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THE MAKEENKO-MIGDAL EQUATION IN A DOMAINED QCD VACUUM

P. OLESEN and J.L. PETERSEN

The Niels Bohr Institute, University of Copenhagen, DK-2100 Copenhagen Ø, Denmark

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We present a solution for simple curves of the Makeenko-Migdal equation which leads to string dynamics. By the non-renormalized nature of the equation, any formal solution requires some extra physics input before it can be meaningfully interpreted. We then propose a "minimal" renormalization both of the Wilson loop for a large number of colors and of the corresponding Makeenko-Migdal equation. This scheme is based on Witten's idea of a master field. The regularized solution contains a scale (related to the Regge slope) which we connect to a gauge-invariant domain size in the QCD vacuum, and the solution can only make sense for loops that are large relative to this scale. Using a vortex condensate model for the QCD vacuum, the length parameter is related to the QCD length scale.

1. Introduction

Makeenko and Migdal [1] have shown that the unrenormalized loop average

$$W[\mathrm{C}]=\frac{1}{N}\left\langle \mathrm{Tr}\,\mathrm{P}\exp\left(\oint_{\mathrm{C}} A_\mu(x)\,\mathrm{d}x_\mu\right)\right\rangle \tag{1}$$

satisfies the equation

$$\frac{\partial}{\partial x_\mu}\frac{\delta W[\mathrm{C}]}{\delta\sigma_{\mu\nu}(x)}=Ng_0^2\oint_{\mathrm{C}}\mathrm{d}y_\nu\,\delta^{(4)}(x-y)\,W[\mathrm{C}_{xy}]\,W[\mathrm{C}_{yx}] \tag{2}$$

in the $N\to\infty$ limit of multicolor SU(N) QCD.

Eq. (2) is a closed equation for W[C] and it is interesting to investigate the solutions. Recently Migdal [2] proposed a solution described in terms of non-linear Fermi strings for which he suggested the name elfin theory.

In sect. 2 we show that a very simple string theory already satisfies the equation in a certain formal sense, at least for curves which are no more complicated than the ones shown in fig. 1. We do not know whether our solution is related to Migdal's.

However, finding formal solutions to the equation can at most be a first step towards obtaining physically interesting results. This is because the equation is derived without regard to renormalization and hence the phenomenon of dimensional transmutation which is crucial to the physics content of the theory is not present. Instead the equation involves very singular contributions proportional to

$$g_0^2 N\delta^{(2)}(0). \tag{3}$$

In sect. 3 we propose to interpret this by the substitution

$$g_0^2 N\delta^{(2)}(0) \to \frac{g^2(a^2)N}{a^2}, \tag{4}$$

where a^2 is the cross section of a domain in the QCD vacuum. Models for such domains have recently been considered by several authors [3,4]. We discuss how the substitution (4) could arise in a particular renormalization scheme based on Witten's master field idea. Finally we use this connection to establish a relation between the Regge slope, the QCD scale parameter and the vacuum energy density (the "bag constant"), using the model of refs. [3] for the QCD vacuum. The numerical agreement is excellent.

We find it interesting that this interpretation of the regularized equation (and its solutions) seems to provide a connection between the loop space formulation of QCD and the idea of flux quantized vacuum structure first considered by 't Hooft [5,6] and Mack [7].

Finally let us comment on the connection between QCD and dual string models which seems to be implied by our work (assuming our solution of the MM equation is also a solution of QCD).

Dual string theory was beset with severe problems in 4 space-time dimensions. However, these arose at the unitarization level which corresponds to $1/N$ corrections in our scheme. Those corrections are strictly outside the MM equation. To treat them, one must include correlations between Wilson loops. This leads to an exact, generalized MM equation [1] which we do not know how to solve.

Another difficulty with dual string theory was that the lowest meson state was a tachyon. One might add that the low-lying mesons do not behave as quarks bounded by a structureless string. A much better approximation appears to be provided by the bag model. For example, the phenomenon of hyperfine splitting seems to have no natural treatment in terms of a structureless string, whereas it is readily taken into account in a bag picture in the form of one-gluon exchange.

