Potential Energy of Yang–Mills Vortices
in Three and Four Dimensions

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Abstract

We calculate the energy of a Yang–Mills vortex as function of its magnetic flux or, else, of the Wilson loop surrounding the vortex center. The calculation is performed in the 1-loop approximation. A parallel with a potential as function of the Polyakov line at nonzero temperatures is drawn. We find that quantized $Z(2)$ vortices are dynamically preferred though vortices with arbitrary fluxes cannot be ruled out.

The hypothesis that $Z(N_c)$ vortices are responsible for confinement has several attractive features. First, one immediately gets the area law for Wilson loops in the fundamental representation of the $SU(N_c)$ gauge group, even under the simplest assumption that vortices are non-interacting and therefore the number of vortices piercing a given loop is Poisson-distributed [1]. Furthermore, Wilson loops in any representation transforming under the group center $Z(N_c)$ will have the same string tension as in the fundamental representation, while the 'center-blind' representations will have, asymptotically, zero string tension. This is what is expected from screening by gluons. The temporal approximate Casimir scaling of the string tensions may be probably explained by the finite sizes of the vortex cores [2].

Second, heating the $d = 3 + 1$ Yang–Mills system towards the high-temperature $d = 3$ case and then heating $d = 2 + 1$ system eventually to the $d = 2$ case, one does not observe abrupt changes in the *spatial* string tensions. This continuity as one goes from four to two dimensions favours $d = 2$ objects as being basic for confinement in all the above cases [1]. In three dimensions vortices form closed lines; in four dimensions they form closed surfaces.

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[1] In $d = 2$ pure Yang–Mills theory the confinement is trivial in axial gauges and may seem to have no relation to vortices. However, it is known that at least on compact $d = 2$ manifolds the theory is exactly equivalent to a sum over vortices, see a recent paper [4] and references therein.

For vortices to be physical objects and not artifacts of a regularization, their cores should be finite in physical units, i.e. to be of the order of $1/\Lambda \simeq M^{-1} \exp(3 \cdot 8\pi^2/11N_c g^2)$ in $d = 4$ and of the order of $1/g_3^2$ in $d = 3$. Correspondingly, the energy of a vortex per unit length should be of the order of g_3^2 in $d = 3$, and the energy per unit surface should be of the order of Λ^2 in $d = 4$. First indications that in might indeed be the case came recently from lattice studies using the smoothing procedure [3].

In order to reveal 'thick' vortices theoretically, one has first of all to integrate out the high-momentum components of the Yang–Mills field. The resulting effective action may then have a stable saddle point of a vortex type, with the core size remaining finite as one takes the UV cutoff to infinity. When and if it happens, one can speak of vortices as physical entities, and further on to try to build a theory of those extended objects in the vacuum.

The main dynamical variable of a vortex is the azimuthal component of the Yang–Mills field $A_\phi^a(\rho) = \epsilon_{\alpha\beta} n_\alpha A_\beta^a$, where n_α is a unit vector in the plane transverse to the vortex. One can always choose a gauge where A_ϕ is independent of the azimuth angle ϕ. Generally speaking, it implies that the radial component $A_\rho(\rho) \neq 0$, however, we shall neglect this component as it can be always reconstructed from gauge invariance by replacing $\partial_\rho \rightarrow \partial_\rho \delta^{ab} + f^{acb} A_\rho^c$. A circle Wilson loop lying in the transverse plane and surrounding the vortex center is then

$$W_{\frac{1}{2}}(\rho) = \frac{1}{2}\mathrm{Tr}\,\mathrm{P}\exp i \oint \rho A_\phi^a t^a d\phi = \cos[\pi\mu(\rho)], \quad \mu(\rho) = \rho\sqrt{A_\phi^a(\rho)A_\phi^a(\rho)}, \qquad (1)$$

taking for simplicity the fundamental representation of the $SU(2)$ gauge group. For an arbitrary representation of $SU(2)$, labelled by spin J, one has

$$W_J(\rho) = \frac{1}{2J+1}\frac{\sin[(2J+1)\pi\mu(\rho)]}{\sin[\pi\mu(\rho)]}. \qquad (2)$$

If $\mu(\rho) \rightarrow integer$ at large distances ρ from the vortex core, the Wilson loop $W_J(\rho) \rightarrow (-1)^{2J}$. This is the definition of the $Z(2)$ vortex. We shall see that integer values of $\mu(\infty)$ are dynamically preferred.

