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QCD in the Axial Gauge Representation*

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Within the canonical Weyl gauge formulation, the axial gauge representation of QCD on a torus is derived. The resolution of the Gauss law constraint is achieved by applying unitary gauge fixing transformations. The result of this formal development is a Hamiltonian explicitly formulated in terms of unconstrained degrees of freedom. Novel features of this Hamiltonian are the non-perturbative dynamics of two-dimensional degrees of freedom appearing in the gauge-fixing procedure, such as Jacobian and centrifugal barrier. These two-dimensional fields appear to be essential for the infrared properties of the theory. The global residual gauge symmetries of QCD are established in this representation. It is shown that $SU(N)$ gauge theories may exhibit at most $N-1$ massless vector (gauge) bosons. The implications for the phase structure of non-abelian gauge theories (QCD, Georgi–Glashow model) are discussed.

1. INTRODUCTION

The canonical formalism of gauge theories is most appropriately developed in the framework of the Weyl (temporal) gauge. In this gauge (cf. [1, 2]), the dynamical variables are given by the vector potentials and the canonically conjugate (chromo)electric fields. The quantization is standard; there are no ghosts and the Hilbert space has a positive norm. The Hamiltonian for these degrees of freedom is supplemented by the Gauss law as a constraint on the physical states. Important formal investigations and dynamical studies of QCD have been performed within this framework [1, 3–7], aimed at providing qualitative insight into the dynamics of strong interaction physics and preparing the ground for more detailed and quantitative studies [8, 9].

With the exception of Feynman's investigation of $(2+1)$ dimensional QCD [7], these studies attempt to resolve explicitly the Gauss law constraint and derive a description in terms of unconstrained variables. The Hamiltonian formulated via unconstrained degrees of freedom only (two instead of three polarization states) has

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to be significantly more complex than the original Hamiltonian containig redundant variables. Nevertheless, the construction of such a Hamiltonian seems to be mandatory for any approximative treatment of the dynamics. Without the resolution of Gauss's law, approximations in general will fail to respect local gauge invariance, and therefore a charge is produced locally in the time evolution governed by those approximate Hamiltonians (this point of view has been particularly stressed in Ref. [10]). Such a scheme is in general not suited for understanding the important phenomenon of confinement. Furthermore, as computations of the running coupling constant of QCD [1, 11, 12] demonstrate, the other important fact of strong interaction physics, asymptotic freedom, results in the canonical formalism from the non-abelian modification of Gauss's law. This reinforces the strategy to incorporate in the first step of a dynamical calculation the Gauss law constraint into the Hamiltonian.

Most of the efforts to date have attempted to implement the Gauss law by the introduction of Coulomb-gauge variables into the canonical formalism of QCD [1, 3, 5, 6]. In QED the radiation gauge is singled out as the gauge in which static charges do not radiate. Since, in QCD, radiation always couples to color spin, this unique dynamical advantage is absent. Moreover, the choice of radiation gauge variables does not conform with the angular momentum algebra of the Gauss law operators, thereby causing significant technical disadvantages. Finally, problems in the gauge-fixing procedure related to the presence of Gribov ambiguities [13] could not be resolved satisfactorily due to the technical complexity of this particular gauge choice.

One therefore might attempt to formulate QCD in terms of unconstrained variables which are less familiar from canonical QED. Here we shall derive a representation of QCD in terms of variables close to those of the axial gauge. The advantages of the axial gauge [14], as well as its difficulties in yielding a proper infrared behavior [15], have been realized early. In order to deal with these formally, as well as physically, expected infrared difficulties in a well-defined framework, we consider QCD on a torus. In the canonical formalism of Weyl-gauge QCD, the reduction in the number of degrees of freedom from three to two polarization states has to be achieved quantum-mechanically, i.e., by respecting the canonical commutation relations between gauge and electric field operators. Thus gauge field variables cannot be assumed to be zero, rather a gauge-fixed formulation is a particular representation of the quantum theory in which the Hamiltonian, if acting in the physical subspace, is independent of certain degrees of freedom. Selection of those degrees of freedom is not completely arbitrary. The Gauss law imposes important restrictions and in particular prevents complete elimination (assumed to be possible in the naive axial gauge) of one of the cartesian components of gauge and electric fields from the Hamiltonian. Gauge variables corresponding to zero eigenvalues of certain (covariant) derivative operators, describing gluons propagating in a prescribed plane and with specified polarization, have to be kept as dynamical degrees of freedom. It is the presence of these degrees of freedom which resolves the problems noted by Schwinger and which renders the

theory well behaved in the infrared. In turn, the major analytical effort in deriving this axial gauge representation will be required for determining the dynamics of the zero modes. The final result of this investigation will be an explicit representation of the Hamiltonian governing the dynamics of the unconstrained degrees of freedom in the space of physical states. To arrive at this result, we shall employ the technique of resolving Gauss's law with the help of unitary gauge fixing transformations. Similar techniques have been investigated in general and applied to the case of the radiation gauge in Refs. [5, 6]. In Ref. [16] we have developed more systematically these methods for QED; the derivation of the axial gauge representation of QED in Ref. [16] contains in a technically much simpler context many of the important elements of the gauge fixing procedure relevant for the non-abelian case. In comparison to the resolution of the Gauss law by the more standard technique of changes of variables (cf., e.g., [3, 6]), the gauge fixing by unitary transformations provides the appropriate setting for the investigation of residual gauge symmetries, i.e., gauge symmetries whose realization in the Wigner–Weyl mode is not enforced by the Gauss law. As for QED (cf. [17–19]), residual symmetries can be expected to be useful also in characterizing the different phases of non-abelian gauge theories [20, 21]. For such a discussion, it is particularly important to formulate the gauge fixing procedure in a way general enough to admit for different coupling of matter and gauge fields. For instance, realization of residual global gauge symmetries must encompass, depending on the nature of the colored matter field, both the possibility of a confined phase as well as the expected Goldstone phase of the Georgi–Glashow model.

Our investigation of the axial gauge representation of QCD is presented on three levels of technical complexity. The main body of the paper describes the most important steps in the formalism leading to the axial gauge representation, i.e., the resolution of the Gauss law, the construction of the unitary gauge fixing transformations, the determination of the axial gauge Hamiltonian, and finally the study of the residual, global gauge symmetry. The technically involved computations necessary for determination of the "zero mode" dynamics have been relegated to a series of appendices. Here we present the details of a new method to calculate (functional) Jacobians from hermiticity defects in the definition of momentum operators. Finally, in the more qualitative concluding Section 9, we summarize the most important structural elements of the derivation, compare our approach to other gauge fixing procedures of QCD, and indicate in a more speculative vein the possible physics implications. In particular, we emphasize the new elements of the formalism associated with the zero-mode dynamics. This rather extended concluding section is intended to provide insight into the gauge fixed formulation of QCD also for readers who are not interested in the formal derivations of Sections 3–8. The introductory section 2 presents the standard Weyl-gauge formulation of QCD as the basis for the whole following discussion.

2. QCD in the Weyl Gauge

The dynamical variables of QCD (with N colors) in the Weyl gauge,

$$A_0^a = 0, \qquad a = 1, ..., N^2 - 1, \tag{2.1}$$

are the three components of the $N^2 - 1$ vector potentials $\mathbf{A}^a$, their conjugate momenta, the (up to the sign) chromoelectric fields,

$$\mathbf{A} = \mathbf{A}^a(\mathbf{x}) \frac{\lambda^a}{2}, \qquad \mathbf{\Pi} = \mathbf{\Pi}^a(\mathbf{x}) \frac{\lambda^a}{2}, \tag{2.2}$$

and the quark fields ψ which carry Dirac and color labels ($\psi_{\alpha, i}(\mathbf{x})$, $\alpha = 1, ..., 4$, $i = 1, ..., N$, flavor labels are suppressed). The configuration space is assumed to be a torus, i.e., periodic boundary conditions for the gauge fields

$$\mathbf{A}(\mathbf{x} + L\mathbf{e}_i) = \mathbf{A}(\mathbf{x}), \qquad i = 1, 2, 3, \tag{2.3}$$

and quasi-periodic ones for the fermion fields are imposed

$$\psi(\mathbf{x} + L\mathbf{e}_i) = e^{i\varphi_i}\psi(\mathbf{x}), \qquad i = 1, 2, 3; \tag{2.4}$$

φ_i are arbitrary phases. In this way, the infrared properties of the system are well defined and translational invariance is preserved at every level of the formulation. The dynamical quark and gluon degrees of freedom are quantized by requiring anti-commutation relations for the fermion fields

$$\{\psi_{\alpha, i}(\mathbf{x}), \psi^\dagger_{\beta, j}(\mathbf{y})\} = \delta_{\alpha\beta}\delta_{ij}\delta_\varphi(\mathbf{x} - \mathbf{y}), \tag{2.5}$$

and commutation relations for the gauge fields,

$$[\Pi_k^a(\mathbf{x}), A_l^b(\mathbf{y})] = \frac{1}{i}\delta_{kl}\delta_{ab}\delta_{\varphi=0}(\mathbf{x} - \mathbf{y}). \tag{2.6}$$

The quasi-periodic δ function,

$$\delta_\varphi(\mathbf{z}) = \frac{1}{V}\sum_{\mathbf{n}} e^{i\mathbf{p}_{n,\varphi}\mathbf{z}}, \qquad V = L^3, \tag{2.3}$$

is determined according to the boundary conditions (2.4) by the wave vectors with components

$$\mathbf{p}_{n,\varphi} = \frac{1}{L}\left(2\pi\mathbf{n} + \sum_{i=1}^{3} \varphi_i \mathbf{e}_i\right). \tag{2.8}$$

In the Weyl gauge, the constrained quark–gluon dynamics is determined by the Hamiltonian density,

$$\mathscr{H}(\mathbf{x}) = -i\psi^\dagger(\mathbf{x})\,\boldsymbol{\alpha}(\nabla - ig\mathbf{A}(\mathbf{x}))\,\psi(\mathbf{x}) + m\psi^\dagger(\mathbf{x})\,\beta\psi(\mathbf{x}) + \mathrm{tr}(\mathbf{\Pi}^2(\mathbf{x}) + \mathbf{B}^2(\mathbf{x})), \tag{2.9}$$

and the constraint on the physical states $|\Phi\rangle$ to satisfy Gauss's law,

$$G^a(\mathbf{x})\,|\Phi\rangle = 0, \qquad a = 1, \ldots, N^2 - 1. \tag{2.10}$$

We have introduced the $N^2 - 1$ chromomagnetic fields $\mathbf{B}^a$ related to the spatial components of the field strength tensor,

$$F_{kl} = \partial_k A_l - \partial_l A_k - ig[A_k, A_l] = (\partial_k A_l^a - \partial_l A_k^a + g f^{abc} A_k^b A_l^c)\frac{\lambda^a}{2}, \tag{2.11}$$

by

$$B_i^a(\mathbf{x}) = \tfrac{1}{2}\varepsilon_{ijk} F_{jk}^a(\mathbf{x}). \tag{2.12}$$

The Gauss law operator is given by

$$G^a(\mathbf{x}) = \operatorname{div} \mathbf{\Pi}^a(\mathbf{x}) + g\rho^a(\mathbf{x}). \tag{2.13}$$

Both gluons and quarks contribute to the color density ρ,

$$\rho^a(\mathbf{x}) = f^{abc}\mathbf{A}^b(\mathbf{x})\,\mathbf{\Pi}^c(\mathbf{x}) + \rho_m^a(\mathbf{x}), \tag{2.14}$$

where ρ_m is the color density associated with the quarks

$$\rho_m^a(\mathbf{x}) = \psi_i^\dagger(x)\frac{\lambda_{ij}^a}{2}\psi_j(\mathbf{x}). \tag{2.15}$$

The operator $G(\mathbf{x})$ defining the constraint commutes with the Hamiltonian,

$$[G(\mathbf{x}), H] = 0, \tag{2.16}$$

and therefore time evolution leaves the system in the space of physical states. The Gauss law operator is closely connected to the generator of the residual gauge transformations. These are time independent but space dependent gauge transformations which preserve the gauge choice equation (2.1) and leave the Hamiltonian invariant. In this quantum mechanical framework of QCD in the Weyl-gauge, finite, residual local gauge transformations are given by the unitary operators,

$$\Omega[\beta] = e^{-iG[\beta]}. \tag{2.17}$$

Here, we have defined

$$G[\beta] = \int d^3x(-\mathbf{\Pi}^a(\mathbf{x})\,\nabla + g f^{abc}\mathbf{A}^b(\mathbf{x})\,\mathbf{\Pi}^c(\mathbf{x}) + g\rho_m^a(\mathbf{x}))\,\beta^a(\mathbf{x}), \tag{2.18}$$

where

$$\beta(\mathbf{x}) = \beta^a(\mathbf{x})\frac{\lambda^a}{2} \tag{2.19}$$

is given by N^2-1 (almost) arbitrary c-number functions (constraints on $\beta(\mathbf{x})$ arising from the boundary conditions will be discussed later).

It is important for the further development to distinguish between Hilbert space operators such as Ω describing quantum mechanical transformations and matrix operations in the color indices as, e.g., described by $\beta(\mathbf{x})$. The unitary operator $\Omega[\beta]$ rotates the color components of the quark fields,

$$\Omega[\beta]\,\psi(\mathbf{x})\,\Omega^{\dagger}[\beta]=e^{ig\beta(\mathbf{x})}\psi(\mathbf{x}), \tag{2.20}$$

and translates and rotates the gauge field operators via the c-number functions $\beta^a(\mathbf{x})$

$$\Omega[\beta]\,\mathbf{A}(\mathbf{x})\,\Omega^{\dagger}[\beta]=e^{ig\beta(\mathbf{x})}\left(\mathbf{A}(\mathbf{x})+\frac{i}{g}\,\nabla\right)e^{-ig\beta(\mathbf{x})}. \tag{2.21}$$

The electric field transforms as a vector in color space,

$$\Omega[\beta]\,\mathbf{\Pi}(\mathbf{x})\,\Omega^{\dagger}[\beta]=e^{ig\beta(\mathbf{x})}\mathbf{\Pi}(\mathbf{x})\,e^{-ig\beta(\mathbf{x})}. \tag{2.22}$$

Existence of this gauge symmetry reflects the presence of superfluous variables in Weyl-gauge QCD. Indeed, similar to QED, implementation of the Gauss law will (apart from $N-1$ zero-mode degrees of freedom) reduce at each space point the $3(N^2-1)$ degrees of freedom described by the vector potential $\mathbf{A}(\mathbf{x})$ to the two polarization states of the N^2-1 gluons. We have been careful in distinguishing between the Gauss law operator and the "generator" of finite gauge transformations. Only for periodic gauge functions $\beta_p(\mathbf{x})$ are these two quantities trivially related,

$$G[\beta_p]=\int d^3x\,G^a(\mathbf{x})\,\beta_p^a(\mathbf{x}). \tag{2.23}$$

The noncommutativity of the Gauss law constraints makes the derivation of formulations of QCD in terms of physical (i.e., unconstrained) variables more difficult than in QED. The commutator of the different components of the Gauss law operator is

$$[G^a(\mathbf{x}),\,G^b(\mathbf{y})]=igf^{abc}G^c(\mathbf{x})\,\delta^{(3)}_{\varphi=0}(\mathbf{x}-\mathbf{y}) \tag{2.24}$$

and permits simultaneous implementation of the N^2-1 Gauss law constraints.

3. Resolution of the Gauss Law Constraint

The axial gauge representation is a formulation of QCD in terms of unconstrained variables, in which both the gauge field component $A_3(\mathbf{x})$ and its conjugate momentum operator $\Pi_3(\mathbf{x})$ do not appear in the Hamiltonian. Elimination of $A_3(\mathbf{x})$ will be achieved by applying a "gauge fixing" unitary transformation

to the Weyl-gauge Hamiltonian. We first consider the elimination of $\Pi_3(\mathbf{x})$ in the space of physical states by implementing the Gauss law constraint. To this end we decompose the Gauss law operator (2.13) into the perpendicular (i.e., 1, 2) and quark contribution,

$$G^a_\perp(\mathbf{x}) = \mathrm{div}_\perp \Pi^a_\perp(\mathbf{x}) + g f^{abc} \mathbf{A}^b_\perp(\mathbf{x})\, \Pi^c_\perp(\mathbf{x}) + g\rho^a_m(\mathbf{x}), \tag{3.1}$$

and the contribution from the three-components of the gauge field variables,

$$G^a_3(\mathbf{x}) = D^{ab}_3 \Pi^b_3(\mathbf{x}) \qquad \text{with} \quad D^{ab}_3 = \partial_3 \delta^{ab} + g f^{acb} A^c_3(\mathbf{x}). \tag{3.2}$$

It will turn out to be important that in this division of the degrees of freedom, the resulting contributions to the Gauss law operator satisfy separately the commutation relations of Eq. (2.24). We now write the constraint, Eq. (2.10), as

$$D_3 \Pi_3(\mathbf{x})\, |\Phi\rangle = -G_\perp(\mathbf{x})\, |\Phi\rangle. \tag{3.3}$$

Equation (3.3) expresses in the space of physical states the covariant three-derivative of Π_3 and not Π_3 itself in terms of other degrees of freedom (it is this fact which would cause the well-known infrared problems, if one would nevertheless attempt to implement the naive axial gauge $A_3 = 0$). To proceed in the resolution of Gauss's law, we define a complete set of eigenfunctions and eigenvalues of the covariant derivative D_3 by the differential equation

$$\frac{1}{i} D^{ab}_3 \zeta^b_{c,n}(\mathbf{x}) = \mu_{c,n}(\mathbf{x}_\perp)\, \zeta^a_{c,n}(\mathbf{x}), \tag{3.4}$$

and the boundary condition,

$$\zeta_{c,n}(\mathbf{x}_\perp, L) = \zeta_{c,n}(\mathbf{x}_\perp, 0). \tag{3.5}$$

The indices c and n specify the eigenvectors and eigenvalues μ (details will be given in Section 4). Periodic boundary conditions have been chosen in order to provide an appropriate basis for expanding the periodic operators $\Pi_3(\mathbf{x})$ and $G_\perp(\mathbf{x})$. We have

$$\Pi_3(\mathbf{x}) = \frac{1}{L} \sum_{c,n} \zeta_{c,n}(\mathbf{x})\, p_{c,n}(\mathbf{x}_\perp), \tag{3.6}$$

with the expansion coefficients given by

$$p_{c,n}(\mathbf{x}_\perp) = (\zeta_{c,n}, \Pi_3)(\mathbf{x}_\perp) = 2\,\mathrm{tr} \int_0^L dx_3\, \zeta^\dagger_{c,n}(\mathbf{x})\, \Pi_3(\mathbf{x}). \tag{3.7}$$

The normalization has been chosen such that the following orthogonality and completeness relations hold:

$$\begin{aligned} &\frac{1}{L} \int_0^L dx_3 \sum_a \zeta^{a*}_{c,n}(\mathbf{x})\, \zeta^a_{c',n'}(\mathbf{x}) = \delta_{cc'} \delta_{nn'}, \\ &\frac{1}{L} \sum_{c,n} \zeta^a_{c,n}(\mathbf{x}_\perp, x_3)\, \zeta^{a'*}_{c,n}(\mathbf{x}_\perp, x'_3) = \delta_{aa'} \delta(x_3 - x'_3). \end{aligned} \tag{3.8}$$

Projection of the Gauss law constraint of Eq. (3.3) onto the eigenvectors of D_3 yields

$$i\mu_{c,n}(\mathbf{x}_\perp)\, p_{c,n}(\mathbf{x}_\perp)\,|\Phi\rangle = -(\zeta_{c,n}, G_\perp)(\mathbf{x}_\perp)\,|\Phi\rangle. \tag{3.9}$$

Thus in the space of physical states and for non-vanishing eigenvalue $\mu_{c,n}$, the Gauss law constraint can be used to eliminate the components $p_{c,n}(\mathbf{x}_\perp)$ of the chromoelectric field operator $\Pi_3(\mathbf{x})$, while for zero eigenvalues the residual Gauss law constraints are left,

$$(\zeta_{c_0,n_0}, G_\perp)(\mathbf{x}_\perp)\,|\Phi\rangle = 0 \qquad \text{if} \quad \mu_{c_0,n_0}(\mathbf{x}_\perp) = 0. \tag{3.10}$$

The action of the operator $\Pi_3(\mathbf{x})$ on physical states is therefore expressed as

$$\Pi_3(\mathbf{x})\,|\Phi\rangle = \frac{1}{L}\Bigg[\sum_{\substack{c_0,n_0 \\ \mu_{c_0,n_0}=0}} \zeta_{c_0,n_0}(\mathbf{x})\, p_{c_0,n_0}(\mathbf{x}_\perp) - \sum_{\substack{c,n \\ \mu_{c,n}\neq 0}} \frac{\zeta_{c,n}(\mathbf{x})}{i\mu_{c,n}(\mathbf{x}_\perp)}(\zeta_{c,n}, G_\perp)(\mathbf{x}_\perp)\Bigg]|\Phi\rangle; \tag{3.11}$$

i.e., the components $p_{c_0,n_0}(\mathbf{x}_\perp)$ of $\Pi_3(\mathbf{x})$ related to the zero modes of the covariant derivative D_3 remain as independent degrees of freedom. This in turn implies the presence of residual Gauss law constraints. These will be implemented at a later stage of the formal development.

