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"Topological" Formulation of Effective Vortex Strings

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Abstract

We present a "topological" formulation of arbitrarily shaped vortex strings in four dimensional field theory. By using a large Higgs mass expansion, we then evaluate the effective action of the closed Abrikosov-Nielsen-Olesen vortex string. It is shown that the effective action contains the Nambu-Goto term and an extrinsic curvature squared term with negative sign. We next evaluate the topological $F_{\mu\nu}\tilde{F}_{\mu\nu}$ term and find that it becomes the sum of an ordinary self-intersection number and Polyakov's self-intersection number of the world sheet swept by the vortex string. These self-intersection numbers are related to the self-linking number and the total twist number, respectively. Furthermore, the $F_{\mu\nu}\tilde{F}_{\mu\nu}$ term turns out to be the difference between the sum of the writhing numbers and the linking numbers of the vortex strings at the initial time and the one at the final time. When the vortex string is coupled to fermions, the chiral fermion number of the vortex string becomes the writhing number (modulo $\boldsymbol{Z}$) through the chiral anomaly. Our formulation is also applied to "global" vortex strings in a model with a broken global $U(1)$ symmetry.

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1 Introduction

The study of string-like objects has been actively pursued from both theoretical and experimental interests in various fields including condensed matter physics and biology. In particle physics and cosmology, the topological vortex string arising in field theory is one of the most interesting string-like objects. In particular, the Abrikosov-Nielsen-Olesen (ANO) vortex string is the simplest one that splendidly shows typical properties of the vortex string [1]. Toward a better understanding of the physics on the vortex string, it is important to examine its geometric and topological properties in four space-time dimensions. We are especially interested in four dimensional extrinsic properties such as the entanglement of the vortex strings. For studying them, we need a systematic formulation of the vortex string. The method used so far in evaluating the effective action of the vortex string in arbitrary shape is based on Förster's parameterization of coordinates [2] and the collective coordinates method [3]. In this method, however, the cut-off dependence of the theory is not so clear and topological structures such as the self-intersection of the world sheet swept by the vortex string cannot be so easily investigated. Therefore, it is desirable to construct a more systematic and efficient formulation satisfying the following: (i) changes of the shapes of the vortex strings can be described, (ii) using perturbative expansions by appropriate parameters such as coupling constants, masses or cut-offs, one can perform systematic approximations, (iii) topological features of the vortex string can be easily examined.

In this paper, we present a "topological" formulation which satisfies the above three requirements and apply it to the arbitrarily shaped vortex strings in field theories with broken local or global $U(1)$ symmetries. This is a relativistic generalization of the "topological" formulation used in the study of quantized vortices in superfluid helium [4]. One of the characteristic features in our formulation is the appearance of an antisymmetric tensor field and a so-called topological BF term [5]. We also adopt a manifestly Lorentz invariant Gaussian-type regularization for the δ-functions in vorticity tensor currents. Using our formulation, we first evaluate the effective action of the ANO vortex string in the Abelian Higgs model, showing that it contains not only the Nambu-Goto term but also an extrinsic curvature squared term with negative sign. Second, we examine the topological $F_{\mu\nu}\tilde{F}_{\mu\nu}$ term, which for example appears as the chiral anomaly and the θ term. The evaluation of this term tells us that there are interesting relations between several geometric or topological quantities: Polyakov's self-intersection number, ordinary self-intersection number, total twist number, self-linking number, writhing number and linking number. The expectation value of the $F_{\mu\nu}\tilde{F}_{\mu\nu}$ term turns out to be the sum of Polyakov's self-intersection number and the ordinary self-intersection number of the world sheet swept by the vortex string at the leading order of our approximation. In addition, the $F_{\mu\nu}\tilde{F}_{\mu\nu}$ term can be written as the difference of the sum of the writhing number and the linking number at the final time and the one at the initial time. Furthermore, we discuss the chiral fermion number of the ANO vortex strings in arbitrary shape by using the chiral anomaly and find it to be the sum of the writhing numbers of each vortex string (modulo $\boldsymbol{Z}$). To make the validity of our formulation clearer, we also study the dynamics of vortex strings in a model with a broken global $U(1)$ symmetry. (In cosmology, the former ANO vortex strings are called "local strings" and the latter ones are called "global strings".) In both models, it is also shown that the large Higgs mass expansions are good approximations. As a whole, it is demonstrated that our "topological" formulation is useful to study the "effective" vortex strings. (The "effective" vortex string means the vortex string remaining after integration over massive fields in field theory.)

Our formulation can be applied to several phenomenologies, although we do not completely discuss them in this paper. First, when we consider grand unified models with extra broken $U(1)$ symmetries, then there can exist vortex strings which are topologically stable. Their string tension is of order a GUT-scale squared or possibly a fundamental string scale squared (because the gauge coupling is smaller than 1). Furthermore, when the models have anomalous $U(1)$ symmetries, it is interesting to investigate whether any fermion number can be violated through the effect of the vortex string. Second, in the Weinberg-Salam theory, there appears the so-called Z string which is equivalent to the ANO vortex string if one neglects other degrees of freedom [6, 7]. It is a kind of sphaleron [8], which perhaps seems to be related to the weak-scale baryogenesis through the chiral anomaly [9, 10]. Therefore, the investigation of the vortex string in arbitrary shape is important from the point of view of the fermion number violation. Third, our theory can be directly applied to the cosmic string model [11] and superconductor systems. Finally, it should be noticed that the study of the effective vortex string would give us a new angle in understanding extrinsic properties of fundamental strings in four dimensional space-time.

The paper is organized as follows: in sect. 2 and 3, we study the Abelian Higgs model with the vortex string. First, in sect. 2, we present our "topological" formulation and evaluate the effective action of the vortex string. Next, in sect. 3, the $F_{\mu\nu}\tilde{F}_{\mu\nu}$ term is examined and geometric or topological relations are shown. We also discuss the chiral fermion number of the vortex string in arbitrary shape. In sect. 4, we examine a model with the broken global $U(1)$ symmetry. In sect. 5, we give conclusions and compare our results with previous ones. In appendix A, we explain the relation between Polyakov's self-intersection number and the total twist number. In appendix B, we derive the relation between the intersection number and the linking number.

2 "Topological" formulation and the effective action of the ANO vortex string

In this section, we consider the Abelian Higgs model with an arbitrarily shaped vortex string in four space-time dimensions. Furthermore, for simplicity we suppose the vortex string to be a closed one with circulation number one. It is easy to extend to the case with many vortex strings with arbitrary circulation numbers. We always use the Euclidean formulation of field theory, so the model is described by the Lagrangian

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}F_{\mu\nu} + |(\partial_\mu - ieA_\mu)\varphi|^2 + 2\lambda\Big(\varphi^\dagger\varphi - \frac{\eta^2}{2}\Big)^2, \tag{2.1}$$

where A_μ is an ordinary $U(1)$ gauge field, φ a complex scalar field and $F_{\mu\nu}$ a field strength tensor of A_μ. The existence of the vortex string with circulation one means that when one takes a turn along any closed contour around the vortex string, the phase of the scalar field changes by 2π. To get such a phase, we use the solid angle subtended by the vortex string

$$\theta(\boldsymbol{x};\boldsymbol{X}) = \frac{1}{2}\int_S d\boldsymbol{S}'\cdot\nabla\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}, \tag{2.2}$$

where S is any surface bounded by the vortex string: $\partial S = \Gamma$, where $\Gamma = \{\boldsymbol{X}(\sigma_1,t); 0 \le \sigma_1 \le 2\pi\}$ and $\boldsymbol{X}(\sigma_1,t)$ denotes the position of the vortex string at time t. In terms of this solid angle, the scalar field with the vortex string is described by

$$\varphi(x) = \exp\{i\theta(\boldsymbol{x};\boldsymbol{X})\}\phi(x), \tag{2.3}$$

where $\phi(x)$ is a regular function. When we substitute $\varphi(x)$ in (2.1), the differential term of $\varphi(x)$ changes and the covariance of (2.1) is apparently broken. So we need an alternative formulation where the covariance is manifest and it is easier to deal with the vortex string.

We propose a manifestly covariant Lagrangian with a topological term and a vorticity tensor current, which is equivalent to the original one (2.1) in the sense explained later:

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}F_{\mu\nu} + |(\partial_\mu - ieA_\mu - ia_\mu)\phi|^2 + 2\lambda\Big(\phi^\dagger\phi - \frac{\eta^2}{2}\Big)^2 + i\epsilon_{\mu\nu\rho\sigma}B_{\mu\nu}f_{\rho\sigma} + iB_{\mu\nu}J_{\mu\nu}, \qquad (2.4)$$

where

$$J_{\mu\nu}(x) = -4\pi\int d^2\sigma\partial_1 X_{[\mu}\partial_2 X_{\nu]}\delta^{(4)}(x - X(\sigma)) \qquad (2.5)$$

is the vorticity tensor current and a_μ another $U(1)$ gauge field for the vorticity, $B_{\mu\nu}$ a rank two antisymmetric tensor field, $f_{\mu\nu}$ a field strength tensor of a_μ, $\partial_a = \partial/\partial\sigma_a$ $(a = 1, 2)$ and $A_{[\mu}B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu$. Here $X_\mu(\sigma)$ denotes the four dimensional location of the vortex string, where $\sigma = (\sigma_1, \sigma_2)$ are the coordinates which parameterize the world sheet swept by the vortex string.

Actually, variations of (2.4) with respect to $B_{\mu\nu}$ lead to the constraint $\epsilon_{\mu\nu\rho\sigma}f_{\rho\sigma} + J_{\mu\nu} = 0$ and if we choose the Coulomb gauge $\partial_i a_i = 0$ and a gauge $\sigma_2 = X_4 = t$, then we get $a_\mu = -\partial_\mu\theta$ and

$$(\partial_\mu - ieA_\mu)\,\varphi = (\partial_\mu - ieA_\mu - ia_\mu)\,\phi, \qquad (2.6)$$

so that this Lagrangian (2.4) turns out to be the original one (2.1) into which the redefined scalar field $\varphi(x)$ is inserted. From now on, let us use the Lagrangian (2.4) as our starting point for the theory.

The Lagrangian (2.4) has some interesting properties. (i) It has two types of gauge symmetries except the usual $U(1)$ symmetry. The first one is another $U(1)$ gauge symmetry: $a_\mu \to a_\mu + \partial_\mu\alpha$ and $\phi \to e^{i\alpha}\phi$ with an arbitrary regular function α. The second one is given by $B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu$, where Λ_μ is also an arbitrary regular function. The corresponding conserved tensor current is $J_{\mu\nu}$, so that the total vorticity $\int d^3x J_{4i}$ is conserved. (ii) It has the topological term $\epsilon_{\mu\nu\rho\sigma}B_{\mu\nu}f_{\rho\sigma}$, which is called a BF term. In general the BF term is used in evaluating linking numbers which are topological numbers counting how many times a string and a two dimensional membrane are entangled in four dimensions [5]. The topological BF term is a generalization of the Chern-Simons term which plays an important role in the study of the quantized Hall effect and anyon systems in $2 + 1$ dimensions [12]. Furthermore, this term appears in various areas of theoretical physics, for example, in models with anomalous U(1) charges in superstring theory [13] and four dimensional 2-form gravity [14]. Through this term, we may be able to search some connections between such theories and the present theory on the vortex string. (iii) Since the vortex string coordinates $X_\mu(\sigma)$ are contained only in the vorticity tensor current $J_{\mu\nu}$, which can describe the vortex string in arbitrary shape, our formulation is useful in many cases, for example, in cases where the vortex core needs regularizing or one derives the equation of motion of the vortex string.

