arXiv:hep-ph/9903443 v1 22 Mar 1999

BUTP–99/2

One–channel Roy equations revisited*

J. Gasser[1] and G. Wanders[2]

[1] Institut für Theoretische Physik, Universität Bern,
Sidlerstrasse 5, CH–3012 Bern, Switzerland
gasser@itp.unibe.ch

[2] Institut de Physique Théorique, Université de Lausanne,
CH–1015 Lausanne, Switzerland
gerard.wanders@ipt.unil.ch

March 1999

Pacs: 11.30.Rd, 11.55.Fv, 11.80.Et, 13.75.Lb

Keywords: Roy equations, Dispersion relations, Partial wave analysis, Meson-meson interactions, Pion-pion scattering, Chiral symmetries

Abstract

The Roy equation in the single channel case is a nonlinear, singular integral equation for the phase shift in the low–energy region. We first investigate the infinitesimal neighborhood of a given solution, and then present explicit expressions for amplitudes that satisfy the nonlinear equation exactly. These amplitudes contain free parameters that render the non–uniqueness of the solution manifest. They display, however, an unphysical singularity at the upper end of the interval considered. This singularity disappears and uniqueness is achieved if one uses analyticity properties of the amplitudes that are not encoded in the Roy equation.

*Work supported in part by Swiss National Science Foundation

1 Introduction

The elastic $\pi\pi$ amplitude has recently been evaluated in the framework of chiral perturbation theory [1] to two loops [2]. The representation is valid in the low–energy region, where the centre–of–mass energy of the pions is less than about 400 MeV. On the other hand, precise experimental data are presently only available above ~ 600 MeV. In order to connect the two regions, one may rely on a set of dispersion relations for the partial wave amplitudes due to Roy [3]. These allow one to extrapolate the data down to threshold and to merge with the chiral expansion [4]. In the present article, we investigate this extrapolation procedure in the one–channel case.

Roy's representation [3] for the partial wave amplitudes t_l^I of $\pi\pi$ scattering reads as follows,

$$\mathrm{Re}\, t_l^I(s) = k_l^I(s) + \sum_{I'=0}^{2} \sum_{l'=0}^{\infty} \fint_4^\infty dx\, K_{ll'}^{II'}(s,x)\, \mathrm{Im}\, t_{l'}^{I'}(x) \;\; ; \;\; 4 \leq s \leq 60 \;\; , \qquad (1.1)$$

where I and l denote isospin and angular momentum, respectively[1]. The linear subtraction polynomials k_l^I are expressed in terms of the two S–wave scattering lengths. The kernels $K_{ll'}^{II'}$ contain a diagonal, singular Cauchy kernel that generates the right–hand cut in the partial wave amplitudes, as well as a logarithmically singular piece that generates the left–hand cut.

The relations (1.1) are consequences of the exact analyticity properties of the $\pi\pi$ scattering amplitude, of the Froissart bound and of crossing symmetry. They demonstrate that the scattering amplitudes are fully determined by the imaginary parts of the partial waves, except for the two scattering lengths that play the role of subtraction constants. Combined with unitarity, (1.1) amounts to an infinite system of coupled, singular integral equations – the Roy equations – for the phase shifts in a low–energy interval from threshold $s = 4$ to a matching point $s_0 < 60$. In this framework, the phase shifts above the matching point s_0, the absorption parameters and the two S–wave scattering lengths are assumed to be externally assigned. The mathematical problem consists in solving the Roy equations with this input.

Soon after the original article of Roy [3] had appeared, extensive phenomenological applications were performed [5], resulting in a detailed analysis and exploitation of the then available experimental data on $\pi\pi$ scattering. For a recent review of those results, we refer the reader to the article by Morgan and Pennington [6]. Parallel to these phenomenological applications, the very structure

[1] We express all energies in units of M_π. Further, $\fint$ denotes a principal value integral.

of the Roy equations was investigated. In [8], extensions of the equations (1.1) were presented, valid in the larger range $-28 \leq s \leq 125.31$. Further, the manifold of solutions of Roy's equations was investigated as well, both in the single channel [9, 10, 11] as well as in the coupled channel case [12]. In the late seventies, Pool [13] provided a proof that the original, infinite set of integral equations does have at least one solution for $s_0 < 23.31$, provided that the driving terms are not too large, see also [14]. Heemskerk and Pool also examined numerically the solutions of the Roy equations, both by solving the N equation [14] and by using an iterative method [15].

It emerged from these investigations that – for a given input of absorptive parts, absorption parameters and S–wave scattering lengths – there are in general many possible solutions to the Roy equations. This non–uniqueness is due to the singular Cauchy kernel in the right–hand side of (1.1). In order to investigate the uniqueness properties of the Roy system, one may – in a first step – keep only this part of the kernels, as a result of which the integral equations decouple: one is left with a single channel problem, i.e. a single partial wave, that has, moreover, no left–hand cut. This mathematical problem was examined by Pomponiu and Wanders [9]. Investigating the infinitesimal neighborhood of a given solution, they found that the multiplicity of the solution increases by one whenever the value of the phase shift at the matching point goes through a multiple of $\pi/2$. By contrast, the number of parameters in the usual partial wave equation increases in general by two whenever the phase shift at infinity passes through a positive integer of π, see e.g. [16, 17] and references cited therein.

Atkinson and Warnock [10] investigated a one–channel problem by using N/D methods, that do not require linearization. Their work may be summarized as follows. First, they find that solutions of the Roy equations with $\delta(s_0) \geq -\frac{\pi}{2}$ are members of solution manifolds depending on m parameters, where m denotes the integer part of $2\delta(s_0)/\pi$. Second, these solutions may be computed through an integral equation in which the m parameters appear explicitly. An exception to these statements – whose proof is rather complex – could occur if a certain Fredholm operator had unit eigenvalue. Although the relevant integral equations in [10] exhibit the m parameters, it had not been shown that these are really effective: an arbitrarily chosen set may lead to ghosts.

After 1980, interest in the Roy equations waned. In recent years, it has however become clear that there are good reasons to revive these techniques. First, new K_{l4} experiments are planned [7] or are already under way [18, 19]. These will provide new information on the low–energy $\pi\pi$ scattering amplitude. A reliable analysis of available low–energy data and high–energy phase shifts

should then again be based on Roy equations. Second, one can establish at this point contact with CHPT, that will allow one [1, 21] to gain insight into the structure of the underlying theory, the Standard Model. First steps in this program have already been performed [22, 23]. Of course, this enterprise requires that one understands the structure of the Roy equations in all details. It is the aim of this and a following article [24] to fill existing gaps in this respect and to provide insights into the problem from an actual point of view, see also Ref. [25].

We concentrate in this work on the uniqueness properties of the solutions in the one–channel case, and on their singularity structure. We start by analyzing the Roy equation with the linearization method proposed in [9] and show how the emerging integral equation can be solved by transforming it into a homogeneous Hilbert problem. This method is very efficient in determining the multiplicity of solutions. It does, however, assume the existence of a solution. Furthermore, it is not easy to show that each solution of the linearized equation approximates a solution of the nonlinear one. For these reasons, we also investigate the Roy equation for a special class of inputs, that allow us to find explicit solutions. These solutions do contain parameters that exhibit their non–uniqueness. We then investigate the role of the unphysical singularity that shows up in the solutions. This singularity manifests itself as a cusp in the real and imaginary parts of the amplitude at the matching point. By making use of analyticity properties of the amplitude that are not explicit in the Roy equation, we find that there is a unique solution, devoid of cusps.

Our article is organized as follows. In section 2, we formulate the mathematical problem we are concerned with here. In section 3, we analyze the infinitesimal neighborhood of a solution. The construction of a second exact solution – once a first solution is known – is reduced to a linear problem in section 4. This allows us to give in section 5 a class of explicit exact solutions. In section 6 we show how uniqueness is obtained by use of analyticity properties of the amplitudes that are not encoded in the Roy equation. A summary and concluding remarks are given in section 7. Details on precise mathematical formulations are relegated to appendix A, where we also construct the general solution of the linearized problem of section 2. Appendix B provides the connection with the N/D method [10], and appendix C contains the proof of the uniqueness property for an analytic input.

In order to help orient the reader, we note that the key statements in this article are summarized in propositions 1, 2, 3 and 4 in sections 3, 4 and 6. Propositions 1 and 3 were established long ago in Refs. [9] and [26], respectively, whereas propositions 2 and 4 are – to the best of our knowledge – new results.

2 The one–channel Roy equation

In order to study the non–uniqueness properties of the Roy equation, we keep only the diagonal, singular Cauchy kernel in (1.1). The partial wave relations then decouple, and the left–hand cut in the amplitudes disappears. We therefore first explore the set of complex amplitudes $f : [4, \infty) \to \mathbb{C}$ with the following properties:

i) In an interval $[4, s_0]$ containing the threshold $s = 4$ and a matching point s_0, the real part is given by a dispersion relation

$$\mathrm{Re}\, f(s) = a + (s-4)\frac{1}{\pi} \fint_4^\infty \frac{dx}{x-4} \frac{\mathrm{Im}\, f(x)}{x-s} \ . \tag{2.1}$$

ii) The imaginary part $\mathrm{Im}\, f$ is a given input function A above s_0,

$$\mathrm{Im}\, f(s) = A(s), \qquad s \geq s_0 \ . \tag{2.2}$$

iii) Elastic unitarity holds below s_0,

$$\mathrm{Im}\, f(s) = \sigma(s)|f(s)|^2 \ \ , \ s \in [4, s_0] \ \ ; \ \ \sigma(s) = [1 - 4/s]^{1/2} \ . \tag{2.3}$$

We relegate a precise formulation of the regularity properties of f to appendix A and simply note that, as a minimal requirement, the imaginary part $\mathrm{Im}\, f$ must be continuous in $[4, s_0]$, in particular,

$$\lim_{s \nearrow s_0} \mathrm{Im}\ f(s) = A(s_0) \ . \tag{2.4}$$

Equations (2.1)–(2.4) constitute the mathematical problem that we discuss in the following: determine the amplitudes f which verify these equations for given scattering length a and given absorptive part A. This one–channel problem allows detailed analytical and numerical calculations and provides useful insight into the solutions of the coupled system considered e.g. in Refs. [4, 5, 12].