To appreciate the essential ingredients of our construction of the axial gauge representation of QCD it is instructive to exhibit the different starting point if one aims at the Coulomb-gauge representation. Here one decomposes the chromoelectric field operator into longitudinal and transverse (including zero-mode) components

$$\boldsymbol{\Pi}(\mathbf{x}) = \boldsymbol{\Pi}^l(\mathbf{x}) + \boldsymbol{\Pi}^t(\mathbf{x}), \tag{3.12}$$

with

$$\nabla \times \boldsymbol{\Pi}^l = 0, \qquad \int d^3x\, \boldsymbol{\Pi}^l(\mathbf{x}) = 0. \tag{3.13}$$

This yields a corresponding separation of the Gauss law constraint

$$(\nabla\delta^{ac} + gf^{abc}\mathbf{A}^b)\,\boldsymbol{\Pi}^{l,c}\,|\Phi\rangle = -G_l^a\,|\Phi\rangle, \tag{3.14}$$

with

$$G_l^a = g(\rho_m^a(\mathbf{x}) + f^{abc}\mathbf{A}^b(\mathbf{x})\,\boldsymbol{\Pi}^{t,c}(\mathbf{x})). \tag{3.15}$$

Representing $\boldsymbol{\Pi}^l$ as the gradient of a scalar field, it is seen that elimination of the longitudinal chromoelectric field, with the help of the Gauss law constraint according to Eq. (3.14), has to proceed by introducing eigenfunctions and eigenvalues of the operator familiar from standard (Coulomb-) gauge-fixing procedures,

$$d_l^{ac} = \Delta\delta^{ac} + gf^{abc}\mathbf{A}^b\nabla. \tag{3.16}$$

Here, as well as in the axial gauge choice, the occurrence of zero modes presents in principle no barrier for further developments. However, unlike the ordinary differential operator D_3 with the perpendicular coordinates entering as parameters, the operator d_l defines an eigenvalue problem associated with a partial differential equation which cannot be solved explicitly. Furthermore, the separation of the Gauss law operator in Eq. (3.14) does not preserve separately the commutation relations (Eq. (2.24)); this limits further the possibilities in proceeding by analytical calculations.

4. The Spectrum of D_3

Resolution of Gauss's law requires knowledge of spectrum and eigenfunctions of the covariant derivative D_3 (cf. Eq. (3.4)). The fundamental quantity for the following calculation is the integral

$$e^{ig\tau(\mathbf{x})} = \mathrm{P}\exp\left(ig\int_0^{x_3} dz\, A_3(\mathbf{x}_\perp, z)\right), \tag{4.1}$$

which is ordered along the path of integration parallel to the x_3-axis. It solves the differential equation

$$(\partial_3 - igA_3(\mathbf{x}))\, e^{ig\tau(\mathbf{x})} = 0. \tag{4.2}$$

The path-ordered integral is not periodic in x_3 and we therefore introduce the operator

$$\tilde{U}(\mathbf{x}) = e^{ig\tau(\mathbf{x})}\, e^{-ig\theta(\mathbf{x}_\perp)\, x_3/L}\, e^{igA(\mathbf{x}_\perp)}, \tag{4.3}$$

which, with the following definition,

$$e^{ig\theta(\mathbf{x}_\perp)} = e^{ig\tau(\mathbf{x}_\perp, L)}, \tag{4.4}$$

is indeed periodic. For later convenience, we have also introduced the unitary matrix e^{igA} which diagonalizes θ,

$$\frac{\theta(\mathbf{x}_\perp)}{L} = e^{igA(\mathbf{x}_\perp)} a_3(\mathbf{x}_\perp)\, e^{-igA(\mathbf{x}_\perp)}, \qquad a_3 = \sum_{c_0=1}^{N-1} a_3^{c_0}\frac{\lambda^{c_0}}{2}. \tag{4.5}$$

In the following, we use the convention that color indices with subscript "0" refer to the generators of the Cartan subalgebra, i.e., to diagonal λ-matrices in the usual representation, and we refer to the associated fields as "neutral" fields. The matrix e^{igA} in Eq. (4.5) is not uniquely determined by the requirement that it diagonalizes a given matrix θ; it can be multiplied by an arbitrary diagonal matrix from the right. This leaves room for additional conditions [22]. In view of the applications

below, the most natural choice is the one where e^{igA} depends on N^2-N variables independent of the $N-1$ variables a_3^{c0}, i.e., we assume

$$\frac{\delta e^{igA(\mathbf{x}_\perp)}}{\delta a_3^{c0}(\mathbf{y}_\perp)}=0. \tag{4.6}$$

It is now straightforward to determine the spectrum and eigenfunctions of D_3. We first observe that the periodic $\tilde{U}$ satisfies a differential equation different from that of the path-ordered integral,

$$(\partial_3 - igA_3)\,\tilde{U} = -ig\tilde{U}a_3. \tag{4.7}$$

This will imply that A_3 can be eliminated from the Hamiltonian only up to the zero-mode a_3. After transforming to the basis

$$\hat{\zeta}_{c,n} = \tilde{U}^\dagger \zeta_{c,n}\tilde{U}, \tag{4.8}$$

which will be seen to be independent of the dynamics, Eq. (3.4) reads in matrix notation

$$\frac{1}{i}[\partial_3 - iga_3, \hat{\zeta}_{c,n}] = \mu_{c,n}\hat{\zeta}_{c,n}. \tag{4.9}$$

The eigenvalue problem defined by the differential equation (4.9) and the requirement of periodic boundary conditions can easily be solved since a_3 is a diagonal matrix; the eigenfunctions are

$$\hat{\zeta}_{c,n} = \hat{z}_c e^{i2\pi n x_3/L}, \qquad n=0,\ \pm1,\ \pm2,\ ..., \tag{4.10}$$

with the vectors of the Weyl basis

$$(\hat{z}_c)_{ij} = \frac{1}{\sqrt{2}}\delta_{ip}\delta_{jq}, \tag{4.11}$$

and the eigenvalues

$$\mu_{c,n}(\mathbf{x}_\perp) = \frac{2\pi n}{L} + g(a_{3,q}(\mathbf{x}_\perp) - a_{3,p}(\mathbf{x}_\perp)). \tag{4.12}$$

Thus $c=c(p,q)$ labels the different basis vectors, and to each basis vector corresponds an infinite sequence of equidistant eigenvalues labelled by n. Here we also have used the notation

$$(a_3)_{ij} = \delta_{ij}a_{3,i}. \tag{4.13}$$

So far, the basis vectors $\hat{z}_c$ still include the unit matrix. When expanding elements

of the $SU(N)$ Lie algebra, we must require $\operatorname{tr} \hat{z}_c = 0$. This is clearly satisfied for $q \neq p$. For $q = p$, it is advantageous to switch to the representation

$$\hat{z}_{c_0} = \tfrac{1}{2}\lambda^{c_0}. \tag{4.14}$$

Such a change of basis is possible because all eigenvalues for a given n and $p = q$ are degenerate. Our choice of the basis is summarized in the representation for the eigenfunctions of D_3,

$$\zeta_{c,n} = \tilde{U}\hat{z}_c\tilde{U}^\dagger e^{i2\pi n x_3/L}, \tag{4.15}$$

with $\tilde{U}$ defined in Eq. (4.3) and $\hat{z}_c$ defined in Eqs. (4.11) and (4.14). For some purposes, it will be more convenient to write them in the equivalent form

$$\zeta_{c,n} = e^{ig\tau} z_c e^{-ig\tau} e^{i\mu_{c,n}x_3}, \tag{4.16}$$

with

$$z_c(\mathbf{x}_\perp) = e^{igA(\mathbf{x}_\perp)}\hat{z}_c e^{-igA(\mathbf{x}_\perp)}. \tag{4.17}$$

Of particular importance are the zero-modes $\zeta_{c_0,0}$ with a vanishing eigenvalue. They play a special role in the resolution of Gauss's law (cf. Eqs. (3.10) and (3.11)). They exhibit a non-trivial x_3-dependence. The corresponding color matrices z_{c_0} are hermitian and commute with θ. Their number is given by the rank of the Lie algebra, $N-1$. This is in contradistinction to the properties of the zero-modes of the ordinary derivative which are relevant for the $g = 0$ limit of QCD. Here, (N^2-1) x_3-independent zero-modes exist. Consequently, the transition to $g = 0$ cannot be expected to be smooth in a non-abelian gauge theory with the Gauss law implemented.

Note that with the definition (4.1) of the path-ordered integral we have introduced explicitly a coordinate system on the torus. In the following, we assume that the three-coordinates are restricted to

$$0 \leqslant x_3 < L. \tag{4.18}$$

Finally, we remark that our definition of the zero-mode fields θ is not yet complete since only the exponential of these fields has been defined (cf. Eq. (4.4)). Also the diagonalization (4.5) involves, in general, additional ambiguities, even with the condition (4.6) observed. The ambiguity in the definition of a_3, which is also seen explicitly in the definition of the eigenvalues $\mu_{c,n}$ of Eq. (4.12), reflects the gauge freedom of the classical Yang–Mills theory of shifting $a_{3,p}$ by multiples of $2\pi/gL$ and permuting the color labels. Most of the following developments will not depend on a complete specification. A discussion of various specific options can therefore be postponed until the end of Section 7.

5. First Unitary Gauge Fixing Transformation

With the resolution of the Gauss law constraint in Eq. (3.11) we have succeeded in expressing the action of the three-component of the chromoelectric field operator (up to the zero mode operator) on physical states by that of operators representing other degrees of freedom of the system. However, at this stage, the Hamiltonian still contains the conjugate variable, the three-component of the gauge field operator and concomitantly still exhibits invariance under the local gauge transformations of Eq. (2.17). The presence of this symmetry forms the basis for elimination of $A_3(\mathbf{x})$ by a succession of two unitary "gauge fixing" transformations. Transformation of the Hamiltonian in two steps reflects the two-step procedure in implementing the Gauss law for finite and zero mode projections, respectively (Eqs. (3.9), (3.10)). The general principle in the construction of the gauge fixing transformations is to perform a gauge transformation in the variables to be kept, with the gauge function being a functional of the gauge field component to be eliminated. Basic to this construction is the decomposition (3.1), (3.2) of the Gauss law operator into contributions which satisfy separately the "angular momentum algebra" (2.24). Thus we define

$$U[\xi] = \exp\left[-i \int d^3x (-\mathbf{\Pi}^a_\perp(\mathbf{x})\, \nabla_\perp + g f^{abc} \mathbf{A}^b_\perp(\mathbf{x})\, \mathbf{\Pi}^c_\perp(\mathbf{x}) + g\rho^a_m(\mathbf{x}))\, \xi^a(\mathbf{x}) \right], \tag{5.1}$$

and assume the gauge function ξ to depend only on the three-component of the gauge field

$$\xi = \xi[A_3]. \tag{5.2}$$

In the definition of the exponential in Eq. (5.1), no operator ordering problem occurs. By construction, $U[\xi]$ acts as a gauge transformation on fermionic and perpendicular gauge degrees of freedom determined by the gauge function $\xi(\mathbf{x})$. The transformed fermion field operators are therefore given by (cf. Eq. (2.20))

$$U[\xi]\, \psi(\mathbf{x})\, U^\dagger[\xi] = e^{ig\xi(\mathbf{x})} \psi(\mathbf{x}). \tag{5.3}$$

The action of $U[\xi]$ on the perpendicular gauge degrees of freedom is (cf. Eqs. (2.21), (2.22))

$$U[\xi]\, \mathbf{A}_\perp(\mathbf{x})\, U^\dagger[\xi] = e^{ig\xi(\mathbf{x})} \left(\mathbf{A}_\perp(\mathbf{x}) + \frac{i}{g} \nabla_\perp \right) e^{-ig\xi(\mathbf{x})}, \tag{5.4}$$

$$U[\xi]\, \mathbf{\Pi}_\perp(\mathbf{x})\, U^\dagger[\xi] = e^{ig\xi(\mathbf{x})} \mathbf{\Pi}_\perp(\mathbf{x})\, e^{-ig\xi(\mathbf{x})}. \tag{5.5}$$

As a consequence of these relations, the three-component of the chromomagnetic field and the perpendicular contributions to the Gauss law operator (3.1) transform as vectors in color space,

$$U[\xi]\, F_{12}(\mathbf{x})\, U^\dagger[\xi] = e^{ig\xi(\mathbf{x})} F_{12}(\mathbf{x})\, e^{-ig\xi(\mathbf{x})}, \tag{5.6}$$

$$U[\xi]\, G_\perp(\mathbf{x})\, U^\dagger[\xi] = e^{ig\xi(\mathbf{x})} G_\perp(\mathbf{x})\, e^{-ig\xi(\mathbf{x})}. \tag{5.7}$$

The unitary transformation, Eq. (5.1), does not affect the three-component of the gauge field,

$$U[\xi]\, A_3(\mathbf{x})\, U^\dagger[\xi] = A_3(\mathbf{x}). \tag{5.8}$$

To continue, we now specify our choice of the gauge function and identify $e^{ig\xi}$ with the periodic matrix $\tilde{U}$ defined in Eq. (4.3),

$$e^{ig\xi(\mathbf{x})} = \tilde{U}(\mathbf{x}). \tag{5.9}$$

This choice eliminates, according to Eq. (4.7), all but the two-dimensional neutral components of A_3 from the corresponding term in the fermion–gauge field coupling,

$$U[\xi]\, \psi^\dagger \alpha_3 (\partial_3 - igA_3)\, \psi U^\dagger[\xi] = \psi^\dagger \alpha_3 (\partial_3 - iga_3)\, \psi. \tag{5.10}$$

The perpendicular components of the chromomagnetic fields can be transformed easily with the help of Eqs. (4.7) and (5.4),

$$-igU[\xi]\, F_{3,i} U^\dagger[\xi] = \tilde{U}[\partial_3 - iga_3,\, D_i]\, \tilde{U}^\dagger. \tag{5.11}$$

Technically involved is the calculation of the transformed three-component of the chromoelectric field operator due to the dependence of the gauge function ξ on A_3. This calculation is significantly simplified if restricted to the physical space, cf. Eq. (3.11). The only quantities to be transformed are then $p_{c_0,0} = (\zeta_{c_0,0}, \Pi_3)$ and $(\zeta_{c,n}, G_\perp)$ arising from the zero-mode and non-zero-mode projections of Π_3, respectively. We find from Eqs. (5.7) and (4.8)–(4.11) the non-zero-mode components of Π_3 in the physical sector,

$$\begin{aligned} U[\xi](\zeta_{c,n}, G_\perp)(\mathbf{x}_\perp)\, U^\dagger[\xi] &= (\hat{\zeta}_{c,n}, G_\perp)(\mathbf{x}_\perp) \\ &= \sqrt{2} \int_0^L dx_3 (G_\perp)_{pq}\, e^{-i2\pi n x_3/L} \qquad (\mu_{c,n} \neq 0), \end{aligned} \tag{5.12}$$

to be, after the transformation, independent of the gauge field A_3. The most laborious part is the unitary transformation of $p_{c_0,0}$, denoted hereafter as

$$p_3^{c_0}(\mathbf{x}_\perp) = p_{c_0,0}(\mathbf{x}_\perp), \qquad p_3 = \sum_{c_0=1}^{N-1} p_3^{c_0} \frac{\lambda^{c_0}}{2}. \tag{5.13}$$

One can show that p_3 is invariant under $U[\xi]$ by verifying that

$$[p_3^{c_0}(\mathbf{y}_\perp),\, \tilde{U}(\mathbf{x})] = 0. \tag{5.14}$$

Since this calculation provides us at the same time with the commutation relation between p_3 and a_3, we shall exhibit it in some detail. The starting point is the identity (recall the restriction (4.18) for the coordinates in three-direction)

$$\frac{1}{i} \frac{\delta e^{ig\tau(\mathbf{x})}}{\delta A_3^a(\mathbf{y})} = g\Theta(x_3 - y_3)\, \delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp)\, e^{ig\tau(\mathbf{x})} \left(e^{-ig\tau(\mathbf{y})} \frac{\lambda^a}{2} e^{ig\tau(\mathbf{y})} \right), \tag{5.15}$$

which can be proven by taking the derivative with respect to x_3 and comparing the result with the functional derivative of Eq. (4.2); $\Theta(z)$ is the Heaviside step function. Projection of this equation onto the zero modes $\zeta_{c_0,0}$ yields (cf. Eq. (4.16))

$$[p_3^{c_0}(\mathbf{y}_\perp), e^{ig\tau(\mathbf{x})}] = gx_3\delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp)\, e^{ig\tau(\mathbf{x})} z_{c_0}(\mathbf{y}_\perp), \tag{5.16}$$

and therefore for $x_3 = L$

$$[p_3^{c_0}(\mathbf{y}_\perp), e^{ig\theta(\mathbf{x}_\perp)}] = gL\delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp)\, e^{ig\theta(\mathbf{x}_\perp)} z_{c_0}(\mathbf{y}_\perp). \tag{5.17}$$

Since the right-hand side of Eq. (5.17) commutes with $\theta(\mathbf{x}_\perp)$, one can evaluate the commutator between $p_3^{c_0}$ and any function of θ by ordinary rules of differentiation. In particular, one obtains

$$[p_3^{c_0}(\mathbf{y}_\perp), e^{-ig\theta(\mathbf{x}_\perp)x_3/L}] = -gx_3\delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp)\, e^{-ig\theta(\mathbf{x}_\perp)x_3/L} z_{c_0}(\mathbf{y}_\perp), \tag{5.18}$$

and, consequently, Eqs. (5.16) and (5.18) yield

$$[p_3^{c_0}(\mathbf{y}_\perp), e^{ig\tau(\mathbf{x})} e^{-ig\theta(\mathbf{x}_\perp)x_3/L}] = 0. \tag{5.19}$$

Similarly, for an arbitrary function of $\theta(\mathbf{x}_\perp)$ the following commutator can be derived

$$[p_3^{c_0}(\mathbf{y}_\perp), F(\theta(\mathbf{x}_\perp))] = -iL\delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp)\, z_{c_0}^a(\mathbf{x}_\perp) \frac{\partial F(\theta(\mathbf{x}_\perp))}{\partial \theta^a(\mathbf{x}_\perp)}. \tag{5.20}$$

For the choice given in Eq. (4.6) of $e^{ig\Delta}$, we have

$$\frac{1}{L}\frac{\partial \theta^a}{\partial a_3^{c_0}} = \frac{1}{2}\operatorname{tr}(e^{ig\Delta}\lambda^{c_0} e^{-ig\Delta}\lambda^a) = z_{c_0}^a, \tag{5.21}$$

where we have used Eqs. (4.14) and (4.16). Thus

$$z_{c_0}^a \frac{\partial}{\partial \theta^a} = \frac{1}{L}\frac{\partial}{\partial a_3^{c_0}} \tag{5.22}$$

and Eq. (5.20) becomes

$$[p_3^{c_0}(\mathbf{y}_\perp), F(\theta(\mathbf{x}_\perp))] = \delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp)\frac{1}{i}\frac{\partial F(\theta(\mathbf{x}_\perp))}{\partial a_3^{c_0}(\mathbf{x}_\perp)}. \tag{5.23}$$

In particular, this result implies that

$$[p_3^{c_0}(\mathbf{y}_\perp), e^{ig\Delta(\mathbf{x}_\perp)}] = 0. \tag{5.24}$$

This last commutator together with Eq. (5.19) yields Eq. (5.14). This derivation also establishes the important fact that p_3 behaves like the momentum conjugate to a_3 when acting on functions of θ. However, the operator $p_3^{c_0}$ is not hermitian; this will be discussed further in Section 7.

Therefore, in the physical sector, the components of Π_3 with respect to the dynamical basis $\zeta_{c,n}$ (cf. Eq. (3.7)) and, consequently, also the associated kinetic energy are expressed in terms of variables which, apart from the zero-mode gauge fields a_3, are independent of A_3. In the space of transformed physical states

$$|\tilde{\Phi}\rangle = U[\xi]\,|\Phi\rangle, \tag{5.25}$$

we have

$$\begin{aligned}
&\langle\Phi|\int d^3x\,\mathrm{tr}\,\Pi_3^2(\mathbf{x})\,|\Phi\rangle \\
&\quad = \langle\tilde{\Phi}|\frac{1}{L}\int d^2x_\perp\left[\frac{1}{2}\sum_{c_0} p_3^{c_0\dagger}(\mathbf{x}_\perp)\,p_3^{c_0}(\mathbf{x}_\perp) + \int_0^L dx_3\int_0^L dy_3\right. \\
&\qquad \left.\times \sum_{p,q,n}{}' \frac{G_{\perp qp}(\mathbf{x}_\perp, x_3)\,G_{\perp pq}(\mathbf{x}_\perp, y_3)}{[2\pi n/L + g(a_{3,q}(\mathbf{x}_\perp) - a_{3,p}(\mathbf{x}_\perp))]^2}\,e^{i2\pi n(x_3-y_3)/L}\right]|\tilde{\Phi}\rangle.
\end{aligned} \tag{5.26}$$

In the last term, the sum does not include $n=0$ for $p=q$; this is indicated by the prime and follows directly from the requirement that $\mu_{c,n}\neq 0$ in Eq. (3.11). To derive the final form of the Hamiltonian, the residual Gauss law of Eq. (3.9) needs to be implemented. In the sector of transformed physical states, it assumes the particularly simple form

$$(\hat{\zeta}_{c_0,0}, G_\perp)\,|\tilde{\Phi}\rangle = \int_0^L dx_3\,G_\perp^{c_0}(\mathbf{x})\,|\tilde{\Phi}\rangle = 0. \tag{5.27}$$

Its actual implementation will be addressed in the next section.

