad (10)$$

with Y_{LM} the usual spherical harmonics, e^a_n orthonormal eigenvectors obeying $T^{ab}_3 e^b_n = ne^a_n$, $T^2e^a = 2e^a$, $\delta^{ab}=\sum_n e^a_n e^{b*}_n$, $e^{a*}_n=\delta_{nn'}$, and $\langle Lm1n|JLM\rangle$ a Clebsch-Gordan coefficient. Since the intrinsic spin T has $T^2=t(t+1)=2$, i.e., $t=1$, $Y^a_{JLM}=0$, unless $L=J$ or $J\pm 1$ with the exception of $J=0$ for which L must equal 1.

From Eq. (10) it follows that

$$\int d\Omega\sum_a Y^{a*}_{JLM}(\Omega)Y^a_{J'L'M'}(\Omega)=\delta_{JJ'}\delta_{LL'}\delta_{MM'} \qquad (11)$$

and

$$\sum_{J=0}^{\infty}\sum_{L=J-1}^{J+1}\sum_{M=-J}^{+J}Y^a_{JLM}(\Omega)Y^{b*}_{JLM}(\Omega')=\delta^{ab}\delta(\Omega-\Omega'), \qquad (12)$$

with the proviso that for $J=0$, $L=1$. Expanding

$$D^{ab}_A(\vec{x},\vec{x}')\equiv\sum_{JLM}Y^a_{JLM}(\Omega)g^L_J(r,r')Y^{b*}_{JLM}(\Omega') \qquad (13)$$

and using Eq. (11), it follows from Eq. (9) that

$$\left[-\left(\frac{1}{r}\frac{\partial}{\partial r}r\right)^2+\frac{1}{r^2}\left(L(L+1)+\frac{\alpha(r)}{2}[J(J+1)-L(L+1)-2]\right)\right]g^L_J(r,r')=\frac{\delta(r-r')}{rr'}. \qquad (14)$$

For arbitrary $\alpha(r)$, Eq. (14) cannot be solved in general, so to get a feel for what is going on we specialize $\alpha(r)=\alpha=$ const and provisionally define

$$N(N+1)\equiv L(L+1)+\frac{\alpha}{2}[J(J+1)-L(L+1)-2]. \qquad (15)$$

In terms of N, formal particular solutions to the homogeneous version of Eq. (14) are r^N, $r^{-(N+1)}$; so provided N is real a standard argument[10] gives

$$g^L_J(r,r')=\frac{1}{2N+1}\frac{(r_<)^N}{(r_>)^{N+1}} \qquad (16)$$

with r indicating the larger (smaller) of r and r', respectively. Hence a formal solution for D_A is

$$D^{ab}_A(\vec{x},\vec{x}')=\sum_{JLM}Y^a_{JLM}(\Omega)\frac{1}{2N+1}\frac{(r_<)^N}{(r_>)^{N+1}}Y^{b*}_{JLM}(\Omega') \qquad (17)$$

with N given by 15. This can be compared with the "free" Coulomb potential

$$\frac{\delta^{ab}}{4\pi|\vec{x}-\vec{x}'|}=\delta^{ab}\sum_{Lm}Y_{Lm}(\Omega)\frac{1}{2L+1}\frac{(r_<)^L}{(r_>)^{L+1}}Y^*_{Lm}(\Omega'), \qquad (18)$$

to which Eq. (17) reduces as $\alpha\to 0$. The entire influence of the "monopole" is isolated in the distinction between N and L.

Now the Coulomb Green's function D_A was originally introduced[2] to enable one to express the longitudinal electric field via Gauss's law as an explicit function of the transverse degrees of freedom A^k_a, E^k_{Ta} $(\nabla^k E^k_{Ta}=0)$ and the charge density k^0_a contributed by other (for example, spinor) degrees of freedom. Formally one has $E^k_L=-\nabla^k\Psi_a(\vec{x},t)$, where

$$\Psi_a(\vec{x},t)=\int d^3X'D^{ab}_A(\vec{x},\vec{x}')\rho^b(\vec{x}',t) \qquad (19)$$

with

$$\rho^b(\vec{x},t)=gA^k_a\epsilon_{abc}E^k_{Tc}+g\bar{\chi}\gamma^0\tfrac{1}{2}\tau^b\chi. \qquad (20)$$