Elastic unitarity allows one to reduce the problem to the determination of a single real function on the real interval $[4, s_0]$, because f is parametrized by its real phase shift δ below s_0:

$$f(s) = \frac{1}{\sigma(s)} \mathrm{e}^{\mathrm{i}\delta(s)} \sin\delta(s), \qquad s \in [4, s_0]. \tag{2.5}$$

We choose the normalization of δ such that it vanishes at threshold, $\delta(4) = 0$. The boundary condition (2.4) becomes

$$\sin^2\delta(s_0) = \sigma(s_0) A(s_0). \tag{2.6}$$

The parametrization (2.5) allows one to write Eqs. (2.1) - (2.3) as a nonlinear, singular integral equation for the phase shift δ,

$$\begin{aligned} \frac{1}{2\sigma(s)} \sin\left(2\delta(s)\right) &= a + \frac{(s-4)}{\pi} \fint_4^\infty \frac{dx}{x-4} \frac{\omega(x)}{x-s} \ , \ s \in [4, s_0] \ , \\ \omega(x) &= \begin{cases} \sigma(x)^{-1} \sin^2 \delta(x) & ; \ 4 \le x \le s_0 \\ A(x) & ; \ x \ge s_0 \ , \end{cases} \end{aligned} \tag{2.7}$$

with boundary condition (2.6). We shall refer to Eqs. (2.1)–(2.4) or Eqs. (2.6) and (2.7) as *Roy equation with input* (a, A). We assume in the following that this input is non–vanishing.

Once a solution of the Roy equation is known, the real part of the amplitude above s_0 is obtained from the dispersion relation (2.1), and f is then defined on $[4, \infty)$.

The above formulation of the Roy equation is used in the following section. There exists an equivalent approach, based on the just mentioned fact that the dispersion relation (2.1) can be extended to the half axis $[4, \infty)$. This implies that the amplitude f is the boundary value of an analytic function, holomorphic in $\mathbb{C} \backslash [4, \infty)$. The Roy equation then amounts to the construction of this analytic function. This method is described in detail in section 4 and used in sections 5 and 6.

The value of the phase shift at the matching point plays a crucial role in the following analysis [9, 10, 12]. However, the input absorptive part fixes $\delta(s_0)$ through the boundary condition (2.6) only modulo π and up to its sign. In Fig. 1, we display $\sin^2 \delta(s_0)$ as a function of the phase shift $\delta(s_0)$, together with the quantity $\sigma(s_0) A(s_0)$, shown with a horizontal line. The filled circles correspond to values of $\delta(s_0)$ that fulfill the condition (2.6). For reasons outlined in [10], we stick to phase shifts that are not too negative. Furthermore, the case where the phase shift at the matching point is a multiple of $\pi/2$ requires special considerations. This complication may be avoided with an appropriate choice of the matching point s_0 in actual calculations. Unless stated otherwise, we therefore assume in the following that

$$\begin{aligned} \delta(s_0) &> -\frac{\pi}{2} \ , \\ \delta(s_0) &\neq n\pi/2 \ , \ n = 0, 1, 2, \ldots \ . \end{aligned} \tag{2.8}$$

The solution of the Roy equation is not unique in general [9, 10, 12], and it is useful to divide the manifold of solutions into classes $C_n^\pm$, parametrized by the value of the phase shift at the matching point,

$$C_n^+ : \qquad \delta(s_0) \in (n\pi, n\pi + \frac{\pi}{2}) \ ,$$

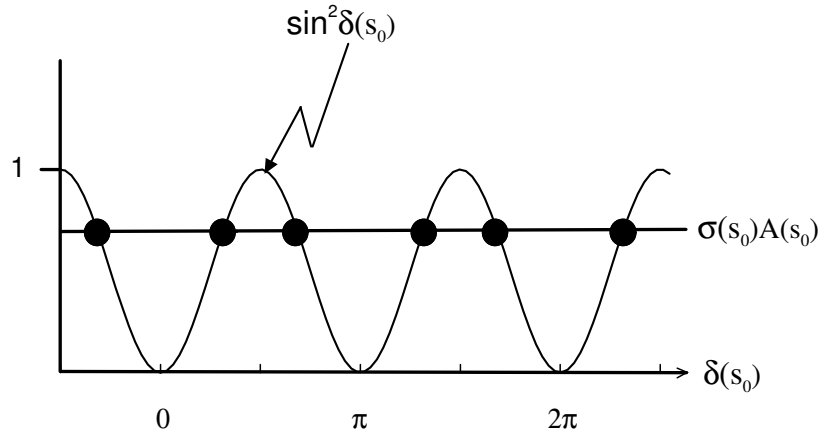

Figure 1: Solutions to Eq. (2.6). The filled circles correspond to values of the phase shift $\delta(s_0)$ that fulfill the condition (2.6).

$$C_n^- : \qquad \delta(s_0) \in (n\pi - \frac{\pi}{2}, n\pi) \ ; \ \ n = 0, 1, 2, \dots \ . \tag{2.9}$$

These classes are indicated in figure 2. To quote an example, a solution f with $\delta(s_0) = \frac{5\pi}{4}$ belongs to the class C_1^+.

Aside from the question of uniqueness, we will be concerned with the singularity structure of the solutions: every solution of (2.7) with arbitrary input (a, A) is regular on $(4, s_0)$, but singular and only Hölder continuous at the end points of that interval. Whereas the singularity at $s = 4$ is due to the threshold behaviour, the one at the matching point is unphysical, because the position of s_0 is arbitrary. It is only for the special class of analytic inputs – which will be specified in section 6 – that there exists a solution that is regular at s_0.

Although the original system of coupled, nonlinear and singular integral equations has been reduced to the simplified equation (2.7), we are unable to solve it explicitly for an arbitrary input. For this reason, we have to rely on alternative tools to achieve our goal. In particular, we i) investigate the infinitesimal neighborhood of a given solution, ii) construct explicit solutions in the case where the matching point is moved to infinity, and iii) investigate the manifold of solutions for a conveniently chosen specific input. A clear picture of the multiplicity of the solutions will emerge in this way, and the role of the unphysical singularity at the matching point can be investigated in a satisfactory manner.

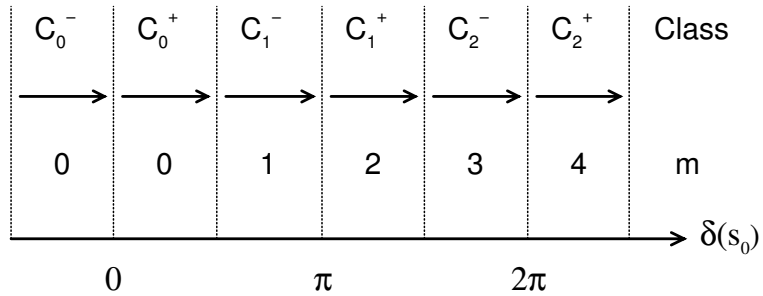

Figure 2: The classes $C_n^{\pm}$ as defined in (2.9). The number m in the lower row equals the dimension of the infinitesimal neighborhood of a given solution, see section 3.

3 Linearization and existence of ambiguities

We assume in this section that the Roy equation with input (a, A) does have a solution δ, and seek for solutions δ' that are nearby. These can be determined by linearizing and explicitly solving the integral equation for the difference of the phase shifts. As already mentioned in the introduction, this method has two apparent drawbacks: first, one does not prove the very existence of the solution δ. Second, it is not shown that the m–dimensional neighborhood of a solution is embedded in an m–dimensional manifold. On the other hand, as we will show in sections 4 and 5, we can construct explicit pairs (a, A) for which a solution is known and for which the ambiguities found below are present. In any case we believe that, despite its shortcomings, the linearization method is very useful and enlightening, in particular so in view of its simplicity.

The solutions of the linearized integral equation show that, if a class with $n > 0$ is non–empty, it contains a continuous family of solutions [9]. This multiplicity structure is identical to the one indicated in the work of Atkinson and Warnock [10], and the investigations of the nonlinear case carried out in later sections support this picture.

We now show how this result is obtained when using the method of [9, 12]. By assumption, both phase shifts δ and δ' satisfy the integral equation (2.7). We wish to determine the difference

$$\Delta(s) = \delta'(s) - \delta(s). \tag{3.1}$$

Equation (2.7) for δ and δ' results in a nonlinear singular integral equation for

Δ. Assuming Δ to be small, linearization of this integral equation gives

$$\cos(2\delta(s))h(s) = (s-4)\frac{1}{\pi}\fint_4^{s_0} dx \frac{1}{x-4}\frac{\sin(2\delta(x))h(x)}{x-s} \quad , \; s \in [4, s_0] \quad , \tag{3.2}$$

where

$$h(s) = \frac{\Delta(s)}{\sigma(s)}. \tag{3.3}$$

To be consistent with the condition (2.6), we require furthermore that $\delta'(s_0) = \delta(s_0)$, or

$$h(s_0) = 0 \; . \tag{3.4}$$

The original Roy equation is replaced by the singular linear integral equation (3.2), to be solved with the boundary condition (3.4). The latter shows that we can determine in this manner only those f' which belong to the same class as f.

Constructing the solution of (3.2) is equivalent to solving a boundary value problem for analytic functions – a so called Hilbert problem [27]. To make this article self–contained, we discuss the procedure in appendix A, where it is shown that the general solution of (3.2) is given by [9]

$$h(s) = (s-4)G(s)P(s), \qquad s \in [4, s_0], \tag{3.5}$$

with

$$G(s) = \frac{1}{(s_0 - s)^m} \exp\left[\frac{2}{\pi}\fint_4^{s_0} dx \frac{\delta(x)}{x-s}\right] \quad , \tag{3.6}$$

and where $P(s)$ is an arbitrary real polynomial of degree $m-1$, with

$$m = \begin{cases} \left[\frac{2\delta(s_0)}{\pi}\right] & \text{if } \delta(s_0) > \frac{\pi}{2}, \\ 0 & \text{if } -\frac{\pi}{2} < \delta(s_0) < \frac{\pi}{2}. \end{cases} \tag{3.7}$$

The symbol $[x]$ in (3.7) denotes the greatest integer not exceeding x. Polynomials of negative degree are considered to be identically zero here and in the following. For $\delta(s_0) > 0$, the number m coincides with the so–called *index* of the Hilbert problem. For a monotonically increasing phase, it counts the number of times $\delta(s)$ goes through multiples of $\pi/2$ as s varies from threshold to the matching point s_0. We indicate in figure 2 some of its values.

The following proposition summarizes the results obtained in this section and in appendix A.

Proposition 1 *Let δ be a solution of Eq. (2.7). It is an isolated solution of that equation if $-\frac{\pi}{2} < \delta(s_0) < \frac{\pi}{2}$. If $\delta(s_0) > \frac{\pi}{2}$, the infinitesimal neighborhood of δ is an m–parameter family of solutions δ' with $\delta'(s_0) = \delta(s_0)$, where m is given in (3.7).*

Comments

1. An obvious question concerns the interpretation of the m parameters in the polynomial P in (3.5). In the case where the phase shift δ is monotonically increasing and where m is an even integer, f exhibits $m/2$ resonances on $(4, s_0)$. The nearby f' has also $m/2$ resonances, with slightly modified positions and widths. These changes are fixed by the m coefficients in P. This is verified in an example that we study in subsection 5.1. The situation is not so simple if m is odd. The case $m = 1$ is illustrated in subsection 5.3.

2. Atkinson and Warnock [10] have studied these parameters in a single channel Roy equation that has a structure which is similar to the one studied here. Using an N/D–method, they find that, when m is an even integer, the parameters are related to the position and residues of the CDD poles between threshold and the matching point. In the case where m is an odd integer, one of these parameters is connected with the singular nature of the N equation.