In our picture we regard all these phenomena to be associated with the short-distance part of the theory, i.e. phenomena happening on a scale shorter than the domain size. Such phenomena cannot be accounted for with our regularization prescription.

It is important to realize that a solution of eq. (2) involves the knowledge of $W[C]$ for *all* curves C. Thus, since we only cover curves of the type shown in fig. 1 our

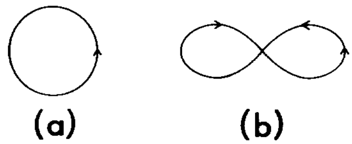

Fig. 1. Simple loops for which the minimal area surface touches the loop only at the boundary. Only for these is the very simple form eq. (10) a solution.

work is much more limited in scope. The key point in our paper is to relate the regularization to a domained vacuum.

In sect. 4 we summarize our findings and give concluding remarks.

2. A string solution of the Makeenko-Migdal equation

Our proposal for a solution (with respect to the curves in fig. 1) of the MM equation (2) together with the Bianchi identity

$$\varepsilon_{\alpha\beta\gamma\delta}\partial_\beta \frac{\delta W[\mathrm{C}]}{\delta\sigma_{\gamma\delta}(x)} = 0, \tag{5}$$

is heavily inspired by QCD_2 [8]. Recently a complete solution of the 2-dimensional MM equation has been obtained [9]. Actually for not too complicated loops it is quite easy to write down the value of the Wilson loop in QCD_2. In this paper our main concern is with loops of the form shown in fig. 1 and simple iterations thereof. We defer any detailed analysis of very complicated loops with many self-intersections till later, although we present a conjecture for a solution of this problem.

Using the axial gauge in QCD_2, the theory becomes "free" (gluons do not interact) and the "gluon-propagator" reduces to the instantaneous form

$$D_{11}^{(2)}(x) = \tfrac{1}{2}|x_2|\delta(x_1), \tag{6}$$

satisfying

$$\partial_2^2 D_{11}^{(2)}(x) = \delta^2(x).$$

From this, one easily obtains the following result for loops of the form shown in fig. 1:

$$\begin{aligned} W[\mathrm{C}]_{\mathrm{QCD}_2} &= \exp\left\{\tfrac{1}{2}g_{2\mathrm{D}}^2 N \oint_{\mathrm{C}} \mathrm{d}x_1 \oint_{\mathrm{C}} \mathrm{d}y_1 D_{11}^{(2)}(x-y)\right\} \\ &= \exp\left\{-g_{2\mathrm{D}}^2 N \tfrac{1}{2} A(\mathrm{C})\right\}, \end{aligned} \tag{7}$$

leading to the familiar expression for the Regge slope α' in QCD_2

$$\alpha' = \frac{1}{\pi g_{2D}^2 N} = \frac{1}{\pi \lambda_{2D}}, \tag{8}$$

where g_{2D} is the (dimensionful) coupling constant in QCD_2, and where $\lambda_{2D} = g_{2D}^2 N$ is kept fixed for $N \to \infty$*. $A(C)$ is the area enclosed by the loop C.

The expression (7) may be written

$$W[C]_{QCD_2} = \exp\left\{ -\tfrac{1}{4}\lambda_{2D} \int_\Sigma \int_\Sigma d\sigma_{ij}(x) d\sigma_{ij}(y) \delta^{(2)}(x-y) \right\}, \tag{9}$$

where Σ is the region enclosed by the curves in fig. 1 and $d\sigma_{ij}$ $(i,j=1,2)$ are elements of area.