This charge density ρ is Hermitian (real) so the physical requirement that E_L be real means that D_A must be real. It follows from the properties of the Y_{JLM} that *this can only be the case if N in Eq. (15) is real.* But is it? Solving Eq. (15) for N we find

$$2N_\pm+1=\pm\left[1+4\left(L(L+1)+\frac{\alpha}{2}[J(J+1)-L(L+1)-2]\right)\right]^{1/2}, \qquad (21)$$

so the reality of N is the condition that

$$\left[1+4\left(L(L+1)+\frac{\alpha}{2}[J(J+1)-L(L+1)-2]\right)\right]\geq 0. \qquad (22)$$

One notices that if the left-hand side in (22) is zero, $2N+1=0$ and Eq. (17) is no longer a well-defined solution to (4). To find the constraints on α that guarantee N be real, we distinguish four cases.

Case 1: $J=0$; $L=T=1$. From (22) it follows that

$$\alpha\leq\tfrac{9}{8}. \qquad (23)$$

Case 2: $J\geq 1$; $L=J+1$. Again from (22),

$$\alpha\leq\frac{1+4(J+1)(J+2)}{4(J+2)}. \qquad (24)$$

Case 3: $J \geq 1$; $L = J$.

$$\alpha \leq 1 + 4J(J+1)\,. \tag{25}$$

Case 4: $J \geq 1$; $L = J - 1$.

$$0 \leq 1 + 4(J-1)(\alpha + J)\,. \tag{26}$$

Thus N in Eq. (15) is guaranteed to be real provided that α belongs to the interval

$$-\tfrac{9}{4} < \alpha < \tfrac{9}{8}\,. \tag{27}$$

The upper bound is already visible in (23), while the lower bound follows from (26) for $J = 2$. It has been noted in Ref. 4 that for $\alpha \geq \frac{9}{8}$, zero modes of the differential operator in Eq. (4) appear. What we have derived here is a more stringent condition that does not implicitly assume that $J = 0$ as is the case in Ref. 4. For α sufficiently negative, zero modes appear, showing up for the first time in the $J = 2$, $L = 1$ partial wave.

ZERO MODES

To see the connection of the restrictions (25) and (27) with the appearance of zero modes, consider the eigenvalue problem for the operator H[Eq. (9)],

$$H\phi_E(\vec{x}) = E\phi_E(\vec{x})\,. \tag{28}$$

Assuming that the ϕ_E's form a complete set one has formally[1,5]

$$D_A(\vec{x},\vec{x}') = \sum_{\{E\}} \frac{\phi_E(x)\bar{\phi}_E(x')}{E}\,, \tag{29}$$

with the $\sum$ over the *entire* spectrum $\{E\}$—bound state plus continuum of H. Equation (29) clearly exhibits the zero-mode problem as the potential divergence of $\sum_{\{E\}}$ depending on the density of eigenvalues near $E = 0$. As before, it is useful to seek solutions corresponding to definite J and L, so

$$\phi_E^a(x) \equiv Y_{JLM}^a(\Omega) f_J^L(r) \tag{30}$$

and introducing two auxiliary functions $U_J^L(r)$ and $h_J^L(r)$ by

$$f(r) = \frac{U(r)}{\sqrt{r}} = \frac{h(r)}{r}\,, \tag{31}$$

where we will not write the J and L dependence any more to save space. Defining λ by

$$\lambda \equiv N(N+1) = L(L+1) + \frac{\alpha}{2}[J(J+1) - L(L+1) - 2]\,, \tag{32}$$

the functions f, U, and h satisfy

$$\left(-\frac{1}{r}\frac{d^2}{dr^2}r + \frac{\lambda}{r^2}\right)f(r) = Ef(r)\,, \tag{33a}$$

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \left(E - \frac{\lambda + 1/4}{r^2}\right)\right]U(r) = 0\,, \tag{33b}$$

$$\left(-\frac{d^2}{dr^2} + \frac{\lambda}{r^2}\right)h(r) = Eh(r)\,. \tag{33c}$$

From (33c) it follows that our problem is equivalent to the one-dimensional Schrödinger problem on the half line $[0,\infty)$ for the *singular potential* λ/r^2. From (33b), which is recognized as the generic form of Bessel's equation, it is clear that a general solution must be linear combinations of the particular solutions $J_{\pm(\lambda+1/4)}(\sqrt{E}r)$. It is, however, well known[11,12] that owing to the strong singularity at $r = 0$ the eigenvalue problem defined by (33c) is underdetermined for sufficiently negative (attractive) λ. This is the quantum-mechanical version of the familiar "spiraling into the center" phenomenon characteristic of the attractive $1/r^2$ potential in classical mechanics. Following Ref. 11 we must distinguish three cases.