We are interested in the effective action of the vortex string defined by

$$\mathcal{S}_{eff}[X] = -\ln \int \mathcal{D}\phi \mathcal{D}A_\mu \mathcal{D}a_\mu \mathcal{D}B_{\mu\nu} \exp\left(-\int d^4x \mathcal{L}\right). \tag{2.7}$$

This path integral representation is suitable for the evaluation of the effective action since systematic perturbative expansions can be easily performed. We do not consider the quantization of $X_\mu(\sigma)$ and the effects from loops of the fields in this paper, although it is of course interesting to study them. Before computing the effective action, let us discuss some points to clarify our procedure.

As well known, in the static solution of the straight ANO vortex, it is satisfied that $\phi - \eta/\sqrt{2} \approx 0$ and $eA_i + a_i \approx 0$ in the distant region from the vortex core, where $a_i = -\partial_i \theta_s$ (θ_s is an azimuthal angle) and "≈ 0" implies "exponentially small". Even in general cases with moving vortex strings, if we require that the energy is finite, it should be satisfied that $\phi - \eta/\sqrt{2} \approx 0$ and $eA_\mu + a_\mu \approx 0$ in the distant region from the vortex core. Furthermore, since $\pi_1(U(1)) = \boldsymbol{Z}$, the vortex string is topologically stable. It is therefore reasonable to expand ϕ and $eA_\mu + a_\mu$ around $\eta/\sqrt{2}$ and 0 respectively in such a region. So, instead of ϕ and A_μ, let us adopt two scalar fields ρ, ω and a vector field C_μ defined by

$$\phi(x) = \frac{1}{\sqrt{2}}(\eta + \rho(x))e^{i\omega(x)}, \tag{2.8}$$

$$C_\mu(x) = eA_\mu(x) + a_\mu(x). \tag{2.9}$$

After these replacements, our action still keeps the gauge invariance under $a_\mu \to a_\mu + \partial_\mu \alpha$, so that we get a gauge invariant effective action for a_μ when we integrate over the fields ρ, ω and C_μ. This fact is convenient for evaluating the effective action of the vortex string.

In the unitary gauge, where $C_\mu - \partial_\mu \omega$ is replaced with V_μ, the Lagrangian becomes

$$\begin{aligned}
\mathcal{L} \;=\; & \frac{1}{4e^2} V_{\mu\nu} V_{\mu\nu} + \frac{1}{2}\eta^2 V_\mu V_\mu \\
& + \frac{1}{2}\partial_\mu \rho \partial_\mu \rho + 2\lambda \eta^2 \rho^2 \\
& - \frac{1}{2e^2} V_{\mu\nu} f_{\mu\nu} + \eta \rho V_\mu V_\mu + \frac{1}{2}\rho^2 V_\mu V_\mu + 2\lambda\eta\rho^3 + \frac{1}{2}\lambda\rho^4 \\
& + \frac{1}{4e^2} f_{\mu\nu} f_{\mu\nu} + i\epsilon_{\mu\nu\rho\sigma} B_{\mu\nu} f_{\rho\sigma} + iB_{\mu\nu} J_{\mu\nu},
\end{aligned} \tag{2.10}$$

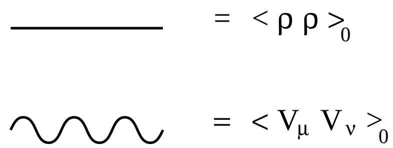

Figure 1: Graphical representation of the propagators of $\rho(x)$ and $V_\mu(x)$.

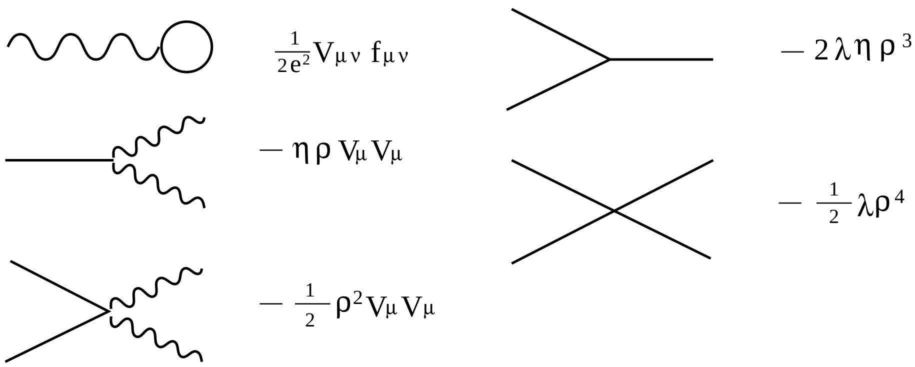

Figure 2: Graphical representation of the vertices in the Lagrangian (2.10). The circle indicating $f_{\mu\nu}$ can be actually replaced with the vorticity tensor current $-2\tilde{J}_{\mu\nu}$ through the constraint $f_{\mu\nu} = -2\tilde{J}_{\mu\nu}$, where $\tilde{J}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} J_{\rho\sigma}$.

where $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$. Propagators of ρ and V_μ, which are denoted as $\langle \rho(x)\rho(y)\rangle_0$ and $\langle V_\mu(x) V_\nu(y)\rangle_0$ respectively, are given by

$$\langle \rho(x)\rho(y)\rangle_0 = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m_H^2} e^{ik\cdot(x-y)}, \tag{2.11}$$

$$\langle V_\mu(x) V_\nu(y)\rangle_0 = e^2 \int \frac{d^4k}{(2\pi)^4} \left(g_{\mu\nu} + \frac{k_\mu k_\nu}{m_V^2} \right) \frac{1}{k^2 + m_V^2} e^{ik\cdot(x-y)}, \tag{2.12}$$

where $m_H^2 = 4\lambda\eta^2$ and $m_V^2 = e^2\eta^2$. The resulting Feynman rules for the propagators and the vertices are summarized in fig. 1 and fig. 2.

Note that since we do not exactly know the inside structure of the vortex core when the vortex string is moving, it is necessary to assume its structure and introduce an appropriate cut-off parameter. The ansatz which we actually apply to our system is that the δ-function in the vorticity tensor current $J_{\mu\nu}$ is regularized smoothly in the cut-off region. In this paper, we make use of the first expression of the following Gaussian-type regularization:

$$\begin{aligned} \delta_\Lambda^{(4)}(x) &= \frac{\Lambda^4}{\pi^2} \exp(-\Lambda^2 x^2) \\ &= \int \frac{d^4k}{(2\pi)^4} \exp\left(ik\cdot x - \frac{k^2}{4\Lambda^2} \right), \end{aligned} \tag{2.13}$$

where the second expression shows that the momentum is effectively cut off at about Λ. In this regularization, the Lorentz invariance is manifest and the conservation of the vorticity tensor current, $\partial_\mu J_{\mu\nu} = 0$, is also preserved. It is one of crucial reasons why our formulation is a convenient one to evaluate fine structures of the effective action of the vortex string and topological properties such as the self-intersection number of the world sheet swept by the vortex string.

In systematic estimations of the order of each tree diagram for the effective action, we can perform the large mass expansion by powers of $1/m_H$ when m_H is larger than other mass scales. Especially, in this large Higgs mass expansion, the propagator of the Higgs field ρ can be treated as

$$\langle \rho(x)\rho(y) \rangle_0 \approx \frac{1}{m_H^2} \delta^{(4)}(x-y). \tag{2.14}$$

Using ordinary relations between numbers of vertices, propagators and external lines for the diagrams, a simple power counting tells us that at the tree level the leading power of $1/m_H$ for Feynman diagrams with N circle ($N \geq 2$), which are corresponding to parts of the effective action with N vorticity tensor currents, is given by

$$\left(\frac{M}{m_H} \right)^{N-2}. \tag{2.15}$$

Here M are m_V, Λ or $1/R$, where R denotes the characteristic length which represents the smoothness of the vortex string, that is, the magnitude of higher order derivatives of X_μ. Hence, it turns out that the diagrams with smaller numbers of vorticity tensor currents are dominant for the effective action.

Let us turn our attention now to the evaluation of the effective action of the vortex string. The leading contribution $\mathcal{S}_0$ to it, which has two vorticity tensor currents and corresponds to Feynman diagrams depicted in fig. 3, can be evaluated in the form

$$\mathcal{S}_0 = \frac{m_V^2}{16e^2} \int d^4x d^4y J_{\mu\nu}(x) D(x-y) J_{\mu\nu}(y), \tag{2.16}$$

where

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m_V^2} e^{ik\cdot(x-y)}. \tag{2.17}$$

Here we have used the conservation law of the vorticity tensor current $J_{\mu\nu}$. The form of $\mathcal{S}_0$ expresses that massive particles propagate between two points on the world sheet swept by the vortex string.

We will now evaluate the dominant effects arising in the case where two vorticity tensor currents approach each other. For that purpose, we adopt the proper time representation of

Figure 3: The diagrams of the leading contribution to the effective action. Touching circles come from the $1/4e^2 \cdot f_{\mu\nu} f_{\mu\nu}$ term in (2.10).

the propagator $D(x-y)$:

$$\frac{1}{k^2+m^2} = \int_0^\infty ds \exp[-s(k^2+m^2)]. \tag{2.18}$$

After inserting it into $\mathcal{S}_0$ and integrating over x, y and k, then we find that

$$\begin{aligned} \mathcal{S}_0 &= \frac{m_V^2}{16e^2} \int_{1/2\Lambda^2}^\infty ds \frac{1}{s^2} \int d^2\sigma d^2\sigma' \partial_1 X_{[\mu} \partial_2 X_{\nu]}(\sigma) \cdot \partial_1 X_{[\mu} \partial_2 X_{\nu]}(\sigma') \\ &\quad \times \exp\left\{ -\frac{1}{4s} |X(\sigma) - X(\sigma')|^2 - s m_V^2 + \frac{m_V^2}{2\Lambda^2} \right\}. \end{aligned} \tag{2.19}$$

Here note that we have used the Gaussian-type regularization of the δ-function (2.13). The factor $e^{-sm_V^2}$ in (2.19) indicates that the region where s is small mainly contributes to $\mathcal{S}_0$ and the factor $e^{-1/4s \cdot |X(\sigma)-X(\sigma')|^2}$ in (2.19) shows that the region where $X(\sigma) \approx X(\sigma')$ is the dominant part of contributions to $\mathcal{S}_0$ when s is small. Putting these together, it turns out that the dominant contribution to $\mathcal{S}_0$ comes from the region where $X(\sigma)$ is near $X(\sigma')$, so let us consider the case where σ' is near σ. In order to investigate the behavior of $\mathcal{S}_0$ in the case where σ' is near σ, we define z as $z = \sigma' - \sigma$ and expand $X_\mu(\sigma')$ in powers of z:

$$X_\mu(\sigma') = X_\mu(\sigma) + z_a \partial_a X_\mu(\sigma) + \frac{1}{2}(z_a \partial_a)^2 X_\mu + \cdots. \tag{2.20}$$