This form suggests an ansatz for the Wilson-loop for similar curves in 4-dimensions:

$$W[C] = \exp\left\{ -\tfrac{1}{4}\lambda_0 \int_{\Sigma_m} \int_{\Sigma_m} d\sigma_{\alpha\beta}(x) d\sigma_{\alpha\beta}(y) \delta^4(x-y) \right\}, \tag{10}$$

where λ_0 is the dimensionless coupling kept fixed in the large-N limit, $\lambda_0 = g_0^2 N$. Here the surface element tensors have indices $\alpha, \beta = 1,2,3,4$ and Σ_m is the *minimal area surface*. As we shall see, the selection of that particular surface is mainly dictated by the Bianchi identity, eq. (5). Equivalently, the form eq. (10) may be written

$$W[C] = \exp\left\{ -\tfrac{1}{2}\lambda_0 \delta^{(2)}(0) A(\Sigma_m) \right\}, \tag{11}$$

so that

$$W[C_{xy}] W[C_{yx}] \delta^{(4)}(x-y) = W[C] \delta^{(4)}(x-y).$$

This gives a very singular expression for the Regge slope. That is not surprising. The MM equation (2) is a non-renormalized quantum equation. Hence one cannot expect the phenomenon of dimensional transmutation to take place in any other way than indicated in eq. (11). Notice that λ_0 is the bare coupling constant which is zero by asymptotic freedom. In sect. 3 we shall give an interpretation of the quantity

$$\lambda_0 \delta^{(2)}(0) \tag{12}$$

leading to a finite number in good agreement with the phenomenological Regge slope.

* Notice that any non-planar diagram vanishes identically by eq. (6).

In this section we treat the expression (12) as a well defined parameter.

The expression (10) suggests the following expression for the area derivative for which we subsequently present independent arguments:

$$\frac{\delta W[\mathrm{C}]}{\delta\sigma_{\mu\nu}(x)} = -\lambda_0 \int_{\Sigma_\mathrm{m}} \mathrm{d}\sigma_{\mu\nu}(y)\,\delta^{(4)}(x-y)\,W[\mathrm{C}]$$

$$= \begin{cases} -\lambda_0\delta^{(2)}(0)\,n_{\mu\nu}(x)\,W[\mathrm{C}], & \text{for } x \in \Sigma_\mathrm{m}, \\ 0, & \text{for } x \notin \Sigma_\mathrm{m}, \end{cases} \tag{13}$$

where $n_{\mu\nu}$ is the orientation tensor of the 2-dim tangent plane at the point x.

We would like to make several comments about this expression. For a general functional of loops we define the area derivative at an arbitrary point x in terms of the wire construction of fig. 2. At the point x a small loop is placed with orientation in the (μ,ν) plane. The loop is connected with "wires" to the main loop C. The difference for loops with opposite orientations is formed, and the result is divided by twice the area of the loop. This guarantees that an antisymmetric tensor is obtained.

Now we understand why the area derivative is zero for $x \notin \Sigma$: the area of the minimal area surface is *increased* for both terms in fig. 2 and the difference is zero.

For $x \in \Sigma$ but the test-loop orthogonal to the original surface, again the area of the deformed surface is changed in the same way for either orientation. This is in accord with eq. (13):

$$\frac{\delta W[\mathrm{C}]}{\delta\sigma_{\mu\nu}(x)} = 0, \qquad \text{for } (\mu,\nu) \text{ orthogonal to surface.} \tag{14}$$

The only case where we get a non-vanishing result is when the test-loop lies in the surface (fig. 3).

In this case we know the result for QCD_2 (see ref. [9]; again this may be easily verified using planar Feynman diagrams in the gauge of eqs. (6)). For the first term

$$W[\mathrm{C}] = \exp\{-\tfrac{1}{2}\lambda_{2\mathrm{D}}[A(\mathrm{C}) + A(\mathrm{C}')]\}(1 - \lambda_{2\mathrm{D}}A(\mathrm{C}')),$$

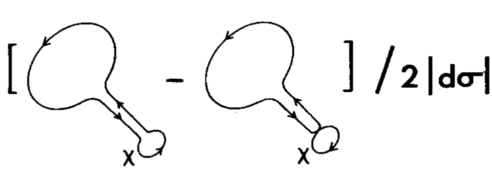

Fig. 2. Definition of the area derivative at an arbitrary point x using the "wire prescription". $|\mathrm{d}\sigma|$ is the numerical value of the area of the test loop.