3. The difference $\triangle$ is singular at the matching point. This is a signal of the singularity of the general solution of the Roy equation mentioned in section 2.

4. Our discussion deals exclusively with restrictions to the low–energy interval $[4, s_0]$. Once Δ is determined on that interval, the value of f' for all real s is obtained from the dispersion relation (2.1). In the linear regime,

$$f'(s) = f(s) + \frac{(s-4)}{\pi} \lim_{\epsilon \searrow 0} \int_4^{s_0} dx \frac{\sin(2\delta(x))h(x)}{x-4} \frac{1}{x-s-\mathrm{i}\epsilon} \,. \tag{3.8}$$

   Consistency with unitarity is then by no means ensured above s_0: if σf stays on or inside the Argand circle, $\sigma f'$ may well be outside. This shows that physical requirements which are not encoded in the Roy equation can reduce the ambiguities.

5. Our method is also applicable if the first inelastic threshold is below s_0, provided that the absorption parameter η is known. The representation (2.5) is replaced by

$$f(s) = \frac{1}{2\mathrm{i}\sigma(s)} \left[\eta(s)\mathrm{e}^{2\mathrm{i}\delta(s)} - 1\right] . \tag{3.9}$$

   The form of the linearized equation (3.2) is unchanged, whereas η enters into the definition of the unknown, $h(s) = \eta(s)\sigma(s)^{-1}\Delta(s)$. With this modification, (3.5) remains valid.

4 The full amplitudes

4.1 Matching at infinity

It is instructive to consider the case where the matching point s_0 is moved to infinity, because the Roy equation can then be solved explicitly. The amplitude vanishes at infinity [17] in this case, as a result of which the input reduces to the scattering length a. Equation (2.7) becomes[2]

$$\frac{1}{2\sigma(s)}\sin\left(2\delta(s)\right) \;=\; a+\frac{(s-4)}{\pi}\fint_4^\infty \frac{dx}{x-4}\frac{\sin^2\delta(x)}{\sigma(x)(x-s)} \;\;,\; s\in[4,\infty] \;\;, \tag{4.1}$$

to be solved for given scattering length a. For $a>0$, a solution is provided by the phase shift of the amplitude

$$f_1(s)=\left[\frac{1}{a}-\rho(s)\right]^{-1} \;\;, \tag{4.2}$$

where ρ is the Chew–Mandelstam function

$$\rho(s)=\frac{1}{\pi}\sigma(s)\left\{\ln\frac{1-\sigma(s)}{1+\sigma(s)}+i\pi\right\} \;\;;\;\; s\geq 4\;. \tag{4.3}$$

In figure 3, we display the phase shift δ_1 of f_1 with a solid line for $a=0.5$. This is not the only solution – further examples are e.g. the phase shifts of

$$\begin{aligned}\frac{1}{f_2} &= \frac{1}{f_1}+\frac{s-4}{s_2-4}\frac{r_2}{s-s_2} \;\;,\\ \frac{1}{f_3} &= \frac{1}{f_2}+\frac{s-4}{s_3-4}\frac{r_3}{s-s_3} \;\;;\;\; s_i>4 \;\;,\; r_i>0\;.\end{aligned} \tag{4.4}$$

The corresponding phase shifts are again displayed in figure 3, for $(s_2,r_2)=(30,40);(s_3,r_3)=(45,40)$. We note that the phase shifts tend to multiples of π at infinity, $\delta_i(\infty)=(i-1)\pi, i=1,2,3$. [The complications with $\delta(s_0)=n\pi/2$, that we mentioned in section 2, disappear when $s_0=\infty$.]

These examples show that the Roy equation with matching point at infinity allows for many solutions. The poles at s_i are CDD poles [28], and one has to specify the phase shift at infinity as well as the CDD parameters r_i, s_i in order to pin down a solution completely. The unsubtracted version of equation (4.1) is discussed in detail in [17].

[2] A subtraction is in fact not needed here [17]. We stick to the present formulation for an easier comparison with the Roy equation at a finite matching point.

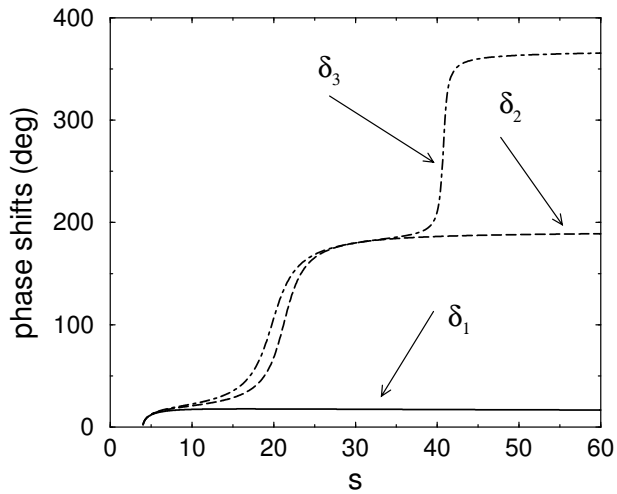

Figure 3: The phase shifts $\delta_{1,2,3}$ that correspond to the solutions $f_{1,2,3}$ of equation (4.1), with parameters $a = 0.5$, $(s_2, r_2) = (30, 40), (s_3, r_3) = (45, 40)$.

4.2 Matching at finite energy

We now present an approach that will allow us to construct exact solutions of the Roy equation with a finite matching point s_0. Although we cannot solve the Roy equation with an arbitrary input, we can construct explicit amplitudes f that satisfy the dispersion relation (2.1) and verify elastic unitarity below some s_0. An amplitude with this property defines an input $a_f \doteq f(4)$, $A_f(s) \doteq \mathrm{Im}\, f(s), s \geq s_0$, and is itself a solution of the Roy equation with this input. This holds true e.g. for the amplitudes f_i in Eqs. (4.2) and (4.4). Our goal is to show that – given f – one can construct other solutions $f' \neq f$, with the same input. In this manner, we find that the Roy equation has in general solutions in different classes C_n^μ. This cannot be seen in the linearization framework discussed in the last section, since there, one has to assume that the phase shifts of the original and of the new solution coincide at the matching point, as a result of which the old and new solutions stay in the same class.

Let us describe the procedure in detail. Until now we worked with amplitudes that are complex functions of the real variable $s \geq 4$. As announced in section 2, we now define our amplitudes as analytic functions in the complex s–plane, cut along the real axis for $s \geq 4$. The corresponding amplitudes verifying Eqs. (2.1) – (2.4) are the boundary values f_+ of f, defined as

$$f_+(s) = \lim_{\epsilon \searrow 0} f(s + \mathrm{i}\epsilon) \ , \qquad s \in [4, \infty) \ . \tag{4.5}$$

In particular, we consider the set of functions f with the following properties:

i) f is holomorphic in $\mathbb{C}\backslash[4,\infty)$ and verifies the dispersion relation (2.1), written for f_+.

ii) f_+ is elastic below the matching point,

$$f_+(s) = \frac{1}{\sigma(s)} e^{i\delta(s)} \sin\delta(s) \quad , \qquad s \in [4, s_0] \; . \tag{4.6}$$

iii) f_+ satisfies the regularity requirements listed in subsection A.2.

Let f satisfy i)–iii). Its boundary value f_+ is a solution of the Roy equation with input

$$\begin{aligned} a_f &\doteq f(4) \;\; , \\ A_f(s) &\doteq \mathrm{Im}\, f_+(s) \;\; , \; s \geq s_0 \; . \end{aligned} \tag{4.7}$$

In this and the following section, we show how to construct functions $f' \neq f$ that satisfy i)–iii) with

$$f'(4) = a_f \;\; , \tag{4.8a}$$

$$\mathrm{Im}\, f'_+(s) = A_f(s) \;\; , \; s \geq s_0 \; . \tag{4.8b}$$

It is clear that f'_+ is then also a solution of the Roy equation with input (4.7), and the existence of an f' with the above mentioned properties therefore establishes the non–uniqueness of the solution of the Roy equation.

In order to construct an f', we use the following Ansatz,

$$\frac{1}{f'(s)} = \frac{1}{f(s)} + (s-4)\frac{H(s)}{D(s)} \; . \tag{4.9}$$

Here, H is an Omnès–type function [9], an analogue of $\bar{G}$ defined in (A.9),

$$H(s) = \left(\frac{s_0}{s-s_0}\right)^m \exp\left[-\frac{2}{\pi} s \int_{s_0}^{\infty} \frac{\mathrm{d}x}{x} \frac{\theta(x)}{x-s}\right] \quad , \tag{4.10}$$

with m given in (3.7). The function θ is Hölder continuous and equal to $\arg f_+$ modulo π, with $\theta(s_0) = \delta(s_0)$. (We assume here in addition that $\mathrm{Im}\; f_+ \geq 0$. As a result of this, we can define $\arg f_+$ such that $0 \leq \arg f_+ \leq \pi$.) The function

D is meromorphic[3] in $\mathbb{C}\backslash[s_0,\infty)$ – which ensures that ii) is fulfilled – and has to be constructed such that also the remaining conditions are satisfied. We write

$$D = D_1 + D_2 \quad , \tag{4.11}$$

where D_1 is a meromorphic component of D, and where D_2 is regular in $\mathbb{C}\backslash[s_0,\infty)$. Condition (4.8a) requires $D(4) \neq 0$. The definition of H has been chosen such that the condition (4.8b) amounts to a simple linear constraint on D_2,

$$\begin{aligned} \mathrm{Im}\ D_{2+}(s) &= \mu(s) \ \ , \ s \geq s_0 \ \ , \\ \mu(s) &= (s-4)|H(s)|A_f(s) \ . \end{aligned} \tag{4.12}$$

Once this condition is fulfilled and $D(s_0)$ is not zero, Im f' is continuous at s_0 if $H(s_0) = 0$. This holds true if $\delta(s_0) > 0$. Choosing a particular function D_2 verifying the condition (4.12), the arbitrariness in f' is entirely contained in D_1. This function has to be such that f' satisfies i) and iii). Therefore, $1/f'$ and $1/f'_+$ have to be nonzero on $\mathbb{C}\backslash[4,\infty)$ and on $[4,\infty)$, respectively. The non–uniqueness of the solution of the Roy equation is due to the very existence of such functions D_1 – we provide explicit examples in the following section. Our discussion leads to

Proposition 2 *Let f be an amplitude verifying conditions i)–iii), with* $\mathrm{Im}\, f_+ \geq 0$, *and $\delta(s_0) > 0$. Let $f' \neq f$ also verify i)–iii), together with the conditions (4.8). Then f' can be written in the form (4.9) – (4.11), where D_1 is a meromorphic component of D, and D_2 is regular in $\mathbb{C}\backslash[s_0,\infty)$, with spectral function (4.12).*

In the N/D approach [10], an important part of the non–uniqueness is due to arbitrariness in the CDD poles. Their relation with the function D in Eq. (4.11) is explained in appendix B.