After substitution of (2.20) in (2.19) and integration over z, the effective action $\mathcal{S}_0$ takes the particularly simple form up to $O(1)$ in powers of $(m_V R)^{-1}$ and $(\Lambda R)^{-1}$:

$$\mathcal{S}_0 = \mu_0 \int d^2\sigma \sqrt{g} + \alpha_0 \int d^2\sigma \sqrt{g} K_{ab}^A K_{ab}^A, \tag{2.21}$$

where $g_{ab} = \partial_a X_\mu \partial_b X_\mu$ and $g = \det(g_{ab})$. Here

$$\mu_0 = \frac{\pi m_V^2}{2e^2} \int_0^\infty du \frac{e^{-u}}{u + \frac{m_V^2}{2\Lambda^2}}, \tag{2.22}$$

which is the tension in the vortex string, and

$$\alpha_0 = -\frac{3\pi}{8e^2}. \tag{2.23}$$

K_{ab}^A is the extrinsic curvature defined by the equation

$$\partial_a \partial_b X_\mu = \Gamma_{ab}^c \partial_c X_\mu + K_{ab}^A n_\mu^A, \tag{2.24}$$

where n^A_μ are two normal unit vectors perpendicular to $\partial_a X_\mu$, satisfying $n^A_\mu n^B_\nu = \delta^{AB}$ and $n^A_\mu \partial_a X_\mu = 0$ $(A = 1, 2)$. Here we have neglected contributions from boundaries and intersection points of the world sheet swept by the vortex string. In evaluating the second term of (2.21), we have used the fact that the scalar curvature defined by g_{ab} is a total divergence. The first term in the right-hand side of (2.21) is the Nambu-Goto action and the second term is the extrinsic curvature squared term which was investigated from the point of view of the QCD string [15].

Let us make a comparison between the string tension μ_0 and the one μ evaluated by using the static ANO vortex solution in the case where m_H is much larger than m_V. Outside the vortex core, one can treat ϕ as a constant $\eta/\sqrt{2}$ and the ANO vortex solution with circulation one is given by

$$A_i(x) = \epsilon_{ij} \frac{x_j}{r^2} \left\{ \frac{1}{e} - \eta r K_1(e\eta r) \right\} \quad (i = 1, 2), \tag{2.25}$$

$$A_3(x) = 0, \tag{2.26}$$

where K_1 is the modified Bessel function and $r = \sqrt{x_1^2 + x_2^2}$ denotes the distance from the center of the vortex string [1, 16]. Here the ANO vortex solution $A_i(x)$ is regular for $r \to 0$. Using a cut-off Λ' such that $m_H \gg \Lambda' \gg m_V$, one can easily obtain the string tension μ, which is the energy per unit length along the third axis:

$$\mu \approx \pi \eta^2 \ln \left(\frac{\Lambda'}{m_V} \right). \tag{2.27}$$

The dominant contribution to the string tension μ comes from the intermediate region satisfying $1/\Lambda' < r < 1/m_V$. In addition, the contribution from the vortex core, which is about η^2, is much smaller than the dominant one. On the other hand, our string tension μ_0 behaves like

$$\mu_0 \approx \pi \eta^2 \ln \left(\frac{\Lambda}{m_V} \right), \tag{2.28}$$

when $\Lambda \gg m_V$. We therefore get $\Lambda \approx \Lambda'$ if we require both string tensions are equal. This shows that our formulation is justified when we take about Λ' as Λ. Note that we should not use the expansion in positive powers of Λ/m_V because $\Lambda \gg m_V$, though it is valid to expand the theory in powers of $1/m_H$ when m_H is large enough. This fact is one of our grounds for using the expansion in powers of $1/m_H$. In addition, it may be worth pointing out that the exponential integral in the string tension μ_0 given by (2.22) depends on only the ratio of m_V^2 to $2\Lambda^2$.

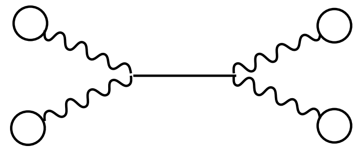

Figure 4: The diagram of the next leading contribution to the effective action.

As concerns the coefficient α_0, there are two remarkable points. First, it takes the non-zero value with negative sign, which indicates that the vortex string prefers curving as far as it is smooth enough. Our approximation is relevant only in the case where the characteristic length representing the curvature of the vortex string is large enough in comparison with the lengths $1/m_H$, $1/m_V$ and $1/\Lambda$, so that the apparent unboundedness of the effective action (2.21) is not a trouble within the approximation which we have used. One may notice that our theory is positive definite by the definition of the Lagrangian (2.4). Second, the α_0 is independent of m_V and Λ unlike the string tension μ_0.

To confirm the validity of the expansion by the large Higgs mass, let us evaluate the next leading contribution whose diagram has four vorticity tensor currents, as illustrated in fig. 4. The next leading effective action $\mathcal{S}_1$ is dominated by the case where the four vorticity tensor currents approach in the same time, and the dominant term becomes the Nambu-Goto term. Indeed, by repeating the procedure used previously, we can calculate $\mathcal{S}_1$:

$$\mathcal{S}_1 = \mu_1 \int d^2\sigma \sqrt{g} + \cdots, \tag{2.29}$$

where

$$\mu_1 = \frac{m_V^2}{e^2 m_H^2} f\left(\frac{m_V^2}{4\Lambda^2}\right). \tag{2.30}$$

Here $f(m_V^2/4\Lambda^2)$ in the next leading string tension μ_1 is a complicated function of $m_V^2/4\Lambda^2$ but finite. As $\Lambda \gg m_V$, this function behaves like $(4\Lambda^2/m_V^2)\{\ln(m_V^2/4\Lambda^2)\}^2$ at most, so that the ratio of μ_1 to μ_0 is at most given by

$$\frac{\mu_1}{\mu_0} \approx \frac{\Lambda^2}{m_H^2} \ln\left(\frac{\Lambda}{m_V}\right). \tag{2.31}$$

It demonstrates that this ratio is small if $m_H^2 \gg \Lambda^2 \ln(\Lambda/m_V)$. Our large Higgs mass expansion is therefore valid under this condition on the Higgs mass. This is consistent with the power counting of (2.15), because $\ln(\Lambda/m_V)$ has been counted as $O(1)$ in (2.15).

It may be worth while to point out that the calculations for μ_0, α_0 and μ_1 are somewhat similar to the ones for effective actions and β functions in "the zero-slope limit" of (fundamental) string theory or non-linear σ models. More detailed investigation of this correspondence will be interesting.

To make our discussion clear, let us summarize the conditions used in our approximation. In the evaluation of the effective action of the ANO vortex string we have supposed the following four conditions, under which our approximation is justified. (i) The coupling constants are small: e, $\lambda < 1$. (This ensures that loop effects are small.) (ii) The Higgs mass is large enough than the mass of the gauge field: $m_H \gg m_V$, meaning $4\lambda \gg e^2$. (iii) The vortex string is long and smooth enough in comparison with $1/m_H$ and $1/m_V$: m_H, $m_V \gg 1/R$. (R denotes the characteristic length which represents the smoothness of the vortex string.) (iv) The cut-off parameter Λ of the δ-function in the vorticity tensor current is smaller than the Higgs mass and larger than $1/R$: $m_H \gg \Lambda \gg 1/R$. (The condition $m_H \gg \Lambda$ is consistent with the analysis in which we have used the classical vortex solution as explained under (2.26).) Under the conditions (ii), (iii) and (iv), the large Higgs mass expansion which we have adopted is allowed. We will also impose the same condition in evaluating the $F_{\mu\nu}\tilde{F}_{\mu\nu}$ term in the next section. If one can construct a formulation for systematic evaluation of quantities on the vortex string without imposing the conditions (iii) and (iv), then one would get more interesting information about the vortex string. For example, it would be possible to understand deep relations between the effective vortex string and the fundamental string.

3 The $F_{\mu\nu}\tilde{F}_{\mu\nu}$ term: geometric and topological properties

The purpose of this section is to examine geometric and topological properties of the Abelian Higgs model with vortex strings. In doing so, our formulation is very useful, since we can treat the vortex string in arbitrary shape. To investigate these properties in detail, we concentrate on the topological term I defined by

$$I = \int d^4x \langle F_{\mu\nu}\tilde{F}_{\mu\nu}\rangle, \tag{3.1}$$

which in general appears as the chiral anomaly and the CP violating θ term in the action. Here the definition of the "dual" is the following: $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$. The expectation value $\langle O\rangle$ indicates the integral value of O over all fields V_μ, ρ, a_μ and $B_{\mu\nu}$ except the vortex string coordinates $X_\mu(\sigma)$ in the path integral representation. As will be clarified below, this topological term is closely related to the geometric and topological properties of the vortex string such as the self-intersection number of the world sheet swept by the vortex string. Notice that the world sheet swept by the vortex string can intersect with itself, that is, the vortex string coordinate $X_\mu(\sigma)$ is an immersion of a two dimensional parameter space into the four dimensional Euclidean space-time $\boldsymbol{R}^4$. In subsequent subsections, first we explicitly evaluate I in the cases where the number of the vortex strings is conserved and the vortex string reconnection does not occur, then discuss its generalization. In addition, in the last part of this section we discuss the chiral fermion number of the ANO vortex string.