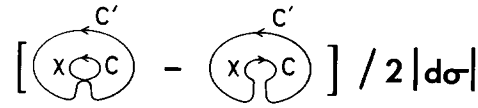

Fig. 3. Loops that need be considered to evaluate the area derivative for $x \in \Sigma_{\mathrm{m}}$ and in the direction of the local tangent plane.

and for the second

$$W[\mathrm{C}] = \exp\{-\tfrac{1}{2}\lambda_{2\mathrm{D}}[A(\mathrm{C}) - A(\mathrm{C}')]\}.$$

This shows that our simple ansatz, eq. (10), cannot be correct for loops corresponding to the first term in fig. 3*. Nevertheless, the expression (13) for the area derivative is correctly reproduced.

It is interesting to compare this expression with Mandelstam's formula:

$$\frac{\delta W[\mathrm{C}]}{\delta \sigma_{\mu\nu}(x)} = \frac{1}{N}\left\langle \mathrm{Tr}\left\{ F_{\mu\nu}(x)\,\mathrm{P}\exp \oint_x^x A_\mu(x')\,\mathrm{d}x'_\mu \right\}\right\rangle. \tag{15}$$

From eq. (13) we deduce that:

(i) the correlation between $F_{\mu\nu}(x)$ and the Wilson operator vanishes except when x belongs to Σ_{m};

(ii) for $x \in \Sigma_{\mathrm{m}}$ only certain components of $F_{\mu\nu}(x)$ have non-vanishing correlation: if Σ_{m} is a purely space-like surface, only the *magnetic* field perpendicular to that surface (at x) contributes; if Σ_{m} is a spacetime-like surface, only the component of the *electric* field along the string contributes. This is very suggestive, and conforms with standard beliefs in this field.

So far we have not had to assume that Σ_{m} is the minimal area surface. Now let us see how the Bianchi identity selects that particular surface. It is convenient to introduce coordinate directions referring to the tangent plane at the point x. Let $t_\mu^{(1)}$ and $t_\mu^{(2)}$ be orthonormal tangent vectors at x and let $n_\mu^{(1)}$ and $n_\mu^{(2)}$ be orthonormal vectors, orthogonal to the tangent plane. The properties of the area derivatives may then be expressed as:

$$\frac{\delta W[\mathrm{C}]}{\delta \sigma_{t_i n_j}(x)} = 0, \qquad \frac{\delta W[\mathrm{C}]}{\delta \sigma_{n_i n_j}(x)} = 0, \tag{16}$$

where $i,j = 1,2$ and t_i refers to components along $t^{(i)}$ and similarly with n_j. The

* This is also clear from the ansatz itself, since $\delta^4(x-y)$ becomes ambiguous for loops of the overlapping type (e.g. the first term in fig. 3).

Bianchi identity becomes the statement

$$\partial_{n_i} \frac{\delta W[\mathrm{C}]}{\delta \sigma_{t_j t_k}(x)} = 0. \tag{17}$$

Thus, moving x by a small amount in a direction normal to the surface, the area derivative (in the surface) should be unchanged.

Now we must remember that to evaluate the area derivative at the displaced x, we should surround x by a test loop and connect by wires to C. Hence, it becomes *necessary* also to *change the surface* correspondingly. From our expressions (13) and (11) we see that condition (17) is that the Wilson loop itself is unchanged or equivalently that the area is stable. But that is just the condition that Σ_{m} be the minimal area surface. To arrive at this result we needed the Bianchi identity for an arbitrary value of x, not just for x in the vicinity of the loop C. This is why we have taken some care in explaining the concept of wire derivatives*.