5 Explicit exact solutions

Here, we illustrate the procedure described in subsection 4.2 with specific examples. An amplitude fulfilling the conditions i)–iii) of that subsection can be written as follows,

$$f(s) = \left[\frac{1}{a} + (s-4)\phi(s) - \rho(s)\right]^{-1} \ , \tag{5.1}$$

[3]In order to simplify the presentation, we shall – without any further mention – use the fact that all our analytic functions satisfy $F(s) = \overline{F(\overline{s})}$.

where ϕ is a suitable function that is meromorphic in the complex s–plane, cut along the real axis for $s \geq s_0$. The amplitudes f_i in Eqs. (4.2) and (4.4) have this form with rational ϕ. To keep our calculations simple, we work with the resonant amplitude f_2 in (4.4). We drop indices and write

$$f(s) = \left[\frac{1}{a} + \frac{s-4}{s_p-4}\frac{r}{s-s_p} - \rho(s)\right]^{-1} \quad , \tag{5.2}$$

where $s_p > 4$, $r > 0$, $a > 0$. As f is elastic on $[4, \infty)$, the argument θ in (4.10) coincides with the phase shift δ. In figures 4 and 5, we display the behaviour of f in the case where

$$a = 0.5 \ \ , \ \ s_p = 40, \ \ r = 25 \ . \tag{5.3}$$

The imaginary part of σf is shown in Fig. 4, whereas its phase shift δ is indicated with a solid line in Fig. 5. Note that the matching point s_0 does not occur in f – it may be chosen at one's convenience. We observe that s_p becomes a CDD pole – as defined in [10] – if $s_p < s_0$.

In the following, we keep the amplitude f fixed and bring it into class C_1^+ or C_1^- by an appropriate choice of s_0. To illustrate, for $s_0 = 50$, f is in class C_1^+, and s_p is a CDD pole.

We notice that f is a solution of the Roy equation with input (a_f, A_f) which is regular at s_0. We are thus dealing with one of the special inputs mentioned in section 2.

5.1 Shift and suppression of a resonance: $f \in C_1^+$, $f' \in C_1^+, C_0^+$

We place s_0 above s_p, as a result of which $\pi < \delta(s_0) < 3\pi/2$, such that f belongs to class C_1^+. We first construct amplitudes f' in the same class C_1^+. This implies that $\delta'(s_0) = \delta(s_0)$. We shall find that, qualitatively, f' is obtained from f by a shift of the position and a change of the width of its resonance. As $\delta'(s_0) \in (\pi, 3\pi/2)$, $1/f'$ has a single pole s_p' on $(4, s_0)$. The second term in the Ansatz (4.9) has to cancel the pole of $1/f$ at s_p and replace it by a new pole at s_p'.

It is convenient to redefine D_1 and D_2 in (4.11) by writing

$$D(s) = (s - s_p)(s - s_p')[D_1(s) + D_2(s)] \ \ , \tag{5.4}$$

where D_1 is meromorphic, and

$$D_2(s) = \frac{1}{\pi}\int_{s_0}^{\infty} \mathrm{d}x \frac{\mu(x)}{(x-s_p)(x-s_p')}\frac{1}{x-s} \ . \tag{5.5}$$

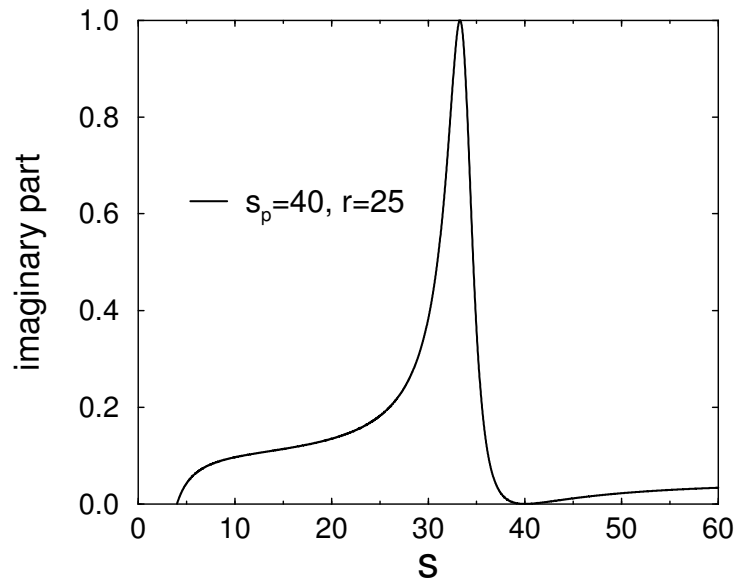

Figure 4: The imaginary part of the function σf in (5.2). The parameters used are the ones in (5.3). The phase shift δ of f is displayed with a solid line in Fig. 5.

The integral converges because $\mu(x) = O(x)$ at infinity. We require regularity of $1/f'$ at s_p and fix the residue r' of its pole at s'_p, with

$$s'_p \neq s_p \ \ , \ 4 < s_p, s'_p < s_0 \ \ , \ r' > 0 \ . \tag{5.6}$$

This gives two conditions which completely determine a two–parameter Ansatz for D_1. One finds that the adequate Ansatz is

$$D_1(s) = \frac{1}{\alpha s + \beta} \ . \tag{5.7}$$

The two constraints on D_1 give

$$\alpha = \frac{\bar{r}'}{R'} - \frac{\bar{r}}{R} \ \ , \qquad \beta = \frac{\bar{r} s'_p}{R} - \frac{\bar{r}' s_p}{R'} \ \ , \tag{5.8}$$

with

$$\begin{aligned} \bar{r} &= \frac{r}{(s_p - 4)H(s_p)}, \qquad & \bar{r}' &= \frac{r'}{(s'_p - 4)H(s'_p)}, \\ R &= 1 + \bar{r}(s_p - s'_p)D_2(s_p), \quad & R' &= 1 + \bar{r}'(s_p - s'_p)D_2(s'_p) \ . \end{aligned} \tag{5.9}$$

The function H is obtained from (4.10) with $m = 2$. Notice that α and β are small if the pair (s'_p, r') is close to (s_p, r). In particular, let

$$\epsilon = \max\left(\frac{|s'_p - s_p|}{s_p} \ , \ \frac{|r' - r|}{r}\right) \ , \tag{5.10}$$

with ϵ small. We then have

$$\alpha = O(\epsilon), \qquad \beta = O(\epsilon) \ . \tag{5.11}$$

Inserting the expressions (5.4) and (5.7) into equation (4.9) we get

$$\frac{1}{f'(s)} = \frac{1}{f(s)} + \frac{(s-4)H(s)}{(s-s_p)(s-s'_p)} \frac{\alpha s + \beta}{1 + (\alpha s + \beta) D_2(s)} \ . \tag{5.12}$$

We see that f' is regular in $\mathbb{C}\backslash[4,\infty)$ and is bounded on its cut if $1/f'$ has no zero. Eqs. (5.12) and (5.11) show that $1/f'$ is close to $1/f$ if ϵ is small and s is outside the vicinity of s_p and s'_p and not too large. The inverse amplitude $1/f'$ is nonzero for such values of s because $1/f$ is nonzero. One finds that $1/f'$ is also nonzero near s_p and s'_p. Since D_2 and H behave at infinity as $s^{-1}\ln|s|$ and $\ln^2|s|$, respectively, the second term in (5.12) is no longer a small correction when $|s|$ becomes large, and a detailed analysis is needed. It is relatively easy to see that no unwanted zeros show up if ϵ is small and β/α is not too large. Consequently we are sure that under these conditions (5.12) provides a two–parameter family of solutions of the Roy equation. This example provides an illustration of the N/D framework outlined in appendix B. In that language, s_p and s'_p correspond to zeros z_j. The CDD pole of f at s_p is removed and replaced by a new one at s'_p.

We have numerically verified that ϵ need not be small for the above conclusions to hold. In particular, there is a finite interval for ϵ such that i) there are no additional singularities in f', and ii) the dispersion relation (2.1) is fulfilled. To illustrate, we choose

$$s'_p = 35 \ , \ r' = 20 \ , \ s_0 = 50 \ . \tag{5.13}$$

In this case one has, using for f the parameters displayed in (5.3),

$$\frac{|s'_p - s_p|}{s_p} = 0.125 \ , \ \frac{|r' - r|}{r} = 0.2 \ . \tag{5.14}$$

These quantities are not small. Nevertheless, f' is a solution of the Roy equation in class C_1^+. We display the phase shifts of f and f' in Fig. 5 with a solid and

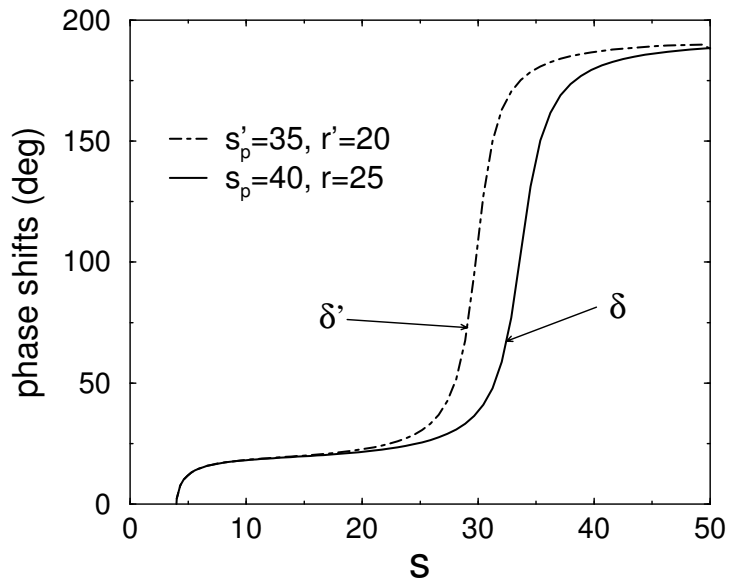

Figure 5: A shift in the class C_1^+. The solid line corresponds to the phase shift δ of the function f shown in Fig. 4, whereas the dot–dashed line displays the phase δ' of f', evaluated from (5.12), with α, β calculated from (5.8, 5.9), using the parameters (5.13).

dot–dashed line, respectively. As required, the two phase shifts agree at the matching point s_0 – the parameters (5.13) really correspond to a shift $C_1^+ \to C_1^+$.

An expression for f' can be obtained in the linear regime from the solution (3.5) of the problem on $[4, s_0]$. It coincides with the result (5.12) to first order in ϵ, if the polynomial in (3.5) is

$$P(s) = C[(\bar{r}' - \bar{r})s + \bar{r}s'_p - \bar{r}'s_p] \ , \tag{5.15}$$

where C is a constant determined by the phase shift δ. This establishes the relation with the coefficients in the arbitrary polynomial P in our example and provides an interpretation of these coefficients.

