Quantum Theory of Strings in Abelian Higgs Model

E.T. Akhmedov[1,2], M.N. Chernodub[1,2], M.I. Polikarpov[1]
and M.A. Zubkov[1]

[1]*Institute of Theoretical and Experimental Physics,*
B.Cheremushkinskaya 25, Moscow, 117259 Russia.
[2]*Moscow Institute of Physics and Technology,*
Dolgoprudny, Moscow region, Russia.

Abstract

Starting from the Abelian Higgs field theory, we construct the theory of quantum Abrikosov–Nielsen–Olesen strings. It is shown that in four space – time dimensions in the limit of infinitely thin strings, the conformal anomaly is absent, and the quantum theory exists. We also study an analogue of the Aharonov–Bohm effect: the corresponding topological interaction is proportional to the linking number of the string world sheet and the particle world trajectory. The creation operators of the strings are explicitly constructed in the path integral and in the Hamiltonian formulation of the theory. We show that the Aharonov–Bohm effect gives rise to several nontrivial commutation relations.

1 Introduction

One of the principal problems of the quantum field theory is the search of the vacuum Ψ function. This problem is especially important for the nonperturbative description of the gluodynamics and chromodynamics. The standard way to obtain the nonperturbative effects is to use some vacuum consisting of instanton like classical solutions. In the present publication we consider the vacuum consisting of Abrikosov–Nielsen–Olesen (ANO) strings [1]. We start from the quantum Abelian Higgs theory, in which ANO strings are classical solutions. This theory can be considered as a relativistic generalization of the effective theory of the superconductor near the critical point (Ginsburg-Landau theory), and we do not pay attention to the zero-charge problem. We work in the Euclidian space and, taking into account the measure, extract from the functional integral the part corresponding to the topological defects which are ANO strings. We can perform all calculations for the case when the world sheets have the topology of the sphere. Actually, we perform in the continuum limit the same transformations that have been used in the lattice compact QED [2] and in the lattice Abelian Higgs model [3]. It was shown that the partition function for the compact fields on the lattice can be factorized: $\mathcal{Z}_{com} = \mathcal{Z}_{ncom} \cdot \mathcal{Z}_{top}$, where $\mathcal{Z}_{ncom}$ is the partition function for the noncompact fields, and $\mathcal{Z}_{top}$ is the partition function for the topological defects[1] (monopoles in compact QED and strings in the Abelian Higgs model.).

In first papers on the quantum ANO strings [7], [8], where the London limit (infinitely massive Higgs boson) was considered, it was shown that in the strong coupling limit (thin and long strings) the strings can be described by the Nambu-Goto action. The exact action for the ANO strings in the London limit is obtained in [9]. It was shown that in the string action there are terms depending on the powers of tensor of extrinsic curvature with exponents > 2. These terms ensure the stability of the classical string. The tree level corrections to the ANO string action were studied in [10]. The duality transformation for the Abelian Higgs model was discussed in [11].

It is impossible to get the quantum theory of the ANO strings from the actions discussed in [7]–[11]. If we consider the limit of infinitely thin strings, the theory becomes conformal, and it is well known that there are difficulties with the quantization of this theory in 4D: there either exists the conformal anomaly [12] (in the case of Hamiltonian or path integral quantization) or Lorentz invariance is absent (in the case of the light cone quantization). So we start from the quantum Abelian Higgs field theory, and it seems that we get the string theory which cannot be quantized. As shown below, the solution of this paradox lies in the accurate change of the field variables to the string variables.

An example of an effective theory of infinitely thin quantum 4D ANO strings was

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[1]The transformation of this type have been considered for the first time for the two-dimensional lattice XY model. It was shown [4, 5] that the partition function of the XY-model is equivalent to the partition function of the Coulomb gas. For the three- and the four-dimensional XY model it is also possible [6] to get the partition function for the topological defects, which are vortex lines and "global strings".

suggested by Polchinski and Strominger [13]. It occurs that if one adds to the Nambu-Goto action an additional term, then this theory can be quantized in the Hamiltonian formalism. For the Nambu-Goto action we have, on the classical level, the Virassoro algebra (algebra of generators of conformal transformations):

$$[L_n^{NG}, L_m^{NG}] = (n - m)L_{n+m}^{NG}. \tag{1}$$

On the quantum level, taking into account the raparametrization ghosts, we have for the pure Nambu-Goto action:

$$
\begin{aligned}
[\mathbb{L}_n^{NG}, \mathbb{L}_m^{NG}] &= (n - m)\mathbb{L}_{n+m}^{NG} + \frac{D - 26}{12}(m^3 - m)\delta_{n+m,0}, \\
\mathbb{L}_n^{NG} &= L_n^{NG} + L_n^{gh},
\end{aligned}
\tag{2}
$$

where D is the dimension of the space-time and L_n^{gh} are Virassoro generators which arise due to the ghost fields. If we add the term suggested in [13] with an arbitrary coefficient γ to the string action, Virassoro algebra for the full generators $\mathbb{L}_n^{tot} = \mathbb{L}_n^{NG} + L_n^{\gamma}$ takes the form[2]:

$$[\mathbb{L}_n^{tot}, \mathbb{L}_m^{tot}] = (n - m)\mathbb{L}_{n+m}^{tot} + \frac{D - 26 + \gamma}{12}(m^3 - m)\delta_{n+m,0}. \tag{3}$$

Therefore, by adjusting γ, we can cancel the conformal anomaly for $D = 4$ [13].

In Section 2 we show that such additional term in the action naturally arises for the ANO strings, if we take into account the Jacobian of the transformation from the field variables to the string variables. A preliminary, not although completely correct, calculation of the this Jacobian is published in [14]. The usual terms are also present. The first two terms in the expansion of the action of the ANO strings [9, 10], in powers of the average inverse string curvature, are the standard term proportional to the area of the string world sheet, and the rigidity term [15, 16] with negative sign.

In the refs. [17, 18, 19, 20, 21] the topological long–range interaction of the strings and charged particles was discussed. This interaction was discussed in [3] for the string representation of the $4D$ *lattice* Abelian Higgs model. In Section 3 we repeat the calculations of [3] in the continuum limit and show explicitly the existence of the Aharonov–Bohm effect in the field theory. The reason for this long–range interaction is that the charges $M = e,\ 2e,\ \ldots (N - 1)e$ cannot be completely screened by the condensate of the field of charge Ne; if M/N is integer, then the screening is complete and there are no long–range forces.

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[2]The algebra (3) was obtained in [13] in the leading order of the expansion in $\bar{R}^{-1}$, where $\bar{R}$ is the mean curvature of the strings.

