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On the large-N_c behaviour of the L_7 coupling in χPT.

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Abstract

It is shown that the usual large-N_c counting of the coupling constant L_7 of the $\mathcal{O}(p^4)$ low-energy chiral $SU(3)$ Lagrangian [3] is in conflict with general properties of QCD in the large-N_c limit. The solution of this conflict within the framework of a chiral $U(3)$ Lagrangian is explained.

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1. Chiral perturbation theory (χPT) is the effective field theory of quantum chromodynamics (QCD) at low energies. It describes the strong interactions of the low-lying pseudoscalar particles in terms of the octet of Nambu-Goldstone fields ($\vec{\lambda}$ are the eight 3×3 Gell-Mann matrices):

$$\Phi(x) = \frac{\overrightarrow{\lambda}}{\sqrt{2}} \cdot \overrightarrow{\varphi}(x) = \begin{pmatrix} \pi^\circ/\sqrt{2} + \eta/\sqrt{6} & \pi^+ & K^+ \\ \pi^- & -\pi^\circ/\sqrt{2} + \eta/\sqrt{6} & K^\circ \\ K^- & \overline{K}^\circ & -2\eta/\sqrt{6} \end{pmatrix}, \tag{1}$$

as explicit degrees of freedom, rather than in terms of the quark and gluon fields of the usual QCD Lagrangian. In the conventional formulation, the Nambu-Goldstone fields are collected in a unitary 3×3 matrix $\mathcal{U}(x)$ with $\det \mathcal{U} = 1$, which under $SU(3)$-chiral transformations (V_L, V_R) is chosen to transform linearly

$$\mathcal{U} \rightarrow V_R \mathcal{U} V_L^\dagger. \tag{2}$$

The possible terms in the effective Lagrangian $\mathcal{L}_{eff}$, with the lowest chiral dimension i.e., $\mathcal{O}(p^2)$ are [1]:

$$\mathcal{L}_{eff}^{(2)} = \frac{1}{4} f_\pi^2 \left\{ tr \partial_\mu \mathcal{U} \partial^\mu \mathcal{U}^\dagger + tr(\chi \mathcal{U}^\dagger + \mathcal{U} \chi^\dagger) \right\}. \tag{3}$$

The term with the matrix χ is the lowest order term induced by the explicit breaking of the chiral symmetry in the underlying QCD Lagrangian, due to the quark masses. For our purposes, it will be sufficient to consider the approximate case where $m_u = m_d = 0$. Then

$$\chi \simeq \mathrm{diag}[0, 0, 2M_K^2]. \tag{4}$$

An explicit representation of $\mathcal{U}$ is

$$\mathcal{U}(x) = \exp\left(-i \frac{1}{f_\pi} \overrightarrow{\lambda} \cdot \overrightarrow{\varphi}(x) \right), \tag{5}$$

and our normalization is such that $f_\pi = 92.5\ MeV$.

The identification of all the independent local terms of $\mathcal{O}(p^4)$, invariant under parity, charge conjugation, and local chiral-$SU(3)$ transformations, as well as the phenomenological determination of the ten physical coupling constants which appear, has been made by Gasser and Leutwyler [2], [3]. We reproduce the terms which will be relevant for our discussion:

$$\mathcal{L}_{eff}^{(4)} \doteq L_5 tr[\partial_\mu \mathcal{U}^\dagger \partial^\mu \mathcal{U} (\chi^\dagger \mathcal{U} + \mathcal{U}^\dagger \chi)] + L_7 [tr(\chi^\dagger \mathcal{U} - \mathcal{U}^\dagger \chi)]^2 + L_8 tr(\chi^\dagger \mathcal{U} \chi^\dagger \mathcal{U} + \chi \mathcal{U}^\dagger \chi \mathcal{U}^\dagger). \tag{6}$$

Notice that in the limit $m_u = m_d = 0$ there are no ambiguities of the type observed by Kaplan and Manohar [4].

It is well known that in the large-N_c limit of QCD [5], the constant f_π is $\mathcal{O}(\sqrt{N_c})$. In ref. [3] it was shown that L_5 and L_8 are of $\mathcal{O}(N_c)$, while the conclusion for L_7 was that it should be considered as of $\mathcal{O}(N_c^2)$. The purpose of this note is to discuss some implications of the large-N_c counting of the L_7 constant. Our arguments implicitly assume that general properties derived from QCD in the large-N_c limit hold order by order in χPT.