The method allows the construction of solutions f' which are no longer resonant. This is simply achieved by setting the residue r' to zero. Equation (5.8) then gives $\alpha s + \beta = -\bar{r}(s - s'_p)/R$, and $1/f'$ becomes

$$\frac{1}{f'(s)} = \frac{1}{a} + \bar{r}\frac{s-4}{s-s_p}\left[H(s_p) - \frac{H(s)}{L(s)}\right] - \rho(s) \ , \tag{5.16}$$

where

$$L(s) = 1 - \bar{r}\left[(s - s_p)D_2(s) + (s_p - s'_p)\left(D_2(s) - D_2(s_p)\right)\right] \ . \tag{5.17}$$

As $1/f'$ has no pole on $[4, s_0]$, $\delta'(s_0) = \delta(s_0) - \pi$, and $f' \in C_0^+$. We have to make sure that $1/f'$ has no unwanted zeros. The conclusions reached in the case $r' \neq 0$ apply here and $1/f'$ is non–vanishing if the residue r is small, i.e. if the resonance in f is narrow. On also finds that $\mathrm{Re}\,(1/f')$ does not have a zero on $[4, s_0]$ either. Therefore the phase shift δ' stays below $\pi/2$, and f' is non–resonant. It is easy to see that L does not depend on the choice of s'_p. This means that the amplitude f' defined in (5.16) effectively contains no free parameter. This amplitude is the unique solution of the Roy equation with input (4.8) belonging to class C_0^+. This uniqueness is in accordance with the results of section 3. In the N/D language, the uniqueness stems from the fact that f' has no CDD pole, and $\delta'(s_0) < \pi/2$. What is new with respect to section 3 is that our example shows explicitly that the same Roy equation has solutions in different classes $C_n^\pm$. At given input (a, A), the number of resonances below s_0 is not fixed unless one imposes the precise value of $\delta'(s_0)$.

5.2 Implantation of a resonance: $f \in C_1^+$, $f' \in C_2^+$

We corroborate our last statement by examining another Ansatz for D_1 in the context of the last subsection. We replace (5.7) by

$$D_1(s) = \frac{1}{\alpha}(s - s_1)\,, \tag{5.18}$$

and again require that $1/f'$ obtained from Eqs. (4.9) and (5.4) be regular at s_p and have a pole at s'_p with residue r', with conditions (5.6). One finds

$$\begin{aligned}
\alpha &= \frac{1}{N}\bar{r}\bar{r}'(s'_p - s_p)^2 \ , \\
s_1 &= \frac{1}{2}(s'_p + s_p) - \frac{1}{2N}(s'_p - s_p)\left\{\bar{r}' + \bar{r} - \bar{r}'\bar{r}(D_2(s'_p) + D_2(s_p))(s'_p - s_p)\right\} \ , \\
N &= \bar{r} - \bar{r}' + \bar{r}\bar{r}'(s_p - s'_p)\left\{D_2(s'_p) - D_2(s_p)\right\} \ .
\end{aligned} \tag{5.19}$$

We see that in general, $\alpha = O(\epsilon)$, whereas s_1 is anywhere on the real axis. Equation (5.12) is replaced by

$$\frac{1}{f'(s)} = \frac{1}{f(s)} + \frac{(s-4)H(s)}{(s - s_p)(s - s'_p)}\,\frac{\alpha}{s - s_1 + \alpha D_2(s)} \ . \tag{5.20}$$

If ϵ is small, $1/f'$ has a pole $\bar{s}_1$ near s_1 with small residue, $O(\epsilon)$. This implies that $1/f'$ exhibits a zero near s_1 which can preclude the Ansatz (5.18). The discussion becomes delicate if s_1 is close to 4 or s_0 and the next statements are valid if $\bar{s}_1$ is outside $O(\epsilon)$ neighborhoods of 4 and s_0:

i) if $\bar{s}_1 < 4$ or $\bar{s}_1 > s_0$, f' has a pole on the real axis near $\bar{s}_1$;

ii) if $4 < \bar{s}_1 < s_0$, f' has a pair of complex conjugate poles close to $\bar{s}_1$. These poles are in the first sheet of the branch point $s = 4$ if $\alpha < 0$; they are in the second sheet if $\alpha > 0$.

We see that the Ansatz (5.18) has to be rejected for pairs (s'_p, r') such that $\bar{s}_1 < 4$ or $\bar{s}_1 > s_0$. It is also inadequate if $4 < \bar{s}_1 < s_0$ and $\alpha < 0$. These requirements on α and $\bar{s}_1$ are fulfilled in a wedge–shaped domain of the (s'_p, r')-plane, with apex at (s_p, r).

If $\alpha > 0$, f' generates a solution of the Roy equation in class C_2^+ ($\delta'(s_0) = \delta(s_0) + \pi$). This solution displays two resonances on $[4, s_0]$: a shifted resonance, $(r, s_p) \to (r', s'_p)$, and an implanted narrow resonance near s_1. We illustrate this transition in figure 6. There, we display the phase shifts of f and f' with a solid and a dot–dashed line, respectively, for

$$s'_p = 47 \ \ , \ r' = 20 \ \ , \ s_0 = 50 \ . \tag{5.21}$$

These parameters are again outside the linear regime. The difference of the phase shifts is exactly π at the matching point $s_0 = 50$. This amounts to a special case where the CDD pole at s_p is replaced by two poles at s'_p and $\bar{s}_1$.

We have pointed out in section 3 that the new solutions exhibit a singular behaviour at the matching point s_0. We illustrate this feature in figure 7, where we display the real and the imaginary part of f and of f' with a solid and a dot–dashed line, respectively, for the situation displayed in figure 6. The singular behaviour of f' is manifest, both in the real and in the imaginary part. Note that, since the imaginary parts of f and f' agree for $s \geq s_0$, the real parts also agree at $s = s_0$ due to unitarity. Above the matching point, they are, however, in general different. Because the real part of f is positive in the present case, the figure shows that f' is inside the Argand diagram, at least in an interval above s_0.

5.3 Shift of a resonance: $f, f' \in C_1^-$

In this example we bring f in (5.2) into C_1^- by pushing s_0 below s_p in such a manner that the resonance position is still below s_0, i.e. $\pi/2 < \delta(s_0) < \pi$. [The

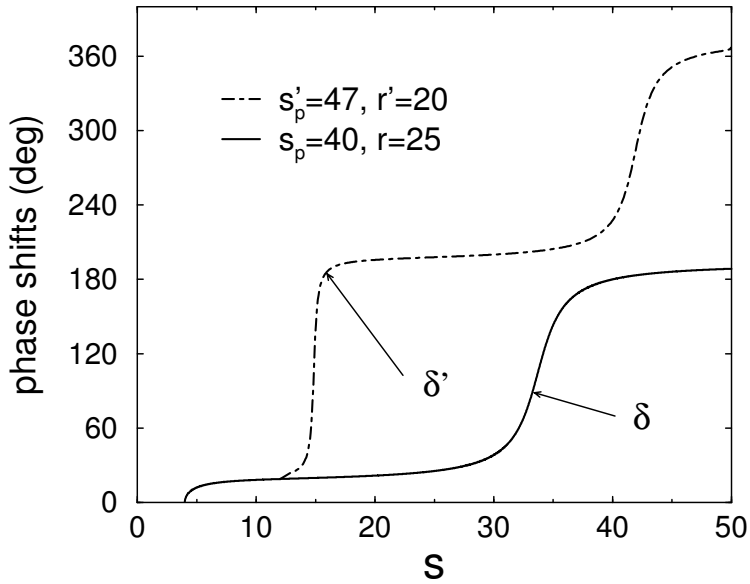

Figure 6: The phase shifts corresponding to a shift $C_1^+ \to C_2^+$. The solid line is the phase shift δ of the function f shown in Fig. 4, whereas the dot–dashed line displays the phase shift δ' of f', evaluated from (5.20). The parameters α and s_1 are evaluated from (5.19), with (5.21).

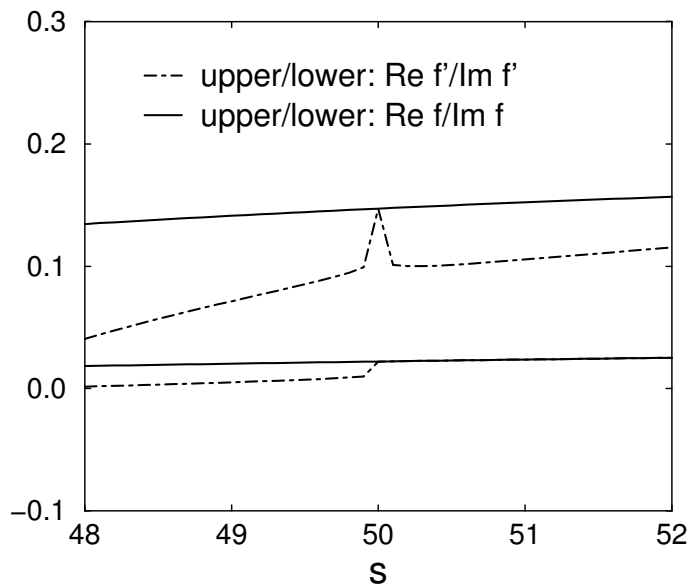

Figure 7: The cusps in f'. We display the real and imaginary parts of f and f', for the situation displayed in Fig. 6.

pole at s_p is no longer a CDD pole.] The function H used in the previous two subsections has now to be replaced by a new one that we call $\hat{H}$. It is given by (4.10), with $m = 1$. Correspondingly, μ defined in (4.12) becomes $\hat{\mu}$. It behaves as s^2 at infinity and 3 subtractions are needed in the construction of D_2. A convenient redefinition of $D_{1,2}$ in (4.11) leads to

$$D(s) = (s - s_p)\left[D_1(s) + (s - s_p)D_2(s)\right] \quad , \tag{5.22}$$

where

$$D_2(s) = s\frac{1}{\pi}\int_{s_0}^{\infty}\frac{\mathrm{d}x}{x}\frac{\hat{\mu}(x)}{(x - s_p)^2}\frac{1}{x - s} \ . \tag{5.23}$$

The integral converges because $\hat{\mu}$ has a second–order zero at s_p. As before, we require $1/f'$ to be regular at s_p. This implies

$$\hat{r} + \frac{\hat{H}(s_p)}{D_1(s_p)} = 0 \quad , \tag{5.24}$$

with $\hat{r} = r/(s_p - 4)$. This equation is consistent with real values of $\hat{r}$ and $D_1(s_p)$, because $\hat{H}(s_p)$ is real. A suitable Ansatz for D_1 is

$$D_1(s) = -\frac{\hat{H}(s_p)}{\hat{r}(s_p - s_p')}(s - s_p'). \tag{5.25}$$

Condition (5.24) is fulfilled with s_p' a free parameter, $s_p' > s_0$. Insertion of Eqs. (5.24) and (5.25) into Eq. (4.9) gives

$$\begin{aligned} \frac{1}{f'(s)} \quad &= \quad \frac{1}{a} + (s - 4)\hat{r}\left\{\left[1 - \frac{\hat{H}(s)}{\hat{H}(s_p)}\right]\frac{1}{s - s_p}\right. \\ &\qquad \left. + \frac{\hat{H}(s)}{\hat{H}(s_p)}\frac{1}{s - s_p' + L(s)}\right\} - \rho(s) \quad , \end{aligned} \tag{5.26}$$

where

$$L(s) = \bar{r}\frac{(s_p' - s_p)^2 D_2(s)}{1 + \bar{r}(s_p' - s_p)D_2(s)} \ . \tag{5.27}$$

Using ϵ defined in (5.10), we have $L = O(\epsilon^2)$ as long as s is not too large. For such values of s, $1/f'$ is close to $1/f$ outside a neighborhood of s_p and s_p'. We notice that Im $L_+ \neq 0$ for $s > s_0$ and $1/f'$ has no real pole whose real residue r' could be imposed (there is a pair of complex conjugate poles near s_p' located in higher sheets of the logarithmic branch point at s_0). One finds that $1/f'$ has no unwanted zeros for any s if ϵ is small enough, and we end up with a family of solutions of the Roy equation in C_1^- indexed by the single parameter s_p'. This

is in accordance with section 3. The shift of the resonance position and the change of its width are now correlated and fixed by the value of $(s'_p - s_p)$. Both f and f' are without CDD poles, and the non–uniqueness is due to the fact that $\pi/2 < \delta(s_0) < \pi$.