In Section 4 we construct the operator which creates the string in a given time slice on the contour $\mathcal{C}$. This operator is the continuum analogue of the lattice operator considered in [6, 3].

In Section 5 we consider the theory in the Hamiltonian formalism and show that the string creation operator has nontrivial commutation relations[3] with the Wilson loop operator; this is a direct consequence of the Aharonov–Bohm effect. We give several other examples of nontrivial commutation relations.

2 From the Abelian Higgs model to the Quantum Strings

The partition function for the four-dimensional Abelian Higgs Model is given by the formula:

$$\mathcal{Z} = \int \mathcal{D}A_\mu \mathcal{D}\Phi \exp\left\{-\int \mathrm{d}^4x \left[\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}|D_\mu\Phi|^2 + \lambda(|\Phi|^2 - \eta^2)^2\right]\right\}, \tag{4}$$

$$D_\mu = \partial_\mu - ieA_\mu, \tag{5}$$

throughout the paper we assume the Pauli–Villars regularization, we discuss some details of the regularization in the Appendix.

In the equation (5) the integration over the complex scalar field $\Phi = |\Phi| \exp(i\theta)$ can be rewritten as:

$$\int \mathcal{D}\Phi... = \int \mathcal{D}Re\Phi \cdot \mathcal{D}Im\Phi... = \int [|\Phi|\mathcal{D}|\Phi|]\mathcal{D}\theta.... \tag{6}$$

The functional integral over θ should be carefully defined, since θ is not defined on the manifolds where

$$Im\Phi = Re\Phi = 0. \tag{7}$$

These two equations define the two–dimensional manifolds in the four dimensional space–time and we should integrate over all functions that are regular everywhere except for these two-dimensional manifolds. These two-dimensional singularities are nothing but the ANO string world sheets, since the Higgs field is zero at the center of the ANO string.

In eq. (6) we integrate over the regular functions $Re\Phi(x)$ and $Im\Phi(x)$, and it can be shown [9, 10, 11, 22] that the singularities in the function $\theta(x)$ defined by eqs.(7) should have the form:

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[3]By a nontrivial commutation relation we mean a relation of the type: $AB - e^{i\xi}BA = 0$.

$$\partial_{[\mu}\partial_{\nu]}\theta^s(x,\tilde{x}) \;=\; 2\pi\epsilon_{\mu\nu\alpha\beta}\Sigma_{\alpha\beta}(x,\tilde{x}), \tag{8}$$

$$\Sigma_{\alpha\beta}(x,\tilde{x}) \;=\; \int_{\Sigma}\mathrm{d}\sigma_{\alpha\beta}(\tilde{x})\delta^{(4)}[x-\tilde{x}(\sigma)],$$

$$\mathrm{d}\sigma_{\alpha\beta}(\tilde{x}) \;=\; \epsilon^{ab}\partial_a\tilde{x}_\alpha\partial_b\tilde{x}_\beta\mathrm{d}^2\sigma = \sqrt{g}\,t_{\alpha\beta}\mathrm{d}^2\sigma,$$

where $\tilde{x}=\tilde{x}(\sigma)$ are the coordinates of the two-dimensional singularities parametrized by σ_a, $a=1,2$; $\partial_a=\frac{\partial}{\partial\sigma_a}$; $\theta^s(x,\tilde{x})$ is the function of x and the functional of $\tilde{x}$; Σ defines the position of the singularities; $g=det||g_{ab}||$; $g_{ab}=\partial_a\tilde{x}_\mu\partial_b\tilde{x}_\mu$ and $t_{\mu\nu}=\frac{\epsilon_{ab}}{\sqrt{g}}\partial_a\tilde{x}_\mu\partial_b\tilde{x}_\nu$ are the tensors of the induced metric and the extrinsic curvature on Σ (we have no intrinsic metric in the theory), $t_{\mu\nu}^2=2$. Note that $\partial_{[\mu}\partial_{\nu]}\theta^s\neq 0$ since θ^s is a singular function.

For simplicity, we consider the London limit ($\lambda\to\infty$ and the radial part of the field Φ is fixed)[4]:

$$\mathcal{Z} = const\cdot\int\mathcal{D}A_\mu\mathcal{D}\theta\exp\left\{-\int\mathrm{d}^4x\left[\frac{1}{4}F_{\mu\nu}^2 + \frac{\eta^2}{2}|\partial_\mu\theta + eA_\mu|^2\right]\right\}, \tag{9}$$

where $\eta^2 =<|\Phi|^2>$. Now we discuss the measure of the integration over θ. From (6) it follows that the norm for the field θ is: $||\delta\theta||^2=\int\mathrm{d}^4x|\Phi|^2(\delta\theta)^2$. In the London limit there are two independent variables: the regular and the singular part of θ, $\theta=\theta^r+\theta^s$, and $||\delta\theta||^2 = const\int\mathrm{d}^4x(\delta\theta^r+\delta\theta^s)^2 = ||\delta\theta^r||^2 + ||\delta\theta^s||^2$. From eqs.(8) it can be easily seen that the interference term $\int\mathrm{d}^4x\,\delta\theta^r\delta\theta^s$ vanishes:

$$\int\mathrm{d}^4x\delta\theta^r\delta\theta^s = const\int\mathrm{d}^4x\int\mathrm{d}^4y\delta\theta^r\partial_{[\mu}\partial_{\nu]}\Delta^{-1}(x-y)\delta\Sigma_{\mu\nu} =$$

$$const\int\mathrm{d}^4x\int\mathrm{d}^4y\left(\partial_{[\mu}\partial_{\nu]}\delta\theta^r\right)\Delta^{-1}(x-y)\delta\Sigma_{\mu\nu} = 0\,, \tag{10}$$

we use the fact that $\partial_{[\mu}\partial_{\nu]}\delta\theta^r = 0$, since $\delta\theta^r$ is a regular function. Therefore, $\int\mathcal{D}\theta... = \int\mathcal{D}\theta^r\mathcal{D}\theta^s...$, and now we can show that the integral over the singular part θ^s can be reduced to the integral over the string world sheets. We have no monopoles in the theory; therefore, due to the conservation of the magnetic flux, the ANO strings are closed, and the singularities, defined by θ^s (Σ in eq.(8)), form closed two-dimensional surfaces. In the infinite space–time $\mathcal{R}^4$ the strings which are closed through the boundary conditions have the infinite action, therefore we do not take them into account.