2. Expanding the Lagrangians $\mathcal{L}^{(2)}_{eff}$ and $\mathcal{L}^{(4)}_{eff}$ in powers of the Nambu-Goldstone fields $\overrightarrow{\varphi}(x)$ gives a string of interaction terms. However, to leading order in the limit $N_c \to \infty$, and due to the fact that the $\overrightarrow{\varphi}(x)$-fields are always normalized to f_π -which itself is $\mathcal{O}(\sqrt{N_c})$-, only kinetic-like terms and mass-like terms survive from the expansion in $\mathcal{L}^{(2)}_{eff}$. With L_5 and L_8 considered as of $\mathcal{O}(N_c)$, the same happens with the terms induced by these couplings. It is easy to check that the omitted couplings in $\mathcal{L}^{(4)}_{eff}$, all lead to terms which vanish in the strict $N_c \to \infty$ limit. This property is in fact in agreement with general arguments which assert that QCD in the $N_c \to \infty$ limit is a theory of non-interacting mesons [6].

The terms generated by the expansion of the trace modulated by the L_7 coupling in eq.(6) require special attention. When restricted to terms relevant to the purpose of the discussion here, one finds

$$L_7[tr(\chi^\dagger \mathcal{U} - \mathcal{U}^\dagger \chi)]^2 = -L_7 \frac{64}{3}\frac{M_K^4}{f_\pi^2}\eta(x)\eta(x) - L_7 \frac{64}{3\sqrt{3}}\frac{M_K^4}{f_\pi^4}\eta(x)\pi^0(x)K^+(x)K^-(x) + \cdots, \tag{7}$$

where the dots denote other 4-meson interactions. With L_7 considered as of $\mathcal{O}(N_c^2)$ in the large-N_c limit we are confronted with (at least) the following serious problems:

i) The quadratic term, a mass like term, diverges in the $N_c \to \infty$ limit contrary to the expected constant behaviour [6]. Furthemore, with $L_7 < 0$ -as suggested from phenomenology [3]- it is tachyonic!

ii) The quartic $\eta\pi^0 K^+ K^-$ term which remains in the limit $N_c \to \infty$ plays the rôle of a $\lambda\varphi^4$-like interaction with a non-vanishing negative ($L_7 < 0$) coupling, again in contradiction with the expected non-interacting behaviour [6]. This would also imply that the effective potential is unbounded from below and the theory, in the limit $N_c \to \infty$, becomes unstable!

In view of these conflicts, it seems mandatory to reconsider the reasons which lead to considering L_7 as of $\mathcal{O}(N_c^2)$ in the large-N_c limit.

3. The reason why it is usually assumed that $L_7 \sim \mathcal{O}(N_c^2)$ is because of the contribution of the $SU(3)$-singlet, the η_0. Indeed, when discussing the large-N_c limit, it is convenient to work with the $U_L(3) \times U_R(3)$ effective Lagrangian which includes nine Nambu-Goldstone fields. To leading order in the chiral expansion and in the $1/N_c$-expansion the Lagrangian is (see refs. [7] to [14]):

$$\mathcal{L}(\tilde{\mathcal{U}}) = \frac{1}{4} f_\pi^2 \left\{ tr\partial_\mu \tilde{\mathcal{U}} \partial^\mu \tilde{\mathcal{U}}^\dagger + tr(\chi \tilde{\mathcal{U}}^\dagger + \tilde{\mathcal{U}}\chi^\dagger) + \frac{a}{4N_c}(tr \log \frac{\tilde{\mathcal{U}}}{\tilde{\mathcal{U}}^\dagger})^2 \right\}, \tag{8}$$

where

$$\tilde{\mathcal{U}} = \exp(-i\sqrt{2/3}\frac{\eta_0(x)}{f_\pi})\mathcal{U}. \tag{9}$$

The constant a has dimensions of mass squared, and with the $1/N_c$ factor pulled out, it is of $\mathcal{O}(1)$ in the large-N_c limit. At the same level of approximations, the expansion in powers of the η_0 field results in the expression:

$$\begin{aligned}\mathcal{L}(\tilde{\mathcal{U}}) &= \frac{1}{2}\partial_\mu\eta_0\partial^\mu\eta_0 - \frac{1}{2}(\frac{3a}{N_c} + \frac{2}{3}M_K^2)\eta_0\eta_0 \\ &\quad -i\sqrt{2/3}\frac{f_\pi}{4}\eta_0 tr(\chi^\dagger\mathcal{U} - \mathcal{U}^\dagger\chi) + \mathcal{L}_{eff}^{(2)}, \end{aligned} \tag{10}$$

where $\mathcal{L}_{eff}^{(2)}$ is the same as in eq.(3), and therefore has no η_0 field couplings.