3.1 Explicit evaluations

First, we study the case in which a single vortex string exists and the vortex string reconnection does not occur. Since the gauge field A_μ is given by $(C_\mu - a_\mu)/e$ as shown in (2.9) and $V_\mu = C_\mu - \partial_\mu \omega$, the topological term I can be written as

$$I = \frac{1}{e^2}\int d^4x \langle f_{\mu\nu}\tilde{f}_{\mu\nu} - 2V_{\mu\nu}\tilde{f}_{\mu\nu} + V_{\mu\nu}\tilde{V}_{\mu\nu}\rangle. \tag{3.2}$$

We divide I into the following three pieces and evaluate them separately:

$$I^{(1)} = \frac{1}{e^2}\int d^4x \langle f_{\mu\nu}\tilde{f}_{\mu\nu}\rangle, \tag{3.3}$$

$$I^{(2)} = -\frac{2}{e^2}\int d^4x \langle V_{\mu\nu}\tilde{f}_{\mu\nu}\rangle, \tag{3.4}$$

$$I^{(3)} = \frac{1}{e^2}\int d^4x \langle V_{\mu\nu}\tilde{V}_{\mu\nu}\rangle. \tag{3.5}$$

At the beginning, we estimate $I^{(1)}$. By using $\tilde{f}_{\mu\nu} = -J_{\mu\nu}/2$, which is derived by variations of the action (2.4) with respect to $B_{\mu\nu}$, the topological term $I^{(1)}$ takes a simpler form

$$I^{(1)} = \frac{1}{4e^2}\int d^4x J_{\mu\nu}(x)\tilde{J}_{\mu\nu}(x). \tag{3.6}$$

Substituting $J_{\mu\nu}$ given by (2.5) for this, we obtain

$$\begin{aligned} I^{(1)} = \frac{4\pi^2}{e^2}\int d^4x \int d^2\sigma d^2\sigma' \Sigma_{\mu\nu}(X(\sigma))\tilde{\Sigma}_{\mu\nu}(X(\sigma')) \\ \times \frac{\Lambda^8}{\pi^4}\exp\left\{-2\Lambda^2 x^2 - \frac{\Lambda^2}{2}|X(\sigma) - X(\sigma')|^2\right\}, \end{aligned} \tag{3.7}$$

where

$$\Sigma_{\mu\nu}(\sigma) = \partial_1 X_{[\mu}(\sigma)\partial_2 X_{\nu]}(\sigma), \tag{3.8}$$

and we have transformed x_μ into $x_\mu + (X_\mu(\sigma) + X_\mu(\sigma'))/2$. Integrating over x, this becomes

$$I^{(1)} = \frac{4\pi^2}{e^2}\int d^2\sigma d^2\sigma' \Sigma_{\mu\nu}(X(\sigma))\tilde{\Sigma}_{\mu\nu}(X(\sigma'))\delta^{(4)}(X(\sigma) - X(\sigma')), \tag{3.9}$$

where the δ-function is regularized as (2.13). (Λ is replaced by $\Lambda/\sqrt{2}$ in the δ-function in (3.9).) One may worry about the contributions from points σ_2, $\sigma_2' = \pm\infty$, since at these points $X_4(\sigma)$ or $X_4(\sigma')$ becomes infinity and then the integral region of x_4 would not run from $-\infty$ to $+\infty$. In order to resolve this difficulty, we introduce a condition $\partial_2 \boldsymbol{X}(\sigma) = \alpha \partial_1 \boldsymbol{X}(\sigma)$ at $\sigma_2 = \pm\infty$, where α is an arbitrary function of σ. By this condition the integrand of (3.7) becomes zero at σ_2, $\sigma_2' = \pm\infty$, so this difficulty disappears. This condition is not unnatural because it includes adiabatic processes satisfying $\partial_2 \boldsymbol{X}(\sigma) = 0$ at $\sigma_2 = \pm\infty$. We therefore require this condition in this subsection.

Eq. (3.9) shows that this integrand takes non-zero values when $X(\sigma) = X(\sigma')$. Since we are considering the immersion $X_\mu(\sigma)$, there are two kinds of contributions arising from coincident points where $X(\sigma) = X(\sigma')$ in space-time: the first contribution $I_1^{(1)}$ comes from points where $\sigma = \sigma'$ in the parameter space and the second one $I_2^{(1)}$ comes from points where $\sigma \neq \sigma'$ but $X(\sigma) = X(\sigma')$. $I^{(1)}$ is the sum of $I_1^{(1)}$ and $I_2^{(1)}$.

Now let us evaluate $I_1^{(1)}$ and $I_2^{(1)}$ separately. The procedure of calculation is essentially the same as that of the previous section. Using the regularized δ-function (2.13) and the expansion of $X_\mu(\sigma')$ (2.20), we have the first contribution

$$I_1^{(1)} = -\frac{2\pi^2}{e^2} PS_i, \tag{3.10}$$

where

$$PS_i = \frac{1}{4\pi} \int d^2\sigma \sqrt{g} g^{ab} \epsilon_{\mu\nu\rho\sigma} \partial_a t_{\mu\nu} \partial_b t_{\rho\sigma} \tag{3.11}$$

is Polyakov's self-intersection number in which

$$t_{\mu\nu} = \frac{1}{\sqrt{g}} \partial_1 X_{[\mu} \partial_2 X_{\nu]}. \tag{3.12}$$

Here we have neglected terms which vanish in the $\Lambda \to \infty$ limit. Indeed, we are considering the case where $1/R\Lambda$ is small, (R is a characteristic length representing the smoothness of the vortex string), so that these terms depending on Λ are not relevant. The PS_i was discussed from the point of view of the QCD string involving its extrinsic geometry [15, 17, 18] and is proper to four dimensional space-time.

Next we will evaluate the second contribution $I_2^{(1)}$. Suppose that the world sheet swept by the vortex string intersects with itself transversally at many isolated points $x = p_1, \ldots, p_n$: $X(\sigma_{p_i}) = X(\sigma'_{p_i})$ at $\sigma = \sigma_{p_i}$ and $\sigma' = \sigma'_{p_i}$, where $\sigma_{p_i} \neq \sigma'_{p_i}$. In the neighborhood of a "self-intersection" point $x = p_i$, we can expand $X_\mu(\sigma)$ and $X_\mu(\sigma')$ around $\sigma = \sigma_{p_i}$ and $\sigma' = \sigma'_{p_i}$

respectively:

$$X_\mu(\sigma) = X_\mu(\sigma_{p_i}) + w_a \partial_a X_\mu(\sigma_{p_i}) + \dots,$$
$$X_\mu(\sigma') = X_\mu(\sigma'_{p_i}) + w'_a \partial_a X_\mu(\sigma'_{p_i}) + \dots, \tag{3.13}$$

where $w = \sigma - \sigma_{p_i}$ and $w' = \sigma' - \sigma'_{p_i}$. Using the fact that at the point $x = p_i$

$$\delta^{(4)}(X(\sigma) - X(\sigma')) = \frac{2}{|\Sigma_{\mu\nu}(X(\sigma_{p_i}))\tilde{\Sigma}_{\mu\nu}(X(\sigma'_{p_i}))|}\delta^{(2)}(w)\delta^{(2)}(w'), \tag{3.14}$$

we find that a local contribution to $I_2^{(1)}$ at $x = p_i$ is $16\pi^2 SI_n(p_i)/e^2$, where

$$SI_n(p_i) = \text{sign}[\Sigma_{\mu\nu}(X(\sigma_{p_i}))\tilde{\Sigma}_{\mu\nu}(X(\sigma'_{p_i}))] \tag{3.15}$$

is a local intersection number at $x = p_i$, namely a suitable sign determined by the relative orientation at $x = p_i$. Here we have neglected terms which vanish in the $\Lambda \to \infty$ limit. The total contribution to $I_2^{(1)}$ is the sum of local contributions at all p_i's:

$$I_2^{(1)} = \frac{16\pi^2}{e^2} SI_n, \tag{3.16}$$

where

$$SI_n = \sum_{i=1}^{n} SI_n(p_i) \tag{3.17}$$

is the self-intersection number of the world sheet swept by the vortex string.

Thus, collecting (3.10) and (3.16), we get

$$I^{(1)} = -\frac{2\pi^2}{e^2} PS_i + \frac{16\pi^2}{e^2} SI_n. \tag{3.18}$$

Next we explain that $I^{(2)}$ and $I^{(3)}$ do not contribute to I. At the leading order of powers of $1/m_H$, $I^{(2)}$ and $I^{(3)}$ are respectively given by

$$I^{(2)} = \frac{2}{e^4}\int d^4x d^4y \epsilon_{\mu\nu\rho\sigma} f_{\mu\nu}(x) f_{\alpha\beta}(y) \partial^x_\rho \partial^x_\alpha \langle V_\sigma(x) V_\beta(y)\rangle_0, \tag{3.19}$$

$$I^{(3)} = \frac{4}{e^2}\int d^4x d^4y d^4z \epsilon_{\mu\nu\rho\sigma} f_{\alpha\beta}(y) f_{\gamma\delta}(z) \partial^y_\mu \partial^y_\alpha \langle V_\nu(x) V_\beta(y)\rangle_0 \times \partial^z_\rho \partial^z_\gamma \langle V_\sigma(x) V_\delta(z)\rangle_0. \tag{3.20}$$

We can compute $I^{(2)}$ in the same way to evaluate the effective action and show that the leading term in powers of $1/\Lambda$, which is proportional of PS_i and SI_n, becomes zero and remaining parts vanishes in the $\Lambda \to \infty$ limit . The difficulty at σ_2, $\sigma'_2 = \pm\infty$ which was mentioned in evaluating $I^{(1)}$ also appears, but this can be removed by imposing the same condition. To show $I^{(3)} = 0$ exactly, it is convenient to rewrite the right-hand side of (3.20) in the following:

$$I^{(3)} = 4 \int d^4xd^4yd^4z\epsilon_{\mu\nu\rho\sigma} f_{\alpha\nu}(y) f_{\gamma\sigma}(z) \partial^y_\mu \partial^y_\alpha D(x-y) \partial^z_\rho \partial^z_\gamma D(x-z), \tag{3.21}$$

where $D(x-y)$ is written as (2.17). Furthermore, using $E(x-y)$ defined as

$$E(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+m_V^2)^2} e^{ik(x-y)}, \tag{3.22}$$

$I^{(3)}$ becomes

$$I^{(3)} = 4 \int d^4yd^4z\epsilon_{\mu\nu\rho\sigma} f_{\alpha\nu}(y) f_{\gamma\sigma}(z) \partial^y_\mu \partial^y_\alpha \partial^y_\rho \partial^y_\gamma E(z-y). \tag{3.23}$$

From this we can easily see $I^{(3)} = 0$.

After all, I is given by

$$I = -\frac{2\pi^2}{e^2} PS_i + \frac{16\pi^2}{e^2} SI_n. \tag{3.24}$$

This indicates that the expectation value of the $F_{\mu\nu}\tilde{F}_{\mu\nu}$ term becomes the sum of Polyakov's self-intersection number and the self-intersection number of the world sheet swept by the vortex string. The interesting relation between the geometric and topological quantities I, PS_i and SI_n may be useful in studying the role of the θ term in systems with the vortex string. Note that PS_i and SI_n are different types of self-intersection numbers.

The next issue is to examine the relation between two kinds of self-intersection numbers (PS_i and SI_n) and geometric and topological quantities defined at the boundary of the world sheet where $\sigma_2 = \pm\infty$. (For simplicity, we have set the initial "time" for $-\infty$ and the final "time" for $+\infty$.) First, as will be proved in appendix A, we can find that PS_i is written as the difference between the initial total twist number and the final one: if we take a gauge such that $\sigma_2 = X_4 = t$ and assume that $\dot{\boldsymbol{X}} = \alpha \boldsymbol{X}'$ at $t = \pm\infty$, then we get

$$PS_i = -4T_w(\boldsymbol{X}(t); \boldsymbol{n}(t)) \Big|_{t=-\infty}^{t=+\infty}, \tag{3.25}$$

where

$$T_w(\boldsymbol{X}(t); \boldsymbol{n}(t)) = \frac{1}{2\pi} \oint d\sigma_1 [\boldsymbol{e}(\sigma_1, t) \times \boldsymbol{n}(\sigma_1, t)] \cdot \boldsymbol{n}'(\sigma_1, t) \tag{3.26}$$

is the total twist number in which $\boldsymbol{e} = \boldsymbol{X}'/|\boldsymbol{X}'|$ and the prime denotes the differentiation by σ_1. Here $\boldsymbol{n}(\sigma_1, t)$ is a smoothly varying unit vector which is perpendicular to $\boldsymbol{X}'$ at each point and is a periodic function of σ_1 with period 2π: $\boldsymbol{n}^2 = 1$, $\boldsymbol{n} \cdot \boldsymbol{X}' = 0$ and $\boldsymbol{n}(\sigma_1 + 2\pi, t) = \boldsymbol{n}(\sigma_1, t)$. This total twist number is a geometric quantity dependent on $\boldsymbol{X}$ locally and on the topological

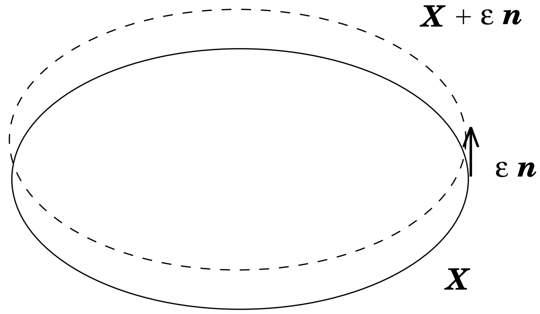

Figure 5: The solid line and the dotted line denote a vortex string $\vec{X}$ and a fictitious one $\vec{X}+\varepsilon\vec{n}$, respectively. The self-linking number of $\vec{X}$ is defined as the linking number of them. In this figure, the self-linking number is zero.