6 How to ensure uniqueness

The solution of the Roy equation is not unique – the behavior of a partial wave below s_0 cannot be predicted in a unique way from an input (a, A). Information on the value of the phase shift at s_0 is needed and free parameters have to be fixed if $\delta(s_0) > \pi/2$. This non–uniqueness restricts severely the efficiency of a Roy equation as a tool for the construction of a low–energy extrapolation. We show in the following, how uniqueness can be restored in principle by imposing additional physical requirements. The unphysical singularity at the matching point is removed at the same time, as it has to be.

6.1 Examples of unique solutions

We start with three illustrative examples.

Example 1 In section 5 we have constructed, starting from the amplitude (5.2), several functions f' that satisfy the Roy equation with matching point $s_0 = 50$ and input generated by f,

$$a = f(4) \;\; ; \;\; A(s) = \mathrm{Im}\, f(s) \;\; , \;\; s \geq s_0 \, . \tag{6.1}$$

Two cases are displayed – in terms of their phase shifts δ' – in Figs. 5 and 6. As we already mentioned, these phase shifts develop singularities at the matching point, see Fig. 7. Below we will show that, had we required the new solution f' to be regular at s_0, there would be exactly one solution of the Roy equation with input (6.1), namely f itself. In other words, all solutions $f' \neq f$ develop singularities at s_0. It turns out that this property of the input (6.1) is due to the fact that the amplitude is elastic above the matching point. Whereas f is elastic on the whole interval $[4, \infty)$, the following example shows that a finite interval containing $[4, s_0]$ suffices to render the solution unique.

Example 2 We consider the function f' constructed in subsection 5.2, see Eqs. (5.20), (5.21) and Fig. 6, dot-dashed line. Suppose we wish to construct solutions of the Roy equation with displaced matching point $s'_0 = 45$ and input

defined by the scattering length and the absorptive part of f',

$$a = f'(4)\,;\; A(s) = \mathrm{Im}\, f'(s)\,, s \geq s_0'\,. \tag{6.2}$$

In the language of section 2, this problem belongs to class C_2^-: the phase shift at the matching point is $3\pi/2 < \delta'(s_0') < 2\pi$, see Fig. 6, and according to proposition 1 in section 3, the Roy equation with input (6.2) has therefore a 3–dimensional manifold of solutions. The uniqueness statement in example 1 is also true here: Suppose we seek for solutions of the Roy equation with input (6.2) and require that the solution is regular at $s = s_0'$. There is again exactly one solution, namelyf' itself.

It is obvious that we are dealing with special inputs – we call them *analytic inputs* below. The following example displays an input that is not analytic.

Example 3 Consider again the amplitude (5.2). We set the residue r to zero,

$$a = f(4) + \epsilon\,;\; A(s) = f(s)\,,\; s \geq s_0\,;\; r = 0\,, \tag{6.3}$$

with $s_0 = 50$. For sufficiently small $\epsilon \neq 0$, one can show that there is *no* solution that is analytic at s_0, see subsection 6.3.

6.2 Analytic input and uniqueness

To generalize our findings, we exploit smoothness properties of the amplitudes which are not explicit in the Roy equations. The partial wave amplitudes enter this equation as boundary values of analytic functions. Boundary values need not be smooth, and this is compatible with the dispersion relation (2.1). However, it is quite remarkable that smoothness is imposed by elastic unitarity:

Proposition 3 *Let f be regular in the complex s–plane, cut along the real axis for $s \geq 4$, and let its boundary value f_+ verify elastic unitarity on $[4, s_1]$. Then the real and imaginary parts of f_+ are separately holomorphic in a complex neighborhood of $(4, s_1)$.*

A proof of the proposition is given in [26]. Notice that the parameter s_1 need not coincide with the first inelastic threshold s_{inel} – the proposition is true for any $s_1 > 4$. Taking $s_1 = s_0$, one concludes that all solutions δ of the Roy equation (2.7) are regular in the interval $(4, s_0)$.

Consider amplitudes that satisfy the conditions of proposition 3 and in addition verify the Roy equation with input $(a_f, A_f(s)) = (f(4), \mathrm{Im}\, f_+(s))$ for some

$s_0 < s_1$. The proposition tells us that f is regular at s_0. A second solution of the same equation may be singular at s_0, because it need not be elastic on $[s_0, s_1]$. The following proposition shows that f is in fact the only solution which is regular at s_0. This is an important feature, allowing one in principle to identify the physical extrapolation within the manifold of solutions.

Proposition 4 *Let f be an amplitude that satisfies the conditions of proposition 3 and that furthermore verifies the Roy equation with input (a_f, A_f) for some $s_0 < s_1$. Let $f' \neq f$ be a second solution with the same input. Then f is regular at s_0, whereas f' is singular.*

We relegate a proof of the proposition to appendix C. It elaborates observations made in [9] and [12]. The examples 1 and 2 given above fulfill the conditions of the proposition with $s_1 = \infty$ and $s_1 = 50$, respectively. Therefore, according to proposition 4, there is exactly one solution of the Roy equation that is regular at the matching point.

Proposition 4 tells us that there is a special class of inputs that allow a unique solution that is regular at the matching point s_0, as announced in section 2. According to proposition 3, the high–energy absorptive part of an input belonging to that class has an analytic continuation from $[s_0, s_1)$ into a complex neighborhood of $(4, s_1)$. For this reason, we say that the members of our special class are *analytic inputs*. A physical amplitude f defines an analytic input (a_f, A_f) if $s_0 < s_{inel}$. Uniqueness is achieved in the sense that the corresponding Roy equation has exactly one solution, coinciding with f, which is regular at s_0.

The existence of a class of inputs ensuring uniqueness and regularity at the matching point has been established in an indirect way. We have no direct and complete characterization of an analytic input. There are involved constraints apart from analyticity of the high–energy absorptive part A, and one cannot decide directly if a given input is an analytic one. In particular, the scattering length a is fixed by A and is not an independent parameter, see example 3 above. We arrived at a unique solution of the Roy equation by choosing inputs which are compatible with elastic unitarity above the matching point. In the physical context, this means $s_{inel} > s_0$. One can prove that uniqueness is also obtained if $s_0 > s_{inel}$, provided that the inelasticity is sufficiently smooth.

6.3 Approximate input and cusps

Although we are working here with model amplitudes without left–hand cut, the results of subsection 6.2 hold true in the physically realistic situation with left–hand cut. As a physically relevant input is analytic, with $s_1 = s_{inel}$, we conclude that it allows a unique solution regular at s_0 if $s_0 < s_{inel}$. However, a physical input is only approximately known, and one is faced in practice with an arbitrary input, as a result of which non–uniqueness and unphysical singularities do occur. This is the reason why we have analyzed in detail the Roy equation in its general setting.

If we know that a high–energy absorptive part A belongs to an analytic input, but the corresponding scattering length is not precisely known, one has to work with a trial input (a', A), with $a' \neq a$. We expect that all the resulting solutions will have cusps at s_0. This can be established if a' is infinitesimally close to a by using the techniques developed in [9, 12]. For instance, if $\delta(s_0) < \pi/2$ and $\delta'(s_0) = \delta(s_0)$, the neighboring solution f' is unique and its phase shift is given by

$$\delta'(s) = \delta(s) + \sigma(s)\frac{G(s)}{G(4)}(a' - a)\,, \tag{6.4}$$

with G defined in (3.6). The ratio $(\delta' - \delta)/(a' - a)$ is shown in Fig. 8 for a neighboring solution of the amplitude f in (5.2), using

$$s_p = 40 \;\;,\; r = 100 \;\;,\; s_0 = 23\,. \tag{6.5}$$

The cusp at s_0 is clearly visible and the effect of a modified scattering length extends over the whole interval $(4, s_0)$. If an incorrect value a' of the scattering length also entails a cusp when it is not close to a, the correct scattering length is specified by the fact that it defines an input allowing a unique solution without a cusp: one solves the Roy equation with several trial a', and uses

$$a \doteq a'_{nocusp}\,. \tag{6.6}$$

In this sense the scattering length of a physically realistic amplitude can be predicted in principle. However, even if we are sure that the high–energy absorptive part is analytic, we do not know it exactly in practice, and we have to use an approximate form A'. We have to work with an input (a', A') which is meant to approximate the analytic (a, A). It is unlikely that there will still be a value of a' which removes the cusp in one of the solutions of the resulting Roy equation. What one can try in practice is to find the value of a' which minimizes the size of the cusp. In the coupled channel case, an analogous procedure may be used to avoid solutions that generate a cusp [4].

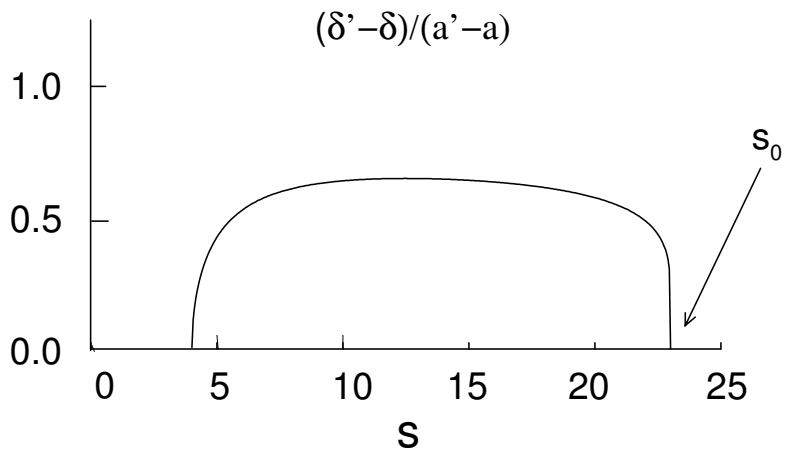

Figure 8: The quantity $(\delta' - \delta)/(a' - a)$ according to Eq. (6.4). The reference amplitude f is the one in (5.2), and the parameters used are given in (6.5). The cusp generated at s_0 is very clearly seen.