Integrating out the η_0 field results in general in a non-local interaction of the form

$$f_\pi^2 \int d^4x d^4y [tr(\chi^\dagger\mathcal{U}(x) - \mathcal{U}(x)^\dagger\chi)] \;\; D(x-y) \;\; [tr(\chi^\dagger\mathcal{U}(y) - \mathcal{U}(y)^\dagger\chi)] \ , \tag{11}$$

with

$$\int d^4x \ e^{ipx} \ D(x) = \left[p^2 - \left(\frac{3a}{N_c} + \frac{2}{3}M_K^2\right)\right]^{-1} \quad .$$

As long as one keeps N_c *finite*, and to the extent that $3a/N_c >> M_K^2$, one can envisage an expansion in powers of momentum that yields a tower of *local* operators. To lowest order in this expansion one finds

$$\mathcal{L}(\tilde{\mathcal{U}}) \Rightarrow \mathcal{L}_{eff}^{(2)} - \frac{f_\pi^2}{48(\frac{3a}{N_c} + \frac{2}{3}M_K^2)}[tr(\chi^\dagger\mathcal{U} - \mathcal{U}^\dagger\chi)]^2. \tag{12}$$

There appears then an L_7 term, with an estimate for the induced coupling constant :

$$L_7^{\eta'} = -\frac{f_\pi^2}{48(\frac{3a}{N_c} + \frac{2}{3}M_K^2)}. \tag{13}$$

In terms of physical masses: $\frac{3a}{N_c} + \frac{2}{3}M_K^2 \simeq M_\eta^2 + M_{\eta'}^2 - \frac{4}{3}M_K^2 \simeq M_{\eta'}^2$, and $L_7^{\eta'} \simeq -2 \times 10^{-4}$.

Let us now discuss the large-N_c limit. Taking the limit $N_c \to \infty$ on the expression (12) invalidates the condition under which eq. (12) was obtained. If, in spite of this fact, one still takes this limit one finds that the answer crucially depends on whether one takes the chiral limit first and $N_c \to \infty$ afterwards or the other way around. The usual result $L_7 \sim \mathcal{O}(N_c^2)$ comes from first neglecting M_K^2 in eq. (13) and then taking $N_c \to \infty$. This faces the problems with the large-N_c counting of QCD that we mentioned at the beginning. If, on the contrary, one takes the limit $N_c \to \infty$ keeping M_K^2 finite one finds that the chiral counting is upset and one can no longer consider L_7 (which now would be of $\mathcal{O}(N_c)$ instead) as a coefficient of the $\mathcal{O}(p^4)$ chiral $SU(3)$ Lagrangian. Of course both situations stem from the fact that in the limit $N_c \to \infty$ the interaction (11) cannot be considered local and therefore, strictly speaking, it cannot be encoded into an L_7 term. The limit $N_c \to \infty$ has to be described by enlarging the chiral $SU(3)$ group to chiral $U(3)$ (i.e. the Lagrangian of eq. (8) plus higher order terms). There will also be an $\mathcal{O}(p^4)$ L_7-*like* term in this chiral $U(3)$ effective Lagrangian, but it will be at most of $\mathcal{O}(N_c)$ at large N_c since the η_0 is an explicit field in the Lagrangian. Then no inconsistencies arise.

There is, however, a sense in which taking the limit $N_c \to \infty$ in eq. (13) is still meaningful. This is when going from $U_L(3) \times U_R(3)$ to the limit $U_L(2) \times U_R(2)$. In this case the kaon is no longer a Nambu-Goldstone particle and can be treated as a heavy particle in an effective theory (of two light flavours) with momenta $p^2 << M_K^2$. Then M_K^2 in eq. (13) is kept finite and the mass term for the η-field in the Lagrangian (12) has two sources, with the result

$$-\frac{1}{2}\,\frac{a}{N_c}\,\frac{4M_K^2}{\frac{3a}{N_c}+\frac{2}{3}M_K^2}\,\eta(x)\,\eta(x)\quad . \tag{14}$$

In the limit $N_c \to \infty$ the Lagrangian (12) reveals the existence of four Nambu-Goldstone particles: the three pions and the η singlet. With $m_u \neq m_d \neq 0$, the same Lagrangian describes the effective theory of four (pseudo) Nambu-Goldstone bosons with an explicit L_7-type interaction. The coupling constant of this interaction term, which is the $U_L(2) \times U_R(2)$ equivalent of the l_7 in refs. [2] [3], appears then as of $\mathcal{O}(N_c)$ in the large-N_c limit.

From the analyses above, we are led to the conclusion that, if the low-energy effective field theory of QCD with three light flavours is to remain compatible with the large-N_c limit of QCD, a safe way to formulate the effective chiral Lagrangian is within the framework of $U_L(3) \times U_R(3)$ instead of $SU_L(3) \times SU_R(3)$. In that respect, a systematic study of the phenomenological implications of low-energy hadron physics within that framework, in particular in the sector of $\eta(\eta')$-decays and, perhaps, non-leptonic K-decays, seems worthwhile.

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