6. Second Unitary Gauge-Fixing Transformation and Hamiltonian

In the representation of QCD obtained after applying the first gauge-fixing transformation, the residual Gauss law constrains only neutral chromoelectric (two-dimensional) fields (cf. Eq. (5.27)). Since these residual constraints commute with each other, the procedure developed for QED [16] can be followed closely and several options are available. The simplest one is the Coulomb-gauge representation. Here, one constructs a further gauge fixing transformation which eliminates the longitudinal neutral gauge fields with vanishing three-momentum from the Hamiltonian and in turn simplifies the residual constraints. In order to arrive at such a two-dimensional Coulomb-gauge representation, we define the fields to be eliminated by

$$\mathbf{a}_\perp^l(\mathbf{x}_\perp) = \frac{1}{L}\int_0^L d^3y\,d(\mathbf{x}_\perp - \mathbf{y}_\perp)\,\nabla_\perp \sum_{c_0=1}^{N-1} \mathrm{div}\,\mathbf{A}_\perp^{c_0}(\mathbf{y})\,\frac{\lambda^{c_0}}{2}. \tag{6.1}$$

Their conjugate momenta are

$$\mathbf{p}^l_\perp(\mathbf{x}_\perp) = \int_0^L d^3y\, d(\mathbf{x}_\perp - \mathbf{y}_\perp)\, \nabla_\perp \sum_{c0=1}^{N-1} \operatorname{div} \mathbf{\Pi}^{c0}_\perp(\mathbf{y}) \frac{\lambda^{c0}}{2}, \tag{6.2}$$

where

$$d(\mathbf{z}_\perp) = -\frac{1}{L^2} \sum_{\mathbf{n}\neq 0} \frac{1}{q_n^2} e^{i\mathbf{q}_n \mathbf{z}_\perp}, \qquad \mathbf{q}_n = \frac{2\pi}{L}(n_1, n_2), \tag{6.3}$$

is the two-dimensional, periodic, scalar Greens function. In terms of these variables, the residual Gauss law, Eq. (5.27), reads

$$[\operatorname{div}^{(2)} \mathbf{p}^l_\perp(\mathbf{x}_\perp) + g\rho^{(2)}(\mathbf{x}_\perp)]\, |\tilde{\Phi}\rangle = 0, \tag{6.4}$$

with the two-dimensional color neutral charge density

$$\rho^{(2)}(\mathbf{x}_\perp) = \int_0^L dx_3 \sum_{c0=1}^{N-1} (f^{c0de} \mathbf{A}^d_\perp(\mathbf{x})\, \mathbf{\Pi}^e_\perp(\mathbf{x}) + \rho^{c0}_m(\mathbf{x})) \frac{\lambda^{c0}}{2}. \tag{6.5}$$

The unitary gauge fixing transformation which eliminates the superfluous degrees of freedom in implementing the residual Gauss law constraints is

$$u^{(2)}[\alpha] = \exp\left(-ig \int d^2x \sum_{c0=1}^{N-1} \rho^{(2)\,c0}(\mathbf{x}_\perp)\, \alpha^{c0}(\mathbf{x}_\perp) \right), \tag{6.6}$$

with $\alpha[\mathbf{a}^l_\perp]$ given by

$$\nabla_\perp \alpha(\mathbf{x}_\perp) = \mathbf{a}^l_\perp(\mathbf{x}_\perp). \tag{6.7}$$

The gauge fixing transformation $u^{(2)}[\alpha]$ acts as a gauge transformation on the fermions and rotates the perpendicular field variables,

$$u^{(2)}[\alpha]\, \psi(\mathbf{x})\, u^{(2)\dagger}[\alpha] = e^{ig\alpha(\mathbf{x}_\perp)} \psi(\mathbf{x}), \tag{6.8}$$

$$u^{(2)}[\alpha]\, A_\perp(\mathbf{x})\, u^{(2)\dagger}[\alpha] = e^{ig\alpha(\mathbf{x}_\perp)} A_\perp(\mathbf{x})\, e^{-ig\alpha(\mathbf{x}_\perp)}. \tag{6.9}$$

As is familiar from the Coulomb-gauge representation of QED [16], this transformation shifts the chromoelectric field by a longitudinal static field $\boldsymbol{\eta}$

$$u^{(2)}[\alpha]\, \mathbf{\Pi}_\perp(\mathbf{x})\, u^{(2)\dagger}[\alpha] = e^{ig\alpha(\mathbf{x}_\perp)} \left(\mathbf{\Pi}_\perp(\mathbf{x}) - \frac{1}{L} \boldsymbol{\eta}(\mathbf{x}_\perp) \right) e^{-ig\alpha(\mathbf{x}_\perp)}. \tag{6.10}$$

This chromoelectric field $\boldsymbol{\eta}$ is generated by the neutral charge density of quarks and gluons,

$$\boldsymbol{\eta}(\mathbf{x}_\perp) = g\nabla_\perp \int d^2y\, d(\mathbf{x}_\perp - \mathbf{y}_\perp)\, \rho^{(2)}(\mathbf{y}_\perp), \tag{6.11}$$

and satisfies the Poisson equation,

$$\operatorname{div}^{(2)} \boldsymbol{\eta}^{c0}(\mathbf{x}_\perp) = g\left(\rho^{(2)\,c0}(\mathbf{x}_\perp) - \frac{1}{L^2} Q^{c0}\right), \tag{6.12}$$

with the neutral components of the total charge given by

$$Q^{c0} = \int d^2x\, \rho^{(2)\,c0}(\mathbf{x}_\perp). \tag{6.13}$$

In this representation, with the transformed physical states

$$|\Phi'\rangle = u^{(2)}[\alpha]\, |\tilde{\Phi}\rangle = U[\xi, \alpha]\, |\Phi\rangle, \tag{6.14}$$

generated by applying both transformations,

$$U[\xi, \alpha] = u^{(2)}[\alpha]\, U[\xi], \tag{6.15}$$

the transformed residual Gauss law (6.4),

$$\left(\operatorname{div}^{(2)} \mathbf{p}_\perp^{c0}(\mathbf{x}_\perp) + \frac{g}{L^2} Q^{c0}\right) |\Phi'\rangle = 0, \tag{6.16}$$

is easily implemented. Periodicity of the conjugate momenta $\mathbf{p}_\perp^{c0}(\mathbf{x}_\perp)$ in $\mathbf{x}_\perp$ implies the neutrality condition

$$Q^{c0}\, |\Phi'\rangle = 0, \tag{6.17}$$

i.e., the vanishing of the neutral components of the total color-charge. Furthermore, in the sector of physical states, the longitudinal neutral conjugate momenta vanish,

$$\mathbf{p}_\perp^{l}(\mathbf{x}_\perp)\, |\Phi'\rangle = 0. \tag{6.18}$$

It is now straightforward to transform the Weyl-gauge Hamiltonian (Eq. (2.9), cf. Eq. (5.26)) with $U[\xi, \alpha]$. We introduce primed (perpendicular) gauge fields by subtracting the longitudinal neutral components

$$\mathbf{A}'_\perp(\mathbf{x}) = \mathbf{A}_\perp(\mathbf{x}) - \mathbf{a}^l_\perp(\mathbf{x}_\perp). \tag{6.19}$$

Thus the fields $\mathbf{A}'_\perp$ describe only physical degrees of freedom, i.e., degrees of freedom which are unconstrained. Their conjugate momenta are

$$\boldsymbol{\Pi}'_\perp(\mathbf{x}) = \boldsymbol{\Pi}_\perp(\mathbf{x}) - \frac{1}{L} \mathbf{p}^l_\perp(\mathbf{x}_\perp). \tag{6.20}$$

Obviously the subtraction of the longitudinal neutral components in Eqs. (6.19) and (6.20) gives rise to a change in the canonical commutation relations. In terms of these variables, the Hamiltonian in the space of transformed physical states reads

$$H' = U[\xi, \alpha]\, H U^\dagger[\xi, \alpha] = \int d^3x\, \mathscr{H}'(\mathbf{x}), \tag{6.21}$$

where

$$\mathscr{H}' = -i\psi^\dagger[\alpha_3(\partial_3 - iga_3) + \boldsymbol{\alpha}_\perp(\nabla_\perp - ig\mathbf{A}'_\perp)]\,\psi + m\psi^\dagger\beta\psi$$
$$+ \mathrm{tr}\left[(\boldsymbol{\Pi}'_\perp)^2 + \left(\frac{1}{L}\boldsymbol{\eta}\right)^2 + \sum_{i<j}(F'_{ij})^2\right] + \frac{1}{2L^2}\sum_{c_0} p_3^{c_0\dagger}(\mathbf{x}_\perp)\, p_3^{c_0}(\mathbf{x}_\perp)$$
$$+ \frac{1}{L^2}\int_0^L dz_3 \int_0^L dy_3 \sum_{p,q,n}{}' \frac{G'_{\perp qp}(\mathbf{x}_\perp, z_3)\, G'_{\perp pq}(\mathbf{x}_\perp, y_3)}{[2\pi n/L + g(a_{3,q}(\mathbf{x}_\perp) - a_{3,p}(\mathbf{x}_\perp))]^2} e^{i2\pi n(z_3 - y_3)/L}. \tag{6.22}$$

The primed chromomagnetic fields are defined as

$$F'_{12} = \partial_1 A'_2 - \partial_2 A'_1 - ig[A'_1, A'_2],$$
$$F'_{i3} = \partial_i a_3 - \partial_3 A'_i - ig[A'_i, a_3] \qquad (i = 1, 2), \tag{6.23}$$

and the summation in Eq. (6.22) is again restricted to $n \neq 0$ if $p = q$.

The various contributions to this Hamiltonian density are easily related to the corresponding ones in the original Weyl-gauge Hamiltonian density of Eq. (2.9). As a result of the first gauge-fixing transformation, the gauge fields A_3 have disappeared from both coupling to the quarks (cf. Eq. (5.10)) and from the field strength tensor (cf. Eq. (5.11)). Complete elimination of A_3 is not possible; the neutral, two-dimensional fields a_3, together with their conjugate momenta p_3, remain. Similarly, as a result of the second gauge-fixing transformation, the neutral, longitudinal gauge field components $\mathbf{a}'_\perp$ have been eliminated and their conjugate momenta $\mathbf{p}'_\perp$ have vanishing eigenvalues in the space of physical states (cf. Eq. (6.18)). Consequently, both the gauge field coupling to the quarks as well as the field strength tensor are given in terms of the primed perpendicular field components (cf. Eqs. (6.19), (6.20)). In both steps of the implementation of the Gauss law "static" chromoelectric fields are generated. Their field energies are given by the $\boldsymbol{\eta}^2$ contribution and the last term in Eq. (6.22), respectively. The latter is determined by the matrix elements of $G'_\perp$,

$$G'_\perp(\mathbf{x}) = \nabla_\perp \boldsymbol{\Pi}'_\perp(\mathbf{x}) + gf^{abc}\frac{\lambda^a}{2}\mathbf{A}'^b_\perp(\mathbf{x})\left(\boldsymbol{\Pi}'^c_\perp(\mathbf{x}) - \frac{1}{L}\boldsymbol{\eta}^c(\mathbf{x}_\perp)\right) + g\rho_m(\mathbf{x}). \tag{6.24}$$

Due to the subtraction in Eq. (6.20) of the corresponding component, there are no interference terms between the longitudinal, neutral field $\boldsymbol{\eta}(\mathbf{x}_\perp)$ and $\boldsymbol{\Pi}'_\perp(\mathbf{x})$. We also note that, for a similar reason, with $\rho^{(2)}$ (Eq. (6.5)) also $\boldsymbol{\eta}$ (Eq. (6.11)) depends apart from the quark degrees of freedom only on the primed-field variables. The chromoelectric field energy of the last term arises from the first step of implementation of Gauss's law (cf. Eq. (5.26)) and becomes further modified by the second unitary transformation. Also here, this transformation eliminates $\mathbf{a}'_\perp(\mathbf{x}_\perp)$ and correspondingly shifts the perpendicular chromoelectric field by the static field $\boldsymbol{\eta}(\mathbf{x}_\perp)$. The y_3, z_3 integration in the last term of $\mathscr{H}'$ eliminates the neutral zero-

mode contribution $\mathrm{div}^{(2)}\,\eta$ obtained in the second gauge-fixing transformation of $G_\perp$ (Eq. (3.1)).

With this result, we have completed the procedure to formulate the dynamics exclusively in terms of unconstrained variables. These variables are: The gauge fields $\mathbf{A}'_\perp(\mathbf{x})$ with the polarization in the (1, 2)-plane and accounting correctly by the subtraction in Eq. (6.19) for the missing second polarization state, if the gluon momentum is in the (1, 2)-plane itself. In addition, the theory still contains color-neutral gluons with polarization along the three-direction, however, only for vanishing momentum in the direction of the polarization. These $N-1$ two-dimensional gluons are described by the fields $a_3(\mathbf{x}_\perp)$. The corresponding conjugate momenta are $\Pi'_\perp(\mathbf{x})$ and $p_3(\mathbf{x}_\perp)$.

7. Jacobian and Boundary Conditions

The final Hamiltonian (6.22) is, in the space of physical states, formulated exclusively in terms of unconstrained variables. Most of these variables are simply related to the original fields appearing in the Weyl-gauge Hamiltonian, Eq. (2.9), before resolving Gauss's law: The fermion fields ψ are identical, the perpendicular components $\mathbf{A}'_\perp$ of the gauge fields differ only by a (kinematical) projection which eliminates certain two-dimensional, longitudinal components. The only exception is the field a_3, the two-dimensional remnant of A_3. Here, the resolution of Gauss's law has forced upon us a complicated non-linear functional of the original field as an unconstrained variable. If one would derive the same Hamiltonian via a change of variables in the Schrödinger representation, one would expect a Jacobian with a non-trivial a_3-dependence and, consequently, a non-standard form for the corresponding kinetic energy. In our formulation via unitary transformations, the question of the Jacobian did not yet arise in deriving the Hamiltonian. Nevertheless, the kinetic energy associated with the a_3 degrees of freedom is non-standard and in fact allows us to "reconstruct" the Jacobian.

The relevant term in the final Hamiltonian H' is

$$\delta H = \frac{1}{2L}\int d^2x \sum_{c_0} p_3^{c_0\dagger}(\mathbf{x}_\perp)\, p_3^{c_0}(\mathbf{x}_\perp). \tag{7.1}$$

As we have shown in Eq. (5.23), $p_3^{c_0}$ acts "to the right" as momentum conjugate to $a_3^{c_0}$:

$$p_3^{c_0}(\mathbf{x}_\perp) = \frac{1}{i}\frac{\delta}{\delta a_3^{c_0}(\mathbf{x}_\perp)}. \tag{7.2}$$

Since $p_3^{c_0}(\mathbf{x}_\perp)$ is a component of $\Pi_3(\mathbf{x})$ with respect to a dynamical basis, this operator is not hermitian. Its hermitian conjugate $p_3^{c_0\dagger}$ can be calculated in a straightforward way. However, since the calculation is technically rather involved,

we refer to Appendix A for the details and quote here only the final result, cf. Eq. (A.17),

$$p_3^{c0\dagger}(\mathbf{x}_\perp) - p_3^{c0}(\mathbf{x}_\perp) = \frac{1}{i}\frac{\delta}{\delta a_3^{c0}(\mathbf{x}_\perp)} \sum_{c(\mu_c \neq 0)} \ln \sin\left(\frac{\mu_c(\mathbf{x}_\perp) L}{2}\right). \tag{7.3}$$

This result has already been brought into a form which is suggestive of a non-trivial Jacobian. Indeed, in quantum mechanics, one can establish quite generally the connection between non-hermitian momentum operators and Jacobians. This is explained in detail in Appendix B, where the generalization to field theory is also briefly indicated. If the Jacobian "factorizes" as function of $\mathbf{x}$, i.e.,

$$\mathscr{J} = \prod_{\mathbf{x}_\perp} J(\mathbf{x}_\perp) = \exp\left(\delta^{(2)}(\mathbf{0}) \int d^2x \ln J(\mathbf{x}_\perp)\right), \tag{7.4}$$

one expects the "hermiticity defect,"

$$p_3^{c0\dagger}(\mathbf{x}_\perp) - p_3^{c0}(\mathbf{x}_\perp) = \frac{1}{i}\frac{\delta}{\delta a_3^{c0}(\mathbf{x}_\perp)} \ln J(\mathbf{x}_\perp). \tag{7.5}$$

(As our notation indicates, proper definition of the Jacobian requires a discretization of the $\mathbf{x}_\perp$ continuum; if l denotes the meshsize, $\delta^{(2)}(\mathbf{0})$ in Eq. (7.4) is then replaced by $1/l^2$.) A comparison of Eq. (7.3) with Eq. (7.5) allows us to simply read off the Jacobian. In the present case, it is also possible to derive the Jacobian by explicit evaluation of a functional determinant. This alternative calculation is presented in Appendix C and yields a result which agrees with the above indirect method. The final outcome of these various calculations has a very simple interpretation: The Jacobian is the invariant group measure for $SU(N)$ which, expressed in terms of our variables, reads

$$\mathscr{J}[a_3] = \exp\left(\delta^{(2)}(\mathbf{0}) \int d^2x \ln J(a_3(\mathbf{x}_\perp))\right), \tag{7.6}$$

$$J(a_3(\mathbf{x}_\perp)) = \prod_{k>l} \sin^2\left(\tfrac{1}{2} g L(a_{3,l}(\mathbf{x}_\perp) - a_{3,k}(\mathbf{x}_\perp))\right). \tag{7.7}$$

In order to represent the kinetic energy via standard second-order functional derivatives, we introduce a radial-like wave functional,

$$\Psi[a_3] = (\mathscr{J}[a_3])^{-1/2}\, \tilde{\Psi}[a_3], \tag{7.8}$$

and obtain

$$\delta H \tilde{\Psi}[a_3] = (\mathscr{J}[a_3])^{-1/2} \left\{\frac{1}{2L} \int d^2x \sum_{c0}\left(-\frac{\delta^2}{\delta a_3^{c0}(\mathbf{x}_\perp)^2}\right) + V_{\text{eff}}\right\} \tilde{\Psi}[a_3], \tag{7.9}$$

with the "effective potential," cf. Appendix B,

$$V_{\text{eff}} = \frac{1}{2L} \int d^2x \, \frac{1}{\sqrt{J(\mathbf{x}_\perp)}} \left(\frac{\delta^2}{\delta a_3^{c0}(\mathbf{x}_\perp)^2} \sqrt{J(\mathbf{x}_\perp)} \right). \tag{7.10}$$

The effective potential is evaluated in Appendix D and found to be simply a constant for $SU(N)$, cf. Eq. (D.17),

$$V_{\text{eff}} = -L^3 [\delta^{(2)}(\mathbf{0})]^2 \frac{g^2}{48} N(N^2 - 1). \tag{7.11}$$

This constant term in the Hamiltonian can be dropped. Formally, the Hamiltonian of the a_3 degrees of freedom is not complicated by the Jacobian, and by use of Eq. (7.8) one is able to work with the standard hermitian momentum operators. Nevertheless the Jacobian affects the dynamics of these degrees of freedom, since the "radial" wave functionals must vanish at the zeroes of the Jacobian,

$$\tilde{\Psi}[a_3] = 0 \qquad \text{if} \quad \mathcal{J}[a_3] = 0. \tag{7.12}$$