Neglecting all components of the Yang–Mills field except the essential azimuthal one, the classical action of the vortex becomes

$$\int d^d x \frac{(F_{\mu\nu}^a)^2}{4g_d^2} = \int d^{d-2}x_\parallel \int d^2 x_\perp \frac{(B_\parallel^a)^2}{2g_d^2} = \frac{1}{2g_d^2}\int d^{d-2}x_\parallel\, 2\pi\int_0^\infty d\rho\, \rho \left[\frac{1}{\rho}\partial_\rho(A_\phi^a\rho)\right]^2. \qquad (3)$$

To get the effective action for $\mu(\rho)$ we integrate over quantum fluctuations about a given background field $A_\phi^a(\rho)\rho$ considered to be a slowly varying field with momenta up to certain k_{min}. Accordingly, quantum fluctuations have momenta from k_{min} up to the UV cutoff k_{max}. Writing $A_\mu = \bar{A}_\mu + a_\mu$ where $\bar{A}_\mu = \delta_{\mu\phi}A_\phi^a(\rho)$ we expand $F_{\mu\nu}^2(A+a)$ in the fluctuation field a_μ up to the second order appropriate for the 1-loop calculation. The term linear in a_μ vanishes due to the orthogonality of high and low momenta. The quadratic form for a_μ is the standard

$$W_{\mu\nu}^{ab} = -[D^2(\bar{A})]^{ab}\delta_{\mu\nu} - 2f^{acb}F_{\mu\nu}^c(\bar{A}), \qquad (4)$$

if one imposes the background Lorentz gauge condition,

$$D_\mu^{ab}(\bar{A})a_\mu^b = 0, \qquad D_\mu^{ab}(\bar{A}) = \partial_\mu \delta^{ab} + f^{acb}\bar{A}_\mu^c. \tag{5}$$

The effective action is

$$S_{eff}[\bar{A}] = \frac{1}{2}\ln\det(W_{\mu\nu}) - \ln\det(-D_\mu^2). \tag{6}$$

In the presence of dynamical fermions in the fundamental representation the Dirac determinant should be added:

$$-\ln\det(\nabla_\mu\gamma_\mu) = -\frac{1}{2}\ln\det\left(\nabla_\mu^2 - \frac{i}{2}[\gamma_\mu\gamma_\nu]F_{\mu\nu}^a t^a\right), \qquad \nabla_\mu = \partial_\mu - i\bar{A}_m u^a t^a. \tag{7}$$

The effective action may be expanded in powers of the (covariant) derivatives of the background field $\bar{A}_\mu$. We are now interested in the first nontrivial term of this expansion, namely in the zero-derivative term, that is in the effective *potential* as function of the flux ρA_ϕ. The next term with two derivatives will renormalize eq. (3), and there will be, generally speaking, further terms. In the zero-derivative order $F_{\mu\nu}(\bar{A}) = 0$, therefore $\det(W_{\mu\nu}) = d\cdot\det(-D^2)$, where d is the full space dimension. With A_ϕ being the only nonzero component of the background field, it is natural to write down D^2 in the cylindric coordinates:

$$D^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\left(\frac{\partial}{\partial\phi} + f^{acb}A_\phi^c\right)^2 + \partial_3^2 + ... + \partial_d^2. \tag{8}$$

The effective action (6) for slowly varying ρA_ϕ is

$$S_{eff}[A_\phi] = -\left(\frac{d}{2} - 1\right)\int_{s_{min}}^{s_{max}}\frac{ds}{s}\ \text{Sp}\ \exp(sD^2) = \int d^{d-2}x_\parallel\int d^2x_\perp\ V^{(d)}(\mu), \qquad \mu = \rho\sqrt{A_\phi^a A_\phi^a}, \tag{9}$$