$0 \leq \lambda$. The potential in (33c) is repulsive; the spectrum consists of a continuum beginning at $E = k^2 = 0$ and extending to $+\infty$ with eigenfunction $J_\nu(kr)$, where $k = +\sqrt{E}$ and $\nu = +(\lambda + 1/4)^{1/2}$. The Bessel functions obey continuum normalization

$$\int_0^\infty r\,dr\,J_\nu(kr)J_\nu(k'r) = \frac{\delta(k-k')}{(kk')^{1/2}} \tag{34a}$$

and are complete[12] in the sense

$$\int_0^\infty k\,dk\,J_\nu(kr)\,J_\nu(kr') = \frac{\delta(r-r')}{(rr')^{1/2}}\,. \tag{34b}$$

By analogy with (28) and (29) we can use (34b) to solve (14). If we call

$$g(r,r') = \frac{\tilde{g}(r,r')}{(rr')^{1/2}}\,, \tag{35}$$

then

$$\left(-\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{\lambda + 1/4}{r^2}\right)\tilde{g}(r,r') = \frac{\delta(r-r')}{(rr')^{1/2}}\,. \tag{36}$$

So expressing the right-hand side of (36) according to (34b) and comparing with (33b) yields

$$g(r,r') = \int_0^\infty \frac{dk}{k}\frac{J_\nu(kr)}{\sqrt{r}}\frac{J_\nu(kr')}{\sqrt{r'}}\,. \tag{37}$$

This discontinuous integral may be evaluated explicitly[13] to yield

$$g(r,r') = \frac{1}{2\nu}\frac{(r_<)^{\nu-1/2}}{(r_>)^{\nu+1/2}}\,, \tag{38}$$

which agrees with (16) provided $\nu = N + \frac{1}{2}$, which also follows from $\nu^2 = \lambda + \frac{1}{4} = N(N+1) + \frac{1}{4}$. So everything is consistent. In particular the spectral integral in (37) exists near $k \approx 0$ because for $\lambda > 0$, $\nu > \frac{1}{2}$, $J_\nu(kr) \sim (kr)^\nu$ for small kr. In fact, the spectral integral is seen to exist as long as $\nu > 0$, which is the clue to a correct treatment of the eigenvalue problem defined by (33c) in the region.

$-\frac{1}{4}<\lambda<0$. Since λ is negative in (33c) it is at least conceivable that some kind of bound state could appear. The existence of a bound state stands or falls with the choice of boundary condition at $r=0$. It is shown in Ref. 12 that the natural boundary condition at $r=0$ involves simply analytically continuing from $\lambda>0$ into the region $-\frac{1}{4}<\lambda<0$. Nothing stands in the way of this continuation, so the spectrum consists of the positive real axis $E>0$ as before with $g(r,r')$ given by (38) with, however, $0<\nu<\frac{1}{2}$.

Thus no zero-mode problem has been encountered as long as $-\frac{1}{4}<\lambda$, which is easily seen to be equivalent to Eq. (22) and hence Eqs. (23)–(26).

$\lambda<-\frac{1}{4}$. As the potential in (33c) becomes even more attractive so that $\lambda<-\frac{1}{4}$, an essentially new phenomenon appears; ν becomes pure imaginary and the two independent solutions to (33b) become $J_{\pm i(|\lambda+1/4|)^{1/2}}(\sqrt{E}r)$. The point $-\frac{1}{4}$ is a branch point in the λ plane so that the analytic continuation recipe for extending the boundary condition information at $r=0$ is ambiguous. Case[11] has shown that a well-defined orthonormal system of eigenfunctions is obtained by specifying the relative phase between $J_{\pm i(|\lambda+1/4|)^{1/2}}(\sqrt{E}r)$ as $r\to 0$. This is accomplished by demanding that all solutions of (33b) behave as

$$U(r)\underset{r\to 0}{\sim}(\text{const})\cos(|\lambda+\tfrac{1}{4}|^{1/2}\ln r+\theta)\,, \tag{39}$$