In the last step of our description of the dynamics of QCD in the axial gauge representation, the role of the boundary conditions to be imposed on the unconstrained variables has to be investigated. This discussion will provide insight into crucial elements of the quantum mechanical gauge-fixing procedure applied here. Starting point of this discussion are the boundary conditions (2.3) and (2.4) for gauge and matter field operators acting in the large Hilbert space of Weyl-gauge QCD. Before proceeding we remark that in a lattice version of this theory, the conditions (2.3), (2.4) simply express the identity of the degrees of freedom labelled by $\mathbf{x}$ and $\mathbf{x} + L\mathbf{e}_i$ and are obviously necessary for the (discrete) translational invariance of the theory. Our formulation of the axial gauge representation of QCD is not manifestly translationally invariant (discrete or continuous) since the gauge fixing unitary transformations introduce coordinate dependences and restrictions of these coordinates to a certain interval (cf. Eq. (4.18)). No particular attention to this problem has to be paid in the lattice formulation, while the normal mode expansion of a continuum formulation requires a careful treatment of the involved changes in the boundary conditions. With the choice of Eq. (2.3), the operators $\mathbf{A}(\mathbf{x}_\perp, x_3 = 0^+)$ and $\mathbf{A}(\mathbf{x}_\perp, x_3 = L^-)$ are associated with neighbouring degrees of freedom, and a similar statement holds for operators differing correspondingly in the perpendicular coordinates. We therefore require (even in the absence of translational invariance) matrix elements of these operators between physical states $|\Phi\rangle$, $|\Psi\rangle$ to be identical,

$$\langle \Phi | \, \mathbf{A}(\mathbf{x}^{(i)}) \, | \Psi \rangle = \langle \Phi | \, \mathbf{A}(\mathbf{x}^{(i)} + L\mathbf{e}_i) \, | \Psi \rangle, \tag{7.13}$$

where $\mathbf{x}^{(i)}$ denotes from now on a point on the surface of the box with vanishing ith component. Similar periodicity requirements hold for matrix elements of matter

field operators. Performing the gauge fixing transformations (6.15), the matrix elements of $\mathbf{A}_\perp$ between transformed states (6.14) satisfy

$$\langle \Phi' | e^{ig\xi(\mathbf{x}^{(i)})} \left[e^{ig\alpha(\mathbf{x}_\perp^{(i)})} \mathbf{A}_\perp(\mathbf{x}^{(i)}) e^{-ig\alpha(\mathbf{x}_\perp^{(i)})} + \frac{i}{g} \nabla_\perp \right] e^{-ig\xi(\mathbf{x}^{(i)})} | \Psi' \rangle$$

$$= \langle \Phi' | e^{ig\xi(\mathbf{x}^{(i)} + L\mathbf{e}_i)} \left[e^{ig\alpha(\mathbf{x}_\perp^{(i)} + L(\mathbf{e}_i)_\perp)} \mathbf{A}_\perp(\mathbf{x}^{(i)} + L\mathbf{e}_i) \right.$$

$$\left. \times e^{-ig\alpha(\mathbf{x}_\perp^{(i)} + L(\mathbf{e}_i)_\perp)} + \frac{i}{g} \nabla_\perp \right] e^{-ig\xi(\mathbf{x}^{(i)} + L\mathbf{e}_i)} | \Psi' \rangle. \tag{7.14}$$

This identity suggests imposing the boundary conditions

$$\mathbf{A}_\perp(\mathbf{x}^{(i)} + L\mathbf{e}_i) = u^{(i)} \left(\mathbf{A}_\perp(\mathbf{x}^{(i)}) + \frac{i}{g} \nabla_\perp \right) u^{(i)\dagger}, \tag{7.15}$$

with

$$u^{(i)} := \begin{cases} \exp\{ig(a_3(\mathbf{x}_\perp^{(i)} + L(\mathbf{e}_i)_\perp) - a_3(\mathbf{x}_\perp^{(i)}))\, x_3\} & \text{for } i = 1, 2, \\ 1 & \text{for } i = 3, \end{cases} \tag{7.16}$$

for the definition of the normal mode expansion of $\mathbf{A}_\perp$ acting in the space of transformed physical states. In this way, matrix elements of each term of such a normal mode expansion satisfy the continuity requirement (7.13). In principle, expansion in terms of standard periodic basis functions is still possible; in this case, however, we expect non-uniform convergence for operators defined on the surface of the box. Similarly it can be seen that the fermionic boundary condition (2.4) for $\psi(\mathbf{x})$ is transformed into

$$\psi(\mathbf{x}^{(i)} + L\mathbf{e}_i) = e^{i\varphi_i} u^{(i)} \psi(\mathbf{x}^{(i)}). \tag{7.17}$$

The change in boundary conditions (7.15)–(7.17) induced here by the transformation of the physical states would be obtained in more standard (or classical) treatments as a consequence of changing variables. The definition of $u^{(i)}$ implies that gauge and matter field operators remain after transformation periodic in the three-direction. This is a consequence of the periodicity in three-direction built into the unitary gauge fixing transformation (6.15). The boundary conditions in the perpendicular directions depend on the precise definition of $a_3(\mathbf{x}_\perp)$. As emphasized above, so far only the exponential of these fields is defined by Eqs. (4.4), (4.5); like the original gauge field A_3, this exponential is periodic. A general property of the variables a_3 is, therefore,

$$a_3(\mathbf{x}_\perp^{(i)} + L(\mathbf{e}_i)_\perp) - a_3(\mathbf{x}_\perp^{(i)}) = n_3^i(\mathbf{x}_\perp^{(i)}) \frac{2\pi}{gL}, \qquad i = 1, 2, \tag{7.18}$$

with n_3^i denoting traceless, diagonal matrices with integer entries, and consequently

$$u^{(i)} = \exp\left\{2i\pi \frac{x_3}{L} n_3^i(\mathbf{x}_\perp^{(i)})\right\}, \qquad i = 1, 2. \tag{7.19}$$

To incorporate the boundary conditions into a dynamical calculation, two different approaches are suggested by this particular form of $u^{(i)}$, treating the dynamical variables $a_3(\mathbf{x}_\perp)$ either as non-compact or as angular variables. In the first case, one has to perform the dynamical calculation with prescribed values for $n_3^i(\mathbf{x}_\perp^{(i)})$ and study the dependence of the results on these values. This method is expected to be particularly appropriate for QED, where the continuum limit can be described by non-compact variables $a_3(\mathbf{x}_\perp)$ [16] which are manifestly periodic in the one- and two-directions (i.e., $n_3^i(\mathbf{x}_\perp^{(i)}) = 0$ in our present notation). On the other hand, one may choose a_3 as angular variables with (for simplicity $\mathbf{x}_\perp$ and color independent) intervals

$$\varphi_0 \leqslant gLa_{3,i}(\mathbf{x}_\perp) < \varphi_0 + 2\pi, \qquad i = 1, \ldots, N-1. \tag{7.20}$$

(Here and below, we consider the first $N-1$ diagonal matrix elements of a_3 as independent variables, whereas $a_{3,N}$ will be determined by the condition $\operatorname{tr} a_3 = 0$.) This yields $u^{(i)} = 1$ and therefore periodic (or quasi-periodic) boundary conditions also in the perpendicular directions. In this case, boundary conditions on the wave functional of these angular variables have to be specified. To this end one proceeds as for a quantum mechanical system of particles on a circle. Since the values of a_3 corresponding to the endpoints of the interval in Eq. (7.20) describe identical states, the wave functional at these values must be phase-invariant (fixed color label i and position $\mathbf{x}_\perp^0$):

$$\langle gLa_{3,i}(\mathbf{x}_\perp^0) = (\varphi_0 + \varepsilon) \mid \Phi\rangle = e^{i\Theta^i(\mathbf{x}_\perp^0)} \langle gLa_{3,i}(\mathbf{x}_\perp^0) = (\varphi_0 + 2\pi - \varepsilon) \mid \Phi\rangle, \qquad \varepsilon \to 0. \tag{7.21}$$

(We use the Dirac bra–ket notation for wave functionals in the Schrödinger representation; as indicated, here the scalar product is taken with respect to one variable only.) Introducing the transformed states $|\Phi'\rangle$ (cf. Eq. (6.14)), the gauge-fixing transformations are evaluated at values of $gLa_{3,i}(\mathbf{x}_\perp^0)$ which differ by 2π. Combining these two different gauge-fixing transformations and projecting the result onto the other dynamical variables yields the following result for the wave functional:

$$\langle a_3', (\mathbf{A}_\perp')', \psi' \mid \Phi'\rangle = e^{i\Theta^i(\mathbf{x}_\perp^0)} \langle a_3, A_\perp', \psi \mid \Phi'\rangle. \tag{7.22}$$

Here, it is assumed that the ith matrix element of $a_3(\mathbf{x}_\perp)$, $1 \leqslant i \leqslant N-1$, takes on the value φ_0 at the point $\mathbf{x}_\perp = \mathbf{x}_\perp^0$, whereas the other matrix elements are arbitrary (within the defining interval). The primed and unprimed variables appearing in the wave functionals, Eq. (7.22), are then related by

$$a_3'(\mathbf{x}_\perp) = v(\mathbf{x})\left(a_3(\mathbf{x}_\perp) + \frac{i}{g}\partial_3\right)v^\dagger(\mathbf{x}),$$

$$(\mathbf{A}_\perp'(\mathbf{x}))' = v(\mathbf{x})\left(\mathbf{A}_\perp'(\mathbf{x}) + \frac{i}{g}\nabla_\perp\right)v^\dagger(\mathbf{x}), \tag{7.23}$$

$$\psi'(\mathbf{x}) = v(\mathbf{x})\,\psi(\mathbf{x}).$$

We have introduced the diagonal matrices

$$v(\mathbf{x})_{kl} = \delta_{kl}\begin{cases}\delta_{ki}\exp\{2i\pi(x_3/L - 1/2)\} + \delta_{kN}\exp\{-2i\pi(x_3/L - 1/2)\} & \text{if } \mathbf{x}_\perp = \mathbf{x}_\perp^0,\\ 1 & \text{if } \mathbf{x}_\perp \neq \mathbf{x}_\perp^0,\end{cases} \tag{7.24}$$

which differ from the unit matrix only for those values of coordinates and color components for which the changes in a_3 occur. As our notation indicates, proper definition of these quantities requires a discretized formulation of the theory. The interpretation of this result is straightforward. Primed and unprimed dynamical variables are related by (in the continuum limit discontinuous) discrete gauge transformations; therefore, the wave functionals for these two sets of values can only differ by a phase, the eigenvalue of the corresponding local, discrete gauge transformation. The $N-1$ independent phases $\Theta^i(\mathbf{x}_\perp)$ can be arbitrarily prescribed at each $\mathbf{x}_\perp$. The relevance of this freedom of choice is not clear to us, in particular, in view of the discontinuities involved with the continuum limit. One therefore might be tempted to assume physical states to be strictly invariant (up to the global component to be discussed in Section 8) under the discontinuous gauge transformations (7.23), (7.24). On the other hand, such discontinuities may be used to incorporate non-trivial topological properties into the formalism, as is suggested by the example of compact QED (cf. [23, 24]).

A priori it is difficult to decide which of the two treatments of boundary conditions is more efficient. As emphasized above, the non-compact version might be most convenient for the continuum limit, provided n_3^i can be chosen as in QED to be $\mathbf{x}_\perp$-independent. In the context of $(1+1)$-dimensional QED (the Schwinger model), the connection between compact and non-compact formulations has been explicitly established [25]. It is remarkable that this possibility of an explicit continuum formulation is much less obvious in the non-abelian case. Here the need for a discretized lattice formulation in the perpendicular coordinates arises independently, as a consequence of a proper definition of the Jacobian associated with the two-dimensional degrees of freedom. It thus appears that even on a symbolic formal level, the construction of a Hamiltonian for QCD with unconstrained degrees of freedom is possible only after some discretization.

The above discussion suggests more general options for proceeding in the gauge fixing by not accounting explicitly for the periodicity of certain variables, provided dynamical boundary conditions are admitted. Such representations might be particularly useful in cases where the dynamics of the boundary conditions can be described by slow variables in the context of a Born–Oppenheimer approximation.

Within the class of axial gauge like representations, one might in particular consider gauge fixing transformations which are not periodic in x_3. For instance, omitting the second of the three unitary transformations in the definition of $\tilde{U}$ of Eq. (4.3), one arrives at a representation in which the fields a_3 appear only in the form e^{igLa_3}. In this case, the Hamiltonian acquires its dependence on these variables only through the resolved Gauss law; additional dependences on these two-dimensional fields appear in the boundary conditions relating the other degrees of freedom in the three-direction. Standard periodic boundary conditions for the perpendicular directions are obtained. This alternative gauge fixing procedure is outlined in Appendix E.

8. Residual Gauge Symmetry

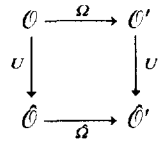
The process of elimination of redundant variables from the Hamiltonian in the physical sector of the Hilbert space has been possible only due to the local gauge invariance of the original Weyl-gauge Hamiltonian (Eq. (2.16)). The dynamics formulated in terms of unconstrained variables in turn does not exhibit anymore this local gauge invariance. However, global residual gauge symmetries might still be present. The possibility for such global gauge symmetries arises since the Gauss law operator defining the physical states and the operator generating the local gauge symmetry transformations are not identical. As a consequence, the local gauge transformations performed in the large Hilbert space do not all reduce necessarily to the identity in the space of physical states. The residual global gauge symmetries of QED have been explicitly constructed [16] and have been shown to be useful in characterizing the different possible phases of matter coupled to radiation. Similarly, discussion of the residual global gauge symmetries of QCD might be expected to be helpful in characterizing the confined phase.

Ideally one would like to calculate the local gauge symmetry transformations $\hat{\Omega}$ in the representation defined by the gauge fixing transformation $U[\xi, \alpha]$ of Eq. (6.15),

$$\hat{\Omega}[\beta] = U[\xi, \alpha]\, \Omega[\beta]\, U^{\dagger}[\xi, \alpha]. \tag{8.1}$$

Here, $\Omega[\beta]$ is the unitary gauge transformation before "gauge fixing" (cf. Eq. (2.17)) and is determined by the classical (c-number) function $\beta(\mathbf{x})$. In QED the reduction of the local gauge symmetry in the large Hilbert space to the global residual symmetries of the physical sector can be followed in detail. In QCD, in the absence of a fully explicit representation of the unitary gauge fixing transformation $U[\xi, \alpha]$ this is not possible. However, we are still able to calculate the effect of the residual gauge transformations on the operators appearing in the Hamiltonian of the physical sector. The procedure of our calculations indicated in the following diagram makes use of the two different ways by which gauge transformed operators

$\hat{\mathcal{O}}'$ in the axial gauge representation can be computed starting with the original operator $\mathcal{O}$.



In the course of this investigation, the general structure of the transformations $\hat{\Omega}$ will be determined; however, in contrast to QED, we shall not be able to relate explicitly the parameters characterizing the action of $\hat{\Omega}$ in the space of physical states with the gauge functions determining Ω. Before proceeding in this calculation, we have to discuss the constraints imposed on the gauge function $\beta(\mathbf{x})$ by the boundary condition on gauge and fermion fields, respectively. These boundary conditions restrict the freedom of locally choosing the gauge. The transformed gauge fields must satisfy the same boundary condition as the original ones (cf. Eq. (2.3)),

$$e^{ig\beta(\mathbf{x}+L\mathbf{e}_i)}\left(\mathbf{A}(\mathbf{x}+L\mathbf{e}_i)+\frac{i}{g}\nabla\right)e^{-ig\beta(\mathbf{x}+L\mathbf{e}_i)}=e^{ig\beta(\mathbf{x})}\left(\mathbf{A}(\mathbf{x})+\frac{i}{g}\nabla\right)e^{-ig\beta(\mathbf{x})}. \tag{8.2}$$

Introducing

$$\Upsilon(\mathbf{x})=e^{-ig\beta(\mathbf{x}+L\mathbf{e}_i)}e^{ig\beta(\mathbf{x})}, \tag{8.3}$$

condition (8.2) reads

$$\nabla\Upsilon(\mathbf{x})-ig[\mathbf{A}(\mathbf{x}),\,\Upsilon(\mathbf{x})]=0. \tag{8.4}$$

Since $\Upsilon(\mathbf{x})$ is not a functional of $\mathbf{A}(\mathbf{x})$ we find

$$[\mathbf{A}(\mathbf{x}),\,\Upsilon(\mathbf{x})]=0 \qquad \text{for all} \quad \mathbf{A}(\mathbf{x}). \tag{8.5}$$

Therefore, $\Upsilon(\mathbf{x})$ must be an element of the center of the group which, furthermore, must be $\mathbf{x}$ independent:

$$\Upsilon(\mathbf{x})=e^{i\Phi^n}, \qquad \Upsilon(\mathbf{x})\in Z, \qquad e^{i\Phi^n}=e^{2i\pi n/N}, \qquad n=0,\,1,\,\ldots,\,N-1. \tag{8.6}$$

Here, Φ^n is an element of the Lie-algebra. The transformation of the fermionic boundary condition (2.4) further restricts the choice of the gauge functions,

$$\Upsilon(\mathbf{x})=\mathbf{1}. \tag{8.7}$$

We will keep the more general option (8.6) for the gauge function $\beta(\mathbf{x})$ which permits discussion of purely gluonic systems. Thus, in summary, the choice of the gauge function is restricted by the periodicity requirement:

$$e^{ig\beta(\mathbf{x}+L\mathbf{e}_i)}=e^{-i\Phi^n}e^{ig\beta(\mathbf{x})}. \tag{8.8}$$

Fundamental to the following analysis is the well-known transformation property of the path-ordered integral (defined in Eq. (4.1)) under gauge transformations. Using the fact that the gauge fixing transformation $U[\xi, \alpha]$ leaves the gauge field $A_3(\mathbf{x})$ unchanged, we have

$$\Omega[\beta]\, e^{ig\tau(\mathbf{x})}\Omega^\dagger[\beta] = \hat{\Omega}[\beta]\, e^{ig\tau(\mathbf{x})}\hat{\Omega}^\dagger[\beta] = e^{ig\beta(\mathbf{x})}e^{ig\tau(\mathbf{x})}e^{-ig\beta(\mathbf{x}_\perp)}, \tag{8.9}$$

with

$$\beta(\mathbf{x}_\perp) := \beta(\mathbf{x}_\perp, x_3 = 0), \tag{8.10}$$

as can be easily derived by using the differential equation (4.2). A special case of this result is (cf. Eqs. (4.4), (8.8))

$$\hat{\Omega}[\beta]\, e^{ig\theta(\mathbf{x}_\perp)}\hat{\Omega}^\dagger[\beta] = e^{-i\Phi^n}e^{ig\beta(\mathbf{x}_\perp)}e^{ig\theta(\mathbf{x}_\perp)}e^{-ig\beta(\mathbf{x}_\perp)}. \tag{8.11}$$

The variable $a_3(\mathbf{x}_\perp)$ defined in Eq. (4.5) transforms as

$$\hat{\Omega}[\beta]\, e^{iga_3(\mathbf{x}_\perp)L}\hat{\Omega}^\dagger[\beta] = e^{-i\Phi^n}u(\mathbf{x}_\perp)\, e^{iga_3(\mathbf{x}_\perp)L}u^\dagger(\mathbf{x}_\perp), \tag{8.12}$$

with

$$u(\mathbf{x}_\perp) = e^{ig\beta(\mathbf{x}_\perp)}e^{-ig\hat{A}}e^{igA}, \tag{8.13}$$

and

$$\hat{\Omega}[\beta]\, e^{igA}\hat{\Omega}^\dagger[\beta] = e^{ig\beta(\mathbf{x}_\perp)}e^{ig\hat{A}}e^{-ig\beta(\mathbf{x}_\perp)}. \tag{8.14}$$

Defining

$$\hat{\Omega}[\beta]\, a_3(\mathbf{x}_\perp)\, \hat{\Omega}^\dagger[\beta] = u(\mathbf{x}_\perp)\, a_3'(\mathbf{x}_\perp)\, u^\dagger(\mathbf{x}_\perp), \tag{8.15}$$

the relation above is written as

$$e^{iga_3'(\mathbf{x}_\perp)L} = e^{-i\Phi^n}e^{iga_3(\mathbf{x}_\perp)L}. \tag{8.16}$$

This equation implies that with $a_3(\mathbf{x}_\perp)$ also $a_3'(\mathbf{x}_\perp)$ is diagonal with the expansion coefficients,

$$a_3^{(\prime)}(\mathbf{x}_\perp) = \sum_{c_0=1}^{N-1} a_3^{(\prime)\,c_0}(\mathbf{x}_\perp)\frac{\lambda^{c_0}}{2}, \tag{8.17}$$

being related by

$$a_3'^{c_0}(\mathbf{x}_\perp) = a_3^{c_0}(\mathbf{x}_\perp) + \delta_3^{c_0}. \tag{8.18}$$