Now, let us transform the partition function of the field theory (9) to the partition function of the string theory. In order to change the integration variables, we substitute the unity into the functional integral (9) (see eq. (8)):

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[4]For an arbitrary λ all the transformations remain the same, but in the final expression for the partition function (13) we have an additional functional integral over the radial part of the field Φ.

$$1 = \tilde{J}[\Sigma_{\mu\nu}] \cdot \int \mathcal{D}\tilde{x} \cdot \delta \left\{ \Sigma_{\mu\nu} - \int_{\Sigma} \mathrm{d}^2\sigma \sqrt{g} t_{\mu\nu} \delta^{(4)}[x - \tilde{x}(\sigma)] \right\}. \tag{11}$$

Here $\tilde{J}[\Sigma_{\mu\nu}]$ is the Jacobian which corresponds to the change of the field variables to the string variables, and in $\int \mathcal{D}\tilde{x}$ we assume summation over the topologies of the string world sheets. Using the δ–function in (11) and the definition of θ^s (8), we integrate over θ^s in the partition function:

$$\int \mathcal{D}\theta e^{-S[\theta, \dots]} = \int \mathcal{D}\theta^r \mathcal{D}\theta^s e^{-S[\theta^r + \theta^s, \dots]} = \tag{12}$$

$$const \cdot \int \mathcal{D}\theta^r \mathcal{D}\Sigma_{\mu\nu} e^{-S[\theta^r + \theta^s(\Sigma_{\mu\nu}), \dots]} = const \cdot \int \mathcal{D}\theta^r \mathcal{D}\tilde{x} J(\tilde{x}) e^{-S[\theta^r + \theta^s(x, \tilde{x}), \dots]},$$

where $J(\tilde{x}) = \tilde{J}[\Sigma_{\mu\nu}]$.

Fixing the gauge $\partial_\mu \theta^r = 0$, it is easy to perform integration over A_μ; the result is:

$$\mathcal{Z} = \int \mathcal{D}\tilde{x} J(\tilde{x}) \cdot \exp \left\{ -\eta^2 \pi^2 \int_{\Sigma} \int_{\Sigma} \mathrm{d}\sigma_{\mu\nu}(\tilde{x}) \mathcal{D}_m^{(4)}(\tilde{x} - \tilde{x}') \mathrm{d}\sigma_{\mu\nu}(\tilde{x}') \right\}, \tag{13}$$

where $(\Delta + m^2)\mathcal{D}_m^{(4)}(x) = \delta^{(4)}(x)$, and $m^2 = e^2\eta^2$ is the mass of the gauge boson. The action which enters the partition function (13) was already discussed in [9, 10], the new object in (13) is the Jacobian $J(\tilde{x})$. It easy to see that $J(\tilde{x})$ defined by (11) and the resulting partition function (13) are invariant under the raparametrization of the coordinate σ on the world sheet. As shown in the Appendix, $J(\tilde{x})$ can be evaluated if the string world sheet Σ has the spherical topology. The calculations are performed in the conformal gauge,

$$g_{12} = g_{21} = 0; \quad g_{11} = g_{22} = \sqrt{g}, \tag{14}$$

and the result is:

$$J(\tilde{x}) = const \cdot \exp \left\{ \int_{\Sigma} \mathrm{d}^2\sigma \left[\frac{11}{48\pi} (\partial_a \ln \sqrt{g})^2 + \mu_1 \sqrt{g} + \frac{\mu_3^2 \ln \Lambda_1 \bar{R}}{\mu_2^2 \pi} \sqrt{g} (\partial_a t_{\mu\nu})^2 \right] \right\}, \tag{15}$$

the parameters μ_i are defined in the Appendix, Λ_1 is the regularization parameter, and $\bar{R}$ is the average curvature of the strings.

Now we study the expansion of the action in powers of $(m\bar{R})^{-1}$. A similar expansion was studied in refs. [9, 10], but we include in the expansion the terms which come from $\ln J(\tilde{x})$. In the leading order, the action is local and, if the surface Σ has the spherical topology, we use the expression (15) for J; the result is:

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$$S = \mu' \int_\Sigma \mathrm{d}^2\sigma \sqrt{g} - \frac{11}{48\pi} \int_\Sigma \mathrm{d}^2\sigma (\partial_a \ln \sqrt{g})^2 - \beta \int_\Sigma \mathrm{d}^2\sigma \sqrt{g}(\partial_a t_{\mu\nu})^2 \,. \tag{16}$$

Here $\mu' = \mu_0 - \mu_1$, the string tension μ_0 comes from the expansion of the action (13) [9, 10], in the regularization scheme accepted in [9] $\mu_0 = 4\pi\eta^2 \ln\left(\frac{\Lambda_2^2}{m^2}\right)$; μ_0 is renormalized by μ_1 which enters the Jacobian (15). $\beta = \frac{\pi}{4e^2} + \frac{\mu_3^2 \ln \Lambda_1 \bar{R}}{\mu_2^2 \pi}$ where the first term in the r.h.s comes from the expansion of the action, the second one is due to the Jacobian.

The first term in (16) is the usual Nambu–Goto action; the second term, as we said, is important for the quantization, and the third one is the rigidity term (see [15, 16]).

If we consider the strings without rigidity, $\beta = 0$, we get the theory studied in [13]. It occurs that the coefficient of the second term in (16) corresponds to $\gamma = 22$ in the Virassoro algebra (3); therefore, the conformal anomaly is absent and the theory can be quantized in $D = 4$. It should be emphasized that this term appears from the Jacobian $J(\tilde{x})$.

It is obvious from the derivation of $J(\tilde{x})$ that it has universal nature, i.e. it is independent on the model under consideration. The Jacobian $J(\tilde{x})$ arises when we pass from the integration over the field variables to the integration over the string variables. Therefore, we expect that any field theory which has the string-like solutions is equivalent to the string theory which can be quantized in $D = 4$. As mentioned in ref. [9], the action which enters the partition function (13) leads to the stable ANO strings, but the dominant vacuum configuration is branched polymers formed by the string world sheets. It would be interesting to study the dominant vacuum configuration of the strings, taking into account the Jacobian $J(\tilde{x})$.

3 The Aharonov–Bohm Effect in the Abelian Higgs Model.

Now we consider the Abelian Higgs model with the Higgs bosons carrying the charge Ne, the partition function now is:

$$\mathcal{Z} = \int \mathcal{D}A_\mu \mathcal{D}\theta \exp\left\{-\int \mathrm{d}^4x \left[\frac{1}{4}F_{\mu\nu}^2 + \frac{\eta^2}{2}(\partial_\mu\theta + NeA_\mu)^2\right]\right\} \,. \tag{17}$$

There exists a nontrivial long–range topological interaction of Nielsen–Olesen strings with particles of charge Me, if $\frac{M}{N}$ is noninteger. This is the four–dimensional analogue [17, 18, 19] of the Aharonov–Bohm effect studied for the lattice Abelian Higgs model in [3]. Now we derive the long range interaction, using the string representation of the theory. Consider the Wilson loop for the particle of the charge Me:

$$W_M(\mathcal{C}) = \exp\left\{iMe \int \mathrm{d}^4x j_\mu^{\mathcal{C}}(x)A_\mu(x)\right\} = \exp\left\{iMe \int_{\mathcal{C}} \mathrm{d}x^\mu A_\mu(x)\right\} \,, \tag{18}$$

where the current is the δ–function on a contour $\mathcal{C}$:

$$j_\mu^{\mathcal{C}}(x) = \int_{\mathcal{C}} \mathrm{d}t \dot{\tilde{z}}^\mu(t) \delta^{(4)}(x - \tilde{z}(t)) \qquad (19)$$

and the function $\tilde{z}_\mu(t)$ parametrizes the contour.