where the multiplicative constant varies from solution to solution but the r dependence is as given. θ is a phase which removes the continuation ambiguity mentioned above. To construct an explicit orthonormal system of eigenfunctions, we first define

$$\nu=+|\lambda+\tfrac{1}{4}|^{1/2}\,, \tag{40a}$$

$$\Gamma(1+i\nu)=|\Gamma(1+i\nu)|\exp(i\gamma_\nu)\,, \tag{40b}$$

$$\beta_\nu(k)=\gamma_\nu-\nu\ln\frac{k}{2}+\theta\,, \tag{40c}$$

$$\beta_\nu(n)=\pi(n+\tfrac{1}{2})-\gamma_\nu,\quad n=0,\pm1,\pm2,\ldots \tag{40d}$$

$$\kappa_n=\exp\left[\frac{\theta+\nu\ln 2+\gamma_\nu-\pi(n+\frac{1}{2})}{\nu}\right], \tag{40e}$$

where $\Gamma(1+i\nu)$ denotes the usual Γ function and γ_ν its real phase. We find[11,12] the continuum and bound-state eigenfunctions $U_\theta(kr)$ and $U_\theta(\kappa_n r)$ to be

$$U_\theta(kr)=\frac{J_{i\nu}(kr)e^{i\beta_\nu(k)}+J_{-i\nu}(kr)e^{-i\beta_\nu(k)}}{\{2[\cos 2\beta_\nu(k)+\cosh\pi\nu]\}^{1/2}} \tag{41}$$

and

$$U_\theta(\kappa_n r)=\frac{\kappa_n}{\sqrt{2\nu}}[\Gamma(1+i\nu)I_{i\nu}(\kappa_n r)e^{i\beta_\nu(n)} + \Gamma(1-i\nu)I_{-i\nu}(\kappa_n r)e^{-i\beta_\nu(n)}]\,, \tag{42}$$

where $I_{i\nu}(\kappa_n r)$ is the Bessel function of pure imaginary index and *pure imaginary argument*.[14] The *continuous spectrum* runs from $E=k^2=0$ to $E=k^2=+\infty$. The k appearing in (41) then may be an arbitrary positive real number.

The *discrete spectrum* $E_n=-\kappa_n^2$ is a sequence of real values beginning at $E=0$ (accumulation point) ($n=+\infty$) and extending to $E=-\infty$ ($n=-\infty$). These numbers are determined by the requirement that as $r\to\infty$, the bound-state wave function $U(\kappa r)$ behaves as $\exp(-\kappa r)$, $\kappa>0$. Thus the spectrum is not bounded below, this being the quantum-mechanical counterpart of the "classical spiraling into the center" phenomenon already mentioned.[15] Note also that the discrete spectrum

$$E_n(\theta)=-\exp\left\{\frac{2}{\nu}[\theta+\nu\ln 2+\gamma_\nu-\pi(n+\tfrac{1}{2})]\right\} \tag{43}$$

is invariant to the substitutions $\theta\to\theta+N\pi$ (N being an arbitrary integer) in the sense that

$$E_n(\theta+N\pi)=E_{n\to N}(\theta) \tag{44}$$

and that (42) implies

$$U_{\theta+N\pi}(\kappa_n(\theta+N\pi)r)=(-1)^N U_\theta(\kappa_{n-N}(\theta)r)\,, \tag{45}$$

while

$$U_{\theta+N\pi}(kr)=(-1)^N U_\theta(kr) \tag{46}$$

follows from (41). Thus, adding $N\pi$ to θ merely reshuffles the eigenvalues and eigenfunctions among themselves, i.e., a trivial renaming and hence may be ignored. In all that follows we assume that $0\leqslant\theta\leqslant\pi$.