7 Summary and conclusions

The following points summarize the content of this article.

1. In view of forthcoming applications [4] of the Roy equations [3] in the analysis of K_{l4} decays, we have considered here the one–channel Roy equation. We have analyzed the multiplicity and singularity structure of its solutions for a given input (a, A) of scattering length a and high–energy absorptive part A.

2. First, we have investigated the infinitesimal neighborhood of a given solution δ [9]. According to proposition 1 in section 3, this neighborhood contains an m–parameter family of solutions, where

$$m = \begin{cases} \left[\frac{2\delta(s_0)}{\pi}\right] & \text{if } \delta(s_0) > \frac{\pi}{2}, \\ 0 & \text{if } -\frac{\pi}{2} < \delta(s_0) < \frac{\pi}{2}. \end{cases} \tag{7.1}$$

The symbol $[x]$ denotes the greatest integer not exceeding x. For a monotonically increasing phase, m counts the number of times $\delta(s)$ goes through multiples of $\pi/2$ as s varies from threshold to the matching point s_0. This result illustrates that a given input (a, A) does not, in general, fix the solution uniquely. One has in addition to fix the phase shift at the matching point, and to determine the corresponding m parameters by other means.

3. Using proposition 2 of section 4, we have constructed in section 5 – starting from a given solution f with input (a_f, A_f) – additional exact solutions $f' \neq f$ with the same input. The function f' contains in general several arbitrary parameters that may be used to either change the position and residues of the poles present in the inverse amplitude $1/f$, or to remove (or implement new) poles. This illustrates that a given input allows for solutions with a different value of the phase shift at the matching point – these phase shifts only have to satisfy the boundary condition (2.6).

4. The solutions $f' \neq f$ so constructed have the property that they contain a cusp in the real and imaginary part at the matching point. An example is displayed in figure 7. The origin of these cusps is made clear in proposition 4 of section 6.

5. In case we know that a given absorptive belongs to an analytic input (see section 6 for this notion), we expect on the basis of proposition 4 that the corresponding scattering length can be determined as the one that results in a solution of the Roy equation without a cusp at s_0.

6. Propositions 1 and 3 in sections 3 and 6 were established long ago in Refs. [9] and [26], respectively. As far as we are aware, propositions 2 and 4 in sections 4 and 6 are, on the other hand, new results.

In conclusion, if we know that we are dealing with an analytic input, the corresponding amplitude is the unique regular solution of the Roy equation. Because we are in general forced to use an approximate input, non–uniqueness and unphysical singularities do show up. A visible cusp in a numerical solution of the Roy equation with $0 < \delta(s_0) < \pi/2$ is a signal that the input is not physical or a poor approximation of a physical one. The deficiency may be hidden in the scattering length a, in the absorptive part A, or in both.

Acknowledgements

We thank G. Colangelo for collaboration in an early stage of this work. We have enjoyed useful discussions with H. Leutwyler, and we are grateful to D. Atkinson for sending us the theses work of T.P. Pool and of A.C. Heemskerk. One of us (J.G.) thanks the mathematicians J. Rätz, M. Reimann and Th. Rychener for useful discussions concerning several aspects of this work.

A Solving the integral equation (3.2)

We first specify the regularity properties of the partial waves and phase shifts considered in the text. Then we construct the general solution of the integral equation (3.2).

A.1 Hölder continuity

The class of Hölder–continuous functions is the appropriate space to consider, because Cauchy–integrals map this space essentially into itself [27]. We explain this notion.

Consider a complex–valued function f of a real variable x. The function f is called Hölder–continuous in the interval $[a, b]$ with exponent μ, where $0 < \mu \leq 1$, if there is a constant C such that

$$|f(x) - f(y)| \leq C\,|x - y|^{\mu} \quad ; \; x, y \in [a, b] \; . \tag{A.1}$$

We call these functions H–continuous for short, and denote the corresponding space with $H^{\mu}_{a,b}$.

Proposition A *Let $f \in H^{\mu}_{a,b}$, with $f(a) = f(b) = 0$ and $\mu < 1$. Then the function*

$$g(y) = \fint_a^b dx \frac{f(x)}{x - y} \tag{A.2}$$

is also an element of $H^{\mu}_{a,b}$, with the same exponent μ. The same is true for

$$g_{\pm}(y) = \lim_{\epsilon \searrow 0} \int_a^b dx \frac{f(x)}{x - y \mp i\epsilon} \; . \tag{A.3}$$

The proof may be found in [27].

A.2 Regularity requirements

The input absorptive part A in (2.1) is assumed to be bounded and Hölder continuous in any finite interval $[s_0, s_0']$ above the matching point s_0. Furthermore,

we seek solutions of the Roy equation that are H–continuous in $[4, s_0]$ and have normal threshold behaviour (which implies $\mu \leq 1/2$):

$$\begin{aligned} &\text{i)} \quad A \in H^{\mu}_{s_0, s_0'} \ , \ s_0' > s_0 \ ; \ A \ \text{ is bounded} \\ &\text{ii)} \quad f \in H^{\mu}_{4, s_0} \\ &\text{iii)} \quad f = a + ia^2\sigma(s) + O(s-4) \ , \ s \searrow 4 \ . \end{aligned} \tag{A.4}$$

The phase shifts can then be chosen H-continuous as well,

$$\begin{aligned} &\text{iv)} \quad \delta \in H^{\mu}_{4, s_0} \\ &\text{v)} \quad \delta = \sigma(s)[a + O(s-4)] \ , \ s \searrow 4 \ . \end{aligned} \tag{A.5}$$

A.3 Solution of the integral equation (3.2)

We first show that solving the integral equation (3.2) with the boundary condition (3.4) is equivalent [27] to solving a boundary value problem known as *Hilbert problem*. We then construct the general solution of the latter.

We start by introducing an auxiliary function Φ [9, 12],

$$\Phi(z) = (z-4)\frac{1}{\pi}\int_4^{s_0} dx \frac{\sin(2\delta(x))h(x)}{x-4}\frac{1}{x-z} \ . \tag{A.6}$$

For convenience, we use in this subsection the variable z to indicate complex values of the variable s. The function Φ has the following properties if h is a solution of (3.2) and (3.4):

i) $\Phi(z) = \overline{\Phi(\overline{z})}$ is regular in $\mathbb{C}\backslash[4, s_0]$.

ii) The boundary values $\Phi_{\pm}(s) = \lim_{\epsilon \searrow 0} \Phi(s \pm \mathrm{i}\epsilon)$ are H-continuous in $[4, s_0]$.

iii) $\Phi_+(s) = e^{4i\delta(s)}\Phi_-(s) \ ; \ s \in [4, s_0]$.

iv) $\Phi(4) = 0$.

v) $\Phi(s_0) = 0$.

vi) Φ is bounded at infinity .

Out of these, we only prove property ii) – the remaining ones are easy to verify. The unknown h in (A.6) is an element of H^{μ}_{4,s_0}, with $0 < \mu \leq 1/2$. This follows from Eq. (A.5) – the same equation shows that $h = O(s-4)$ at $s = 4$.

Together with $h(s_0) = 0$ and with proposition A in subsection A.1, it follows that $\Phi_\pm(s) \in H^\mu_{4,s_0}$, with the same exponent μ.

The problem to determine functions $\Phi(z)$ with i)–vi) is called a *(homogeneous) Hilbert problem*. We conclude that each solution of the original integral equation generates via (A.6) a solution of i)–vi). Vice versa, each solution of the Hilbert problem has a representation of the form (A.6), where h is given by

$$h(s) = e^{-2i\delta(s)}\Phi_+(s) \ . \tag{A.7}$$

Furthermore, the function h is real and solves the integral equation (3.2). To prove this, we write a Cauchy representation for $[\Phi(z) - \Phi(4)]/(z-4)$, with a path that wraps around the cut $[4, s_0]$. We then deform the outer part of the path towards infinity, in such a manner that only the integral above and below the cut $[4, s_0]$ survives. By use of condition iv), we find

$$\Phi(z) = \frac{(z-4)}{2\pi i}\int_4^{s_0} \frac{dx}{x-4}\frac{\Phi_+(x) - \Phi_-(x)}{x-z} \ . \tag{A.8}$$

Using properties i) and iii), the claim is easily proven, and we conclude that solving (3.2) with (3.4) indeed is equivalent to solving the Hilbert problem i)–vi).

It remains to construct the general solution of i)–vi). First, we observe that the Omnès–type function [9]

$$\bar{G}(z) = \frac{1}{(s_0 - z)^m}\exp\left[\frac{2}{\pi}\int_4^{s_0} dx \frac{\delta(x)}{x-z}\right] \ , \tag{A.9}$$

with m defined in (3.7), satisfies property i). The behaviour at s_0 is given by

$$\bar{G}(z) \sim (s_0 - z)^\gamma, \qquad \gamma = \frac{2}{\pi}\delta(s_0) - m \ , \tag{A.10}$$

$-1 < \gamma < 1$. The H-continuity and threshold behavior of δ imply that $\bar{G}$ satisfies ii) and iii) except, possibly, at s_0. Outside s_0, $(s_0 - z)^m\bar{G}$ and its boundary values are nonzero and the function

$$F(z) = (s_0 - z)\frac{\Phi(z)}{\bar{G}(z)} \tag{A.11}$$

is regular in $\mathbb{C}\backslash\{s_0\}$. For $z \neq s_0$ it is given by its Laurent series. The principal part of this series is identically zero because condition v) and Eq. (A.10) imply that $F(s_0) = 0$. $F(z)$ is therefore an entire function. Condition vi) tells us that $F(z) \sim z^{m+1}$ at infinity, as a result of which

$$F(z) = (s_0 - z)Q(z), \tag{A.12}$$

where $Q(z)$ is a polynomial of degree m. Condition iv) imposes $Q(4) = 0$, and this gives $Q = 0$ if $m = 0$: the conditions i)–vi) allow only for the trivial solution in this case. If $m > 0$, $Q(z) = (z-4)P(z)$, with P a polynomial of degree $m-1$. We conclude that the general solution of i)–vi) is given by

$$\Phi(z) = (z-4)\bar{G}(z)\sum_{n=0}^{m-1} c_n z^n \ \ , \ c_n \in \mathbb{R} \ \ , \tag{A.13}$$

with $\Phi = 0$ for $m = 0$. Using (A.7), the result (3.5) follows.