Substituting (18) into the path integral (17) and changing the field variables to the string variables, as described in the previous section, we obtain:

$$< W_M(\mathcal{C}) >=$$

$$\frac{1}{\mathcal{Z}} \int \mathcal{D}\tilde{x} J(\tilde{x}) \exp\left\{ - \int \mathrm{d}^4 x \int \mathrm{d}^4 y \left[\pi^2 \eta^2 \Sigma_{\mu\nu}(x) \mathcal{D}_m^{(4)}(x - y) \Sigma_{\mu\nu}(y) \right. \right.$$

$$\left. + \frac{M^2 e^2}{2} j_\mu^{\mathcal{C}}(x) \mathcal{D}_m^{(4)}(x - y) j_\mu^{\mathcal{C}}(y) + \pi i \frac{M}{N} j_\mu^{\mathcal{C}}(x) \mathcal{D}_m^{(4)}(x - y) \partial_\nu \epsilon_{\mu\nu\alpha\beta} \Sigma_{\alpha\beta}(y) \right]$$

$$\left. + 2\pi i \frac{M}{N} I\!\!L(\Sigma, \mathcal{C}) \right\}, \qquad (20)$$

where $m = Ne\eta$ is the boson mass, and

$$I\!\!L(\Sigma, \mathcal{C}) = \frac{1}{2} \int \mathrm{d}^4 x \int \mathrm{d}^4 y \epsilon_{\mu\nu\alpha\beta} \Sigma_{\mu\nu}(x) j_\alpha^{\mathcal{C}}(y) \partial_\beta \mathcal{D}_0^{(4)}(x - y) =$$

$$= \frac{1}{4\pi^2} \int \mathrm{d}^4 x \int \mathrm{d}^4 y \, \epsilon_{\mu\nu\alpha\beta} \, \Sigma_{\mu\nu}(x) \, j_\alpha^{\mathcal{C}}(y) \frac{(x - y)_\beta}{|x - y|^4} \qquad (21)$$

is the linking number of the string world sheet Σ and the trajectory of the charged particle $\mathcal{C}$, this formula represents a four–dimensional analogue of the Gauss linking number for loops in three dimensions. The first three terms in the exponent in (20) are short range interactions and self–interactions of strings and the tested particle. The forth term is the long–range interaction which describes the four–dimensional analogue [17, 18, 19, 20] of the Aharonov–Bohm effect: strings correspond to solenoids which scatter charged particles. $I\!\!L$ is an integer, and if M/N is an integer too, then there is no long–range interaction; this situation corresponds to such a relation between the magnetic flux in the solenoid and the charge of the particle when the scattering of the charged particle is absent.

Another consequence of the Aharonov–Bohm effect can be obtained, if we consider the operator $\mathcal{F}_N(\mathcal{S})$ [20] which creates the string with the magnetic flux $\frac{2\pi}{Ne}$ moving along a fixed closed surface $\mathcal{S}$. $\mathcal{F}_N(\mathcal{S})$ is the analogue of the Wilson loop which creates the particle moving along the closed loop $\mathcal{C}$. An explicit form of $\mathcal{F}_N(\mathcal{S})$ is [20]:

$$\mathcal{F}_N(\mathcal{S}) = \exp\left\{ -\frac{\pi}{Ne} \int_{\mathcal{S}} \mathrm{d}\sigma_{\mu\nu} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}(x) \right\}. \qquad (22)$$

There exists an operator which can be calculated *exactly*, [19]; this operator is the normalized product of the Wilson loop $W_M(\mathcal{C})$ and $\mathcal{F}_N(\mathcal{S})$:

$$A_{NM}(\mathcal{S},\mathcal{C}) = \frac{\mathcal{F}_N(\mathcal{S})W_M(\mathcal{C})}{< \mathcal{F}_N(\mathcal{S}) >< W_M(\mathcal{C}) >}. \tag{23}$$

Here $< \mathcal{F}_N(\mathcal{S}) >$ is a constant which depends on the regularization scheme. Substituting this operator into the functional integral (17) and integrating over the fields A and θ, we obtain the following result:

$$< A_{NM}(\mathcal{S},\mathcal{C}) >= e^{2\pi i \frac{M}{N} \mathbb{L}(\mathcal{S},\mathcal{C})}. \tag{24}$$

The meaning of this result is very simple. If the surface $\mathcal{S}$ lies in a given time slice, then $< A_{NM}(\mathcal{S},\mathcal{C}) >= \exp\left\{\frac{2\pi i}{Ne}Q_{\mathcal{S}}\right\}$ (see [20, 19]), where $Q_{\mathcal{S}}$ is the total charge inside the volume bounded by the surface $\mathcal{S}$; if $\mathbb{L}(\mathcal{S},\mathcal{C}) = n$, then there is the charge Mne in the volume bounded by $\mathcal{S}$.

4   The String Creation Operator.

In Section 2 we have derived the partition function of the Abelian Higgs model as a sum over the closed world sheets of the ANO strings. Now we construct the operator which creates the string on a closed loop at a given time; after a while the string shrinks. The vacuum expectation value of this operator is the sum over all surfaces spanned on a given loop. A similar operator for the lattice theory was suggested in [6, 3]. The construction is quite the same as that of the soliton creation operator suggested by Fröhlich and Marchetti [23]. First we consider the model [11] which is dual to the original Abelian Higgs model. It contains the gauge field B_μ dual to the gauge field A_μ, and also the hypergauge field $h_{\mu\nu}$ dual to θ^r. As in eq. (12), we change the integration in θ^s to the integration in $\tilde{x}$. The details of the duality transformation are given in [11]. Taking into the account the Jacobian, we get:

$$\mathcal{Z} = \int \mathcal{D}h\mathcal{D}B\mathcal{D}\tilde{x}J(\tilde{x}) \exp\left\{- \int \mathrm{d}^4x\left[\frac{1}{3\eta^2}H_{\mu\nu\alpha}^2 + \frac{e^2N^2}{2}(h_{\mu\nu} - \partial_\mu B_\nu + \partial_\nu B_\mu)^2\right.\right.$$
$$\left.\left. + 2\pi i h_{\mu\nu}\Sigma_{\mu\nu}(x,\tilde{x})\right]\right\}, \tag{25}$$

where $H_{\mu\nu\sigma} = \partial_\mu h_{\nu\sigma} + \partial_\nu h_{\sigma\mu} + \partial_\sigma h_{\mu\nu}$ is the field strength of the hypergauge field $h_{\mu\nu}$. The action of the dual theory is invariant under the gauge transformations: $B_\mu(x) \rightarrow B_\mu(x) + \partial_\mu\alpha(x)$, $h_{\mu\nu}(x) \rightarrow h_{\mu\nu}(x)$, and under the hypergauge transformations: $B_\mu(x) \rightarrow B_\mu(x) - \gamma_\mu(x)$, $h_{\mu\nu}(x) \rightarrow h_{\mu\nu}(x) + \partial_\mu \gamma_\nu(x) - \partial_\nu \gamma_\mu(x)$.

The ANO string carries magnetic flux, and in order to construct the creation operator, it is natural to use the dual Wilson loop: $\mathcal{W}_D(\mathcal{C}) = \exp\{i \int \mathrm{d}^4x B_\mu(x)j_\mu^{\mathcal{C}}(x)\}$, where

the current $j_\mu^\mathcal{C}(x)$ defines the loop $\mathcal{C}$ (19). This operator is gauge invariant but it is not hypergauge invariant, and its vacuum expectation value is zero. To construct the hypergauge invariant operator [6, 3], we follow an idea of Dirac [24], who suggested the gauge invariant creation operator of a particle with the charge M:

$$\Phi_M^c(\mathbf{x}) = \Phi_M(\mathbf{x}) \exp \left\{ iMe \int \mathrm{d}^3 y G_l(\mathbf{x} - \mathbf{y}) A_l(\mathbf{y}) \right\}, \tag{26}$$

here $\partial_i G_i(\mathbf{x}) = \delta^{(3)}(\mathbf{x})$, and the gauge variation of the matter field $\Phi(x) \to \Phi_M(x) \exp\{iMe\alpha(x)\}$ is compensated by the gauge variation of cloud of photons A_μ. Now we use a similar construction, namely, we surround $\mathcal{W}_D(\mathcal{C})$ by the cloud of the Goldstone bosons:

$$U(\mathcal{C}) = \mathcal{W}_D(\mathcal{C}) \exp \left\{ \frac{i}{2} \int \mathrm{d}^3 y G_\mathcal{C}^{ij}(x - y) h_{ij}(y) \right\}. \tag{27}$$

It is easy to see that $U(\mathcal{C})$ is hypergauge invariant if the skew–symmetric tensor $G_\mathcal{C}^{ij}(x)$ satisfies the equation[5] $\partial_i G_\mathcal{C}^{ik}(x) = j_k^\mathcal{C}(x)$. It is convenient to choose $G_\mathcal{C}^{ik}(x)$ as the surface, spanned on the loop $\mathcal{C}$: $G_\mathcal{C}^{ij} = \int_{\mathcal{S}_\mathcal{C}} \mathrm{d}\sigma^{ij}(\tilde{x}) \delta^{(4)}[x - \tilde{x}(\sigma)]$ (cf. eq.(8)). Since the string creation operator should act at a definite time slice, the surface defined by $G_\mathcal{C}^{ij}(x)$ and the loop $\mathcal{C}$ should belong to that time slice[6].

Substituting the operator (27) into the dual partition function (25) and performing the inverse duality transformation, we get the vacuum expectation value of the string creation operator in terms of the original fields A and θ:

$$< U(\mathcal{C}) >= \frac{1}{\mathcal{Z}} \int \mathcal{D}A\mathcal{D}\theta \exp \left\{ - \int \mathrm{d}^4 x \left[\frac{1}{4} \left(F_{\mu\nu} + \frac{2\pi}{Ne} \epsilon_{\mu\nu\lambda\sigma} G_\mathcal{C}^{\lambda\sigma}(x) \right)^2 \right. \right.$$
$$\left. \left. + \frac{\eta^2}{2} (\partial_\mu \theta + NeA_\mu)^2 \right] \right\}, \tag{28}$$

where the tensor $G_\mathcal{C}^{\mu\nu}$ is equal to $G_\mathcal{C}^{ij}$ if $\mu = i$ and $\nu = j$ are spatial indices, and $G_\mathcal{C}^{0\mu} = G_\mathcal{C}^{\mu 0} = 0$ for any μ. If we change the field variables in (28) to the string variables, we get a sum over closed surfaces Σ:

$$< U(\mathcal{C}) >= \frac{1}{\mathcal{Z}} \int \mathcal{D}\tilde{x} J(\tilde{x}) \exp \left\{ -\eta^2 \pi^2 \int \mathrm{d}^4 x \int \mathrm{d}^4 y \left[(\Sigma^{\mu\nu}(x, \tilde{x}) + \right. \right. \tag{29}$$
$$\left. \left. G_\mathcal{C}^{\mu\nu}(x)) \mathcal{D}_m^{(4)}(x - y) (\Sigma^{\mu\nu}(y, \tilde{x}) + G_\mathcal{C}^{\mu\nu}(y)) \right] \right\}.$$

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[5]In this and in the next sections, Latin indices vary from 1 to 3 and Greek ones vary from 0 to 3.

[6]The solution of the equation $\partial_i G_\mathcal{C}^{ik}(x) = j_k^\mathcal{C}(x)$ is non–unique, moreover we choose a two dimensional surface as the support of G^{ik}, the solution which has three–dimensional support can be of the form: $G_\mathcal{C}^{ik} = \int \mathrm{d}^3 y \partial^{[i} j_\mathcal{C}^{k]}(\mathbf{y}) \mathcal{D}_0^{(3)}(\mathbf{x} - \mathbf{y})$, where $\mathcal{D}_0^{(3)} = -\frac{1}{4\pi|\mathbf{x} - \mathbf{y}|}$. It is easy to find that all these ambiguities do not change physical results.

The summation over all closed surfaces $\Sigma^{\mu\nu}$, plus the open surface $G^{\mu\nu}$ with the boundary $\mathcal{C}$, is equivalent to the summation over all closed surfaces and over *all* surfaces spanned on the loop $\mathcal{C}$. Therefore, the operator $U(\mathcal{C})$ creates a string on the loop $\mathcal{C}$. Using the string creation operators, it is easy to construct the operators which correspond to the processes of decay and scattering of the strings.