The multiplicative factors in (41) and (42) have been chosen so that $U_\theta(kr)$ and $U_\theta(\kappa_n r)$ are *real* and *orthonormal* in the sense

$$\int_0^\infty r\,dr\,U_\theta(\kappa_n r)U_\theta(\kappa_{n'}r)=\delta_{nn'}\,, \tag{47a}$$

$$\int_0^\infty r\,dr\,U_\theta(kr)U_\theta(k'r)=\frac{\delta(k-k')}{(kk')^{1/2}}\,, \tag{47b}$$

$$\int_0^\infty r\,dr\,U_\theta(kr)U_\theta(\kappa_n r)=0\,. \tag{47c}$$

The analog of (34b) (completeness) reads

$$\frac{\delta^\theta(r-r')}{(rr')^{1/2}}=\sum_{n=-\infty}^{\infty}U_\theta(\kappa_n r)U_\theta(\kappa_n r') + \int_0^\infty k\,dk\,U_\theta(kr)U_\theta(kr')\,, \tag{48}$$

where the θ on δ^θ is to remind us that the right-hand side acts like a δ function on the space of functions obeying (39). To obtain a formal expression for $g(r,r')$ we proceed as in (35), (36), and (37) and use (48) to find

$$(rr')^{1/2}g(r,r') = \sum_{n=-\infty}^{\to\infty} \frac{U_\theta(\kappa_n r)U_\theta(\kappa_n r')}{-\gamma_n^{\,2}} + \int_{\downarrow 0}^{\infty} \frac{dk}{k} U_\theta(kr)U_\theta(kr') , \qquad (49)$$

where the notation $\to\infty$ and $\downarrow 0$ is to remind us that contributions from the zero "energy" region are potentially divergent and may have to be cut off. To estimate the degrees of divergence, note that

$$U_\theta(\kappa_n r) \xrightarrow[\kappa_n r\to 0]{} \sqrt{2}\,\frac{\kappa_n}{\nu}\cos[\nu \ln r + \theta] \qquad (50a)$$

and

$$U_\theta(kr) \xrightarrow[kr\to 0]{} \frac{\sqrt{2}}{[\cos 2\beta_\nu(k) + \cosh\pi\nu]^{1/2}}\cos[\nu \ln r + \theta] , \qquad (50b)$$

where cos··· is as required by (39) and the respective constants have been determined by the normalization constraints (47a) and (47b). This means the following.

Case 1. The contributions $\Delta^c(r,r')$ to $(rr')^{1/2} g(r,r')$ from the small-k region of the continuum $\int$ for *fixed* finite r,r' is

$$\Delta^c(r,r') = \int_{\downarrow 0} \frac{dk}{k}\,\frac{2}{\cosh\pi\nu + \cos 2\beta_\nu(k)} \times \cos[\nu \ln r + \theta]\cos[\nu \ln r' + \theta] . \qquad (51)$$

For $\nu > 0\,(\lambda < -\frac{1}{4})\cosh\pi\nu > |\cos 2\beta_\nu(k)|$ so that $\Delta^c(r,r')$ is *logarithmically divergent*.

This logarithmic divergence persists to the point $k=0$ since

$$\Delta^c(r,r') = \int_{\downarrow 0} \frac{dk}{k}\,\frac{2\cos^2\theta}{1+\cos 2\theta} , \qquad (52)$$

so we conclude that the generic degree of divergence *in* $(rr')^{1/2}g(r,r')$ *from the small-k regions of the continuum is logarithmic*.

Case 2. The contribution $\Delta^{BS}(r,r')$ to $(rr')^{1/2} g(r,r')$ from the small-E_n region of the discrete spectrum is

$$\Delta^{BS}(r,r') = -\frac{2}{\nu^2}\sum_{n=\mathfrak{N}}^{\to\infty} \cos[\nu \ln r + \theta]\cos[\nu \ln r' + \theta] , \qquad (53)$$

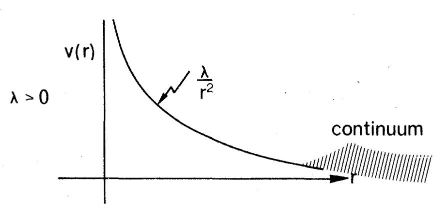

FIG. 1. $V(r)$ for the case $0<\lambda$.