Writing the MM equation (2) at the point $x \in \Sigma_{\mathrm{m}}$ using the coordinate system $(t^{(i)}, n^{(j)})$ and the properties (16) it takes the form

$$\partial_{t_i} \frac{\delta W[\mathrm{C}]}{\delta \sigma_{t_i t_j}(x)} = \lambda_0 \oint_{\mathrm{C}} \mathrm{d} y_{t_j} W[\mathrm{C}_{xy}] W[\mathrm{C}_{yx}] \delta^{(4)}(x-y)$$

$$= \lambda_0 \delta^{(2)}(0) \oint_{\mathrm{C}} \mathrm{d} y_{t_j} W[\mathrm{C}_{xy}] W[\mathrm{C}_{yx}] \delta_{\mathrm{s}}^{2}(x-y), \qquad \text{for } x \in \mathrm{C}. \tag{18}$$

This is our central form. The right-hand side is zero except for $x \in \mathrm{C}$. It gives a non-trivial contribution only when x is a self-intersection of the curve. In that case we have the split-up

$$\delta^{(4)}(x-y) = \delta((x-y)_{t_1}) \delta((x-y)_{t_2}) \delta((x-y)_{n_1}) \delta((x-y)_{n_2})$$

$$= \delta_{\mathrm{s}}^{(2)}(x-y) \delta^{(2)}(0), \qquad \text{for } x, y \in \mathrm{C}.$$

We see that due to the properties (16) of the area derivative we get a dimensional reduction of the MM equation to the 2-dimensional form, eq. (18), with the substitution

$$\lambda_0 \delta^{(2)}(0) \to \lambda_{2\mathrm{D}}. \tag{19}$$

* We are indebted to B. Felsager for discussions. Notice that we have taken the displacement of x to zero prior to taking the area of the test loop to zero. That is justified within our interpretation of sect. 3 that any area should be larger than the domain size.

This suggests the following conjecture: for any loop C, however complicated, introduce the minimal area surface (thus, if C is so singular that a minimal surface cannot be constructed, our conjecture does not work). Thereby curves with similar patterns of self-intersections are defined in QCD_2. For those we know the value of the Wilson loop from ref. [9]. It is given as a combination of exponential and polynomial functions of the *areas* of the "windows", and it satisfies the 2-dimensional MM equation. Then by the principle of general covariance, the same function of the areas satisfies a 2-dimensional "curved" MM equation in the curved space of the minimal area surface. Eq. (18) is just this equation written down in the local "inertial frame" of the tangent plane.

To summarize: For simple curves, eq. (10) is valid. For more complicated curves the conjectured solution for $W[C]$ is obtained by evaluating the planar graphs for $W[C]$ in QCD_2 for the curved surface Σ_m. According to ref. [9], the result may be expressed in terms of the areas of all the "windows" of the curve.

We realize that our prescription may become ambiguous for very complicated curves (knots and the like), but we do not want to pursue these complications here.

In the appendix we present an elementary detailed proof that the equation is satisfied for the simple curves of fig. 1.

3. Regularization and renormalization of the Makeenko-Migdal equation

As pointed out by Witten [10], $N \to \infty$ QCD behaves as if the functional integration defining the quantum theory is dominated by a single field configuration (up to gauge transformation). This single field is Witten's master field $A_\mu^{\mathrm{Cl}}(x)$. The Wilson loop average in $N \to \infty$ QCD could then be expressed in terms of the master field with no functional averaging involved.

The simplest conceivable ("minimal") renormalization would then give a finite value for the Wilson loop if we demand that *the master field A_μ^{Cl} be renormalization-group invariant*. Notice that in our notation A_μ includes the coupling constant. The above requirement is what one finds in the background field method. In that case, however, the background field must be a solution of the classical equations of motion. This is not true for the master field and hence our requirement cannot be justified by the background field method. Indeed our prescription might lead to a renormalized $W[C]$ which differs radically from the result of a more conventional renormalization.

By eq. (11) we see that a finite value of W implies a finite value of the Regge-slope parameter α' which, of course, must be required. Further we see that the quantity

$$g_0^2 N \delta^{(2)}(0) \equiv \frac{g_0^2 N}{a_0^2} \tag{20}$$

is required to be renormalization-group invariant. In eq. (20), a_0^2 is a "bare" area ($=$ zero) corresponding to the reciprocal value of $\delta^{(2)}(0)$.