Note that from the eq.(28) it follows that the vacuum expectation value $\langle U(\mathcal{C}) \rangle$ in the euclidean theory is positively defined. The fact does not mean the existence of the string condensate, the situation is similar to the case of the Fröhlich–Marchetti monopole creation operator [23], see discussion in ref. [25].

If the string condensate is not zero then the infinitely large strings contribute to the vacuum state. Formally, the string condensate exists if in the limit $|x - y| \to \infty$:

$$< U(\mathcal{C}_1) \cdot U^+(\mathcal{C}_2) > \to A + B\, e^{-\mu r} + \cdots , \tag{30}$$

$A \neq 0$. In eq.(30) $\mathcal{C}_1$ and $\mathcal{C}_2$ are finite loops at which we create and annihilate string and x (y) is any point on the loop $\mathcal{C}_1$ ($\mathcal{C}_2$).

5 Aharonov–Bohm Effect In The Hamiltonian Formalism.

In this section we consider the ANO strings in the framework of the canonical quantization. We start with the standard commutation relations: $[\pi^i(\mathbf{x}), A^j(\mathbf{y})] = -i\delta_{ij}\delta(\mathbf{x} - \mathbf{y})$, $\pi^i = F^{0i}$ and $[\pi_\phi(\mathbf{x}), \phi(\mathbf{y})] = -i\delta(\mathbf{x} - \mathbf{y})$, $\pi_\phi = (D^0\phi)^*$. Using the string creation operators (22) and (27), we construct several operators, which satisfy the commutator relations of the type: $A \cdot B - B \cdot A e^{i\xi} = 0$. Similar operators are known for $3D$ Abelian models, see for example refs.[26]. The physical phenomenon leading to the nontrivial commutation relations in the nonabelian theories was discussed by 't Hooft [27].

First, let us consider the operator $U_{str}(\mathcal{C})$ which creates the ANO string on the loop $\mathcal{C}$:

$$U_{str}(\mathcal{C}) = \exp\left\{\frac{2\pi i}{Ne}\int \mathrm{d}^3 x \frac{1}{2}\epsilon_{ijk}G_{\mathcal{C}}^{ij}(x)\pi^k(x)\right\} , \tag{31}$$

here $G_{\mathcal{C}}^{ij}(x)$ is the same function as in eq.(27). The operator (31) is a special case of the creation operator:

$$U[A^{cl}] = \exp\left\{i\int \mathrm{d}^3 x A_k^{cl}(x)\pi^k(x)\right\} , \tag{32}$$

where $A^{cl}(x)$ is a classical field. It is easy to see, that $U[A^{cl}]|A(x) >= |A(x) + A^{cl}(x) >$. In (31) we have $A_k^{cl}(x) = \frac{2\pi}{Ne}\epsilon_{ijk}G_{\mathcal{C}}^{ij}(x)$, and the magnetic field corresponds to the infinitely thin string on the loop $\mathcal{C}$: $B_i(x) = \frac{2\pi}{Ne}j_i^{\mathcal{C}}(x)$; the current $j_i^{\mathcal{C}}$ is defined by eq. (19).

The commutation relations for the operator (31) with the operators of the electric charge $Q = \int \mathrm{d}^3 x \partial_i \pi^i(\mathbf{x})$ and the magnetic flux $\Phi_i = \int \mathrm{d}^3 x \epsilon_{ijk} \partial^j A^k(\mathbf{x})$ also show that $U_{str}(C)$ creates a string which carries the magnetic flux $\frac{2\pi}{Ne}$ on the contour $\mathcal{C}$:

$$[Q(x_0, \mathbf{x}), U_{str}(\mathcal{C})] = 0, \quad \left[\Phi^j(x_0, \mathbf{x}), U_{str}(\mathcal{C})\right] = \frac{2\pi}{Ne} j_i^{\mathcal{C}}(x) U_{str}(\mathcal{C}), \tag{33}$$

Note that, the string creation operator (28) considered in the previous section can be rewritten in the following way:

$$U(\mathcal{C}) = \exp\left\{-\int \mathrm{d}^4 x \left[\frac{1}{4}\left(F_{\mu\nu} + \frac{2\pi}{Ne}\epsilon_{\mu\nu\lambda\sigma} G_{\mathcal{C}}^{\lambda\sigma}(x)\right)^2 - \frac{1}{4}F_{\mu\nu}^2\right]\right\}, \tag{34}$$

and it is clear that, up to an inessential factor, it coincides with the definition (31).

Now we consider the commutator of the operator $U_{str}(\mathcal{C}_1)$ and the Wilson loop $W_M(\mathcal{C}_2)$ (18), the contours $\mathcal{C}_1$ and $\mathcal{C}_2$ belong to the same time slice. Using the relation $e^A e^B = e^B e^A e^{[A,B]}$, which is valid if $[A, B]$ is a c–number, it is easy to get:

$$U_{str}(\mathcal{C}_1) W_M(\mathcal{C}_2) - e^{i\xi(\mathcal{C}_1, \mathcal{C}_2)} W_M(\mathcal{C}_2) U_{str}(\mathcal{C}_1) = 0, \tag{35}$$

where $\xi(\mathcal{C}_1, \mathcal{C}_2) = \frac{2\pi M}{N} \mathbb{L}(\mathcal{C}_1, \mathcal{C}_2)$, and $\mathbb{L}(\mathcal{C}_1, \mathcal{C}_2) = \frac{1}{4\pi} \int_{\mathcal{C}_1} \mathrm{d}x_i \int_{\mathcal{C}_2} \mathrm{d}y_i \epsilon^{ijk} \frac{(x-y)_k}{|\mathbf{x}-\mathbf{y}|^3}$ is the linking number of the loops $\mathcal{C}_1$ and $\mathcal{C}_2$. The commutation relation (35) is the direct consequence of the Aharonov–Bohm effect; the wave function of the particle of the charge Me acquires the additional phase $e^{\frac{2\pi i M}{N}}$ if it goes around a solenoid with the magnetic flux $\frac{2\pi}{Ne}$.