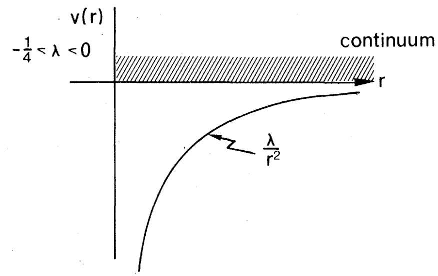

FIG. 2. $V(r)$ for the case $-\frac{1}{4}<\lambda<0$.

where $\mathfrak{N}$ is a large positive integer and for all $n \geq \mathfrak{N}$ we have replaced $U_e(\kappa_n r)$ by its small $\kappa_n r$ behavior (50a). The $\sum$ in (53) *diverges linearly* as all dependence of the summand on n cancels for n sufficiently large. Thus *the accumulation of discrete states at $E=0$ implies a linear divergence in $(rr')^{1/2}g(r,r')$ for fixed, finite r,r'.*

Note also the relative minus sign between (53) and (51). This intrinsic sign difference is already visible in (49) as the eigenvalues of discrete states are negative relative to the continuum.

Before discussing the physical relevance of the above, let us summarize the results. Write (33c) as

$$\left[-\frac{d^2}{dr^2} + V(r)\right]h(r) = Eh(r) , \qquad (54)$$

which is one-particle quantum mechanics on $[0,\infty)$. Up until now we have concentrated on potentials $V(r) = r^{-2}\lambda(r)$, with $\lambda(r)$ given in (32). For $\lambda=$ const two essentially different situations present themselves. The case $0<\lambda$ possesses as expected a continuum beginning at $E=k^2=0$ and extending to ∞ (Fig. 1). For $-\frac{1}{4}<\lambda<0$, the potential $V(r)$ is attractive, but not strong enough to produce a bound state (Fig. 2). For $\lambda<-\frac{1}{4}$, all resistance to bound-state formation is broken and one obtains the usual continuum plus an infinite sequence of negative eigenvalues extending to $-\infty$ and having a point of accumulation at 0 (Fig. 3).

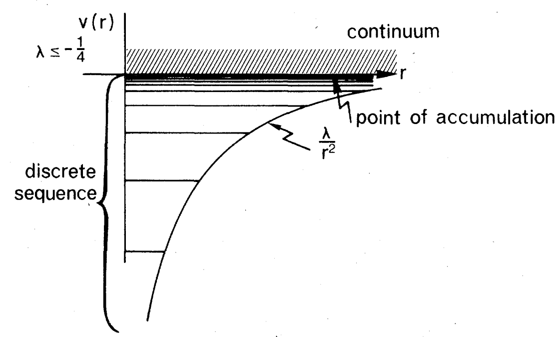

FIG. 3. $V(r)$ for the case $\lambda<-\frac{1}{4}$.

The accumulation of discrete states at zero energy is responsible for the leading (linear) divergence in $D(\vec{x},\vec{x}')$. However, $\lambda=\text{const}$ implies $\alpha=\text{const}$ which implies that the magnetic configurational energy (3) is ∞ unless $\alpha=0,2$. A necessary condition for finite $\mathcal{H}_M$ is therefore $\alpha(0)=0$ [the $\alpha(0)=2$ case being subsumed in the cases already treated]. As an illustration write

$$V(r)=\frac{L(L+1)}{r^2}+\frac{\alpha(r)}{r^2}\,\frac{J(J+1)-L(L+1)-2}{2}\,, \tag{55}$$

demand $\alpha(0)=0$ (finite energy) and $\alpha(\infty)=0$ (no monopole tail at large distances), and choose

$$\alpha(r)=-\frac{r^2 r_0^{\,2}}{(r^2+r_0^{\,2})^2}\,\frac{2(2L+1)(2L+3)}{[J(J+1)-L(L+1)-2]}\,,\quad L\geq 1 \tag{56}$$

so that

$$V(r)=\frac{L(L+1)}{r^2}-\frac{(r_0)^2}{(r^2+r_0^{\,2})^2}(2L+1)(2L+3)\,, \tag{57}$$

where r_0 is an arbitrary length which is necessary if the above $\alpha(r)$ is to lead to finite $\mathcal{H}_M$. $V(r)$ is depicted in Fig. 4. One verifies that the wave function

$$h(r)=(\text{const})\frac{r^{L+1}}{(r^2+r_0^{\,2})^{L+1/2}} \tag{58}$$

is a normalizable (square integrable) solution to (54) with (r) given by (57) having eigenvalue $E=0$. In fact, $V(r)$ [and hence $\alpha(r)$] is constructed by using (58) as an Ansatz, demanding that $E=0$ and using (54) to "solve for the potential." This example[16] *precludes* the possibility that a monopole tail $[\alpha(\infty)\neq 0]$ is necessary if $D_A(\vec{x},\vec{x}')$ is to receive (divergent) enhanced contributions from the $E\approx 0$ region of the spectrum in (29). Such discrete zero modes imply (at least) as divergent contributions as the accumulation point phenomenon in the case $\alpha=\text{const}$. Obvious generalizations to (56)