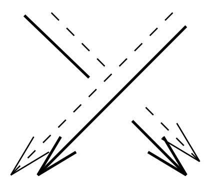

Figure 6: The vortex string (the solid line) crosses the fictitious one (the dotted line) twice at a crossing point in a projecting plane.

class of $\boldsymbol{n}$, but not a topological invariant. If we modify $\boldsymbol{n}$ globally, then the total twist number changes by some integer. This ambiguity will be settled when we consider the self-linking number together, as will be discussed later. Note that this total twist number can be interpreted as the spin factor in three dimensions, which is related to the torsion of a path [19].

We next consider the relation between the self-intersection number and the self-linking number. The self-linking number $SL_k(\boldsymbol{X}(t);\boldsymbol{n}(t))$ at t is defined as the linking number of a vortex string $\boldsymbol{X}(\sigma_1,t)$ and a fictitious vortex string $\boldsymbol{X}(\sigma_1,t)+\varepsilon\boldsymbol{n}(\sigma_1,t)$, where $\boldsymbol{n}$ is equal to the one used in the definition of the total twist number (see fig. 5). This self-linking number is independent of ε and is a topological invariant which can take only integer values. One can think of this "framing" as a thickening of the vortex string into a ribbon bounded by $\boldsymbol{X}$ and $\boldsymbol{X}+\varepsilon\boldsymbol{n}$. As shown in appendix B, the intersection number of two world sheets swept by two different vortex strings coincides with the difference between the linking number at $t=-\infty$ and that at $t=+\infty$. When the vortex string $\boldsymbol{X}$ crosses the fictitious one $\boldsymbol{X}+\varepsilon\boldsymbol{n}$ at a crossing point in a projecting plane, as illustrated in fig. 6, they crosses twice each other at the point. Thus, the self-linking number turns out to be twice the linking number of $\boldsymbol{X}$ and $\boldsymbol{X}+\varepsilon\boldsymbol{n}$. On the other hand, the self-intersection number of the vortex string $X_\mu(\sigma)$ is equal to the intersection number of $X_\mu(\sigma)$ and $X_\mu + \varepsilon n_\mu(\sigma)$ as $\varepsilon \to 0$. Therefore, we obtain

$$SI_n = -\frac{1}{2} SL_k(\boldsymbol{X}(t); \boldsymbol{n}(t))\Big|_{t=-\infty}^{t=+\infty} \quad : \tag{3.27}$$

the self-intersection number of $X_\mu(\sigma)$ is the half of the difference of the self-linking number at $t = -\infty$ and the one at $t = +\infty$.

Using the relations between the quantities defined in the "bulk" and those defined at the "boundary" as in (3.25) and (3.27), we obtain

$$I = -\frac{8\pi^2}{e^2}\Big[SL_k(\boldsymbol{X}(t); \boldsymbol{n}(t)) - T_w(\boldsymbol{X}(t); \boldsymbol{n}(t))\Big]_{t=-\infty}^{t=+\infty}. \tag{3.28}$$

Furthermore, as was shown in [20] (see also [21]), the combination $SL_k - T_w$ is free from the ambiguity of the definition of $\boldsymbol{n}$ and is called the writhing number W_r:

$$SL_k(\boldsymbol{X}(t); \boldsymbol{n}(t)) - T_w(\boldsymbol{X}(t); \boldsymbol{n}(t)) = W_r(\boldsymbol{X}(t)), \tag{3.29}$$

where

$$W_r(\boldsymbol{X}(t)) = \frac{1}{4\pi} \epsilon_{ijk} \oint_\Gamma dx_i \oint_\Gamma dy_i \frac{(\boldsymbol{x} - \boldsymbol{y})_k}{|\boldsymbol{x} - \boldsymbol{y}|^3}. \tag{3.30}$$

Here Γ denotes the configuration of the vortex string $\boldsymbol{X}(\sigma_1, t)$ (see also appendix B). Consequently, we can represent I by using the writhing number:

$$I = -\frac{8\pi^2}{e^2}[W_r(\boldsymbol{X}(+\infty)) - W_r(\boldsymbol{X}(-\infty))]. \tag{3.31}$$

When the vortex string intersects with itself transversally, the writhing number changes by 2 before and after the time of intersection. Note that the writhing number is a geometric quantity, which is a continuous function except at the time of intersection.

Although we have considered the case with a single vortex string as yet, it is easy to extend the result on a single vortex string to the case with many vortex strings. In the case with n vortex strings $X_p(\sigma)$ $(p = 1, \cdots, n)$, there appears a new contribution to I given by

$$\frac{4\pi^2}{e^2} \sum_{\substack{p,q=1 \\ (p \neq q)}}^{n} \int d^2\sigma d^2\sigma' \Sigma_{\mu\nu}(X_p(\sigma)) \tilde{\Sigma}_{\mu\nu}(X_q(\sigma')) \delta^{(4)}(X_p(\sigma) - X_q(\sigma')). \tag{3.32}$$

By using the same argument in evaluating the self-intersection number, this contribution can be computed to be the sum of intersection numbers between different vortex strings:

$$\frac{8\pi^2}{e^2} \sum_{\substack{p,q=1 \\ (p \neq q)}}^{n} I_n(X_p, X_q), \tag{3.33}$$

where $I_n(X_p, X_q)$ denotes the intersection number of two different vortex strings X_p and X_q. Furthermore, as shown in appendix B, the intersection number is represented as the difference between the linking number at $t = -\infty$ and that at $t = +\infty$, so we get

$$I = -\frac{8\pi^2}{e^2} \left[\sum_{p=1}^{n} W_r(\boldsymbol{X}_p(t)) + \sum_{\substack{p,q=1 \\ (p \neq q)}}^{n} L_k(\boldsymbol{X}_p(t), \boldsymbol{X}_q(t)) \right]_{t=-\infty}^{t=+\infty}. \tag{3.34}$$

The first term comes from "self-interactions" of vortex strings and the second one comes from "mutual interactions" between different vortex strings. Note that the writhing number serves as a measure of the right-left asymmetry of the vortex string, that is a measure of its chirality [21].

3.2 Generalization

Until now, we have considered the case in which the creation, annihilation and reconnection of the vortex string do not occur. However, if we adopt several natural assumptions, we can get more general results which include our results (3.34) and the ones in the cases where the number of vortex strings is not conserved. The first key to the generalization is the observation that in the previous subsection the massive gauge field C_μ (or V_μ) had no effect in evaluating I at least at the leading order. It is therefore perhaps natural to assume that this is the case even in more general cases at the leading order. The second key is that the $F_{\mu\nu}\tilde{F}_{\mu\nu}$ term can be rewritten as a total divergence: $F_{\mu\nu}\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\partial_\mu(A_\nu F_{\rho\sigma})$. We assume as usual that $\epsilon_{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}$ is continuous and $F_{\mu\nu}$ becomes zero in the spatial infinity, so that we get

$$I = -\int d^3x \langle \epsilon_{ijk} A_i F_{jk} \rangle \Big|_{t=-\infty}^{t=+\infty}. \tag{3.35}$$

Furthermore, using the first assumption that C_μ (or V_μ) does not contribute to I and the fact that $A_\mu = (C_\mu - a_\mu)/e$, I can be written as

$$I = -\frac{1}{e^2} \int d^3x \epsilon_{ijk} a_i f_{jk} \Big|_{t=-\infty}^{t=+\infty}. \tag{3.36}$$

As an example, let us consider the case where no vortex string exists at the initial time and only one vortex string $\boldsymbol{X}$ exists at the final time. Since a_i satisfies $f_{\mu\nu} = -2\tilde{J}_{\mu\nu}$, a_i is easily solved in the Coulomb gauge $\partial_i a_i = 0$ at the final time $t = +\infty$:

$$a_i(\boldsymbol{x}, t = +\infty) = \frac{1}{2}\epsilon_{ijk} \oint_\Gamma dy_j \frac{(\boldsymbol{x} - \boldsymbol{y})_k}{|\boldsymbol{x} - \boldsymbol{y}|^3}, \tag{3.37}$$

where Γ denotes the position of the vortex string $\boldsymbol{X}(\sigma_1, t = +\infty)$. Here we have taken the $\Lambda \to \infty$ limit . As a result, we obtain

$$I = -\frac{8\pi^2}{e^2} W_r(\boldsymbol{X}(t = +\infty)), \tag{3.38}$$

where the path of the integral in this writhing number ranges over the position of the vortex string at the final time $\boldsymbol{X}(\sigma_1, t = +\infty)$. This result (3.38) can be easily extended to more general cases where m vortex strings exist at the initial time and n vortex strings exist at the final time:

$$I = -\frac{8\pi^2}{e^2} \Big[W_r(t) + L_k(t) \Big]_{t=-\infty}^{t=+\infty} \tag{3.39}$$

where $W_r(t)$ and $L_k(t)$ denote the sum of the writhing numbers of each vortex string at t and the sum of the linking numbers between vortex strings at t, respectively (see (B.7)). Taking $n = m$, one can easily see that this generalized result (3.39) includes the previous result (3.34) which has evaluated exactly at least at the leading order. We can examine the chiral fermion number of the ANO vortex string by using (3.39) or (3.31), as will be shown in the next subsection.

It is worth emphasizing that the explicit evaluation of I such as (3.24), (3.31) and (3.34) is important and necessary itself because through it we can find several interesting relations between geometric or topological quantities and without ambiguities we can realize the cases in which the creation, annihilation and reconnection of the vortex string do not occur. The explicit evaluation is of course necessary to justify the generalization where several assumptions are made.

3.3 The chiral fermion number of a closed ANO vortex string

It was pointed out that if topological defects are coupled to fermions, they might have fermion numbers [22]. In this subsection, using the results obtained in sect. 3.1 and sect. 3.2, we derive the chiral fermion number of the ANO vortex string in arbitrary shape.