We now get inspiration from models of the QCD vacuum suggesting the existence of domains having a characteristic size [3,4]. In the vortex condensate model of ref. [3], domains are characterized by a certain gauge-invariant flux and in a particular frame by a (renormalization-group invariant) magnetic field strength gH. The domains are tubes with cross-sectional area a^2 given by

$$a^2 = \frac{2\pi}{gH}. \tag{21}$$

This area corresponds to a gauge-invariant flux quantum (see the fourth paper in ref. [3]).

The vacuum energy is minimized for*

$$\frac{g^2(a^2)}{4\pi} = 0.19, \qquad \text{and} \qquad \sqrt{gH} = 1.28\Lambda_{\mathrm{p}}, \tag{22}$$

where $g^2(a^2)$ is the QCD running coupling constant minimizing the vacuum energy in the gauge group SU(3) and Λ_{p} is the QCD scale defined in propagator renormalization.

These considerations induce us to suggest that we regularize the MM equation as well as our solution by the substitution

$$\frac{g_0^2 N}{a_0^2} \to \frac{g^2(a^2) N}{a^2}. \tag{23}$$

This prescription is similar in spirit to one considered by Greensite [11] in a different context**.

From eqs. (11),(20),(22) we get

$$\alpha' = \left[0.29\Lambda_{\mathrm{p}}^{-1}\right]^2. \tag{24}$$

In principle we could insert a value for Λ_{p} obtained from the analysis of deep inelastic data or from $e^+e^- \to q\bar{q}g$ data. However, such a procedure is of limited value since our expressions come from 1-loop calculations. Instead we prefer to compare with the following expression for the vacuum energy density (the "bag constant", ref. [3])***

$$B^{1/4} = 0.42\Lambda_{\mathrm{p}},$$

* There is a factor 2 difference between our definition of g^2 and the one used in ref. [3].

** We are grateful to J.P. Greensite for discussions on this point.

*** Thus B measures the vacuum energy density. We need not assume the general validity of the bag model.

or, using eq. (24),

$$B^{1/4} = \frac{0.12}{\sqrt{\alpha'}} . \tag{25}$$

Using $\alpha' = 0.9\ \mathrm{GeV}^{-2}$ we find

$$B^{1/4} = 130\ \mathrm{MeV},$$

a surprisingly good value considering the crudeness of the estimate. The estimate was based on a model for $N = 3$. Recently, progress has been made in generalizing the vortex condensate model to $N = \infty$ [12]. In particular, the expectation that the value of gH and the domain size has a smooth $N \to \infty$ limit, has been confirmed.

In the arguments given above it is crucial that *the domains occur in an* O(4)-*invariant way*, since the curve C can have any orientation in euclidean space. This is possible in the *disordered* phase (see second paper in ref. [3]), but, of course, not in an ordered phase, where our arguments thus break down. This is satisfactory, since we do not expect confinement in the ordered phase.

Also, to preserve gauge invariance the area a^2 has the minimal value given by eq. (21). In general a^2 can be an integer times $2\pi/gH$ (corresponding to an increased α')*, so that eq. (24) represents the minimal value of α'.

4. Discussion and conclusion

The formal solution of sect. 2 seems to establish a connection between $N \to \infty$ QCD (with no light quarks) and a dual string model. It is worth emphasizing in this connection that unitarity corrections are down by $1/N$, and hence are not covered by the MM equation (2). Hence unphysical dimensions are not needed for quantization.