The symbol $\delta_3^{c_0}$ denotes the three-component of the vector

$$\boldsymbol{\delta}^{c_0} = \frac{1}{gL}\sum_{i=1}^{N}\lambda_{ii}^{c_0}(2\pi\mathbf{k}_i - \boldsymbol{\Phi}_i^n), \tag{8.19}$$

and the integers of the vectors $\mathbf{k}_i$ satisfy

$$\sum_{i=1}^{N} \mathbf{k}_i = 0. \tag{8.20}$$

Similarly, the vector

$$\mathbf{\Phi} = (\Phi^{n_1}, \Phi^{n_2}, \Phi^{n_3}) \tag{8.21}$$

denotes a set of three generators of center elements of the group (cf. Eq. (8.6)). We furthermore can read off from Eq. (8.15), together with Eq. (8.17), the general structure of the operator $u(\mathbf{x}_\perp)$. According to Eq. (8.15) u is a $SU(N)$ matrix which transforms the diagonal into diagonal $SU(N)$ matrices. These transformations form a group $\hat{G}$, and it is seen easily that each element of this group can be written as a product,

$$u(\mathbf{x}_\perp) = Re^{ig\omega^0(\mathbf{x}_\perp)}. \tag{8.22}$$

Here $\omega^0(\mathbf{x}_\perp)$ is an element of the Cartan subalgebra,

$$\omega^0(\mathbf{x}_\perp) = \sum_{c_0=1}^{N-1} \omega^{c_0}(\mathbf{x}_\perp) \frac{\lambda^{c_0}}{2}, \tag{8.23}$$

and R is a discrete transformation which generates a permutation of the basis vectors. In $SU(2)$, R is either $\mathbf{1}$ or $i\sigma_1$. In $SU(N)$, R is a matrix of the N-dimensional representation of the symmetric group S_N, with the matrix elements multiplied, if necessary, by $\exp(i\pi/N)$ to ensure the unit determinant. Thus in $SU(3)$ the transformations can be visualized as the symmetry transformations of an equilateral triangle with the three colors associated with the vertices. The transformations can also be characterized by the matrix elements $R^{c,d}$ defined by

$$R^\dagger \frac{\lambda^c}{2} R = R^{c,d} \frac{\lambda^d}{2}. \tag{8.24}$$

With these results for the transformation of $\tau(\mathbf{x})$, $a_3(\mathbf{x}_\perp)$, and $\Delta(\mathbf{x}_\perp)$, it is straightforward to calculate the transformation of $\xi(\mathbf{x})$ defined in Eqs. (5.9) and (4.3),

$$e^{ig\hat{\xi}(\mathbf{x})} = \hat{\Omega}[\beta]\, e^{ig\xi(\mathbf{x})} \hat{\Omega}^\dagger[\beta] = e^{ig\beta(\mathbf{x})} e^{ig\xi(\mathbf{x})} u^\dagger(\mathbf{x}), \tag{8.25}$$

with

$$u(\mathbf{x}) = u(\mathbf{x}_\perp) \exp\left\{ ig \sum_{c_0=1}^{N-1} \delta_3^{c_0} \frac{\lambda^{c_0}}{2} x_3 \right\}. \tag{8.26}$$

Next we transform the basis states $\zeta_{c_0,n}$ given in Eqs. (4.15), (4.16). Using Eqs. (4.14) and (4.17), we easily show that

$$\hat{\Omega}[\beta]\, z_{c_0} \hat{\Omega}^\dagger[\beta] = e^{ig\beta(\mathbf{x}_\perp)} R^{c_0,d_0} z_{d_0} e^{-ig\beta(\mathbf{x}_\perp)}, \tag{8.27}$$

which yields

$$\hat{\Omega}[\beta]\,\zeta_{c_0,n}(\mathbf{x})\,\hat{\Omega}^{\dagger}[\beta] = e^{ig\beta(\mathbf{x})}R^{c_0,d_0}\zeta_{d_0,n}(\mathbf{x})\,e^{-ig\beta(\mathbf{x})}. \tag{8.28}$$

Using this result and Eqs. (5.14), (5.24), finally the transformed momentum operator $p_3^{c_0}$ defined in Eq. (5.13) can be computed with the result that

$$\hat{\Omega}[\beta]\,p_3^{c_0}(\mathbf{x}_\perp)\,\hat{\Omega}^{\dagger}[\beta] = R^{c_0 d_0}p_3^{d_0}(\mathbf{x}_\perp). \tag{8.29}$$

The transformation properties of the two-dimensional gauge fields $a_3(\mathbf{x}_\perp)$ (Eqs. (8.15), (8.18)) and of the conjugate chromoelectric fields $p_3(\mathbf{x}_\perp)$ (Eq. (8.29)) are the main results of this first step of construction of the residual gauge symmetry. With regard to its effect on these degrees of freedom, an arbitrary local gauge transformation becomes converted in the process of gauge-fixing via unitary transformations into a gauge transformation of specific structure, consisting of a space independent transformation (R) and a spatially varying one generated by the Cartan subalgebra.

We now turn to the transformation of the other operators appearing in the Hamiltonian relevant for the physical sector. These operators are also affected by the two-dimensional Coulomb-gauge-fixing transformation (6.6). Using the transformation properties of $\mathbf{A}_\perp(\mathbf{x})$, Eqs. (2.21), (5.4), (6.9), and following the general scheme outlined above, we find that

$$\hat{\Omega}[\beta]\,e^{ig\alpha(\mathbf{x}_\perp)}\mathbf{A}_\perp(\mathbf{x})\,e^{-ig\alpha(\mathbf{x}_\perp)}\hat{\Omega}^{\dagger}[\beta] = u(\mathbf{x})\left(e^{ig\alpha(\mathbf{x}_\perp)}\mathbf{A}_\perp(\mathbf{x})\,e^{-ig\alpha(\mathbf{x}_\perp)} + \frac{i}{g}\nabla_\perp\right)u^{\dagger}(\mathbf{x}). \tag{8.30}$$

Since the gauge function $\alpha(\mathbf{x}_\perp)$ is neutral (cf. Eq. (6.7)), the transformation properties of the c_0 color components of $\mathbf{A}_\perp$ are particularly simple; the effect of the transformation arises exclusively from the x_3 independent parts of the gauge fields. According to the definition equation (6.1) of the gauge fields $\mathbf{a}_\perp$, we obtain

$$\hat{\Omega}[\beta]\,\mathbf{a}'_\perp(\mathbf{x}_\perp)\,\hat{\Omega}^{\dagger}[\beta] = R[\mathbf{a}'_\perp(\mathbf{x}_\perp) + \nabla_\perp(\omega^0(\mathbf{x}_\perp) - \omega_l^0(\mathbf{x}_\perp))]\,R^{\dagger}. \tag{8.31}$$

We have separated $\omega^0(\mathbf{x}_\perp)$ into a linear part $\omega_l^0(\mathbf{x}_\perp)$ and the rest. The subtraction of the linear part is necessary, since the longitudinal fields do not have a zero-mode (cf. the definition of the Greens function d in Eq. (6.3)). Thus, with Eq. (6.7), the transformation property of the gauge function α follows:

$$\hat{\alpha}(\mathbf{x}_\perp) = \hat{\Omega}[\beta]\,\alpha(\mathbf{x}_\perp)\,\hat{\Omega}^{\dagger}[\beta] = R[\alpha(\mathbf{x}_\perp) + \omega^0(\mathbf{x}_\perp) - \omega_l^0(\mathbf{x}_\perp)]\,R^{\dagger}. \tag{8.32}$$

We now can complete our calculation of the transformation of the transverse fields $\mathbf{A}'_\perp(\mathbf{x})$ defined in Eq. (6.19). As Eq. (8.30) shows, the effective gauge functions are

$$e^{-ig\hat{\alpha}(\mathbf{x}_\perp)}u(\mathbf{x})\,e^{ig\alpha(\mathbf{x}_\perp)} = Re^{ig\omega_l^0(\mathbf{x})}, \tag{8.33}$$

with (cf. Eqs. (8.26), (8.19))

$$\omega_I^0(\mathbf{x}) = \sum_{c_0=1}^{N-1} \delta_3^{c_0} \frac{\lambda^{c_0}}{2} x_3 + \omega_I^0(\mathbf{x}_\perp) = \mathbf{x} \cdot \boldsymbol{\delta}^{c_0} \frac{\lambda^{c_0}}{2}. \tag{8.34}$$

Incorporated into this parametrization of the linear function $\omega_I^0(\mathbf{x}_\perp)$ are the restrictions imposed by the boundary conditions on gauge and possibly fermion fields. In terms of this linear function with values in the Cartan subalgebra, the transformation of $\mathbf{A}'_\perp$ reads

$$\hat{\Omega}[\beta]\, \mathbf{A}'_\perp(\mathbf{x})\, \hat{\Omega}^\dagger[\beta] = R e^{ig\omega_I^0(\mathbf{x})} \left(\mathbf{A}'_\perp(\mathbf{x}) + \frac{i}{g} \nabla_\perp \right) e^{-ig\omega_I^0(\mathbf{x})} R^\dagger. \tag{8.35}$$

Following the same general scheme, the transformed fermion fields can be evaluated with the result:

$$\hat{\Omega}[\beta]\, \psi(\mathbf{x})\, \hat{\Omega}^\dagger[\beta] = R e^{ig\omega_I^0(\mathbf{x})} \psi(\mathbf{x}). \tag{8.36}$$

Finally, the conjugate momenta transform covariantly under these transformations generated by linear gauge functions,

$$\hat{\Omega}[\beta]\, \boldsymbol{\Pi}'_\perp(\mathbf{x})\, \hat{\Omega}^\dagger[\beta] = R e^{ig\omega_I^0(\mathbf{x})} \boldsymbol{\Pi}'_\perp(\mathbf{x})\, e^{-ig\omega_I^0(\mathbf{x})} R^\dagger. \tag{8.37}$$

These results complete the construction of the gauge transformed operators. In summary, the gauge transformations acting in Hilbert space on the unconstrained degrees of freedom as appearing in the Hamiltonian (6.15)–(6.22), i.e., the gauge fields $a_3(\mathbf{x}_\perp)$, $\mathbf{A}'_\perp(\mathbf{x})$ and their conjugate momenta $p_3(\mathbf{x}_\perp)$, $\boldsymbol{\Pi}'_\perp(\mathbf{x})$ reduce in the process of gauge-fixing on the one hand to discrete transformations R, describing permutations of the basis vectors of $SU(N)$ with associated unitary transformations in Hilbert space, Ω_R, and on the other hand to (for $L \to \infty$) continuous abelian transformations generated by the displacement vectors (defined in analogy with electrodynamics, cf. [19]),

$$\mathbf{D}^{c_0} = \int d^3x \left\{ -\frac{1}{L} p_3^{c_0}(\mathbf{x}_\perp)\, \mathbf{e}_3 - \boldsymbol{\Pi}'^{c_0}_\perp(\mathbf{x}) + g\mathbf{x}\rho^{c_0}(\mathbf{x}) \right\}, \qquad c_0 = 1, \ldots, N-1. \tag{8.38}$$

The Hamiltonian H' of Eq. (6.15) is invariant under both discrete transformations Ω_R and quasi continuous displacements,

$$\Omega_R H' \Omega_R^\dagger = H', \qquad e^{-i\mathbf{D}^{c_0}\delta^{c_0}} H' e^{i\mathbf{D}^{c_0}\delta^{c_0}} = H'. \tag{8.39}$$

In general it might not be practical to implement in an actual calculation from the very beginning the neutrality condition (Eq. (6.17)); in this case, the system, in addition, exhibits global gauge symmetries associated with the charges of the Cartan subalgebra

$$[Q^{c_0}, H'] = 0. \tag{8.40}$$

While this abelian global gauge symmetry commutes with the displacements,

$$[Q^{c_0}, \mathbf{D}^{d_0}] = 0, \tag{8.41}$$

the discrete transformations do not commute with the continuous ones,

$$\Omega_R \mathbf{D}^{c_0} \Omega_R^\dagger = R^{c_0 d_0} \mathbf{D}^{d_0}, \qquad \Omega_R Q^{c_0} \Omega_R^\dagger = R^{c_0 d_0} Q^{d_0}. \tag{8.42}$$

In color space, the group $\hat{G}$ of residual transformations consists of neutral displacements, neutral, global gauge transformations, and discrete transformations related to the permutations of the basis vector. Therefore, every element of $\hat{G}$ can be written as

$$u(\mathbf{x}) = R \exp\left\{ ig \frac{\lambda^{c_0}}{2} (\beta^{c_0} + \mathbf{x} \cdot \boldsymbol{\delta}^{c_0}) \right\}. \tag{8.43}$$

The neutral displacements and neutral global gauge transformations form an invariant subgroup H of $\hat{G}$; the factor group $\hat{G}/H$ is isomorphic to the symmetric group S_N.

In the process of gauge-fixing the original continuous non-abelian local gauge transformations are reduced to the simple transformations of the group $\hat{G}$. For this significant reduction in complexity to occur, the restriction of our computations to operators which are relevant for the dynamics in the physical sector was essential. In other words, the full complexity of the original local gauge transformations is carried only by the field operators $A_3(\mathbf{x}) - a_3(\mathbf{x}_\perp)$ and their conjugate momenta which, however, have no influence on the dynamics of physical states and which therefore did not have to be considered. The relation between the original gauge transformation (cf. the diagram above) $\Omega[\beta]$ and the residual gauge transformation appropriate for the physical sector $\hat{\Omega}[\beta]$ is not an explicit one. Although we have been able to determine the general structure of $\hat{\Omega}$, our computation did not yield the relation of the parameters characterizing the residual symmetry transformations (such as $\boldsymbol{\delta}^{c_0}$, cf. Eq. (8.19)) with properties of the original gauge functions $\beta(\mathbf{x})$.

The global residual gauge transformations derived in this section can be verified explicitly to be symmetry transformations of the gauge-fixed Hamiltonian of Eq. (6.22). In addition to such an explicit verification, our derivation shows that there are no other residues of the infinity of symmetries of the original Weyl-gauge Hamiltonian (2.9).

9. Summary of the Formalism and Interpretation

In the first part of this concluding discussion, we shall summarize our derivation of the axial gauge representation of QCD and emphasize the relevant structural elements of the method. In the quantum mechanical framework of the Weyl-gauge formulation of QCD ($A_0 = 0$), elimination of redundant variables is achieved by implementation of the Gauss law constraint. The first step consists in selecting among the degrees of freedom appearing in the Gauss law those which are supposed to be eliminated from the dynamics in the physical sector. Success of this

procedure is guaranteed by the local gauge invariance of the Hamiltonian. In $SU(N)$ gauge theory there are N^2-1 such redundant variables per space point which, by implementing Gauss's law, can be eliminated from the Hamiltonian. Thereby the degrees of freedom are reduced to two polarization states of the gluons per space point or momentum. Although to a large extent arbitrary, this choice of redundant variables may be restricted by certain topological constraints as, for instance, embodied in the boundary conditions. In the axial gauge representation, gauge fields of a fixed cartesian component, the N^2-1 three-components, are declared as redundant variables. This choice implies the decomposition of the Gauss law constraint

$$D_3 \Pi_3 |\Phi\rangle = -G_\perp |\Phi\rangle \tag{9.1}$$

into the contribution of the three-component of the chromoelectric field and the corresponding perpendicular components including the quark color charge density. Consequently, the operator $G_\perp$,

$$G_\perp = \mathbf{D}_\perp \mathbf{\Pi}_\perp + \rho_m, \tag{9.2}$$

appears as one of the fundamental building blocks in the construction of the axial gauge Hamiltonian. In the second step the action of Π_3 on the physical states has to be expressed by that of the operator $G_\perp$. To this end, the three-component of the covariant derivative in the adjoint representation

$$D_3 = \partial_3 - ig[A_3, \tag{9.3}$$

has to be inverted. Despite the dependence of D_3 on the gauge field component A_3, this inversion is possible and is facilitated after performing a unitary gauge-fixing transformation. As a result of this transformation, the N^2-1 gauge field variables A_3^a are eliminated from both the Hamiltonian and the covariant derivative in favor of $N-1$ two-dimensional gauge fields which can be represented as diagonal $SU(N)$ matrices,

$$A_3 \to \sum_{c_0=1}^{N-1} a_3^{c_0} \frac{\lambda^{c_0}}{2}. \tag{9.4}$$

The unitary transformation acts as a gauge transformation on the perpendicular and quark degrees of freedom, with gauge functions depending on the three-components of the gauge fields. It rests upon the decomposition (9.1) in which both terms $G_\perp$ and $D_3 \Pi_3$ satisfy the "angular momentum algebra" of the full Gauss law operator. The significant simplification of the covariant derivative

$$D_3 \to d_3 = \partial_3 - ig[a_3, \tag{9.5}$$

does not come as a surprise. As is well known, transformation of the variables by the standard path-ordered exponential $P\exp(ig\int dz A_3)$ effectively converts the covariant derivative D_3 into the ordinary derivative ∂_3. Here, this transformation is

not legitimate since it does not preserve the boundary conditions. However, this defect can be cured at the expense of keeping the two-dimensional, diagonal gauge fields a_3. Concomitantly, in the space of physical states, the three-component of the chromoelectric field operator can be replaced by

$$\Pi_3 \to \sum_{c0} p_3^{c0} \frac{\lambda^{c0}}{2} - \frac{1}{d_3} G_\perp . \tag{9.6}$$

Here, the $N-1$ two-dimensional chromoelectric field operators p_3^{c0} are the components of Π_3 along the eigenvectors of D_3 or d_3 with vanishing eigenvalue which, according to Eq. (9.1), are not constrained by Gauss law. As could be expected on the basis of a similar effect in electrodynamics [16], the purely transverse gluons propagating in the one-two plane with polarization in the three-direction cannot be eliminated; less obvious is the fact that after the choice of an appropriate basis only neutral gluons can propagate in the chosen one-two plane if they are polarized in the three-direction. Present together with these neutral two-dimensional gluons are, at this stage, redundant variables among the perpendicular field components. Physically, they correspond to neutral gluons propagating in the one-two plane with polarization parallel to the direction of propagation. Indeed the projection of Eq. (9.1) on the eigenvectors of D_3 with vanishing eigenvalue also displays the presence of residual, two-dimensional Gauss law constraints which remain to be implemented. Since these residual Gauss laws constrain only neutral degrees of freedom, standard gauge choices applied in the electrodynamics can be implemented. In particular, by a further unitary gauge fixing transformation, the neutral, two-dimensional, longitudinal gauge fields $\mathbf{a}'_\perp(\mathbf{x}_\perp)$ can be eliminated from the Hamiltonian; in the implementation of the residual Gauss law the conjugate chromoelectric field operators $\mathbf{p}'_\perp(\mathbf{x}_\perp)$ are replaced by chromoelectric static fields $\boldsymbol{\eta}(\mathbf{x}_\perp)$ which are generated by the two-dimensional neutral color charge density of gluons and quarks. With this last step, the Gauss law is implemented—up to an overall color neutrality constraint. The physical degrees of freedom $\mathbf{A}', \boldsymbol{\Pi}'$ of this axial gauge representation are therefore obtained from the constrained variables $\mathbf{A}, \boldsymbol{\Pi}$ by the replacements,

$$\begin{aligned} &A_3(\mathbf{x}) \to a_3(\mathbf{x}_\perp), \qquad && \mathbf{A}_\perp(\mathbf{x}) \to \mathbf{A}_\perp(\mathbf{x}) - \mathbf{a}'_\perp(\mathbf{x}_\perp) \\ &\Pi_3(\mathbf{x}) \to \frac{1}{L} p_3(\mathbf{x}_\perp), \qquad && \boldsymbol{\Pi}_\perp(\mathbf{x}) \to \boldsymbol{\Pi}_\perp(\mathbf{x}) - \frac{1}{L} \mathbf{p}'_\perp(\mathbf{x}_\perp). \end{aligned} \tag{9.7}$$

The Hamiltonian for these unconstrained variables is the original Weyl-gauge Hamiltonian evaluated with the unconstrained variables and supplemented by the field energies of the two chromostatic fields generated in the two-step elimination of the redundant variables,

$$\begin{aligned} \mathscr{H}' = &- i\psi^\dagger \boldsymbol{\alpha}(\boldsymbol{\nabla} - ig\mathbf{A}')\,\psi + m\psi^\dagger\beta\psi + \operatorname{tr}\left(\boldsymbol{\Pi}'^2_\perp + \frac{1}{L^2} p_3^\dagger p_3 + \mathbf{B}'^2\right) \\ &+ \operatorname{tr}\left(G'_\perp \frac{1}{d_3^2} G'_\perp + \frac{1}{L^2}\boldsymbol{\eta}^2\right). \end{aligned} \tag{9.8}$$

At this point, the formal structure and the physics content of the axial gauge representation of QCD can be clarified further by comparison with other approaches. Salient features of the formalism are exhibited by a first and most obvious comparison with the standard continuum formulation of the axial gauge representation. This representation is obtained if all the two-dimensional gauge and chromoelectric fields are neglected. This much simpler formulation would have occurred naturally had we not properly taken into account in the gauge fixing procedure the boundary conditions on the gauge and fermion fields. We reiterate that in this case, the covariant derivative D_3 would have been replaced by the ordinary derivative ∂_3 which in turn would determine the chromostatic field energy appearing in Eq. (9.8) (this naive procedure, however, leaves open the question of a proper infrared definition of ∂_3^{-1}). One could therefore suspect that the complexity of the derivation is due to a large extent to subtleties which become irrelevant in the continuum limit. The results, however, indicate that it might be just the role of these two-dimensional fields to properly define the continuum limit. Simple replacement of d_3 by ∂_3 is not possible as is made explicit by the representation,

$$\langle \mathbf{z}| \left(\frac{1}{d_3}\right)^{ab} |\mathbf{y}\rangle = \frac{1}{2iL}\delta^{(2)}(\mathbf{z}_\perp - \mathbf{y}_\perp) \sum_{i,j,n}{}' \frac{e^{i2\pi n(z_3 - y_3)/L}}{[2\pi n/L + g(a_{3,i}(\mathbf{y}_\perp) - a_{3,j}(\mathbf{y}_\perp))]} \lambda_{ij}^a \lambda_{ji}^b. \tag{9.9}$$

Here i, j are color matrix labels and the $n=0$ term has to be omitted for $i=j$. For $i \neq j$ the $n=0$ term contributes and therefore the sum is infrared finite only due to the presence of the two-dimensional gauge fields. It was Schwinger [15] who, soon after the discussion of the advantages of the axial gauge in Ref. [14], pointed out the infrared difficulties associated with the ambiguities in the definition of the Greens function ∂_3^{-1} and recognized the need for additional gauge fixing, as is realized in our formalism by the implementation of the residual Gauss law constraints.