We consider models with anomalous global $U(1)$ symmetries and local $U(1)$ symmetries which are spontaneously broken. Although our consideration is model-independent, for definiteness, let us consider the following model defined by

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + |(\partial_\mu - ieA_\mu)\varphi|^2 + 2\lambda \Big(\varphi^\dagger \varphi + \frac{\eta^2}{2} \Big)^2 + i\bar{\psi}(\partial\!\!\!/ - ieA\!\!\!/)\psi. \tag{3.40}$$

This Lagrangian has a chiral symmetry $\psi \to e^{i\gamma_5\theta}\psi$ which is anomalous, so that the conservation law of the chiral current becomes

$$\partial_\mu \langle j^5_\mu \rangle = -i\frac{e^2}{8\pi^2}\langle F_{\mu\nu}\tilde{F}_{\mu\nu}\rangle, \tag{3.41}$$

where $j^5_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi$. Thus the chiral fermion number

$$Q_5 = -i\int d^3x \langle j^5_4 \rangle \tag{3.42}$$

satisfies

$$Q_5(+\infty) - Q_5(-\infty) = -\int d^4x[\nabla_i\langle j^5_i\rangle + \frac{e^2}{8\pi^2}\langle F_{\mu\nu}\tilde{F}_{\mu\nu}\rangle]. \tag{3.43}$$

In order to calculate the chiral fermion number of a closed ANO vortex string, we consider a field configuration which starts at the trivial vacuum when $t = -\infty$ and arrives at a closed ANO vortex string configuration when $t = +\infty$. We assume that the field configuration at spatially infinity is topologically trivial all the time. This is possible because we are considering a closed vortex string. As a result the first term of the right-hand side of (3.43) does not contribute to Q_5. Substituting (3.39) for the second term of the right-hand side of (3.43), we get

$$Q_5(+\infty) = W_r(\boldsymbol{X}), \tag{3.44}$$

where $\boldsymbol{X}$ denotes the position of the vortex string at $t = +\infty$. In general, $Q_5(+\infty)$ has two kinds of contributions: the first comes from the fermion production and the second comes from the chiral fermion number of the ANO vortex string. However, since the first contribution affects only integer part of Q_5 and $W_r(\boldsymbol{X})$ can take non-integer values, we can conclude that the closed ANO vortex has a non-zero chiral fermion number related to the writhing number:

$$Q_5^{\text{vortex}} = W_r(\boldsymbol{X}) \qquad (\text{mod } \boldsymbol{Z}). \tag{3.45}$$

Here we make some comments. (i) To derive (3.45), we have used the relation (3.39) which needs several assumptions. If we only use the explicitly evaluated result (3.31), we obtain $\Delta Q_5^{\text{vortex}} = \Delta W_r(\boldsymbol{X})$ (mod $\boldsymbol{Z}$). Namely, the change of the chiral fermion number of the ANO vortex string is equal to the change of its writhing number. Thus $Q_5^{\text{vortex}} = W_r(\boldsymbol{X}) + const$ (mod $\boldsymbol{Z}$), where "*const*" does not depend on the shapes of the vortex string. (ii) We have imposed the condition "$\partial_2\boldsymbol{X} = \alpha\partial_1\boldsymbol{X}$ at $t = \pm\infty$" to obtain (3.31), but in evaluating the chiral fermion number this condition is automatically satisfied since in this case we have only to treat the vortex string which moves adiabatically [23].

There may be interesting phenomena when the vortex string passes through itself, because the writhing number W_r suffers discontinuities and changes by ± 2 at that time. Anyway, we realize that the shapes of the vortex string must be important to investigate the anomalous fermion production through the ANO vortex string.

4 The vortex string in a model with a broken global $U(1)$ symmetry

The application of our formulation is not restricted to only the Abelian Higgs model. Using the "topological" formulation, we can also treat the vortex string in a model with a broken global $U(1)$ symmetry. Although this model is simpler than the Abelian Higgs model, some differences appear, which are briefly sketched in this section. The Lagrangian with a broken global $U(1)$ symmetry in the Euclidean formulation is given by

$$\mathcal{L} = \partial_\mu \varphi^\dagger \partial_\mu \varphi + 2\lambda \Big(\varphi^\dagger \varphi - \frac{\eta^2}{2} \Big)^2, \tag{4.1}$$

where φ is a complex scalar field. As explained in sect. 2, it is convenient to use the following Lagrangian in considering the dynamics of the vortex string:

$$\mathcal{L} = |(\partial_\mu - i a_\mu)\phi|^2 + 2\lambda \Big(\phi^\dagger \phi - \frac{\eta^2}{2} \Big)^2 + i\epsilon_{\mu\nu\rho\sigma} B_{\mu\nu} f_{\rho\sigma} + i B_{\mu\nu} J_{\mu\nu}. \tag{4.2}$$

Here a_μ is a $U(1)$ gauge field for the vorticity, $B_{\mu\nu}$ a rank two antisymmetric tensor field, $f_{\mu\nu}$ a field strength tensor of a_μ. The definitions of the complex scalar field ϕ and the vorticity tensor current $J_{\mu\nu}$ are given by (2.3) and (2.5), respectively. This Lagrangian is equivalent to (4.1) in the same sense that (2.4) is equivalent to (2.1).

In order to estimate the effective action of the vortex string, we take the unitary gauge and replace the fields as follows:

$$\phi(x) = \frac{1}{\sqrt{2}} (\eta + \rho(x)) e^{i\omega(x)}, \tag{4.3}$$

$$U_\mu(x) = a_\mu(x) - \partial_\mu \omega(x). \tag{4.4}$$

Thus the Lagrangian becomes

$$\begin{aligned} \mathcal{L} \;=\; & \frac{1}{2}\partial_\mu \rho \partial_\mu \rho + 2\lambda \eta^2 \rho^2 + \frac{1}{2}\eta^2 U_\mu U_\mu \\ & + \eta \rho U_\mu U_\mu + \frac{1}{2}\rho^2 U_\mu U_\mu + 2\lambda \eta \rho^3 + \frac{1}{2}\lambda \rho^4 \\ & + i\epsilon_{\mu\nu\rho\sigma} B_{\mu\nu} U_{\rho\sigma} + i B_{\mu\nu} J_{\mu\nu}, \end{aligned} \tag{4.5}$$

where $U_{\mu\nu} = \partial_\mu U_\nu - \partial_\nu U_\mu$. At first sight the propagating mode looks to be only the ρ field, but the field U_μ contains a massless mode implicitly, as will be explained below. The evaluation of the effective action can be performed as in the Abelian Higgs model. As before the propagator of the ρ field is given by

$$\langle \rho(x)\rho(y)\rangle_0 = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m_H^2} e^{ik(x-y)}, \tag{4.6}$$

where $m_H^2 = 4\lambda\eta^2$. Furthermore, we can use the large mass expansion by the powers of $1/m_H$. After integrating over the ρ field, we get at the tree level

$$\mathcal{L} = \frac{1}{2}\eta^2 U_\mu U_\mu + i\epsilon_{\mu\nu\rho\sigma} B_{\mu\nu} U_{\rho\sigma} + iB_{\mu\nu} J_{\mu\nu} + \cdots, \tag{4.7}$$

where "$\cdots$" in (4.7) denote terms with more than one piece of $U_\mu U_\mu$. Using ordinary relations between numbers of vertices, propagators and external lines for diagrams, one can find that the terms with N pieces of $U_\mu U_\mu$ are suppressed by $2(N-1)$-th powers of $1/m_H$. Therefore, if U_μ satisfies a condition $U_\mu \ll m_H$, we can perform a systematic approximation by using the large Higgs mass expansion. In the last part of this section, we will show that this condition is satisfied if cut-off parameters in the present model are smaller enough than m_H.

Now let us explain how a massless mode, which corresponds to the Goldstone mode, is derived from the above Lagrangian (4.7). Integrating over $B_{\mu\nu}$, the partition function Z becomes the following form:

$$Z = \int \mathcal{D}U_\mu \delta(\epsilon_{\mu\nu\rho\sigma} U_{\rho\sigma} + J_{\mu\nu}) \exp(-\int d^4x \frac{1}{2}\eta^2 U_\mu U_\mu + \cdots), \tag{4.8}$$

where "$\cdots$" in (4.8) are the same terms as in (4.7). To integrate over U_μ, the constraint $\epsilon_{\mu\nu\rho\sigma} U_{\rho\sigma} + J_{\mu\nu} = 0$ must be solved to be

$$U_i(x) = \frac{1}{8\pi} \int d^3y \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} \epsilon_{ijk} \partial_j^y J_{4k}(y) + \frac{\sqrt{2}}{\eta} \partial_i \pi(x), \tag{4.9}$$

$$U_4(x) = \frac{1}{16\pi} \int d^3y \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} \epsilon_{ijk} \partial_i^y J_{jk}(y) + \frac{\sqrt{2}}{\eta} \partial_4 \pi(x), \tag{4.10}$$

where $\pi(x)$ is an arbitrary function and $y_4 = x_4$. This arbitrary function $\pi(x)$ reflects the "gauge symmetry" of the above constraint for U_μ. However $\pi(x)$ cannot be fixed through this "gauge symmetry" because the integrand $\exp(-\int d^4x \eta^2 U_\mu U_\mu/2 + \cdots)$ in (4.8) is not gauge invariant. Indeed, after integrating over U_μ, we get

$$Z = \int \mathcal{D}\pi \exp(-\int d^4x \partial_\mu \pi \partial_\mu \pi + \cdots). \tag{4.11}$$

One can easily find that the terms "$\cdots$" in (4.11) do not contain a quadratic term of $\pi(x)$, so $\pi(x)$ represents a massless physical mode.

At the leading order of the large Higgs mass expansion, the Lagrangian (4.5) is reduced to a more familiar form, which can be written as

$$\mathcal{L}_0 = \frac{1}{2}\eta^2 U_\mu U_\mu + i\epsilon_{\mu\nu\rho\sigma} B_{\mu\nu} U_{\rho\sigma} + iB_{\mu\nu} J_{\mu\nu}. \tag{4.12}$$

If we first integrate over U_μ in (4.12), then the Lagrangian becomes

$$\mathcal{L}_0 = \frac{4}{3\eta^2} H_{\mu\nu\lambda} H_{\mu\nu\lambda} + iB_{\mu\nu} J^{\mu\nu}, \tag{4.13}$$

where

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\lambda B_{\mu\nu} + \partial_\nu B_{\lambda\mu}. \tag{4.14}$$

Thus in the large Higgs mass limit, this model gives rise to the Kalb-Ramond model coupling to the vortex string through $J_{\mu\nu}$ [24]. This fact was pointed out by other authors [25]. In the total action obtained in [25], however, there appears a term with the inverse of $|\phi|$, which is proportional to $1/|\phi|^2 \cdot H_{\mu\nu\rho} H_{\mu\nu\rho}$, so that an infinit number of vertices of ρ comes out in the perturbative expansion around the non-zero expectation value η of the Higgs field. On the other hand, in our formulation where the starting Lagrangian is (4.5) (or (4.2)), there appears only a finite number of vertices of ρ, so that it is easy to perform the perturbative expansion in a systematic way. This advantage of our formulation becomes clearer in the evaluation of higher order corrections. In addition, it is also remarkable that we have two different descriptions of the model (4.11) and (4.13), depending on the order of integration over U_μ and $B_{\mu\nu}$ in (4.7) or (4.5). This can be regarded as a kind of "dual transformations".