B Connection with the N/D approach

The N/D method [10] transforms the nonlinear Roy equation into a linear N equation, whereas the method displayed in section 4 linearizes the construction of the solution f', once a first solution f is known. In this appendix, we establish the relationship between the two methods.

We write the N/D representation of f as follows,

$$f(s) = \frac{n(s)}{d(s)} \ . \tag{B.1}$$

The N–function n is holomorphic in $\mathbb{C}\backslash[s_0, \infty)$, and the D–function d is holomorphic in $\mathbb{C}\backslash[4, s_0]$, with possible CDD poles on the cut. Similarly, $f' = n'/d'$. We now determine the relation between the pairs (n, d) and (n', d').

Equation (4.9) gives

$$f'(s) = \frac{D(s)}{F(s)} f(s) \ \ , \tag{B.2}$$

where

$$F(s) = D(s) + (s-4)H(s)f(s) \ . \tag{B.3}$$

Whereas D is meromorphic in $\mathbb{C}\backslash[s_0, \infty)$, F is meromorphic in $\mathbb{C}\backslash[4, s_0]$. This crucial point is a consequence of Eqs. (4.10)–(4.12), which imply that $\mathrm{Im}\, F_+(s) = 0$ for $s > s_0$. In order to turn (B.2) into a N/D representation, we need the set $\{p_i\}$ of poles of D, as well as the set $\{z_j\}$ of its zeros on $[4, s_0]$, and define

$$\bar{D}(s) = \prod_i (s - p_i) \prod_j \frac{1}{s - z_j} D(s) \ . \tag{B.4}$$

To simplify the argument, we assume in the following that the N–functions n and n' have no zeros on $[4, s_0]$ – the discussion is more involved and the definitions (B.5) have to be modified if this is not the case. The N/D representation of f' is then obtained by eliminating D in favour of $\bar{D}$ in the expression (B.2), and writing

$$n'(s) = Cn(s)\bar{D}(s) \;\; ; \;\; d'(s) = Cd(s)\bar{F}(s) \;\; , \tag{B.5}$$

where C is a constant and where

$$\bar{F}(s) = \bar{D}(s) + (s-4)H(s)f(s)\prod_j \frac{1}{s-z_j}\prod_i (s-p_i) \; . \tag{B.6}$$

The functions n' and d' defined in (B.5) have the correct analyticity properties. Denominator functions are normalized to one at infinity [10]. We have checked in our examples that $\bar{F}$ has a finite, nonzero limit at infinity. Therefore, d' is properly normalized if $C = 1/\bar{F}(\infty)$. For $\delta(s_0) > 0$, d can be written as

$$d(s) = \prod_{k=1}^{r}\left(\frac{s-s_0}{s-s_k}\right)\exp\left[-\frac{1}{\pi}\int_4^{s_0} dx \frac{\delta(x)}{x-s}\right] \;\; , \tag{B.7}$$

where the s_k are the r CDD poles of f, $s_k \in (4, s_0), r = [\delta(s_0)/\pi]$. We see that d' is finite at s_k if s_k coincides with a zero z_j of D, and if $\bar{F}(z_j) = 0$. Otherwise, every s_k and z_j is a CDD pole of f'. Within our assumptions on n and n', the fate of the CDD poles is dictated by the zeros of D: these produce new CDD poles in f' or remove CDD poles present in f. The poles of D are points where $f' = f$.

C Proof of proposition 4

Using the framework of section 4, we show that f' coincides with f, if $\mathrm{Re}\, f'_+$ is regular at s_0 [in the sense that it has a holomorphic extension into a circle of radius ϵ and center s_0]. See figure 9 for the analyticity domains used in the proof.

i) The dispersion relation (2.1) written for f' determines Re f'_+ on $\mathbb{R}$. Inversion of this relation gives

$$\mathrm{Im}\; f'_+(s) = -(s-4)\frac{1}{\pi}\fint_{\mathbb{R}} dx \frac{\phi(x)}{x-s} \;\; , \tag{C.1}$$

where $\phi(s) = (\mathrm{Re}\; f'_+(s) - f'_+(4))/(s-4)$. The holomorphy of Re f'_+ in C implies the holomorphy of Im f'_+ in the same circle.

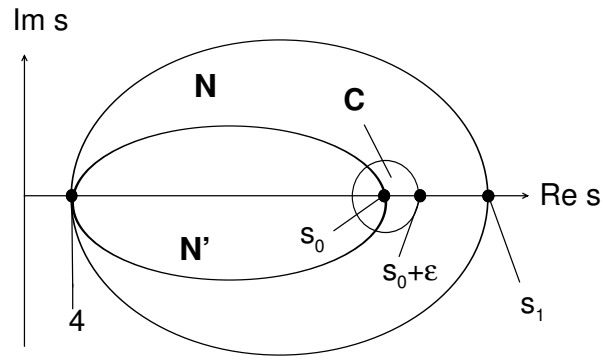

Figure 9: Analyticity domains referred to in the proof of proposition 4.

ii) According to proposition 3, Im f'_+ has a holomorphic extension into a neighborhood N' of $(4, s_0)$, with s_0 on its boundary. Combining this with the previous result we see that Im f'_+ is holomorphic in $\bar{N} = N' \cup C$, a domain extending up to $s_0 + \epsilon$.

iii) Proposition 3 tells us that the high–energy absorptive part A, originally defined on $[s_0, \infty)$, has an analytic continuation into a neighborhood N of $(4, s_1)$, which coincides with $\mathrm{Im}\, f_+$ on $[4, s_0]$. As f' is a solution of the Roy equation with input (a, A), we have

$$\mathrm{Im}\ f'_+(s) = A(s) \tag{C.2}$$

for $s_0 \leq s \leq s_0 + \epsilon$. In view of this equality and of the regularity of Im f'_+ in $\bar{N}$, Im f'_+ has to be equal to A on $[4, s_0]$. As A is equal to Im f_+ on that interval, we conclude that $f' = f$.

iv) Similarly, if Im f'_+ is assumed to be regular at s_0, one concludes that $\mathrm{Re}\, f_+$ is regular at s_0 and $f' = f$.

Therefore, f' has to be singular at s_0 if $f' \neq f$. This is the content of proposition 4.

References

[1] S. Weinberg, Physica A 96 (1979) 327; J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) 158 (1984) 142; Nucl. Phys. B 250 (1985) 465

[2] J. Bijnens et al., Phys. Lett. B 374 (1996) 210; Nucl. Phys. B 508 (1997) 263; a dispersive representation of the amplitude at order p^6 is given in M. Knecht et al., Nucl. Phys. B 457 (1995) 513

[3] S.M. Roy, Phys. Lett. 36B (1971) 353

[4] B. Ananthanarayan, G. Colangelo, J. Gasser, and H. Leutwyler, work in progress

[5] J.L. Basdevant, J.C. Le Guillou, and H. Navelet, Nuovo Cimento 7A (1972) 363; M.R. Pennington and S.D. Protopopescu, Phys. Rev. D 7 (1973) 1429; ibid. D 7 (1973) 2591; J.L. Basdevant, C.D. Froggatt, and J.L. Petersen, Phys. Lett. 41B (1972) 173; ibid. 41B (1972) 178; Nucl. Phys. B 72 (1974) 413; J.L. Petersen, Acta Phys. Austriaca Suppl. 13 (1974) 291; C.D. Froggatt and J.L. Petersen, Nucl. Phys. B 91 (1975) 454; ibid. B 104 (1976)(E) 186; J.L. Petersen, *The $\pi\pi$ interaction*, Lectures given in the Academic Training Program of CERN, 1975–1976, Yellow CERN report CERN 77-04; C.D. Froggatt and J.L. Petersen, Nucl. Phys. B 129 (1977) 89

[6] D. Morgan and M.R. Pennington, in [7], p. 193

[7] *The Second* DAPHNE *Physics Handbook* (INFN–LNF–Divisione Ricerca, SIS–Ufficio Publicazioni, Frascati 1995, ISBN 88-86409-02-8; L. Maiani, G. Pancheri, and N. Paver, Eds.)

[8] G. Mahoux, S.M. Roy, and G. Wanders, Nucl. Phys. B 70 (1974) 297

[9] C. Pomponiu and G. Wanders, Nucl. Phys. B 103 (1976) 172

[10] D. Atkinson and R.L. Warnock, Phys. Rev. D 16 (1977) 1948.

[11] D. Atkinson, T.P. Pool, and H.A. Slim, J. Math. Phys. 18 (1977) 2407

[12] L. Epele and G. Wanders, Phys. Lett. 72B (1978) 390; Nucl. Phys. B 137 (1978) 521

[13] T.P. Pool, Nuovo Cimento 45A (1978) 207

[14] A.C. Heemskerk, Ph.D. thesis, University of Groningen, 1978

[15] A.C. Heemskerk and T.P. Pool, Nuovo Cimento 49A (1979) 393

[16] C. Lovelace, Commun. math. Phys. 4 (1967) 261

[17] O. Brander, Commun. math. Phys. 40 (1975) 97

[18] J. Loewe, in [20], p. 368, and hep-ph/9711361

[19] D. Počanić, in [20], p. 352, and hep-ph/9801366; Talk given at the International Workshop on Hadronic Atoms and Positronium in the Standard Model, Dubna, Russia, 1998, to appear in the Proceedings, and hep-ph/9809455

[20] *Chiral Dynamics: Theory and Experiment*, Proceedings of the Workshop Held in Mainz, Germany, 1997 (Lecture Notes in Physics, Vol. 513; A. Bernstein, D. Drechsel, and T. Walcher, Eds.)

[21] M. Knecht, B. Moussallam, and J. Stern, in [7], p. 221, and hep-ph/9411259; J. Stern, in [20], p. 26, and hep-ph/9712438; see also the references cited in these works

[22] O.O. Patarakin, V.N. Tikhonov, and K.N. Mukhin, Nucl. Phys. A 598 (1996) 335; O.O. Patarakin (for the CHAOS collaboration), in [20], p. 376, and hep-ph/9711361; The CHAOS collaboration, πN Newsletter 13 (1997) 27

[23] P. Büttiker, Ph.D. thesis, University of Bern, 1996; B. Ananthanarayan and P. Büttiker, in [20], p. 370; Phys. Lett. B 415 (1997) 402; Phys. Rev. D 54 (1996) 1125; ibid. D 54 (1996) 5501

[24] G. Wanders, in preparation

[25] B. Ananthanarayan, Phys. Rev. D 58 (1998) 036002

[26] W. Zimmermann, Nuovo Cimento 51 (1961) 249, and references cited therein

[27] N.J. Muskhelishvili, *Singular Integral Equations* (Noordhoff, Groningen, 1953); S. Fenyö and H.W. Stolle, *Theorie und Praxis der linearen Integralgleichungen*, Band 3 (Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften: Math. Reihe, Band 76; Birkhäuser, Basel, 1984)

[28] L. Castillejo, R.H. Dalitz, and F.J. Dyson, Phys. Rev. 101 (1956) 453