One may ask why the string is structureless, since in QCD at short distances the Coulomb potential dominates and provides, for example, a hyperfine structure in charmonium. The reason for this is that the scale of the MM equation (2) is singular, given by the $\delta^{(4)}(x-y)$ function, and it is only through the regulation of this singular scale that one obtains a non-trivial α'. In our case the regulation of the δ-function is done by associating it with the domain size. Such a procedure, however, does imply that one gives up saying anything about phenomena at distances below the domain size, i.e. below ≈ 1 fm. Thus, with our regulator we can only obtain information over large distances (where, for example, the hyperfine structure cannot be seen), and hence it is not unnatural to obtain a structureless string.

* Thus, it is not possible, e.g. to have $a^2 = 0.7 \cdot 2\pi/gH$.

We emphasize that we do not know whether this solution coincides with Migdal's solution [2]*. Further investigations are necessary in order to find the complete set of solutions. If there are several solutions presumably only one of these is also a solution of QCD.

Our second main point in this paper is the interpretation of the singular quantity

$$g_0^2 N \delta^{(2)}(0)$$

in terms of a QCD vacuum domain size. In order for the domain size to be gauge invariant, it must presumably contain a Z_N flux. This therefore suggests an interesting connection between the loop space formulation of QCD and the approach to confinement based on the center of the gauge group [5–7].

After completion of this work we have noticed that a remark in ref. [13] indicates that Migdal also has considered eq. (10)**.

We have profited greatly from discussions with B.J. Durhuus, H.B. Nielsen and B. Felsager.

Appendix

The purpose of this appendix is to verify the MM equation (2) for the ansatz (10),(11),(13) in the form

$$-\partial_\mu^x \int_{\Sigma_m} d\sigma_{\mu\nu}(y)\,\delta_s^{(2)}(x,y) = \oint_C dy_\nu\,\delta_s^{(2)}(x,y) \tag{A.1}$$

(using eqs. (11)). We introduce conformal coordinates (σ,τ) for the surface with the following properties

$$(\sigma,\tau) \to y_\mu(\sigma,\tau),$$

$$y'_\mu \dot{y}_\mu = 0, \qquad y'_\mu y'_\mu = \dot{y}_\mu \dot{y}_\mu, \tag{A.2}$$

where we have used the notation $\partial f/\partial\sigma = f', \partial f/\partial\tau = \dot{f}$.

At any point y on the surface, we introduce the tangent vectors along coordinate lines y'_μ and $\dot{y}_\mu$. Then

$$\delta_s^{(2)}(x,y) = \delta(y'(x-y))\,\delta(\dot{y}(x-y))\,J, \tag{A.3}$$

where

$$J = |y'||\dot{y}| = y'^2 = \dot{y}^2. \tag{A.4}$$

* If so, the elf mass must be related to the inverse domain size.

** We are grateful to A. Patkós for drawing our attention to this work.

Also

$$\mathrm{d}\sigma_{\mu\nu}(y)=\mathrm{d}\sigma\,\mathrm{d}\tau\left(y'_\mu\dot{y}_\nu-\dot{y}_\mu y'_\nu\right). \tag{A.5}$$

Then for the left-hand side of eq. (A.1) we get

$$\begin{aligned}
-\int_\Sigma &\mathrm{d}\sigma\,\mathrm{d}\tau\left[y'_\mu\dot{y}_\nu-\dot{y}_\mu y'_\nu\right]\left[y'_\mu\delta'(y'(x-y))\delta(\dot{y}(x-y))\right.\\
&\qquad\left.+\dot{y}_\mu\delta(y'(x-y))\delta'(\dot{y}(x-y))\right]J\\
&=-\int_\Sigma \mathrm{d}\sigma\,\mathrm{d}\tau\left(\dot{y}_\nu\delta'(y'(x-y))\delta(\dot{y}(x-y))\right.\\
&\qquad\left.-y'_\nu\delta(y'(x-y))\delta'(\dot{y}(x-y))\right)J^2.
\end{aligned} \tag{A.6}$$