Understanding the role of these two-dimensional degrees of freedom is sharpened by comparison of the axial gauge with the Coulomb-gauge representation. Starting from the Weyl-gauge Hamiltonian, the Coulomb-gauge representation for continuum QCD has been derived in Ref. [3]; this work has formed the basis for formulations of QCD on a torus [8, 9]. The crucial step in the derivation is the decomposition of the Gauss law operator into a piece containing the longitudinal chromoelectric field $\mathbf{\Pi}^l$ and the remainder. To express the action of $\mathbf{\Pi}^l$ in terms of the other variables, the operator

$$d_l = \mathbf{D} \cdot \mathbf{\nabla} \tag{9.10}$$

has to be inverted. Since the full vector potential is still present in this operator as well as in the Hamiltonian, a change of basis is necessary which eliminates the longitudinal gauge field $\mathbf{A}^l$ from the Hamiltonian. This is achieved in Ref. [3] by a change of coordinates which resembles the change from cartesian to polar coordinates in one particle quantum mechanics. The effect of this coordinate change is

to replace in the Hamiltonian and Gauss law the full vector potential by the transverse one, including the zero mode operators on the torus. Although formally very similar, the net result of these manipulations is quite different for Coulomb- and axial gauges, respectively. The axial gauge Fadeev–Popov operator d_3 can be explicitly inverted; note that the expression in Eq. (9.9) involves an algebraic inverse. The same is not possible for the differential operator d_l. As a consequence, the expression for the field energy of the chromostatic field contains the symbolic operator inverse d_l^{-1} in the Coulomb-gauge representation. In actual calculations one has to resort to perturbative evaluations of this operator inverse (cf. [8, 9]). In this procedure the well-known "Gribov ambiguities" arise; i.e., field configurations with zero eigenvalue of d_l are present [13] which invalidate a perturbative expansion. In the axial gauge representation, such field configurations are also present; indeed the variables $a_{3,i}$ and $a_{3,j}$ may have identical values, in which case d_3^{-1} becomes singular (cf. Eq. (9.9)). However, this singularity is as little disturbing as is a centrifugal barrier in quantum mechanics.

Thus, unlike QED, QCD surprisingly appears to be described in simpler terms in the axial gauge rather than in the Coulomb-gauge representation. Manifest rotational invariance of the Coulomb gauge seems to be less of an advantage than the explicit inversion of the Fadeev–Popov operator in the axial gauge. Formally, the axial gauge is favoured since it is based on a decomposition of the Gauss law which preserves the algebraic structures typical for gauge symmetries. Physically, radiation of static charges, which is conveniently eliminated in the Coulomb-gauge representation of QED, occurs in QCD and has to be described by the complicated coupling of the longitudinal chromostatic field with the transverse gluons.

A completely different class of representations of gauge theories is constructed if the longitudinal chromoelectric field is fully retained as a degree of freedom, and the redundant variables are chosen among the degrees of freedom which determine the (color) charge density. In scalar QED this procedure yields the unitary gauge representation with the (local) phases of the Higgs field eliminated from the Hamiltonian as redundant variables [19]. In QCD such representations have first been derived in Ref. [26] and later in Refs. [27, 28]. In Ref. [26], the Gauss law is used to solve (within $SU(2)$) for three particular, local, non-linear functions of the nine gauge field components A_i^a. Since no differential operator has to be inverted, this representation—unlike the Coulomb- and axial gauge representations—preserves the locality of the original Weyl-gauge Hamiltonian. In scalar QED, the locality of the Hamiltonian in the unitary gauge representation is intimately related to the generation of massive photons in the Higgs phase. One therefore might expect that a study of the relation between local and non-local representations of the QCD Hamiltonian will help to understand the nature of the confining phase of QCD.

We now return to the discussion of the axial gauge representation of QCD. The comparison with other approaches points to the particular dynamical role of the two-dimensional gauge fields. Resolution of the Gauss law introduces non-perturbative features into the Hamiltonian, and we emphasize that these non-perturbative

infrared properties are exclusively carried by the two-dimensional degrees of freedom. It is the presence of these degrees of freedom which prevents expansion of the Hamiltonian in powers of the coupling constant as well as a simple transition to the continuum limit. The singular dependence of the Hamiltonian, Eq. (9.8), on the color neutral gauge fields a_3 is that of a centrifugal barrier (cf. Eq. (9.9)) which yields infinite repulsion for field configurations which approach the "Gribov horizon." In full analogy with quantum mechanics of a particle formulated in polar coordinates, at this point of infinite repulsion, the Jacobian which is associated with the "radial momentum," vanishes. This Jacobian appears when defining components of the chromoelectric field operator Π_3 with respect to the dynamical basis such as is generated by the eigenvectors of the covariant derivative D_3, or equivalently d_3. The appearence of such Jacobians when using the curvilinear coordinates of the Coulomb-gauge representation has been emphasized in Ref. [3]. In the axial gauge representation, this Jacobian appears only in the definition of the kinetic energy of a_3 and does not, as happens in the Coulomb-gauge, affect the other degrees of freedom. In $SU(2)$, where only one independent, neutral two-dimensional gauge and conjugate chromoelectric field exists ($a_{3,2} = -a_{3,1}$), the Jacobian associated with this degree of freedom at fixed discretized $\mathbf{x}_{\perp,i}$ is

$$J_i = \sin^2(gLa_{3,1}(\mathbf{x}_{\perp,i})), \tag{9.11}$$

and the kinetic energy appears as

$$\mathrm{tr}(p_3^\dagger(\mathbf{x}_{\perp,i})\, p_3(\mathbf{x}_{\perp,i})) \to -\frac{1}{J_i}\frac{\partial}{\partial a_{3,1}(\mathbf{x}_{\perp,i})} J_i \frac{\partial}{\partial a_{3,1}(\mathbf{x}_{\perp,i})}. \tag{9.12}$$

As for polar coordinates, this Jacobian can be removed totally from the Hamiltonian by a redefinition of the wave functionals of physical states. In general this yields an effective potential. For the radial wave function (in three dimensions) it is zero; here it is constant and, consequently, irrelevant. Furthermore, the analogy with polar coordinates suggests vanishing of the reduced wave functionals $\tilde{\Psi}[a_3]$ at the zeroes of the Jacobian,

$$\tilde{\Psi}[a_3] = 0 \qquad \text{if} \quad gLa_{3,1}(\mathbf{x}_{\perp,i}) = n\pi \qquad \text{for one } i. \tag{9.13}$$

In comparison, the axial gauge representation of electrodynamics does not exhibit any of these non-perturbative features [16]. Zero-mode gauge and electric fields also appear in this case; they are, however, defined as components of Π_3 with respect to the eigenfuctions of the ordinary derivative ∂_3 and consequently neither a centrifugal barrier nor a non-trivial Jacobian emerges from this projection. In contradistinction to QCD, the axial gauge representation of the QED Hamiltonian exhibits also after implementation of the Gauss law the trivial perturbative limit of free photons and, furthermore, the continuum limit can be straightforwardly approached. Therefore, the particular infrared properties of the Hamiltonian, Eq. (9.8), are exclusively due to the gluonic self-interactions. One thus may

speculate that confinement is related to the appearance of the Jacobian (9.11) which effectively makes, together with the boundary condition (cf. Section 7), the variables $a_{3,i}(\mathbf{x}_\perp)$ compact (the physics of such Jacobians has been emphasized, e.g., in Ref. [29]). Such a speculation is supported by numerical results from lattice simulations which indicate the absence of confinement in non-compact QCD, cf. [30–32].

Intuition about possible mechanisms for dynamical infrared regularization through degrees of freedom of lower dimension can be gained by a study of $(1+1)$-dimensional QCD. In this case, the zero momentum neutral gluons play the role of the two-dimensional fields a_3, p_3 and these quantum mechanical degrees of freedom have been suggested to be necessary for a proper infrared behavior of QCD_{1+1} [33]. Indeed, investigation of $SU(2)$ QCD_{1+1} in the presence of static charges shows [34] the disappearance of colored hadrons to be a result of the dynamics of the neutral zero-momentum gluons. After projection onto the subspace of singlet quark states, the zero-momentum gluons can be neglected in the continuum limit. Thus these zero-momentum gluons play a crucial role in eliminating certain states; however, being of lower dimension, they never manifest themselves as physical degrees of freedom.

The structure of the Hamiltonian, Eq. (9.8), similarly suggests associating with the two-dimensional gauge field degrees of freedom infrared properties such as confinement; in this case one again might expect the perturbative regime of QCD not to be affected by degrees of freedom of lower dimension. Such an analysis could be carried through, for instance, in the effective Hamiltonian approach developed in Ref. [8] (see also [9]) in the context of the Coulomb-gauge representation. Unlike the effective Hamiltonian for the zero-momentum gluons in the Coulomb-gauge representation with its perturbative expansion of the inverse of the differential operator d_l (cf. Eq. (9.10)), an effective Hamiltonian for the two-dimensional gauge fields in principle has the chance for describing confinement in the continuum limit.

Independently of such a more detailed and approximative dynamical study, an investigation of the symmetries of the axial gauge Hamiltonian, Eq. (9.8), might reveal the possible phases of the quark gluon system. The axial gauge Hamiltonian with its unconstrained degrees of freedom no longer exhibits the local gauge symmetry of the Weyl-gauge Hamiltonian. However, as in electrodynamics, the gauge symmetry does not disappear completely: residual gauge symmetries are left. Not all of the gauge transformations acting in the large Hilbert space of the Weyl-gauge Hamiltonian reduce to the identity in the physical sector. Determination of the residual symmetries in the space of physical states is an important issue and can either be carried out by an analysis of the Hamiltonian, Eq. (9.8), or by following the change in the local gauge transformations when applying the gauge fixing transformations. The result of this investigation is in close analogy to the corresponding one in electrodynamics.

The residual global gauge transformations leaving the Hamiltonian (9.8) invariant read in component notation:

$$
\begin{aligned}
&a_{3,i}(\mathbf{x}_\perp) \to a_{3,i}(\mathbf{x}_\perp) + \frac{2\pi}{gL}\left(k_{3,i} + \frac{n_{3,i}}{N}\right),\\
&p_{3,i}(\mathbf{x}_\perp) \to p_{3,i}(\mathbf{x}_\perp),\\
&(\mathbf{A}'_\perp(\mathbf{x}))_{ij} \to e^{i[2\pi(\mathbf{k}_i - \mathbf{k}_j + (\mathbf{n}_i - \mathbf{n}_j)/N)\,\mathbf{x}/L + (\beta_i - \beta_j)]}(\mathbf{A}'_\perp(\mathbf{x}))_{ij}\\
&\qquad\qquad + \delta_{ij}\frac{2\pi}{gL}\left(\mathbf{k}_{\perp,i} + \frac{\mathbf{n}_{\perp,i}}{N}\right),\\
&(\mathbf{\Pi}'_\perp(\mathbf{x}))_{ij} \to e^{i[2\pi(\mathbf{k}_i - \mathbf{k}_j + (\mathbf{n}_i - \mathbf{n}_j)/N)\,\mathbf{x}/L + (\beta_i - \beta_j)]}(\mathbf{\Pi}'_\perp(\mathbf{x}))_{ij},\\
&\psi_i(\mathbf{x}) \to e^{i(2\pi\mathbf{k}_i\mathbf{x}/L + \beta_i)}\psi_i(\mathbf{x}).
\end{aligned}
\tag{9.14}
$$

The integer vectors $\mathbf{k}_i$ specify the $N-1$ vector symmetries, the "displacements" ($\sum_{i=1}^{N}\mathbf{k}_i = 0$). The continuous parameters β_i specify residual global phase rotations. In order for the Gauss law constraint to be satisfied on the torus the physical states must be invariant under these global rotations. Furthermore, in the absence of quarks, the QCD Hamiltonian is invariant under symmetry transformations given by the elements of the center of $SU(N)$ [35]. These "central conjugations" [8] are characterized by the integer vector

$$
\begin{aligned}
&\mathbf{n}_i = (n_{1i}, n_{2i}, n_{3i}),\\
&n_{ki} = \nu_k \qquad \text{for} \quad i = 1, \ldots, N-1; \qquad n_{kN} = -(N-1)\,\nu_k,\\
&\nu_k \in \{0, 1, \ldots, N-1\}, \qquad k = 1, 2, 3.
\end{aligned}
\tag{9.15}
$$

In the presence of quarks this symmetry is absent and only $\mathbf{n}_i = 0$ generates symmetry transformations (9.14) of the system.

In addition to the displacements and central conjugations, the "gauge fixed" Hamiltonian (9.8) is invariant under discrete symmetry transformations given by permutations of the basis vectors,

$$
\begin{aligned}
a_{3,i} &\to a_{3,P(i)},\\
p_{3,i} &\to p_{3,P(i)},\\
(\mathbf{A}'_\perp)_{ij} &\to (\mathbf{A}'_\perp)_{P(i),P(j)},\\
(\mathbf{\Pi}'_\perp)_{ij} &\to (\mathbf{\Pi}'_\perp)_{P(i),P(j)},\\
\psi_i &\to \psi_{P(i)},
\end{aligned}
\tag{9.16}
$$

with $(P(1), P(2), \ldots, P(N))$ denoting a permutation of the N matrix indices of $SU(N)$. In general, discrete and quasi-continuous symmetries do not commute.

The following discussion focuses on the symmetry transformations involving the two-dimensional degrees of freedom $a_{3,i}(\mathbf{x}_\perp)$. The effect of the symmetry transformations on these gauge variables is illustrated in Fig. 1 for $SU(2)$ and $SU(3)$, where also the points and lines of singularities of the Hamiltonian or, equivalently, the

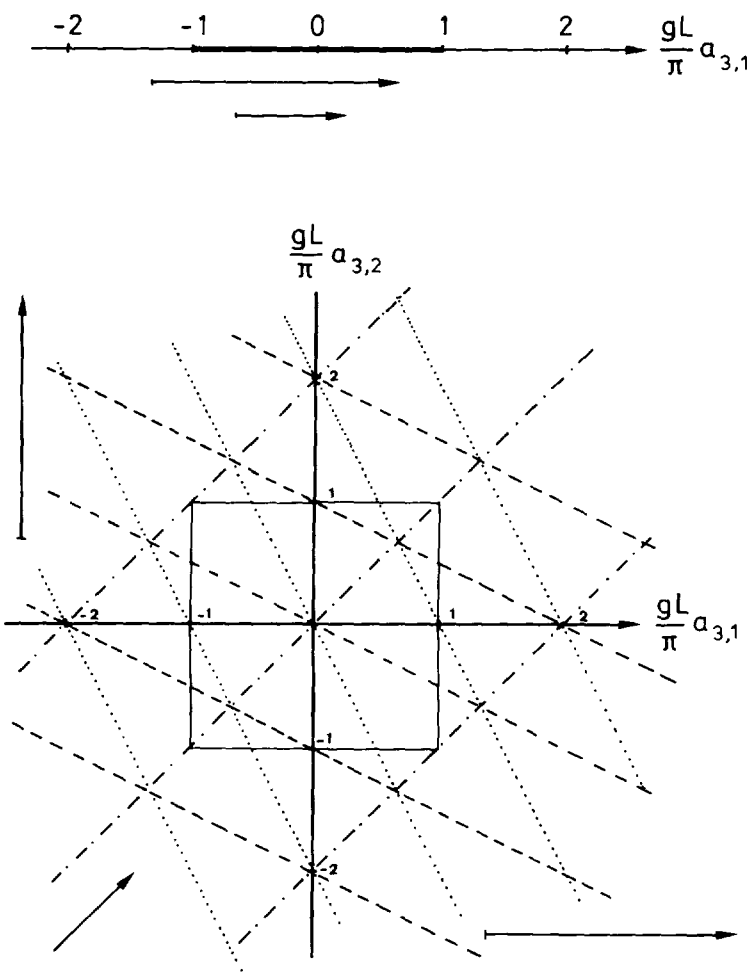


FIG. 1. Points ($SU(2)$) and lines ($SU(3)$) of singularities of the Hamiltonian (9.8) (or, equivalently, zeroes of the Jacobian (9.11), (9.12)). The arrows indicate the symmetry transformations discussed in the text. Also shown is the fundamental interval ($SU(2)$) and square ($SU(3)$) for the variables $a_{3,i}$, corresponding to the choice $\varphi_0 = -\pi$ (cf. Eq. (7.20)).

zeroes of the Jacobian are shown. It is the interplay between symmetry transformations and these restrictions from the dynamics which are typical for the non-abelian gauge theories and may be crucial for determination of the phases of QCD.

Displacements shift the fields $a_{3,i}$ by multiples of the fundamental value $2\pi/gL$ and are accompanied by rotations of quarks and off-diagonal gauge fields. Central conjugations are correlated shifts in the values of $a_{3,i}$ which are accompanied by corresponding rotations of the off-diagonal gauge fields. The central conjugations are in $SU(2)$ a shift in $a_{3,1}$ by the value π/gL, i.e., half of the displacement, and

in $SU(3)$ a shift of the fields $a_{3,1}$ and $a_{3,2}$ along the diagonal by the value $(2\pi/3gL)\sqrt{2}$. The discrete transformations (9.16) describe reflections of $a_{3,1}$ at $a_{3,1}=0$ for $SU(2)$ and can be generated in $SU(3)$ by the two symmetry transformations,

$$\begin{aligned} P_3&: a_{3,1} \to a_{3,2}, & a_{3,2} &\to a_{3,1}, \\ P_2&: a_{3,1} \to -(a_{3,1}+a_{3,2}), & a_{3,2} &\to a_{3,2}. \end{aligned} \tag{9.17}$$

The effect of these transformations is to interchange the set of dotted and dashed, and the set of dashed and dash-dotted lines of singularities, respectively. Also indicated in Fig. 1 are the elementary interval (for $SU(2)$) and the square (for $SU(3)$) to which the gauge variables may be restricted. The discrete symmetry P_2 does not leave invariant the square.

The origin of the displacement symmetries is most easily seen, in close analogy to electrodynamics, by rewriting the equations of motion for the electric field with the help of the continuity equation for the color current as a set of $3\times(N^2-1)$ continuity equations

$$\frac{\partial}{\partial t}\mathscr{D}_k^a(\mathbf{x},t)+\partial_i(F_{ik}^a(\mathbf{x},t)+gx_k j_i^a(\mathbf{x},t))=0. \tag{9.18}$$

Here, in analogy with Maxwell's displacement vector, the color displacement field operator has been introduced,

$$\mathscr{D}^a=-\mathbf{\Pi}^a+g\mathbf{x}\rho^a. \tag{9.19}$$

The total color current is given by

$$j_i^a=\psi^\dagger\alpha_i\frac{\lambda^a}{2}\psi+f^{abc}F_{ik}^b A_k^c. \tag{9.20}$$

Consequently, the space integral of the displacement field operator

$$\mathbf{D}^a=\int d^3x\,\mathscr{D}^a(\mathbf{x}) \tag{9.21}$$

generates N^2-1 vector symmetries of the quark gluon system. In the process of gauge fixing, it is the set of color neutral displacements which do not become reduced to the identity in the space of physical states. In electrodynamics, both possible realizations of the displacement symmetry are known. Scalar and spinor QED are realized in the Goldstone mode with the photons as Goldstone particles, while the Wigner–Weyl mode is realized in systems with macroscopic charge densities (or equivalently condensates) like the plasma or the abelian Higgs model [19].

The appearance of the $N-1$ global abelian symmetries is reminiscent of the appearance of the $U(1)^{N-1}$ residual local gauge group in the abelian projection [36, 37]. Indeed, these $N-1$ local $U(1)$ gauge symmetries can be reduced exactly to the $N-1$ displacement symmetries by completely fixing these local gauges by unitary transformations, as has been shown for QED [16]. Equivalently, these $N-1$ local abelian degrees of freedom would have appeared in our formalism, had we reinstated, in complying with the boundary conditions, a complete diagonal gauge field $A_3(\mathbf{x})$ instead of the zero-modes $a_3(\mathbf{x}_\perp)$. This amounts to replacing

$$a_3(\mathbf{x}_\perp)\, x_3 \to \int_0^{x_3} dz\, A_3^{c_0}(\mathbf{x}_\perp, z)\frac{\lambda^{c_0}}{2} \tag{9.22}$$

in $\tilde{U}$, cf. Eqs. (4.3)–(4.5). Thus, following 't Hooft notation, one may consider the $N-1$ neutral components of $\mathbf{A}$ as $N-1$ photons; they are simply shifted globally under displacements (cf. Eq. (9.14)). On the other hand, the phases of the off-diagonal components of $\mathbf{A}$—the fields $(\mathbf{A}'_\perp)_{ij}$—and of the quarks are rotated by the residual gauge transformations (Eq. (9.14)). Consequently, both quarks and off-diagonal gauge fields may be considered as electrically charged with respect to the $N-1$ photons. As in the abelian projection, QCD in the axial gauge representation can thus be interpreted as $N-1$ photons $(\mathbf{A}_\perp^{c_0}, a_3^{c_0})$ coupled to a system of N^2-N interacting charged gluons $(\mathbf{A}_\perp'^{c}, c \neq c_0)$ and N charged quarks. The advantage of this formulation is the common basis for describing the dynamics of quarks and gluons and its potential for understanding confinement of both types of degrees of freedom via their electric charge.