Finally, we comment on the cut-off parameters which are necessary for justifying the large Higgs mass expansion. In order to satisfy $U_\mu \ll m_H$, we have to introduce two types of cut-off parameters: the first is the cut-off Λ for the δ-function in $J_{\mu\nu}$ which was introduced in the ANO vortex string case (see (2.13)) and the second is the "cut-off" restricting the momentum of the massless mode $\pi(x)$. Now let us imagine a circle C which is on a plane perpendicular to a tangent direction at a point on the vortex string and has a radius r from that point. When $r > 1/\Lambda$, the first terms in (4.9) and (4.10) roughly coincide with $-\partial_\mu\theta$ (θ is a solid angle subtended by the vortex string) and this $\partial_\mu\theta$ takes almost the same value on the circle C, thus satisfying $\oint_C dx^\mu \partial_\mu\theta = 2\pi$ and $\oint_C dx^\mu \partial_\mu\theta \approx \partial_\mu\theta \times 2\pi r$. As a result, we find $\partial_\mu\theta \approx 1/r$ on the circle C. On the other hand, when $r < 1/\Lambda$, the first terms in (4.9) and (4.10) are smaller than Λ on the circle C because we are regularizing the δ-function in $J_{\mu\nu}$ by the cut-off Λ: for example, in the case of a straight vortex string, the first term at $r = 0$ becomes zero. Putting both cases together, the first terms turn out to be smaller than Λ in the whole region. Furthermore, since the momentum of $\pi(x)$ is limited by the second "cut-off", the second terms in (4.9) and (4.10) are smaller than the second "cut-off". Therefore, if the cut-off parameters are smaller enough than m_H, the condition $U_\mu \ll m_H$ is satisfied.

5 Conclusions

In the preceding sections, we have developed the "topological" formulation which allows the systematic analysis of the effective vortex string in arbitrary shape and have applied to the Abelian Higgs model and the model with a broken global $U(1)$ symmetry. Using our formulation, in particular, we have evaluated the effective action of the vortex string and the expectation value of the topological $F_{\mu\nu}\tilde{F}_{\mu\nu}$ term. As a result, many geometric and topological quantities concerning the vortex string have been derived. From the effective action of the ANO vortex string including the Nambu-Goto term and the extrinsic curvature squared term with negative sign, one can realize the motion of the vortex string, which indicates that the vortex string prefers curving as far as it is smooth enough. Furthermore, we have found that the ANO vortex string has a non-zero chiral fermion number related to the writhing number (modulo $\boldsymbol{Z}$) and have suggested that interesting phenomena such as the anomalous fermion production might occur through intersection processes of the ANO vortex string. It should be emphasized that the chiral fermion number of the ANO vortex string depends on its "shape". In addition, we have shown remarkable relations between I, PS_i, SI_n, T_w, SL_k and W_r in (3.24), (3.25), (3.27), (3.28), (3.31), (3.34) and (3.39). They must be useful themselves in studying geometric or topological properties of the vortex string and the role of the θ term.

In our "topological" formulation, there have appeared non-zero extrinsic quantities of the ANO vortex string such as the extrinsic curvature squared term, Polyakov's self-intersection number and the writhing number. On the other hand, it was argued that there appears no extrinsic curvature squared term in the effective action of the ANO vortex string in the formulation based on Förster's parameterization of coordinates [27], which we call the Förster-Gregory (FG) formulation. Furthermore, if one evaluates the topological $F_{\mu\nu}\tilde{F}_{\mu\nu}$ term by using the FG formulation, this term turns out to be zero at least at the leading order. This is due to the fact that the electric field vanishes in the static ANO vortex solution, which is applied at the leading order in the FG formulation. Note that $F_{\mu\nu}\tilde{F}_{\mu\nu} \approx \boldsymbol{B}\cdot\boldsymbol{E}$, where $\boldsymbol{B}$ is a magnetic field and $\boldsymbol{E}$ an electric field. In our formulation, as we have said, we have found the $F_{\mu\nu}\tilde{F}_{\mu\nu}$ term to take a non-zero value at the leading order. It is not easy to compared the FG formulation with ours, because in the FG formulation the equation of motion is used instead of the path integral representation which we have adopted. However, these discrepancies between the FG formulation and ours likely come from the differences between the parameterizations of coordinates and the regularizations of the vortex core in each formulation.

We would like to stress the following points. (i) In our formulation, the Lorentz invariance and the conservation of the vorticity are manifestly satisfied all the time. Indeed, we have used the Lorentz invariant Guassian-type regularization for the δ-function in the vorticity tensor current. Furthermore, the static ANO vortex solution, which is not Lorentz invariant at first sight, is not used at all. (ii) Our perturbative calculation (e.g. by the large Higgs mass expansion) is systematic and efficient. In addition, the path integral representation is convenient to evaluate physical quantities on the vortex string in a systematic manner. (iii) Our formulation can be applied to the model with a broken global $U(1)$ symmetry as shown in sect. 4, while it is difficult to adopt the FG formulation for that model. (iv) Using our formulation, the dynamics of quantized vortices in superfluid can be examined [4]. In this case, the effective action of the vortex string turns out to be of the same form as the action of a vortex string in an incompressible perfect fluid, so that we can explain the phenomena in experiments on the quantized vortex by applying our formulation. On the grounds mentioned above, our "topological" formulation is reliable.

Our formulation can be directly applied to superconductor systems, the cosmic string model and grand unified models with extra $U(1)$ symmetries. In particular, it is of interest to investigate the possibility of the fermion number violation by the vortex string in more detail in various cases including the Weinberg-Salam theory. It will be also suggestive to examine strings and gravity from the point of view of the effective string.

Acknowledgements

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Appendices

A Relation between Polyakov's self-intersection number and the total twist number

In this appendix A, we derive the relation (A.1) under the assumption $\dot{\boldsymbol{X}} = \alpha \boldsymbol{X}'$ at $t = \pm\infty$, where α is some arbitrary function of (σ_1, t):

$$\frac{1}{16\pi} \int d^2\sigma \sqrt{g} g^{ab} \epsilon_{\mu\nu\rho\sigma} \partial_a t_{\mu\nu} \partial_b t_{\rho\sigma} = -T_w(\boldsymbol{X}(t); \boldsymbol{n}(t))\Big|_{t=-\infty}^{t=+\infty}. \tag{A.1}$$

Here $t_{\mu\nu} = \partial_1 X_{[\mu} \partial_2 X_{\nu]}/\sqrt{g}$ and the total twist number T_w is defined as

$$T_w(\boldsymbol{X}(t); \boldsymbol{n}(t)) = \frac{1}{2\pi} \oint d\sigma_1 [\boldsymbol{e}(\sigma_1, t) \times \boldsymbol{n}(\sigma_1, t)] \cdot \boldsymbol{n}'(\sigma_1, t), \tag{A.2}$$

where $\boldsymbol{e} = \boldsymbol{X}'/|\boldsymbol{X}'|$. Since the assumption $\dot{\boldsymbol{X}} = \alpha \boldsymbol{X}'$ at $t = \pm\infty$ coincides with the condition $\partial_2 \boldsymbol{X} = \alpha \partial_1 \boldsymbol{X}$ at $t = \pm\infty$ which has been required in sect. 3.1 (see under (3.9)), it is not unnatural.

Let us first rewrite the left-hand side of (A.1) by the extrinsic curvature. The extrinsic curvature of the world sheet $X^\mu(\sigma)$ is defined as

$$\partial_a \partial_b X_\mu = \Gamma^c_{ab} \partial_c X_\mu + K^A_{ab} n^A_\mu. \tag{A.3}$$

Here Γ^c_{ab} is the Christoffel symbol in terms of derivatives of g_{ab} and n^A_μ is a unite vector satisfying $n^A_\mu n^B_\mu = \delta^{AB}$ and $\partial_a X_\mu n^A_\mu = 0$. In addition, we choose such a direction of n^A_μ as $\epsilon_{\mu\nu\rho\sigma} n^1_\mu n^2_\nu \partial_1 X_\rho \partial_2 X_\sigma > 0$. Using (A.3), $\epsilon_{\mu\nu\rho\sigma} \partial_a X_\mu \partial_b X_\nu \partial_c X_\rho = 0$ and $\epsilon_{\mu\nu\rho\sigma} n^A_\mu n^B_\nu \partial_a X_\rho \partial_b X_\sigma = \sqrt{g}\, \epsilon^{AB} \epsilon_{ab}$, we obtain

$$\frac{1}{16\pi} \int d^2\sigma \sqrt{g} g^{ab} \epsilon_{\mu\nu\rho\sigma} \partial_a t_{\mu\nu} \partial_b t_{\rho\sigma} = -\frac{1}{4\pi} \int d^2\sigma g^{ab} \epsilon^{AB} \epsilon_{cd} K^A_{ac} K^B_{bd}. \tag{A.4}$$

There is another expression of the right-hand side of (A.4):

$$-\frac{1}{4\pi} \int d^2\sigma g^{ab} \epsilon^{AB} \epsilon_{cd} K^A_{ac} K^B_{bd} = -\frac{1}{4\pi} \int d^2\sigma \epsilon^{AB} \epsilon_{ab} \partial_a (n^A_\mu \partial_b n^B_\mu). \tag{A.5}$$

The above relation is derived by using $\epsilon^{AB} \epsilon_{ab} (n^A_\mu \partial_a n^C_\mu)(n^B_\nu \partial_b n^C_\nu) = 0$ and

$$\partial_a n^A_\mu = -(n^A_\sigma \partial_a n^B_\sigma) n^B_\mu - K^A_{ab} g^{bc} \partial_c X_\mu. \tag{A.6}$$

Eq. (A.6) is proved easily by (A.3) and the completeness of the vectors $\partial_a X_\mu$ and n^A_μ. Combining (A.4) and (A.5), we get [18]

$$\frac{1}{16\pi}\int d^2\sigma\sqrt{g}g^{ab}\epsilon_{\mu\nu\rho\sigma}\partial_a t_{\mu\nu}\partial_b t_{\rho\sigma} = -\frac{1}{4\pi}\int d^2\sigma\epsilon^{AB}\epsilon_{ab}\partial_a(n^A_\mu\partial_b n^B_\mu). \tag{A.7}$$

We can construct n^A_μ explicitly by using a vector $n_\mu = (\boldsymbol{n}, 0)$ satisfying $\boldsymbol{n}^2 = 1$, $\boldsymbol{X}'\cdot\boldsymbol{n} = 0$ and $\boldsymbol{n}(\sigma_1 + 2\pi, t) = \boldsymbol{n}(\sigma_1, t)$:

$$\begin{aligned} n^1_\mu &= \frac{1}{\sqrt{1-\frac{(n\cdot\partial_2 X)^2 g_{11}}{g}}}\left[n_\mu - \frac{(n\cdot\partial_2 X)}{g}(g_{11}\partial_2 X_\mu - g_{12}\partial_1 X_\mu)\right], \\ n^2_\mu &= -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}n^1_\nu t_{\rho\sigma}. \end{aligned} \tag{A.8}$$