where we have introduced the 'vortex potential energy' in d dimensions $V^{(d)}(\mu)$. Here Sp denotes the functional and the colour traces, and integration over the proper time s has been introduced. The limits $s_{min} \simeq 1/k_{max}^2$ and $s_{max} \simeq 1/k_{min}^2$ are gauge-invariant UV and IR cutoffs, respectively. In saturating the functional trace one can use any complete set of functions. Naturally, one uses the plane wave basis for the longitudinal components $x_3...x_d$, and the polar basis $\exp(im\phi)f_{k,m}(\rho)$ for the transverse ones, where the functions $f_{k,m}(\rho)$ must form a complete set, see below. In what follows we shall consider only the $SU(2)$ case. The transverse part of the covariant Laplacian (8), after acting on the polar harmonics $\exp(im\phi)$, $m = integer$, has three eigenvalues as a 3×3 colour matrix:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{1}{\rho^2}\cdot\left\{\begin{array}{l} m^2, \\ (m+\mu)^2, \\ (m-\mu)^2. \end{array}\right. \tag{10}$$

The first one cancels out when one subtracts the free determinant (without the external field); the last two are actually coinciding, given that m ranges from minus to plus infinity. The differential operator (eq. (10), last line) has eigenfunctions $J_{\pm(m-\mu)}(k\rho)$ with eigenvalues

$-k^2$. The index of Bessel functions must be non-negative to ensure regularity at the origin. We thus choose the functions $F_{m,k}(\phi, \rho) = \exp(im\phi)J_{|m-\mu|}(k\rho)$ as a complete functional basis in the transverse plane:

$$\frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} \int_0^\infty dk\, k\, F_{m,k}(\phi, \rho)F_{m,k}^\star(\phi', \rho') = \frac{1}{\rho}\delta(\rho - \rho')\,\delta(\phi - \phi')|_{\text{mod } 2\pi}. \qquad (11)$$

These functions satisfy also the ortho-normalization condition:

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^\infty d\rho\, \rho\, F_{m,k}(\phi, \rho)F_{m',k'}^\star(\phi, \rho) = \frac{1}{k}\delta(k - k')\,\delta_{mm'}. \qquad (12)$$

Eqs.(11, 12) can be checked using the formula 6.541.1 from Gradshteyn and Rhyzhik [5]. Eq. (9) can be thus rewritten as

$$V^{(d)}(\mu) = -(d-2)\int \frac{dp_3...dp_d}{(2\pi)^{d-2}} \int_{s_{min}}^{s_{max}} \frac{ds}{s} \int_0^\infty k\, dk\, \exp[-s(k^2 + p_3^2 + ... + p_d^2)]$$

$$\cdot \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \left[J_{|m-\mu|}^2(k\rho) - J_{|m|}^2(k\rho) \right]. \qquad (13)$$

We see that the dependence on the flux μ enters only through the indices of the Bessel functions and that the potential is explicitly periodic in μ with period 1. Integration over momenta can be easily performed using eq. 6.633.2 from [5], yielding

$$V^{(d)}(\mu) = -(d-2)\int_{s_{min}}^{s_{max}} \frac{ds}{s}\left(\frac{1}{4\pi s}\right)^{\frac{d}{2}} \exp\left(-\frac{\rho^2}{2s}\right) \sum_{m=-\infty}^{\infty} \left[I_{|m-\mu|}\left(\frac{\rho^2}{2s}\right) - I_{|m|}\left(\frac{\rho^2}{2s}\right) \right]. \qquad (14)$$

We next sum over m using the integral representation for modified Bessel functions, eq. 8.431.5 of [5]. To write explicit formulae we imply that $\mu \in (0, 1)$. If μ is outside this interval $V^{(d)}(\mu)$ should be continued by periodicity. We get

$$V^{(d)}(\mu) = \frac{d-2}{(2\pi\rho^2)^{\frac{d}{2}}} \cdot \frac{\sin(\pi\mu)}{\pi} \int_0^\infty dx \frac{\cosh\left(\frac{1}{2} - \mu\right)x}{\cosh\frac{x}{2}} \int_{t_{min}}^{t_{max}} dt\, t^{\frac{d}{2}-1} \exp\left[-t(\cosh x + 1)\right], \qquad (15)$$

where we have introduced a new variable $t = \rho^2/(2s)$ instead of s. Correspondingly, the integration limits become $t_{min} = \rho^2/2s_{max}$ and $t_{max} = \rho^2/2s_{min}$. It should be stressed that the above expression is finite in the limit when one removes both the UV cutoff ($t_{max} \to \infty$) and the IR cutoff ($t_{min} \to 0$). In this limit both integrations in (15) become elementary, and we finally get:

$$V^{(d)}(\mu) = \frac{d-2}{(\pi\rho^2)^{\frac{d}{2}}} \cdot \frac{\sin(\pi\mu)}{\pi} \cdot \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2} + \mu\right)\Gamma\left(\frac{d}{2} + 1 - \mu\right)}{\Gamma(d+1)}$$

$$= \begin{cases} \left.\frac{1}{\rho^4}\frac{1}{12\pi^2}\, \mu(1 - \mu^2)(2 - \mu)\right|_{\text{mod } 1} & \text{for } d = 4, \\[2ex] \left.\frac{1}{\rho^3}\frac{1}{96}\frac{\tan(\pi\mu)}{\pi}(1 - 4\mu^2)(3 - 2\mu)\right|_{\text{mod } 1} & \text{for } d = 3. \end{cases} \qquad (16)$$
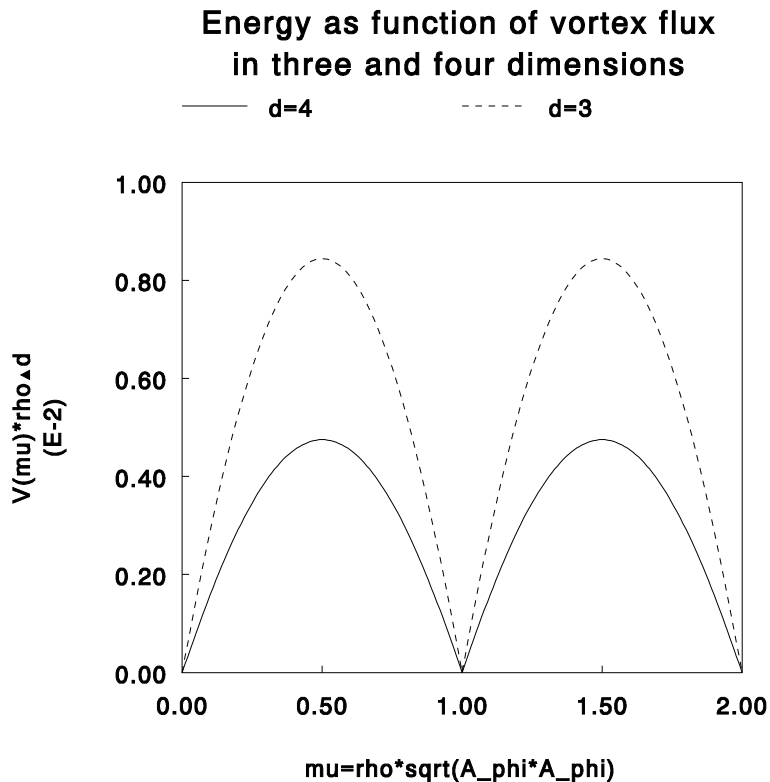
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Figure 1: The vortex potential, eq. (16).

At $d = 2$ the vortex potential is identically zero as there are no transverse gluons in two dimensions. Though the potential is given just by a polynomial (in case $d = 4$), it is actually a periodic function of μ with a unit period, as seen explicitly from eq. (13). At integer μ the potential is zero but has a jump in the first derivative; it is depicted in Fig.1.

The problem we have solved has certain similarity with that of the potential energy as function of the d-th component of the Yang–Mills field $A_{\mu=d}^a$, in case of nonzero *temperatures*. In that problem one integrates over all Matsubara frequences in the background of a static A_d^a, where $d = 4$ for $3 + 1$ dimensions, and $d = 3$ for $2 + 1$ dimensions. The two problems are ideologically similar, only the topology is different: in the nonzero-temperature case the topology is that of a cylinder $S^{(1)} \times R^{(d-1)}$, with the compact dimension in the 'temperature' direction, while in the problem considered here the topology is that of a plane with a deleted point at the origin, $R^{(2)} \setminus \{0\} \times R^{(d-2)}$. The role of Matsubara frequencies is played by the polar harmonics $\exp(im\phi)$. In the first case one finds the potential energy as function of the Polyakov line winding in the compact dimension; in the second case one finds the potential energy as function of the Wilson loop winding around the point at the origin. In both cases the potential energy is a periodic function, however the calculations are easier in the nonzero-temperature case as one can evaluate the functional trace (9) in the plane-wave basis. A simple calculation of eq. (9) gives the following form of the nonzero-temperature potential energy as function of $\nu = \sqrt{A_d^a A_d^a}/(2\pi T)$, valid for any number of dimensions:

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$$V^{(d)}(\nu) = (d-2)T \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \ln \frac{\cosh \frac{p}{T} - \cos 2\pi\nu}{\cosh \frac{p}{T} - 1}. \tag{17}$$

It is explicitly periodic in ν. At $d = 4$ the integration can be performed with the help of eq. 3.533.3 [5], and we get the well-known result [6]:

$$V^{(4)}(\nu) = \left. \frac{(2\pi T)^4}{12\pi^2}\nu^2(1-\nu)^2 \right|_{\text{mod } 1} = \frac{T^2}{3}(A_4^a)^2 + O(A^3). \tag{18}$$

At $d = 3$ the integral (17) cannot be expressed through elementary functions; however it can be compactly written as

$$V^{(3)}(\nu) = \frac{2T^3}{\pi}\sin^2 \pi\nu \int_0^\infty dx \frac{x^2 \cosh x}{\sinh x(\sinh^2 x + \sin^2 \pi\nu)}$$

$$= \pi T^3 \nu^2 \ln \frac{1}{\nu^2} + O(\nu^4) = \frac{T}{4\pi}(A_3^a)^2 \ln \frac{T^2}{(A_3^a)^2} + O(A^4). \tag{19}$$

To our best knowledge, this is a new result.

It is worth mentioning that for $d = 3$ the Debye mass is infrared-divergent, hence the non-analiticity in the ν^2 term. In the case of the vortex potential (16) the infrared divergency is even stronger: as a result the expansion of the potential starts from the non-analytic $|\mu|$ term both for $d = 3$ and 4. It is interesting that in all cases above the non-analytic terms are due to the contribution of zero Matsubara frequencies (in case of the vortex "zero frequency" means the azimuthal harmonic with $m = 0$). Indeed, the contributions of zero frequencies to the 'temperature' potentials (18) and (19) are

$$V_{\omega=0}^{(4)}(\nu) = -\frac{(2\pi T)^4}{12\pi^2}2\nu^3 + \text{UV divergence}, \qquad V_{\omega=0}^{(3)}(\nu) = \pi T^3 \nu^2 \ln \frac{1}{\nu^2} + \text{UV divergence}. \tag{20}$$

The $m = 0$ contribution to the vortex potential (16) is

$$V_{m=0}^{(4)}(\mu) = \frac{1}{\rho^4}\frac{1}{12\pi^2}(\mu - \mu^3) + \text{UV divergence}. \tag{21}$$

These are exactly the non-analytic terms in all the cases. The UV divergences are cancelled by contributions from nonzero frequencies.

If there are dynamical fermions in the problem, they should be treated differently in the nonzero-temperature case and in the vortex one. In the first case fermions are antiperiodic in the 'temperature' direction, hence one has to sum over half-integer Matsubara frequencies. In the latter case fermion wave functions are periodic functions of the azimuthal angle ϕ. The resulting fermion contribution to the vortex potential (in the fundamental representation) is obtained from eq. (16) by substituting $\mu \to \mu/2$ and multiplying the result by -2. The fermion contribution is periodic in μ with period 2, and not 1 as in the boson case. It can be easily checked that in the supersymmetric case, with Majorana gluinos belonging to the adjoint representation, the fermion contribution cancels exactly with the boson one, so that the vortex potential is zero.

Returning to the bosonic contribution to the effective vortex potential (16) in the pure glue case, one concludes that integer values of $\mu(\rho)$ are a must at $\rho \to 0$, otherwise the integral over ρ,

$$\text{potential energy} = 2\pi \int_0^\infty d\rho \, \rho \, V^{(d)}(\mu), \tag{22}$$

diverges. Integer values of $\mu(\rho)$ at $\rho \to \infty$ are, clearly, energetically favourable though the energy (22) remains finite at noninteger values of $\mu(\infty)$ as well because of the convergence of the integration over ρ at large ρ. It means that the quantized $Z(2)$ vortices are dynamically preferred but noninteger fluxes are not altogether ruled out by the energetics. This should be contrasted to the case of nonzero temperatures where quantized values of A_d at spatial infinity are necessary to make the energy finite.

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