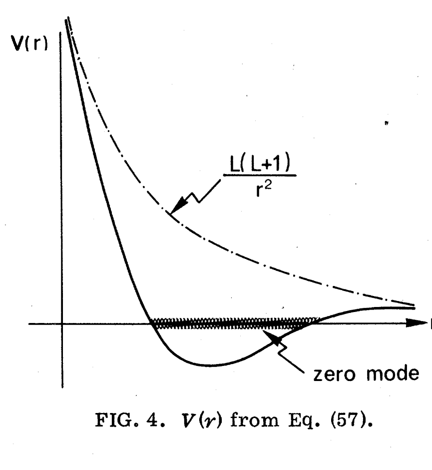

FIG. 4. $V(r)$ from Eq. (57).

(56), (57), and (58) corresponding to $\alpha(\infty)\neq 0$ are possible but do not bring anything new so we will not discuss them here (see, however, Ref. 16).

DISCUSSION

We motivated this note by referring to the growing suspicion that quark confinement is caused by a magnetic degeneracy of the ground state.[6,7] Configurations of the type given by Eq. (2) cost relatively little magnetic energy if we take $\alpha(r)=2$ (pure gauge) or the $\alpha(r)$ given by (56). In fact, we may generalize to the configurations

$$A_a^k(\vec{x})=-\frac{1}{g}\sum_{i=1}^{N}\epsilon^{kaj}\frac{(\vec{x}-\vec{x}_{(i)})^j}{|\vec{x}-\vec{x}_{(i)}|^2}2\,, \tag{59a}$$

$$A_a^k(\vec{x})=-\frac{1}{g}\sum_{i=1}^{N}\epsilon^{kaj}\frac{(\vec{x}-\vec{x}_{(i)})^j}{|\vec{x}-\vec{x}_{(i)}|^2}\alpha(|\vec{x}-\vec{x}_{(i)}|) \tag{59b}$$

with $\{x_{(i)}\}$ a set of N three-vectors—the positions of the cores of A and α in (59b) are given by (56). In both cases the magnetic energy is finite, the contributions from the core regions converging since A approaches a pure gauge (59a) near any of the $\{x_{(i)}\}$ and (59b) has $\alpha(|\vec{x}-\vec{x}_{(i)}|)=0$ there. Furthermore, by taking the $\text{Min}_{i\neq j}|x_{(i)}-\hat{x}_{(j)}|$ larger and larger, we can make M from (59a) as small as we please, in particular smaller than $\mathcal{H}_M$ from (59b). Thus from the point of view of energetics, (59a) would seem to be preferred. One must not forget, however, that the (relatively) short-range character in (59b) $[\alpha(\infty)=0]$ means that we can pack the points $\{\vec{x}_{(i)}\}$ closer together at relatively little cost in energy while the $\mathcal{H}_M$ from (59a) grows dramatically as any $x_{(i)}$ approaches another $x_{(i)}$. Thus one would expect more configurations of type (59b) and this "density of states" advantage may offset the pure energy advantage of (59a). We suggest that both types of configurations will be important for understanding vacuum structure in the Coulomb gauge[1,17-19] and that pure-gauge boundary conditions at *small* distances should be reexamined. In fact,

$$\frac{1}{i}\frac{\tau^a}{2}\left(-2\epsilon^{kai}\frac{xi}{|\vec{x}|^2}\right)=U(\vec{x})\partial^k U^1(\vec{x}) \tag{60}$$

if

$$U(\vec{x})=\exp\left(i\frac{\pi}{2}\frac{\vec{x}}{|\vec{x}|}\cdot\vec{\tau}\right), \tag{61}$$

showing that the case $\alpha(r)=2$ is derived from a singular gauge transformation since $\vec{x}/|\vec{x}|$ is not defined at $\chi=0$. One usually dismisses the behavior (61) at small $\vec{x}$ in order to secure a topologically well-defined classification of pure gauges.[19] However, the increased prominence of such singular gauge transformations[6] in recent years suggests this to be unwise. This would mean incidentally that the Coulomb propagator $D_A(\vec{x},\vec{x}')$ would be expected to show anomalous *small*-distance behavior due to the presence of the discrete spectrum for $\alpha(0)=2$. This anomaly is above and beyond that associated with zero modes. This small-distance behavior would depend on θ introduced in (39), while those contributions arising from the $E\approx 0$ region in (29) are largely independent of θ.