The next example is the commutation relation of the Dirac operator $\Phi_M^c(\mathbf{x})$ (26) which creates the particle with charge M at the point $\mathbf{x}$, and the operator $\mathcal{F}_N(\mathcal{S})$ which creates the string on the surface $\mathcal{S}$. In Minkowsky space, the operator $\mathcal{F}_N(\mathcal{S})$ has the form (an analogue of eq.(22)):

$$\mathcal{F}_N(\mathcal{S}) = \exp\left\{\frac{i\pi}{Ne} \int_{\mathcal{S}} \mathrm{d}\sigma_{\mu\nu} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}(x)\right\}. \tag{36}$$

If the surface $\mathcal{S}$ belongs to the same time slice as the point $\mathbf{x}$, then:

$$\mathcal{F}_N(\mathcal{S}) \Phi_M^c(\mathbf{x}) - \Phi_M^c(\mathbf{x}) \mathcal{F}_N(\mathcal{S}) e^{i\theta(\mathcal{S}, \mathbf{x})} = 0, \tag{37}$$

where $\theta(\mathcal{S}, \mathbf{x}) = \frac{2\pi M}{N} \Theta(\mathcal{S}, \mathbf{x})$. The function $\Theta(\mathcal{S}, \mathbf{x})$ is the "linking number" of the surface $\mathcal{S}$ and the point $\mathbf{x}$:

$$\Theta(\mathcal{S}, \mathbf{x}) = \begin{cases} 1 & \text{if } x \text{ lies inside volume bounded by } \mathcal{S}; \\ 0 & \text{otherwise} \end{cases}. \tag{38}$$

It is obvious that the commutation relation (37) is also a consequence of the Aharonov–Bohm effect.

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Now consider the composite operator

$$H_{MN}(\mathbf{x}, \mathcal{S}) = \Phi_M^c(\mathbf{x})\, \mathcal{F}_N(\mathcal{S})\,, \tag{39}$$

where the surface $\mathcal{S}$ lies at the same time slice as the point $\mathbf{x}$. Using commutation relation (37) it is easy to find that:

$$H_{M_1,N}(\mathbf{x}_1, \mathcal{S}_1)\, H_{M_2,N}(\mathbf{x}_2, \mathcal{S}_2) - H_{M_2,N}(\mathbf{x}_2, \mathcal{S}_2)\, H_{M_1,N}(\mathbf{x}_1, \mathcal{S}_1)\, e^{i\zeta_{12}} = 0\,, \tag{40}$$

where $\zeta_{12} = \frac{2\pi M_1}{N}\Theta(\mathbf{x}_1, \mathcal{S}_2) - \frac{2\pi M_2}{N}\Theta(\mathbf{x}_2, \mathcal{S}_1)$. If the point $\mathbf{x}_1$ lies in the volume bounded by $\mathcal{S}_2$, the point $\mathbf{x}_2$ lies out of the volume $\mathcal{S}_1$, $M_{1,2} = 1$ and $N = 2$, then eq.(40) leads to the fermion–like commutation relation

$$H(\mathbf{x}_1)H(\mathbf{x}_2) + H(\mathbf{x}_2)H(\mathbf{x}_1) = 0\,, \tag{41}$$

where $H(\mathbf{x}_i) = H_{M_i}(\mathbf{x}_i, \mathcal{S}_i)$.

The commutation relations (40) and (41) can be explained as follows. The operator $\mathcal{F}_N(\mathcal{S})$ creates the closed world sheet of the ANO string and the configuration space of the (charged) particles becomes not simply connected. Similar reasons lead to nontrivial statistics in $2+1$ dimensions [28]. Note that all operators and commutation relations considered in the present section can be constructed in the free theory, but the states created by the operators $U_{str}(\mathcal{C}_1)$ and $\mathcal{F}_N(\mathcal{S})$ are very unstable in this case. In the Abelian Higgs theory, the ANO strings exist as a solution of the classical equations of motion, and this fact justifies the study of the commutation relations which contain string creation operators.

Acknowledgments

Appendix A

Below we show how to derive the expression (15) for $J(\tilde{x})$. We start from the following definition of $J(\tilde{x})$ (see eqs. (11) and (12)):

$$[J(\tilde{x})]^{-1} = \int \mathcal{D}\tilde{y}_\mu \cdot \delta \left\{ \Sigma_{\mu\nu}(x, \tilde{y}) - \Sigma_{\mu\nu}(x, \tilde{x}) \right\}, \tag{A.1}$$

where $\Sigma_{\mu\nu}(x, \tilde{x})$ and $\Sigma_{\mu\nu}(x, \tilde{y})$ are defined by (8).

First we represent functional δ–function in (A.1) as:

$$\delta \left\{ \Sigma_{\mu\nu}(x, \tilde{y}) - \Sigma_{\mu\nu}(x, \tilde{x}) \right\} =$$
$$const. \cdot \int \mathcal{D}k_{\mu\nu}(x) \exp \left\{ i \int d^4x k_{\mu\nu} \left(\Sigma_{\mu\nu}(x, \tilde{y}) - \Sigma_{\mu\nu}(x, \tilde{x}) \right) \right\} =$$
$$const. \cdot \int \mathcal{D}k_{\mu\nu}(x) \exp \left\{ i \int_\Sigma d^2\sigma k_{\mu\nu}[\tilde{x}(\sigma)]\sqrt{g}t_{\mu\nu} - i \int_{\Sigma'} d^2\sigma' k_{\mu\nu}[\tilde{y}(\sigma')]\sqrt{h}\tau_{\mu\nu} \right\}, \tag{A.2}$$

where g and $t_{\mu\nu}$ are the same as in (8), $h_{ab} = \partial_a \tilde{y}_\mu \partial_b \tilde{y}_\mu$, $h = det||h_{ab}||$ and $\tau_{\mu\nu} = \frac{\epsilon_{ab}}{\sqrt{h}} \partial_a \tilde{y}_\mu \partial_b \tilde{y}_\nu$. Functional integral over $k_{\mu\nu}$ leads to:

$$[J(\tilde{x})]^{-1} = const \cdot \int \mathcal{D}\tilde{y}_\mu \prod_{\mu<\nu} \delta \left(\sqrt{h}\tau_{\mu\nu} - \sqrt{g}t_{\mu\nu} \right). \tag{A.3}$$

Consider now the following functional integral:

$$I(\tilde{x}) = \cdot \int \mathcal{D}\tilde{y}_\mu \cdot \Delta \cdot \delta \left(h^3 - g^3 \right) \prod_{\mu<\nu} \delta(\tau_{\mu\nu} - t_{\mu\nu}) =$$
$$const \cdot \int \mathcal{D}\tilde{y}_\mu \cdot \Delta \cdot \delta \left(1 - \left(\frac{g}{h} \right)^3 \right) \prod_{\mu<\nu} \delta(\sqrt{h}\tau_{\mu\nu} - \sqrt{g}t_{\mu\nu}). \tag{A.4}$$

Due to the second δ–function, $\sqrt{h}\tau_{\mu\nu} = \sqrt{g}t_{\mu\nu}$, and we should assume some regularization[7] of the first δ–function: $\delta \left(1 - \left(\frac{g}{h} \right)^3 \right) = \delta^{reg}(0)$. The next transformations can be accurately performed in the discreetized space[8], but we simply introduce the parameter Λ which plays the role of the inverse thickness of the string or the ultraviolet cut–off.