If $g \neq 0$ and $\boldsymbol{n}^2 \neq 0$, $g - (n\cdot\partial_2 X)^2 g_{11}$ is not zero, so the above n^A_μ are well-defined. One can easily see that these n^A_μ satisfy the conditions such that $n^A_\mu n^B_\mu = \delta^{AB}$, $\partial_a X_\mu n^A_\mu = 0$ and $\epsilon_{\mu\nu\rho\sigma}n^1_\mu n^2_\nu\partial_1 X_\rho\partial_2 X_\sigma > 0$. Since $\boldsymbol{n}$ and X_μ are periodic functions of σ_1, n^A_μ also become periodic functions of σ_1. Thus the integral of $\epsilon^{AB}\partial_1(n^A_\mu\partial_2 n^B_\mu)$ vanishes, and the right-hand side of (A.7) becomes

$$\frac{1}{2\pi}\oint d\sigma_1(n^1_\mu\partial_1 n^2_\mu)\Big|^{\tau=+\infty}_{\tau=-\infty}. \tag{A.9}$$

Substituting (A.8) for (A.9), the integrand $n^1_\mu\partial_1 n^2_\mu$ becomes

$$\frac{1}{\sqrt{g}\,(1-\frac{(n\cdot\partial_2 X)^2 g_{11}}{g})}\epsilon_{\mu\nu\rho\sigma}n_\mu\left[\partial_1 n_\nu - \frac{n\cdot\partial_2 X}{g}(g_{11}\partial_1\partial_2 X_\nu - g_{12}\partial_1\partial_1 X_\nu)\right]\partial_1 X_\rho\partial_2 X_\sigma. \tag{A.10}$$

Now we take $\sigma_2 = X_4 = t$ gauge. By the assumption $\dot{\boldsymbol{X}} = \alpha\boldsymbol{X}'$ at $t = \pm\infty$, $n\cdot\partial_2 X$ vanishes and g becomes $|\boldsymbol{X}'|^2$ as $t\to\pm\infty$. Therefore, we obtain

$$\frac{1}{2\pi}\oint d\sigma(n^1_\mu\partial_1 n^2_\mu)\Big|^{t=+\infty}_{t=-\infty} = \frac{1}{2\pi}\oint d\sigma_1[\boldsymbol{e}(\sigma_1,t)\times\boldsymbol{n}(\sigma_1,t)]\cdot\boldsymbol{n}'(\sigma_1,t)\Big|^{t=+\infty}_{t=-\infty}. \tag{A.11}$$

From (A.11) and (A.7), (A.1) holds.

B Relation between the intersection number and the linking number

In this appendix B, we explain the relation between the intersection number and the linking number.

There are many methods for defining the linking number, all of which are equivalent. Our definition in this paper is the following (see [26]). Let J and K be two disjoint oriented knots in $\boldsymbol{R}^3$, which are corresponding to vortex strings at a fixed time t. Consider a "regular" projection of J and K to any plane. At each point c_i ($i = 1, \cdots, m$) where J crosses K in the projecting plane, $\text{sign}(c_i)$ is defined as $+1$ for the case depicted in the left picture of fig. 7 and -1 for the case depicted in the right one of fig. 7. The linking number of J and K,

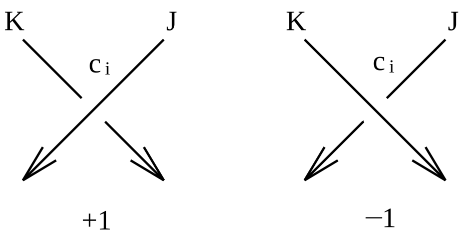

Figure 7: The definition of $\text{sign}(c_i)$. J crosses K at a point c_i in a projecting plane.

$L_k(J, K)$, is defined by the half of the sum of all signs at c_i's:

$$L_k(J, K) = \frac{1}{2}\sum_{i=1}^{m} \text{sign}(c_i). \tag{B.1}$$

The linking number satisfies $L_k(J, K) = L_k(K, J)$ and $L_k(-J, K) = -L_k(J, K)$ where $-J$ is J with the inverse orientation.

Next consider two different world sheets $X_\mu(\sigma)$ and $Y_\mu(\sigma')$ which intersect transversally at p_i ($i = 1, \cdots, n$). We take a gauge such that $\sigma_2 = X_4$ and $\sigma'_2 = Y_4$, and identify X_4 and Y_4 as "time": $X_4 = Y_4 = t$. Let us look at the neighborhood of a intersection point $p = (\boldsymbol{p}, p_4)$ among p_i's. (We can get the total result by collecting the local contributions at p_i's.) If two vortex strings move from a state such as the left picture of fig. 8 to a state such as the right one of fig. 8 as time goes from $t < p_4$ to $t > p_4$, the linking number of $\boldsymbol{X}(\sigma_1, t)$ and $\boldsymbol{Y}(\sigma'_1, t)$ changes by -1:

$$L_k(\boldsymbol{X}, \boldsymbol{Y}; t > p_4) - L_k(\boldsymbol{X}, \boldsymbol{Y}; t < p_4) = -1, \tag{B.2}$$

where $L_k(\boldsymbol{X}, \boldsymbol{Y}; t)$ indicates the linking number of $\boldsymbol{X}$ and $\boldsymbol{Y}$ at t.

On the other hand, in our gauge, the local intersection number of $X_\mu(\sigma)$ and $Y_\mu(\sigma')$ at p is given by

$$I_n(p) = \text{sign}[\Sigma_{\mu\nu}(X)\tilde{\Sigma}_{\mu\nu}(Y)] = \text{sign}[\epsilon_{ijk}X'_iY'_j(\dot{Y}_k - \dot{X}_k)]. \tag{B.3}$$

When the location of two vortex strings changes from the left picture to the right one in fig. 8, then the above sign becomes $+1$, namely the intersection number at p is $+1$. Therefore, it turns out that the local intersection number at p coincides with the difference between the

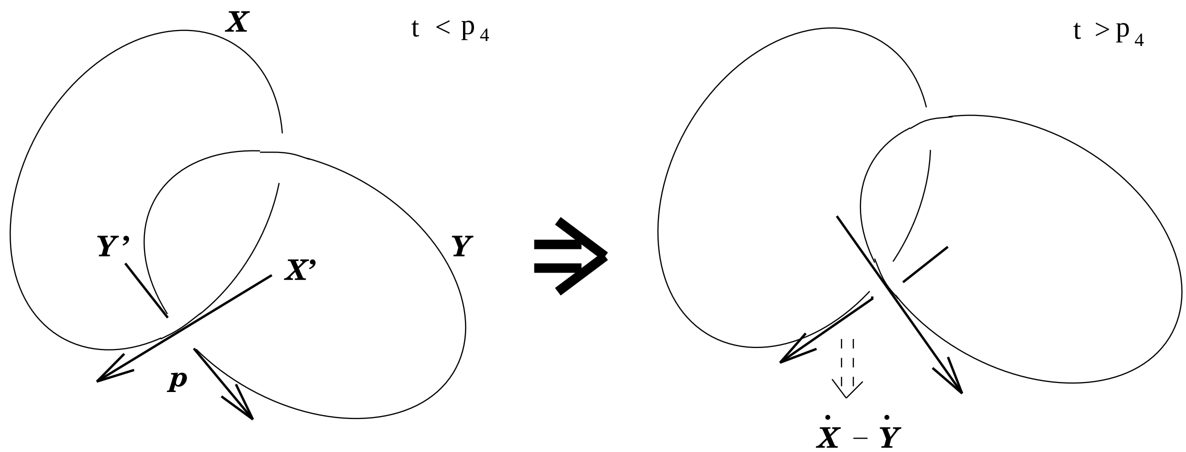

Figure 8: A example of the two different world sheets $X_\mu(\sigma)$ and $Y_\mu(\sigma')$ which intersect transversally at $(\vec{p}, p_4)$. The configurations of $\vec{X}(\sigma_1, t)$ and $\vec{Y}(\sigma_1', t)$ at $t < p_4$ are illustrated in the left picture, and the ones at $t > p_4$ in the right picture. The solid arrows denote the tangent vectors $\vec{X}'$ and $\vec{Y}'$, and the dotted arrow denotes the vector $\dot{\vec{X}} - \dot{\vec{Y}}$.

linking numbers of $\boldsymbol{X}$ and $\boldsymbol{Y}$ before and after the time $t = p_4$:

$$I_n(p) = -L_k(\boldsymbol{X}, \boldsymbol{Y}; t)\Big|_{t<p_4}^{t>p_4}. \tag{B.4}$$

Repeating the consideration above at each local intersection point p_i, we obtain

$$I_n[X, Y] = -L_k(\boldsymbol{X}, \boldsymbol{Y}; t)\Big|_{t=-\infty}^{t=+\infty}, \tag{B.5}$$

where

$$I_n[X, Y] = \sum_{i=1}^{n} I_n(p_i) \tag{B.6}$$

is the intersection number of $X_\mu(\sigma)$ and $Y_\mu(\sigma')$. We conclude that the intersection number of the world sheets is equal to the difference between the linking number at $t = -\infty$ and that at $t = +\infty$.

On the other hand, the linking number of J and K defined by (B.1) is equivalent to the Gauss linking number, which can be written as

$$\frac{1}{4\pi}\epsilon_{ijk} \oint_J dx_i \oint_K dy_j \frac{(\boldsymbol{x} - \boldsymbol{y})_k}{|\boldsymbol{x} - \boldsymbol{y}|^3}, \tag{B.7}$$

where $\boldsymbol{x}$ ranges over J and $\boldsymbol{y}$ over K. Eq. (B.7) has used in sect. 3.2. Assuming $J = K$, this Gauss linking number becomes the writhing number of J.

Finally, we comment on the writhing number W_r defined by (3.30). Let us parameterize $\boldsymbol{x}$ and $\boldsymbol{y}$ ranging over the path Γ in (3.30) as $\boldsymbol{X}(\sigma_1)$ and $\boldsymbol{X}(\sigma_1')$ respectively, where $0 \le \sigma_1 \le 2\pi$

and $0 \le \sigma_1' \le 2\pi$. Furthermore, define the domain $D = \{\sigma_1' | \sigma_1 - \delta \le \sigma_1' \le \sigma_1 + \delta\}$ where δ is a small constant, and then expand $\boldsymbol{X}(\sigma_1')$ around $\sigma_1' = \sigma_1$. Inserting the expanded $\boldsymbol{X}(\sigma_1')$ into the numerator and the denominator in the integrand of the writhing number (3.30), we find that the numerator and the denominator behave like $|\sigma_1' - \sigma_1|^4$ and $|\sigma_1' - \sigma_1|^3$ respectively. The behavior $|\sigma_1' - \sigma_1|^4$ of the numerator is due to the antisymmetric property of the ϵ tensor. Therefore, the writhing number is not singular in the region where $\boldsymbol{X}(\sigma_1) = \boldsymbol{X}(\sigma_1')$ (i.e. $\boldsymbol{x} = \boldsymbol{y}$) since the integrand of the writhing number behaves like $|\sigma_1' - \sigma_1|$ in D, that is in the domain where $\sigma_1' \approx \sigma_1$. Here we have assumed that the path Γ does not intersect with itself.

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