Pursuing this analogy with the abelian projection, we also recognize the similarity in origin of the singularities in the Hamiltonian of the axial gauge representation when $a_{3,i} = a_{3,j}$ (cf. Eqs. (6.22), (9.8), (9.9)) and of the magnetic monopoles in the abelian projection. In both cases these singularities arise when the diagonalized gauge condition contains identical matrix elements. Keeping in mind the obvious differences in the choice of gauge, this comparison suggests a relation between the singularities of the Hamiltonian (9.8) and monopole configurations in the abelian projection. It is also remarkable, although not completely unexpected, that the kinetic energy via the zeroes of the Jacobian prevents the system from reaching these singular configurations even for vanishing centrifugal barrier.

Such a description of QCD emphasizes the similarities to QED, and we shall use this analogy further to gain understanding of the symmetries and dynamics of QCD. First, and as a mere consequence of symmetry considerations, we note that the presence of the $N-1$ global, quasi-continuous symmetries implies that $SU(N)$ QCD will exhibit at most $N-1$ massless vector particles arising from the spontaneous breakdown of the $N-1$ displacement symmetries. There is no residual gauge symmetry which by its spontaneous breakdown would leave the "charged" gluons massless after switching on the coupling constant. This is in contradistinction to QED, where it is exactly the displacement symmetry which guarantees the existence of massless photons even after turning on the coupling to the electron-

positron field. In the non-abelian case, the symmetry of the free theory ($g=0$) does not continuously carry over into the interacting theory. This fact should be relevant when characterizing the phase changes in the confinement–deconfinement transition.

The issue of emergence of massless, neutral gluons via spontaneous breakdown of the displacement symmetries is ultimately decided dynamically. Indeed one expects, in close analogy to electrodynamics, also in the non-abelian case, different realizations of the displacement symmetry. In the confined phase with its absence of massless Goldstone particles the displacement symmetry is expected to be realized in the Wigner–Weyl mode. At this point the dynamical origin of the Wigner–Weyl realization is not understood. In analogy with the abelian Higgs model, one might expect the presence of the gluon condensate to dynamically favour the Wigner–Weyl realization. In order to display this possible connection between the gluon condensate and confinement, we consider the magnetic field energy associated with the two-dimensional fields (for $SU(2)$ color),

$$\int d^3x \,\mathrm{tr}(\mathbf{B}_1'^2+\mathbf{B}_2'^2)=2\int d^3x \sum_{k=1}^{2}[(\partial_k a_{3,1})^2+(\partial_3 A_{k,11})^2]$$
$$+2\int d^3x \sum_{k=1}^{2}[(\partial_3+2iga_{3,1})\phi_k][(\partial_3+2iga_{3,1})\phi_k]^\dagger. \tag{9.23}$$

In order to emphasize the analogy with the abelian Higgs model, we have introduced the complex scalar fields describing the charged gluons

$$\phi_k(\mathbf{x})=(A_k'(\mathbf{x}))_{12}, \qquad k=1,2. \tag{9.24}$$

Significant changes in the dynamics occur if the "Higgs-field" develops a non-vanishing expectation value,

$$2\sum_{k=1,2}\langle\phi_k^\dagger\phi_k\rangle=\sum_{\substack{k=1,2\\ i\neq j}}\langle A_{k,ij}A_{k,ji}\rangle\neq 0. \tag{9.25}$$

Since the operator $\sum_{i\neq j}\mathbf{A}_{\perp,ij}\mathbf{A}_{\perp,ji}$ is invariant under the residual transformations (9.14), formation of the condensate is not ruled out by gauge invariance and a dynamical issue. Furthermore, this operator is part of the $\mathbf{B}^2$ operator and the presence of the gluon condensate therefore suggests also the non-vanishing of the vacuum expectation value (9.25). Thus the presence of the gluon condensate and the absence of the massless excitations are properties which are likely to be both related to the Wigner–Weyl realization of the displacement symmetry. By these symmetry considerations we could not distinguish the Wigner–Weyl realization in the Higgs-phase with massive gluons from that of the confined phase. This is in agreement with general results on the phase structure of non-abelian gauge theories with bosons in the fundamental representation obtained in the context of the lattice formulation [38, 39]. On the other hand, the presence of a gap in pure Yang–Mills

theories as, according to the above arguments, is suggested by the presence of the QCD gluon condensate, might be sufficient for causing confinement (cf. [40]).

Our discussion of the displacement symmetry indicates the possibility of Goldstone phases in non-abelian theories with $N-1$ rather than N^2-1 massless gauge particles. This possibility seems to be realized in the Georgi–Glashow model [41]. Our formalism straightforwardly applies to this case if the color density of bosons in the adjoint representation,

$$\rho_m^a = f^{abc}\phi^b\dot{\phi}^c, \tag{9.26}$$

is substituted for the quark color density and if the minimally coupled fermionic Hamiltonian is replaced by the bosonic one. This model is supposed to contain (in color $SU(2)$) one massless particle, the "photon," which within our formalism is naturally interpreted as the one Goldstone boson associated with the spontaneous breakdown of the displacement symmetry. Thus, following this line of arguments, in the ground state of a pure Yang–Mills theory, the displacement symmetry is realized in the Wigner–Weyl mode and exhibits confinement. Coupling of matter in the adjoint representation may lead to a spontaneous breakdown of the displacement symmetry with the emergence of a "photon" as a Goldstone boson; in agreement with these symmetry considerations, the N^2-N charged vector particles do not become massless. In contradistinction to QED, the Goldstone phase in non-abelian theories is realized only as a result of the interaction with matter.

Beyond these symmetry considerations, the axial gauge representation of QCD presented here opens new perspectives for dynamical investigations. In this context a study of the dynamics of the two-dimensional degrees of freedom $a_{3,i}(\mathbf{x}_\perp)$ appears to be particularly promising. In the axial gauge, these degrees of freedom are singled out and the description of their dynamics is most radically affected by the unitary gauge-fixing transformations. Also in the axial gauge representation of QED, these degrees of freedom play a special role. They describe, for vanishing coupling ($e=0$), free photons propagating in the one-two plane with the electric field pointing along the three-axis. It is remarkable that such a simple limit of freely propagating gluons does not exist in QCD. The presence of boundary conditions, Jacobians, and centrifugal barriers reflect the non-linearities of the original Weyl-gauge Hamiltonian and Gauss law constraint. Moreover, they prevent simple appearence of non-interacting gluons in the weak coupling limit. Therefore, study of these novel aspects in the axial gauge representation of QCD could be useful for the issue of confinement. The axial gauge representation of QCD also opens new possibilities for studying other non-perturbative issues like the θ-vacua and the $U(1)$-anomaly. In general, these phenomena are investigated before gauge fixing [2]; there one studies topologically non-trivial field configurations or gauge transformations which act in the large Hilbert space of Weyl-gauge QCD. The projection of such transformations onto the physical subspace is not fully understood. Consequently, the physics implications of such constructions are not straightforwardly derived (cf. [20]). New tools for performing such investigations are provided by the technique of gauge-fixing via unitary transformations developed here.

Finally, the potential of quantum mechanical gauge-fixing by unitary transformations can be used to extend these investigations towards other choices of the redundant variables to be eliminated with the help of Gauss's law. In this context, the established relation to QCD in the abelian projection suggests a choice of the gauge where the neutral gluons are kept completely as degrees of freedom, whereas part of the charged gluons are eliminated. Furthermore, insight into the relation between non-abelian theories with a confining or Goldstone phase, respectively, can be expected from the construction of the unitary gauge representation of the Georgi–Glashow model.

APPENDIX A: EVALUATION OF $p_3^{c_0\dagger} - p_3^{c_0}$

As a result of the dynamical basis used in Eqs. (3.6)–(3.7), the projection $p_3^{c_0}(\mathbf{x}_\perp)$ of Π_3 onto $\zeta_{c_0,0}$ is not hermitian. In this appendix, we evaluate the hermiticity defect,

$$p_3^{c_0\dagger}(\mathbf{x}_\perp) - p_3^{c_0}(\mathbf{x}_\perp) = \int_0^L dx_3 \, \frac{1}{i} \frac{\delta \zeta^a_{c_0,0}(\mathbf{x})}{\delta A_3^a(\mathbf{x})}. \tag{A.1}$$

Our starting point is Eq. (5.15) which, together with the general operator identity

$$e^B[A, e^{-B}] = \int_0^1 dt \, e^{tB}[B, A] \, e^{-tB}, \tag{A.2}$$

can be used to derive

$$\frac{\delta \tau^b(\mathbf{x})}{\delta A_3^a(\mathbf{y})} = \Theta(x_3 - y_3) \, \delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp) \, S^{ac}(\mathbf{y})(T^{-1}(\mathbf{x}))^{cb}. \tag{A.3}$$

Here, we have introduced the matrices

$$S^{ab}(\mathbf{z}) = \tfrac{1}{2}\mathrm{tr}(e^{-ig\tau(\mathbf{z})}\lambda^a e^{ig\tau(\mathbf{z})}\lambda^b), \tag{A.4}$$

$$T^{ab}(\mathbf{z}) = \left(e^{-ig\tau(\mathbf{z})} \frac{1}{ig} \frac{\partial}{\partial \tau^a(\mathbf{z})} e^{ig\tau(\mathbf{z})}\right)^b$$

$$= \frac{1}{2}\,\mathrm{tr} \int_0^1 dt (e^{-igt\tau(\mathbf{z})}\lambda^a e^{igt\tau(\mathbf{z})}\lambda^b). \tag{A.5}$$

Specializing Eq. (A.3) to $x_3 = L$ allows us to convert Eq. (A.1) into

$$p_3^{c_0\dagger}(\mathbf{x}_\perp) - p_3^{c_0}(\mathbf{x}_\perp) = L\delta^{(2)}(\mathbf{0}) \frac{1}{i} \frac{\partial z^b_{c_0}(\mathbf{x}_\perp)}{\partial \theta^a(\mathbf{x}_\perp)} (T^{-1}(\mathbf{x}_\perp, L))^{ba}. \tag{A.6}$$

In order to evaluate the derivative of $z_{c_0}^b$ with respect to θ^a, it is useful to differentiate the eigenvalue equation (4.9), written in terms of z_c (cf. Eqs. (4.5), (4.17))

$$[z_c, \theta] = \frac{L}{g}\mu_c z_c \qquad (\mu_c = \mu_{c,n=0}), \tag{A.7}$$

with respect to θ^a:

$$\left[\frac{\partial z_c}{\partial \theta^a}, \theta\right] + \left[z_c, \frac{\lambda^a}{2}\right] = \frac{L}{g}\left(\frac{\partial \mu_c}{\partial \theta^a} z_c + \mu_c \frac{\partial z_c}{\partial \theta^a}\right). \tag{A.8}$$

Expanding

$$\frac{\partial z_c}{\partial \theta^a} = \sum_{c'} z_{c'} \left(z_{c'}, \frac{\partial z_c}{\partial \theta^a}\right) = \sum_{c'} z_{c'} 2\,\mathrm{tr}\left(z_{c'}^\dagger \frac{\partial z_c}{\partial \theta^a}\right) \tag{A.9}$$

and using again the "unperturbed" equation (A.7), we obtain

$$\sum_{c'} \frac{L}{g}(\mu_{c'} - \mu_c) z_{c'} \left(z_{c'}, \frac{\partial z_c}{\partial \theta^a}\right) + \left[z_c, \frac{\lambda^a}{2}\right] = \frac{L}{g}\frac{\partial \mu_c}{\partial \theta^a} z_c. \tag{A.10}$$

Projecting this equation onto $z_c^\dagger$ (with the same subscript c) yields the useful relation

$$\frac{L}{g}\frac{\partial \mu_c}{\partial \theta^a} = [z_c^\dagger, z_c]^a. \tag{A.11}$$

On the other hand, choosing $c = c_0$ in Eq. (A.10) and projecting it onto $z_c^\dagger$ with arbitrary c, we obtain

$$\frac{L}{g}\mu_c \left(z_c, \frac{\partial z_{c_0}}{\partial \theta^a}\right) = -[z_c^\dagger, z_{c_0}]^a. \tag{A.12}$$

Equations (A.9) and (A.12) can be combined into

$$\frac{\partial z_{c_0}}{\partial \theta^a} = \sum_{c(\mu_c \neq 0)} \frac{g z_c}{L \mu_c} [z_{c_0}, z_c^\dagger]^a + \sum_{d_0} z_{d_0} \left(z_{d_0}, \frac{\partial z_{c_0}}{\partial \theta^a}\right), \tag{A.13}$$

where the last term (i.e., the zero-mode contribution) can easily be shown to vanish. For $x_3 = L$, the matrix $T^{ab}(\mathbf{x})$ defined in Eq. (A.5) has the spectral representation

$$T^{ab}(\mathbf{x}_\perp, L) = \sum_c z_c^a(\mathbf{x}_\perp)\, t_c(\mathbf{x}_\perp)\, z_c^{b*}(\mathbf{x}_\perp), \tag{A.14}$$

with the eigenvalues

$$t_c = \frac{1 - e^{-i\mu_c L}}{i\mu_c L} \qquad \text{for} \quad p \neq q, \quad t_{c_0} = 1. \tag{A.15}$$

Equations (A.13)–(A.15) yield

$$\frac{\partial z^b_{c_0}}{\partial \theta^a}(T^{-1})^{ba} = \sum_{c(\mu_c \neq 0)} \frac{g}{L\mu_c}\left(\frac{i\mu_c L}{e^{i\mu_c L}-1}\right) 2\,\mathrm{tr}([z^\dagger_c, z_c]\, z_{c_0})$$

$$= \frac{L}{2} \sum_{c(\mu_c \neq 0)} \cot\left(\frac{\mu_c L}{2}\right) z^a_{c_0} \frac{\partial \mu_c}{\partial \theta^a}. \tag{A.16}$$

In the last step, we have made use of Eq. (A.11) and the fact that the eigenvalues μ_c appear in pairs $(\pm\mu)$, so that $\sum_c \mu_c = 0$. Since $z^a_{c_0}$ can be replaced by a derivative, cf. Eq. (5.21), the hermiticity defect (A.6) can be written in the following compact form:

$$p_3^{c_0\dagger} - p_3^{c_0} = \delta^{(2)}(\mathbf{0}) \frac{L}{2i} \sum_{c(\mu_c \neq 0)} \cot\left(\frac{\mu_c L}{2}\right) \frac{\partial \mu_c}{\partial a_3^{c_0}}$$

$$= \delta^{(2)}(\mathbf{0}) \frac{1}{i} \frac{\partial}{\partial a_3^{c_0}} \sum_{c(\mu_c \neq 0)} \ln \sin\left(\frac{\mu_c L}{2}\right). \tag{A.17}$$

Inserting the eigenvalues μ_c, Eq. (4.12), we alternatively obtain the more explicit form

$$p_3^{c_0\dagger} - p_3^{c_0} = \delta^{(2)}(\mathbf{0}) \frac{1}{i} f_{c_0}, \tag{A.18}$$

where

$$f_{c_0} = \frac{Lg}{2} \sum_{l<k} \cot\left(\frac{1}{4} gL a_3^{d_0}(\lambda_l^{d_0} - \lambda_k^{d_0})\right)(\lambda_l^{c_0} - \lambda_k^{c_0}) \tag{A.19}$$

and

$$\lambda_k^{c_0} = (\lambda^{c_0})_{kk} \qquad \text{(no sum)}. \tag{A.20}$$

APPENDIX B: Hermiticity Defects, Measure, Jacobian, and Effective Potential

In this appendix we describe a method to construct Jacobians and, if possible, effective potentials from hermiticity defects of momentum operators. Application to our work yields an indirect evaluation of the functional determinant.

Hermiticity Defect and Weight Function

Consider a quantum mechanical system of N degrees of freedom described by a Hamiltonian which is expressed in the canonical coordinates and momenta q_k, p_k $(k = 1, ..., N)$, which satisfy standard commutation relations

$$[q_k, q_l] = [p_k, p_l] = 0,$$
$$[q_k, p_l] = i\delta_{kl}. \tag{B.1}$$

This means that one has in the Schrödinger representation,

$$p_k = -i\frac{\partial}{\partial q_k}. \tag{B.2}$$

We study the case where the momenta may be non-hermitian,

$$p_k - p_k^\dagger = if_k(q), \tag{B.3}$$

where we assume the real functions f_k to be independent of the momenta. One easily constructs hermitian momentum operators,

$$\tilde{p}_k = p_k - \frac{i}{2}f_k(q) + r_k(q), \tag{B.4}$$

with $r_k(q)$ real functions of the coordinates; we take them to be identically zero. Of course, then these hermitian momenta satisfy, together with the coordinates, the standard commutation relations given above. In the Schrödinger representation the identity is in general given by

$$\int \cdots \int |q_1 \cdots q_N\rangle \rho(q)\, dq_1 \cdots dq_N \langle q_1 \cdots q_N| = 1, \tag{B.5}$$

where ρ is the measure or weight function. It immediately follows from Eqs. (B.4) and (B.2) that

$$\frac{\partial \ln \rho(q)}{\partial q_k} = f_k(q). \tag{B.6}$$

For one variable one explicitly finds

$$\rho(q) = \exp\left(\int^q f(q')\, dq'\right). \tag{B.7}$$

For more variables, Eq. (B.6) can only be integrated if the conditions

$$\frac{\partial f_l}{\partial q_k} = \frac{\partial f_k}{\partial q_l} \tag{B.8}$$

are satisfied.

Effective Potentail

If the kinetic energy has the form

$$T = \tfrac{1}{2} p_k^\dagger p_k, \tag{B.9}$$

then it is convenient to absorb the corresponding weight factor in the wave function by redefining it as

$$u(q) = \sqrt{\rho(q)}\, \psi(q). \tag{B.10}$$

The price which in general then has to be paid is the occurrence of an effective potential; it is given by

$$V_{\text{eff}} = \frac{1}{2\sqrt{\rho(q)}} \frac{\partial^2 \sqrt{\rho}}{\partial q_k^2}. \tag{B.11}$$

In terms of the functions f_k we obtain

$$V_{\text{eff}} = \frac{1}{4} \frac{\partial f_k}{\partial q_k} + \frac{1}{8} (f_k)^2. \tag{B.12}$$

Justification via Coordinate Transformations (See also [4])

Let us start with (cartesian) coordinates and conjugate momenta x_k, π_k, $k = 1, ..., N$, satisfying canonical commutation relations of the unconstrained variables. In this case the momenta are hermitian,

$$\pi_k^\dagger = \pi_k. \tag{B.13}$$

A general coordinate transformation,

$$x_k \to q_l = q_l(x_k), \tag{B.14}$$

defines a matrix M (the metric),

$$M_{ij} = \frac{\partial x_k}{\partial q_i} \frac{\partial x_k}{\partial q_j}. \tag{B.15}$$

The determinant of this matrix yields the Jacobian of this coordinate transformation, i.e., the measure in the new coordinates

$$\rho = J = \sqrt{\det(M)}. \tag{B.16}$$

The kinetic energy in Schrödinger representation transforms as

$$\frac{1}{2} \frac{\partial^2}{\partial x_i^2} = \frac{1}{2} \frac{1}{\sqrt{\det(M)}} \frac{\partial}{\partial q_i} M_{ij}^{-1} \sqrt{\det(M)} \frac{\partial}{\partial q_j}. \tag{B.17}$$

Identifying $-i(\partial/\partial q_l) = p_l$ yields the quantum kinetic energy operator in the new canonical variables q_k, p_k. They satisfy standard commutation relations; however, the momenta are non-hermitian. This follows from the relation between the old and new momenta,

$$\pi_j = \left(\frac{\partial q_k}{\partial x_j}\right) p_k, \tag{B.18}$$

and Eq. (B.13). We obtain

$$\begin{aligned} p_j^\dagger &= \frac{\partial q_k}{\partial x_l} p_k \frac{\partial x_l}{\partial q_j} \\ &= p_j - \frac{i}{2\det(M)} \frac{\partial \det(M)}{\partial q_k} \\ &= p_j - i \frac{\partial \ln(\rho)}{\partial q_j}. \end{aligned} \tag{B.19}$$

This relation indeed confirms the result obtained earlier, Eq. (B.6) (use Eq. (B.3)). Moreover, the kinetic energy operator can be written as

$$T = \tfrac{1}{2} p_k^\dagger M_{kl}^{-1}(q)\, p_l. \tag{B.20}$$

Constraints (Gauss Laws)

We are interested in applying the method described above for a constrained system, in particular, gauge theories where the constraints are the Gauss laws. In this section the following labelling will be used: $k, l, m = 1, ..., n_0$; $r, s = n_0 + 1, ..., N$ $(n > N_0)$. Assume that the Hamiltonian has the form

$$H = \tfrac{1}{2} p_k^\dagger M_{kl}^{-1}(q_m)\, p_l + \tfrac{1}{2} p_r^\dagger M_{rs}^{-1}(q)\, p_s + V(q_k). \tag{B.21}$$

Furthermore, the Gauss laws allow us to replace, in the sector of physical states, the second term by some operator (or c-number) $\tilde{V}(q_k)$. The hermiticity defects of the momenta p_k are given by Eq. (B.3); if one has

$$\partial f_k / \partial q_s = 0, \tag{B.22}$$

then (cf. Eq. (B.6))

$$\partial \ln \rho / \partial q_k = f_k(q_l). \tag{B.23}$$

As a consequence, the measure (and volume-element) factorize and we obtain an effective Hamiltonian and measure only depending on the variables q_k, p_k. Absorbing the weight function into the wave function, which implies an effective trivial (flat) measure is possible for

$$M_{kl}^{-1}(q_m) = \delta_{kl}. \tag{B.24}$$

As shown above it still can give rise to an effective potential.