$$I(\tilde{x}) = \int \mathcal{D}\tilde{y}_\mu \cdot \Delta \cdot \left(\prod_{x \in \Sigma} \delta^{reg}(0) \right) \prod_{\mu<\nu} \delta(\sqrt{h}\tau_{\mu\nu} - \sqrt{g}t_{\mu\nu})$$
$$= const \cdot \int \mathcal{D}\tilde{y}_\mu \cdot \Delta \cdot \exp \left\{ \mu S(\Sigma) \right\} \prod_{\mu<\nu} \delta(\sqrt{h}\tau_{\mu\nu} - \sqrt{g}t_{\mu\nu}), \tag{A.5}$$

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[7] For example $\delta^{reg}(x) = \frac{M}{\sqrt{2\pi}} \exp\{-M^2|x|^2\}$, $M \to \infty$.
[8] The analogous trick was used in [29].

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where $\mu = \Lambda^2 \ln\left(\delta^{reg}(0)\right)$, $S(\Sigma) = \int_\Sigma d^2\sigma' \sqrt{h(\sigma')}$. The term $\exp\{\mu S(\Sigma)\}$ in (A.5) is due to the infinite product of $\delta^{reg}(0)$ over all the points on the surface Σ. If we now set

$$\Delta = const \cdot \exp\{-\mu S(\Sigma)\}\,, \tag{A.6}$$

then $I(\tilde{x}) = [J(\tilde{x})]^{-1}$, and

$$[J(\tilde{x})]^{-1} = const \cdot \int \mathcal{D}\tilde{y}_\mu \cdot \exp\{-\mu S(\Sigma)\}\,\delta\left(h^3 - g^3\right) \prod_{\mu<\nu} \delta(\tau_{\mu\nu} - t_{\mu\nu})\,. \tag{A.7}$$

The transformations (A.4) and (A.5) seem to be not very strict: we have to use $\delta^{reg}(0)$ and the regularization parameter Λ. A more accurate derivation of (A.7) can be done if we notice that (A.1) and (A.2) is the theory of the Kolb–Ramond field $k_{\mu\nu}(\tilde{y})$, which interacts with the Nambu–Goto string, the bare string tension being equal to zero. It is important that (A.7) is the string theory in which, as we show, the conformal anomaly naturally arises. This conformal anomaly cancels the conformal anomaly of the original theory (9), (13).

Substituting into (A.7) the unity of the form:

$$1 = \int \mathcal{D}h_{ab} \cdot \delta\left(h_{ab} - \partial_a\tilde{y}_\mu\partial_b\tilde{y}_\mu\right)\mathcal{D}\tau_{\mu\nu} \cdot \delta\left(\tau_{\mu\nu} - \frac{\epsilon^{ab}}{\sqrt{h}}\partial_a\tilde{y}_\mu\partial_b\tilde{y}_\nu\right) \tag{A.8}$$

we obtain:

$$[J(\tilde{x})]^{-1} = const. \cdot \int \mathcal{D}\tilde{y} \cdot \mathcal{D}h_{ab} \cdot \delta\left(h_{ab} - \partial_a\tilde{y}_\mu\partial_b\tilde{y}_\mu\right)\mathcal{D}\tau_{\mu\nu}$$

$$\cdot\delta\left(\tau_{\mu\nu} - \frac{\epsilon^{ab}}{\sqrt{h}}\partial_a\tilde{y}_\mu\partial_b\tilde{y}_\nu\right) \cdot \delta(g^3 - h^3) \cdot \delta(t_{\mu\nu} - \tau_{\mu\nu})e^{-\mu\int_\Sigma d^2\sigma'\sqrt{h}}, \tag{A.9}$$

It is possible to make the following transformation:

$$\int \mathcal{D}h_{ab} \cdot \delta(h_{ab} - \partial_a\tilde{y}_\mu\partial_b\tilde{y}_\mu)\mathcal{D}\tau_{\mu\nu} \cdot \delta\left(\tau_{\mu\nu} - \frac{\epsilon_{ab}}{\sqrt{h}}\partial_a\tilde{y}_\mu\partial_b\tilde{y}_\nu\right)\cdot$$

$$\exp\left\{-\mu\int_\Sigma d^2\sigma'\sqrt{h(\sigma')}\right\}\ldots = const \cdot \int \mathcal{D}h_{ab}\mathcal{D}\tau_{\mu\nu}$$

$$\cdot\exp\left\{-\int_\Sigma d^2\sigma'\left[\frac{\mu_1}{2}\sqrt{h}\tau_{\mu\nu}^2 + \mu_2\sqrt{h}h^{ab}\partial_a\tilde{y}_\mu\partial_b\tilde{y}_\mu + \mu_3\tau_{\mu\nu}\epsilon_{ab}\partial_a\tilde{y}_\mu\partial_b\tilde{y}_\nu\right]\right\}\ldots \tag{A.10}$$

The fields h_{ab} and $\tau_{\mu\nu}$ have no kinetic terms in the action, and therefore they acquire their classical values, (see [12], or chapter 9 of [30]). If surface Σ has spherical topology we can

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fix the conformal gauge globally on the surface. To this end, we substitute the following unity into the integral:

$$1 = \int \mathcal{D}f \delta(h_{12}^f)\, \delta(h_{11}^f - h_{22}^f) \cdot \Delta_{FP} \tag{A.11}$$

where $\mathcal{D}f$ means integration over all possible reparametrizations, and h_{ab}^f is the change of the metric under a particular reparametrization f. In the last formula, the Faddeev–Popov determinant Δ_{FP} appears, which is the exponent of the Liouville action in conformal gauge with the central charge -26 (see [30]). The next step is the integration over $\tilde{y}_\mu$. The first two terms in the exponent (A.10) give the Liouville action with the central charge equal to $D = 4$ (see [30]). And the third term leads to the term $\frac{\mu_3^2 \ln(\Lambda \bar{R})}{\mu_2^2 \pi} \int_\Sigma \mathrm{d}^2\tau \sqrt{h}(\partial_a \tau_{\mu\nu})^2$ in the Jacobian. The integration over h and $\tau_{\mu\nu}$ leads to the expression (15).

Note that all transformations can be performed in an arbitrary gauge, if instead of (A.11) we use the gauge fixing term in the form:

$$1 = \int \mathcal{D}f \prod_{a=b} \delta(h_{ab}^f - g_{ab}) \cdot \Delta_{FP} \,, \tag{A.12}$$

here Δ_{FP} is the exponent of the Liouville action in the considered gauge.

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