We want to show that this expression equals

$$\begin{aligned}
\int_\Sigma &\mathrm{d}\sigma\,\mathrm{d}\tau\left\{\frac{\partial}{\partial\sigma}\left[\dot{y}_\nu J\delta(y'(x-y))\delta(\dot{y}(x-y))\right]-\frac{\partial}{\partial\tau}\left[y'_\nu J\delta(y'(x-y))\delta(\dot{y}(x-y))\right]\right\}\\
&=\oint_C\left(\mathrm{d}\tau\frac{\partial y_\nu}{\partial\tau}+\mathrm{d}\sigma\frac{\partial y_\nu}{\partial\sigma}\right)\delta_s^2(x,y)\\
&=\oint_C dy_\nu\delta_s^{(2)}(x,y),
\end{aligned} \tag{A.7}$$

which is the right-hand side of eq. (A.1).

Let us prove that (A.7) and (A.6) are equivalent. Consider the first term in the integrand of eq. (A.7):

$$\frac{\partial}{\partial\sigma}\left[\dot{y}_\nu J\delta(y'(x-y))\delta(\dot{y}(x-y))\right]=\dot{y}'_\nu\delta_s^{(2)}(x,y) \tag{a}$$

$$+\dot{y}_\nu 2y'y''\delta(y'(x-y))\delta(\dot{y}(x-y)) \tag{b}$$

$$\begin{aligned}
&+\dot{y}_\nu J\left[\left(y''(x-y)-y'^2\right)\delta'(y'(x-y))\delta(\dot{y}(x-y))\right.\\
&\left.+\left(\dot{y}'(x-y)-\dot{y}y'\right)\delta(y'(x-y))\delta'(\dot{y}(x-y))\right].
\end{aligned} \tag{c}$$

Here the term (a) is cancelled against a similar term from doing the τ-differentiation in eq. (A.7). The (c) part contains the term

$$-y'^2 J\dot{y}_\nu \delta'(y'(x-y))\delta(\dot{y}(x-y)),$$

which we recognize as the first term in the integrand in eq. (A.6). To see that the remaining terms in (c) cancel the term (b), we expand y''_μ and $\dot{y}'_\mu$ on $y'_\mu, \dot{y}_\mu$ and two vectors orthogonal to the tangent plane. We then use that $x\delta'(x) = -\delta(x)$. For the $y''(x-y)$ term we get

$$y''(x-y) = y'(x-y)\frac{y''y'}{y'^2} + \text{terms giving vanishing contributions.}$$

Similarly

$$\dot{y}'(x-y) = \dot{y}(x-y)\frac{\dot{y}'\dot{y}}{\dot{y}^2} + \cdots .$$

But from eq. (A.4) $2y'y'' = 2\dot{y}\dot{y}'$ and one sees that the claimed cancellations take place.

Notice that we did not use the condition for the surface to be the minimal area one.

References

[1] Yu.M. Makeenko and A.A. Migdal, Phys. Lett. 88B (1979) 135; ITEP-23 preprint (1980); A.A. Migdal, Ann. of Phys. 126 (1980) 279
[2] A.A. Migdal, Landau Inst. preprint (1980)
[3] N.K. Nielsen and P. Olesen, Nucl. Phys. B144 (1978) 376; H.B. Nielsen and P. Olesen, Nucl. Phys. B160 (1979) 380; J. Ambjørn and P. Olesen, Nucl. Phys. B170 [FS1] (1980) 60; 265
[4] K. Johnson, SLAC preprint 2436 (1979)
[5] G. 't Hooft, Nucl. Phys. B138 (1978) 1
[6] G. 't Hooft, Nucl. Phys. B153 (1979) 141
[7] G. Mack, Cargese lectures (1979)
[8] G. 't Hooft, Nucl. Phys. B72 (1974) 461; B75 (1974) 461
[9] V.A. Kazakov and I.K. Kostov, Nucl. Phys. B176 (1980) 199
[10] E. Witten, Cargese lectures (1979)
[11] J.P. Greensite, Nucl. Phys. B158 (1979) 469
[12] H. Flyvbjerg and J.L. Petersen, NBI-HE-80-57 preprint (1980)
[13] Yu.M. Makeenko, ITEP preprint 141 (1979)