Generalization to Field Theory

All of the above can be generalized to quantum field theory by considering a continuum of variables, $\{q_k\} \to q(x)$ $(x \in R^n)$. For our purpose, it is sufficient to

consider the case where the Jacobian factorizes for different values of x. If we first think of discretizing the n-dimensional space, we have

$$\mathcal{J}[q] = \prod_i J(q(x^{(i)})) = \exp\left(\sum_i \ln J(q(x^{(i)}))\right). \tag{B.25}$$

The continuum limit is performed as

$$\begin{aligned}\sum_i \ln J(q(x^{(i)})) &= \lim_{\Delta x \to 0} \frac{1}{\Delta^n x} \sum_i \Delta^n x \ln J(q(x^{(i)})) \\ &= \delta^{(n)}(0) \int d^n x \ln J(q(x)).\end{aligned} \tag{B.26}$$

The momenta conjugate to $q(x)$ are given in the Schrödinger representation by the functional derivatives

$$p(x) = \frac{1}{i} \frac{\delta}{\delta q(x)}. \tag{B.27}$$

Using Eqs. (B.25)–(B.27), we can generalize the relation between hermiticity defect and Jacobian of Eqs. (B.3) and (B.6) to field theory as

$$p(x) - p^{\dagger}(x) = i\delta^{(n)}(0) f(q(x)), \tag{B.28}$$

$$f(q(x)) = \frac{\partial \ln J(q(x))}{\partial q(x)}. \tag{B.29}$$

Similarly, an effective potential can be derived by introducing new wave functionals

$$U[q] = \sqrt{\mathcal{J}[q]}\, \Psi[q]. \tag{B.30}$$

The result corresponding to Eqs. (B.11) and (B.12) is

$$V_{\text{eff}}[q] = [\delta^{(n)}(0)]^2 \int d^n x\, \mathcal{V}_{\text{eff}}(q(x)), \tag{B.31}$$

with (suppressing the argument of $q(x)$)

$$\begin{aligned}\mathcal{V}_{\text{eff}}(q) &= \frac{1}{2\sqrt{J(q)}} \frac{\partial^2}{\partial q^2} \sqrt{J(q)} \\ &= \frac{1}{4} \frac{\partial f(q)}{\partial q} + \frac{1}{8} (f(q))^2.\end{aligned} \tag{B.32}$$

The further generalization to the case of several fields is trivial.

APPENDIX C: DIRECT CALCULATION OF THE JACOBIAN

In this appendix, we evaluate the Jacobian directly via a functional determinant. For this purpose, we interpret the result of our unitary transformation as a change of variables from $A_3^a(\mathbf{x})$ to $\tau^a(\mathbf{x})$, defined (locally) in Eq. (4.1), followed by a diagonalization of $\theta(\mathbf{x}_\perp)=\tau(\mathbf{x}_\perp, L)$. Only the eigenvalues of $\theta(\mathbf{x}_\perp)$ (denoted by $La_{3,i}$ in the main text) appear in the final Hamiltonian restricted to the physical sector; therefore we can disregard all factors in the Jacobian which are independent of a_3.

The Jacobian of the first transformation is

$$\mathscr{J}=\left|\det\left(\frac{\delta\tau^b(\mathbf{x})}{\delta A_3^a(\mathbf{y})}\right)\right|, \tag{C.1}$$

where the (continuous) "matrix" has been given in Eq. (A.3). This matrix is diagonal in $x_\perp$ and "triangular" in x_3, so that the spatial part of the determinant reduces to a product of diagonal matrix elements,

$$\det\left(\frac{\delta\tau^b(\mathbf{x})}{\delta A_3^a(\mathbf{y})}\right)=\prod_{\mathbf{x}}\det(S(\mathbf{x})\,T^{-1}(\mathbf{x})). \tag{C.2}$$

Only the factors with $x_3=L$ on the right-hand side contain the θ degrees of freedom and need to be retained. Since $S(\mathbf{x})$ is orthogonal, cf. Eq. (A.4), the relevant part of the integration measure is, therefore,

$$\prod_{\mathbf{x}_\perp,\,a} d\theta^a(\mathbf{x}_\perp)\,|\det T(\mathbf{x}_\perp, L)|. \tag{C.3}$$

Recalling that the matrix $T^{ab}(\mathbf{x}_\perp, L)$ in Eq. (A.5) can be represented as

$$T^{ab}(\mathbf{x}_\perp, L)=\left(e^{-ig\theta(\mathbf{x}_\perp)}\,\frac{1}{ig}\,\frac{\partial}{\partial\theta^a(\mathbf{x}_\perp)}\,e^{ig\theta(\mathbf{x}_\perp)}\right)^b, \tag{C.4}$$

one recognizes that the measure appearing in Eq. (C.3) is the invariant group measure (Haar measure) of the gauge group (see, e.g., [42]).

In the second step, we reparametrize $e^{ig\theta}$ as in Eq. (4.5), i.e.,

$$e^{ig\theta}=e^{ig\Delta}e^{iga_3L}e^{-ig\Delta}. \tag{C.5}$$

$e^{ig\Delta}$ can be assumed to depend on N^2-N variables ϕ^a, and we denote the full set of N^2-1 variables $\{a_3^{c_0}L, \phi^a\}$ by $\{\chi^a\}$. We then have to evaluate the new integration measure

$$\prod_a d\chi^a\,|\det T|\left|\det\left(\frac{\partial\theta^b}{\partial\chi^c}\right)\right|=\prod_a d\chi^a\,|\det\tilde{T}|, \tag{C.6}$$

with

$$\begin{aligned}\tilde{T}^{ab} &= \left(e^{-ig\theta}\frac{1}{ig}\frac{\partial}{\partial\chi^a}e^{ig\theta}\right)^b \\ &= \left(e^{igA}e^{-iga_3L}e^{-igA}\frac{1}{ig}\frac{\partial}{\partial\chi^a}(e^{igA}e^{iga_3L}e^{-igA})\right)^b \\ &= w^{ac}u^{cb},\end{aligned} \tag{C.7}$$

where the matrices w and u introduced in the last line are given by

$$w^{ac} = \left(e^{-iga_3L}\Lambda^a e^{iga_3L} - \Lambda^a + e^{-iga_3L}\frac{1}{ig}\frac{\partial}{\partial\chi^a}e^{iga_3L}\right)^c, \tag{C.8}$$

$$\Lambda^a = e^{-igA}\frac{1}{ig}\frac{\partial}{\partial\chi^a}e^{igA}, \tag{C.9}$$

$$u^{cb} = \tfrac{1}{2}\mathrm{tr}(e^{igA}\lambda^c e^{-igA}\lambda^b); \tag{C.10}$$

u is again orthogonal and can be dropped. The matrix w has the following structure: If χ^a refers to $a_3^{c0}L$, only the last term in Eq. (C.8) survives and it yields

$$\left(e^{-iga_3L}\frac{1}{ig}\frac{\partial}{\partial a_3^a L}e^{iga_3L}\right)^b = \delta^{ab}. \tag{C.11}$$

If χ^a refers to ϕ^a, only the first two terms in Eq. (C.8) contribute. Owing to the resulting block structure of the matrix w, the determinant is given by

$$\det\tilde{T} = \det\Lambda\,\det(\hat{w}-1), \tag{C.12}$$

where

$$\Lambda^{ab} = (\Lambda^a)^b \tag{C.13}$$

(cf. Eq. (C.9)) and

$$\hat{w}^{ab} = \tfrac{1}{2}\mathrm{tr}(e^{-iga_3L}\lambda^a e^{iga_3L}\lambda^b). \tag{C.14}$$

Note that a and b run over the non-diagonal λ-matrices only. The factor $\det\Lambda$ in Eq. (C.12) is independent of a_3. The second factor can be evaluated by diagonalization of the matrix $\hat{w}$,

$$\hat{w}^{ab}\hat{z}_c^b = e^{-i\mu_c L}\hat{z}_c^a, \tag{C.15}$$

cf. Eq. (4.11). Here, the restriction to non-diagonal λ-matrices removes zero eigenvalues. Up to factors which do not depend on a_3, we obtain the final result for the integration measure,

$$\prod_{\mathbf{x}_\perp,\,c0} da_3^{c0}(\mathbf{x}_\perp)\,J(a_3(\mathbf{x}_\perp)), \tag{C.16}$$

with

$$J=\prod_{c(\mu_c\neq 0)}(e^{-i\mu_c L}-1)=\prod_{c(\mu_c>0)}|e^{-i\mu_c L}-1|^2. \tag{C.17}$$

Using the explicit form of μ_c, Eq. (4.12), this can equivalently be written as

$$J=\prod_{k>l}\sin^2\left(\frac{gL(a_{3,l}-a_{3,k})}{2}\right), \tag{C.18}$$

in the notation of Eq. (4.13). This yields the well-known Haar measure of the group $SU(N)$ for the case that the integrand depends only on the invariants [22].

APPENDIX D: Effective Potential for $SU(N)$

We evaluate the effective potential

$$\mathscr{V}_{\text{eff}}^{[N]}=\frac{1}{8}\sum_{c_0=1}^{N-1}\left(2\frac{\partial f_{c_0}^{[N]}}{\partial a_3^{c_0}}+f_{c_0}^{[N]}f_{c_0}^{[N]}\right), \tag{D.1}$$

with

$$f_{c_0}^{[N]}=\frac{gL}{2}\sum_{k=2}^{N}\sum_{l=1}^{k-1}\cot\left(\frac{1}{4}gLa_3^{d_0}(\lambda_l^{d_0}-\lambda_k^{d_0})\right)(\lambda_l^{c_0}-\lambda_k^{c_0}), \tag{D.2}$$

in a recursive way. The superscript $[N]$ refers to $SU(N)$, and we have dropped a divergent, constant term. For $SU(2)$, $\mathscr{V}_{\text{eff}}$ can be easily calculated with the result

$$\mathscr{V}_{\text{eff}}^{[2]}=-\frac{g^2L^2}{8}. \tag{D.3}$$

Using the standard form of the diagonal λ-matrices with, in particular,

$$\lambda^{N-1}=\sqrt{2/N(N-1)}\,\text{diag}(1,1,\ldots,-(N-1)), \tag{D.4}$$

we first relate $f_{c_0}^{[N]}$ to $f_{c_0}^{[N-1]}$ as

$$f_{c_0}^{[N]}-f_{c_0}^{[N-1]}=\delta f_{c_0}=\frac{gL}{2}\sum_{k=1}^{N-1}(\lambda_k^{c_0}-\lambda_N^{c_0})\cot\Phi_k. \tag{D.5}$$

Here, Φ_k is defined as

$$\Phi_k=\tfrac{1}{4}gLa_3^{d_0}(\lambda_k^{d_0}-\lambda_N^{d_0}), \tag{D.6}$$

and we note that

$$\lambda_k^{c_0}-\lambda_N^{c_0}=\begin{cases}\lambda_k^{c_0} & \text{for } c_0<N-1\\ \sqrt{2N/(N-1)} & \text{for } c_0=N-1.\end{cases} \tag{D.7}$$

Inserting the definition of δf_{c_0}, Eq. (D.5), into Eq. (D.1), we find that

$$\mathcal{V}_{\text{eff}}^{[N]} - \mathcal{V}_{\text{eff}}^{[N-1]} = \frac{1}{8} \sum_{c_0=1}^{N-1} \left(2 \frac{\partial \delta f_{c_0}}{\partial a_3^{c_0}} + 2 f_{c_0}^{[N-1]} \delta f_{c_0} + (\delta f_{c_0})^2 \right) \tag{D.8}$$

$$= R_1 + R_2 + R_3. \tag{D.9}$$

We proceed to evaluate the separate contributions R_i. Using

$$\lambda_k^{c_0} \lambda_l^{c_0} = 2(\delta_{kl} - 1/N), \tag{D.10}$$

which follows from the usual completeness-relation for λ-matrices and the fact that the non-diagonal matrices have no diagonal entries, it is straightforward to determine R_1,

$$R_1 = -\frac{g^2 L^2}{8} \sum_{k=1}^{N-1} \frac{1}{\sin^2 \Phi_k}. \tag{D.11}$$

Similarly, R_3 can be evaluated easily with the result

$$R_3 = \frac{g^2 L^2}{8} \sum_{k=1}^{N-1} \cot^2 \Phi_k + \frac{g^2 L^2}{16} \sum_{k \neq l} \cot \Phi_k \cot \Phi_l. \tag{D.12}$$

The only non-trivial part is the interference term R_2. Performing the c_0-sum and writing

$$\cot(\tfrac{1}{4} g L a_3^{d_0} (\lambda_l^{d_0} - \lambda_k^{d_0})) = \cot(\Phi_l - \Phi_k), \tag{D.13}$$

we obtain

$$R_2 = \frac{g^2 L^2}{8} \sum_{k<l} \cot(\Phi_k - \Phi_l)(\cot \Phi_k - \cot \Phi_l) \tag{D.14}$$

$$= -\frac{g^2 L^2}{16} \sum_{k \neq l} \cot \Phi_k \cot \Phi_l - \frac{g^2 L^2}{16} (N-1)(N-2), \tag{D.15}$$

where we have made use of the addition theorem for the cotangent in the last step. Collecting the results, Eqs. (D.11), (D.12), and (D.15), we arrive at the simple formula

$$\mathcal{V}_{\text{eff}}^{[N]} - \mathcal{V}_{\text{eff}}^{[N-1]} = -\frac{g^2 L^2}{16} N(N-1). \tag{D.16}$$

Together with the initial condition for $SU(2)$, Eq. (D.3), this recursion relation determines $\mathcal{V}_{\text{eff}}^{[N]}$ to be

$$\mathcal{V}_{\text{eff}}^{[N]} = -\frac{g^2 L^2}{48} N(N^2-1). \tag{D.17}$$

APPENDIX E: ALTERNATIVE FORMULATION

In the first unitary gauge-fixing transformation (Section 5), the matrix $e^{ig\zeta(\mathbf{x})}$ has been constructed such as to preserve manifest translational invariance in the three-direction, cf. Eqs. (4.3), (5.9). This requires specification of the two-dimensional field variables a_3 beyond the "periodic" form $e^{iga_3 L}$ in which they appear originally in the formalism. Here, we reformulate the problem entirely in terms of $e^{iga_3 L}$, thereby keeping manifest translational invariance in the one-, two-directions.

In this alternative formulation, we do not insist on the periodicity of $e^{ig\xi}$ in the three-direction. We repeat the succession of two unitary transformations leading to the axial gauge representation of QCD, but omitting the factor $e^{-ig\theta(\mathbf{x}_\perp)x_3}$ in Eq. (4.3), i.e., using

$$e^{ig\xi(\mathbf{x})} = e^{ig\tau(\mathbf{x})} e^{ig\Delta(\mathbf{x}_\perp)}. \tag{E.1}$$

As a result, the Hamiltonian density of Eq. (6.22) is replaced by

$$\begin{aligned}
\mathscr{H}' = & -i\psi^\dagger[\alpha_3\partial_3 + \boldsymbol{\alpha}_\perp(\boldsymbol{\nabla}_\perp - ig\mathbf{A}'_\perp)]\psi + m\psi^\dagger\beta\psi \\
& + \mathrm{tr}\left[(\boldsymbol{\Pi}'_\perp)^2 + \left(\frac{1}{L}\boldsymbol{\eta}\right)^2 + (F'_{12})^2 + (\partial_3\mathbf{A}'_\perp)^2\right] \\
& + \frac{1}{2L^2}\sum_{c0}\left(p_3^{c0\dagger}(\mathbf{x}_\perp) + \int_0^L dz_3 z_3 G''^{c0}_\perp(\mathbf{x}_\perp, z_3)\right)\left(p_3^{c0}(\mathbf{x}_\perp) + \int_0^L dy_3 y_3 G''^{c0}_\perp(\mathbf{x}_\perp, y_3)\right) \\
& + \frac{1}{L^2}\int_0^L dz_3 \int_0^L dy_3 \sum_{p,q,n}{}' \frac{G'_{\perp qp}(\mathbf{x}_\perp, z_3)\, G'_{\perp pq}(\mathbf{x}_\perp, y_3)}{[\mu_{c,n}(\mathbf{x}_\perp)]^2} e^{i\mu_{c,n}(\mathbf{x}_\perp)(z_3 - y_3)}.
\end{aligned} \tag{E.2}$$

As compared to our former result (6.22), we note the following modifications: $A_3(\mathbf{x})$ has been eliminated completely from the minimal coupling to the fermions and the color magnetic energy. The momenta $p_3^{c0}(\mathbf{x}_\perp)$ are no longer invariant under the unitary transformation, but become shifted by a first moment of the operator $G''^{c0}_\perp$ defined as

$$G''^{c0}_\perp(\mathbf{x}) = \boldsymbol{\nabla}_\perp \boldsymbol{\Pi}'^{c0}_\perp(\mathbf{x}) + g\rho'^{c0}(\mathbf{x}), \tag{E.3}$$

where ρ' is the color density without zero-mode in the three-direction,

$$\rho'^{c0}(\mathbf{x}) = \int_0^L dz \left(\delta(z - x_3) - \frac{1}{L}\right)(f^{c0de}\mathbf{A}'^d_\perp(\mathbf{x}_\perp, z)\,\boldsymbol{\Pi}'^e_\perp(\mathbf{x}_\perp, z) + \rho_m^{c0}(\mathbf{x}_\perp, z)). \tag{E.4}$$

In this way, a coupling of perpendicular gluons and fermions to the residual a_3 field appears which does not explicitly involve the variables a_3. Finally, the non-zero-mode part of the original Π_3^2 term is modified; here, a_3 enters only through the expression

$$\sum_n \frac{1}{[\mu_{c,n}(\mathbf{x}_\perp)]^2} e^{i\mu_{c,n}(\mathbf{x}_\perp)(z_3 - y_3)}$$

which is manifestly periodic in $a_{3,i}$, cf. Eq. (4.12). Thus on the level of the Hamiltonian, we do not have to specify how to take the logarithm of e^{igaL}. The same is true for the relevant commutation relation which can be formulated as (cf. Eq. (5.17))

$$[p_3^{c0}(\mathbf{x}_\perp), e^{iga_3(\mathbf{y}_\perp)L}] = gL\delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp)\, e^{iga_3(\mathbf{y}_\perp)L}\frac{\lambda^{c0}}{2}. \tag{E.5}$$

Following the same line of reasoning which has led us to the modified boundary conditions (7.15) in Section 7, we would now impose the dynamical boundary conditions

$$\begin{aligned}\psi(\mathbf{x}_\perp, L) &= e^{i\phi_3} e^{-iga_3(\mathbf{x}_\perp)L}\psi(\mathbf{x}_\perp, 0),\\ \mathbf{A}_\perp(\mathbf{x}_\perp, L) &= e^{-iga_3(\mathbf{x}_\perp)L}\left(\mathbf{A}_\perp(\mathbf{x}_\perp, 0) + \frac{i}{g}\nabla_\perp\right) e^{iga_3(\mathbf{x}_\perp)L},\end{aligned} \tag{E.6}$$

when constructing the appropriate normal mode expansion of the field operators. Here again, only the exponential of a_3 appears.

Finally, we display the unitary transformation which relates the present formulation to the one used in the main text,

$$U[\xi, \alpha]_{\text{old}} = \exp\left\{i\int d^3x\, G''^{c0}_\perp(\mathbf{x})\, x_3 a_3^{c0}(\mathbf{x}_\perp)\right\} U[\xi, \alpha]_{\text{new}}. \tag{E.7}$$

It can be used to rederive the boundary conditions (7.22), (7.23) for the wave functionals, if one chooses the compact definition of the variables $a_{3,i}(\mathbf{x}_\perp)$.

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