Our last remark is that for x sufficiently far away from all $x_{(i)}$ in (59a) one has

$$A_a^k(\vec{x}) \rightarrow -\frac{1}{g}\epsilon^{kaj}\frac{x^j}{|x|^2}2N\,, \tag{62}$$

indicating that configurations with arbitrarily high $\alpha(\infty)$ are relevant. From (23)–(26) it follows that higher and higher partial waves begin developing accumulation points at $E=0$. Any attempt to estimate the vacuum expectation value of the longitudinal-electric-field energy [see (19)] must take into account these higher partial waves and not just those corresponding to $J=0$.[4,16] We will return to this question elsewhere.

INTERPRETATION AND SUMMARY

We have investigated the Wu-Yang configurations of Eq. (2). For the case α constant and $-\frac{9}{4}<\alpha<\frac{9}{8}$, there are no zero modes and an explicit multipole expansion can be given. For $\alpha<-\frac{9}{4}$ or $\frac{9}{8}<\alpha$ zero modes appear, and make formally divergent contributions to $D_A(\vec{x},\vec{x}')$. As $|\alpha|$ increases, more and more partial waves of definite J and L develop this pathology and the precise conditions under which it happens are determined. The Wu-Yang monopole ($\alpha=1$) has no zero modes while the pure gauge ($\alpha=2$) has precisely one in the $J=0$, $L=1=T$ channel. The small-distance boundary conditions for this last case are elucidated and found to depend on an angle θ with values between 0 and π. For physical applications one may have to imagine averaging over θ as there is no obvious physical argument to determine it. This remains an open question.

We close this note by tentatively answering the following question: How are the zero-mode divergences in $D_A(\vec{x},\vec{x}')$, that we have gone through all the trouble of isolating, to be interpreted?

The answer lies in the expression for the longitudinal-electric-field energy[2]

$$\mathcal{H}_E^L=\frac{1}{2}\int d^3x\sum_{a,k}\left[-\int d^3x'\,\nabla_x^k D_A^{ab}(\vec{x},\vec{x}')\rho^b(\vec{x}',t)\right]^2, \tag{63}$$

with the total effective charge density

$$\rho^b(\vec{x},t)=gA_c^b\epsilon_{cbd}E_d^{Tk}+g\bar{\chi}\gamma^0\tfrac{1}{2}\tau^b\chi\,. \tag{20}$$

If D_A in (63) has zero modes, $\mathcal{H}_E^L$ will be infinite if $\rho^b(\vec{x}',t)$ is not orthogonal to those modes. Thus we see the role of zero modes not to be that of rendering the Coulomb-gauge formulation[2] inconsistent,[5] but rather as specifying a set of conditions which a given configuration $A, E^T, \chi, \bar{\chi}$ must satisfy if it is to be represented in the vacuum wave functional with nonvanishing probability. The effective charge density ρ_A^a formed in that configuration must obey

$$\int d^3x\,\phi_{A0}^a(\vec{x})\rho_A^a(\vec{x},t)=0 \tag{64}$$

with ϕ_{A0}^a a zero mode belonging to A. Thus the zero modes define a kind of projection operator onto physically relevant configurations. Of course, the practical utility of this notion remains minimal until an efficient complete characterization of the zero-mode subspace of all configurations has been made and this analysis does not lie before us (see, however, Ref. 20). But two things are sure. (i) Adding a quark point charge distribution [the limit where $g\bar{\chi}\gamma^0\frac{1}{2}\tau^b\chi$ in (20) is replaced by $n^b(g/2)\delta^3(\vec{x})$, with n^b a unit isovector] into the vacuum is energetically impossible due to the divergences coming from (63), as there is no way that $\delta^3(x)$ can be orthogonal to *all* zero modes. (ii) Charge *fluctuations* originating in the terms $gA_c^k\epsilon_{cbd}E_d^{Tk}$ are also suppressed indicating a strong tendency for the transverse electric field fluctuations to be parallel to A (in isospace). We shall return to these intriguing questions elsewhere.

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