Resolution of Gauss' law in Yang-Mills theory by Gauge Invariant Projection:

Topology and Magnetic Monopoles†

H. Reinhardt§
Institut für Theoretische Physik, Universität Tübingen,
Auf der Morgenstelle 14, D-72076 Tübingen, Germany

Abstract

An efficient way of resolving Gauss' law in Yang-Mills theory is presented by starting from the projected gauge invariant partition function and integrating out one spatial field variable. In this way one obtains immediately the description in terms of unconstrained gauge invariant variables which was previously obtained by explicitly resolving Gauss' law in a modified axial gauge. In this gauge, which is a variant of 't Hooft's Abelian gauges, magnetic monopoles occur. It is shown how the Pontryagin index of the gauge field is related to the magnetic charges. It turns out that the magnetic monopoles are sufficient to account for the non-trivial topological structure of the theory.

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§Electronic address: reinhardt@uni-tuebingen.de

1 Introduction

Quantization of gauge theory can be accomplished in two basically different ways: canonical quantization and path integral quantization. The equivalence of both approaches has been established e.g. in refs. [1], [2]. The fundamental problem in dealing with quantum Yang-Mills theory is the elimination of the unphysical gauge degrees of freedom. This can be accomplished by either reformulating the theory in manifestly gauge invariant variables [3] or by explicitly resolving the Gauss law constraint [4].

Recently, for Yang-Mills theory on a torus, a complete resolution of Gauss' law has been accomplished in a modified axial gauge in both the canonical operator approach [5] and in the Hamilton functional integral approach [6]. These approaches end up with the description of Yang-Mills theory in terms of two transverse degrees of freedom (for each group generator) and a reduced Abelian field, living in the Cartan subgroup and in $D-1$ dimensions. This seems to be the minimum number of gauge invariant degrees of freedom necessary to describe Yang-Mills theory, at least on a torus. In the present paper I show that, by starting from the projected gauge invariant partition function, a more efficient resolution of Gauss' law is achieved by applying the Weyl integration formula to the integration over the gauge group. The result agrees with that obtained by a resolution of Gauss' law in modified axial gauge [5], [6]. In this gauge, which is equivalent to the so-called Polyakov gauge, and represents a variant of 't Hooft's Abelian gauge [7], magnetic monopoles arise. Lattice calculations [8] indicate that magnetic monopoles are possibly the dominant infrared degrees of freedom, at least in the so-called maximal Abelian gauge [9] and also in the Polyakov gauge [10]. There are, however, also critical remarks on Abelian projection [11] which favour center dominance, where vortices rather than monopoles are the relevant infrared degrees of freedom.

Recently much work has been devoted to the relation between magnetic monopoles and instantons [12]. In ref. [13] it was found that in the maximal Abelian gauge a monopole trajectory goes around an instanton.[1] In the Polyakov gauge a monopole trajectory was found to pass through the center of the instanton [15]. The distribution of monopoles in a dilute instanton gas was determined in ref. [16] in a different Abelian gauge. Recent lattice calculations [12], [17] indicate that there is an intricate relation between instantons and magnetic monopoles.

The essential features of the instantons are their topological properties, which are measured by the Pontryagin index. Since monopoles are the sources of long range fields they should influence the topology of the gauge fields. In this paper I derive an exact relation between the Pontryagin index of the gauge field and the magnetic charges of the monopoles contained in the gauge field.

The organization of the paper is as follows: In sect. 2 I start from Yang-Mills theory in the Weyl gauge and define the gauge invariant partition function. Gauge invariance

[1] In ref. [14] it was found that a monopole trajectory passes through the center of each instanton. However, in this case a gauge was adopted for which the pertinent gauge functional diverges rather than becoming minimal.

is achieved here by integrating over all gauge equivalent initial states with the Haar measure of the gauge group. I consider then the path integral representation of the gauge invariant partition function and convert it to the standard Yang-Mills functional integral by a time-dependent gauge transformation, where the latter needs to have zero winding number. This is shown in sect. 3, where the topology of the gauge field is considered. In sect. 4, I perform a Cartan decomposition of the gauge group. By applying the Weyl integration formula, the integration over the coset $SU(N)/U(1)^{N-1}$ can be explicitly performed, leaving from the gauge invariant projection a residual integration over the Cartan subgroup. Integrating out one spatial component of the gauge field I obtain the desired representation of the Yang-Mills partition function in gauge fixed, unconstrained variables, which was previously derived by resolving Gauss' law in a modified axial gauge [5], [6]. In sect. 5, I discuss the emergence of magnetic monopoles as a consequence of the Cartan decomposition of the gauge group and performing a coset gauge transformation. Finally, in sect. 6 the relation between the Pontryagin index and the magnetic charges of the monopoles is derived. A short summary and some concluding remarks are given in sect. 7. A few generic examples of monopole type of fields induced by singular gauge transformations are presented in the appendix.

2 The Gauge Invariant Partition Function

We consider Yang-Mills theory with the gauge group $G = SU(N)$. The quantum theory is defined in the Weyl gauge $A_0 = 0$ by the Hamiltonian

$$H = \int d^3x \left(\frac{g^2}{2} E_i^a(x) E_i^a(x) + \frac{1}{2g^2} B_i^a(x) B_i^a(x) \right) . \tag{1}$$

Here, the electric field $E_k^a(x) = \frac{1}{i} \frac{\delta}{\delta A_k^a(x)}$ is the canonical momentum conjugate to the field coordinate $A_k^a(x)$ and $B_k^a = \epsilon_{kij}(\partial_i A_j^a + \frac{1}{2} f^{abc} A_i^b A_j^c)$ is the magnetic field. Furthermore, g is the (bare) coupling constant and f^{abc} is the structure constant of the gauge group.

Let $|C\rangle$ denote an eigenstate of $A_i(x)$, i.e.

$$A_i(x)|C\rangle = C_i(x)|C\rangle , \tag{2}$$

where $C_i(x)$ is some classical field function. The gauge invariant partition function of Yang-Mills theory can then be defined as (see e.g. refs. [1], [18], [19])

$$Z = \int \mathcal{D}C_i(x) \langle C| e^{-HT} P |C\rangle , \tag{3}$$

where the (functional) integration runs over all classical field functions $C_i(x)$ and P projects onto gauge invariant states:

$$P|C\rangle = \sum_n e^{-in\Theta} \int\limits_G \mathcal{D}\mu(\Omega_n) |C^{\Omega_n}\rangle \tag{4}$$

Here, $A_i^\Omega(x)$ denotes the gauge transform of the gauge field defined by[2]

$$A_i^\Omega = \Omega A_i \Omega^\dagger + \Omega \partial_i \Omega^\dagger \,. \tag{5}$$

Furthermore, Θ is the vacuum angle [21] and $\mu(\Omega)$ denotes the Haar measure of the gauge group. The integration runs over all time-independent gauge transformations Ω_n with winding number n, which is defined by

$$n[\Omega] = -\frac{1}{24\pi^2}\int d^3x\, \epsilon^{ijk} \mathrm{Tr} L_i L_j L_\kappa \,, \quad L_\kappa = \Omega \partial_\kappa \Omega^\dagger \,. \tag{6}$$

As usual we assume here that the gauge function $\Omega(\mathbf{x})$ approaches a unique value Ω_∞ for $|\mathbf{x}| \to \infty$, so that R^3 can be compactified to S^3 and $n[\Omega]$ is a topological invariant. For later convenience we will choose

$$\lim_{|\mathbf{x}|\to\infty} \Omega_n(\mathbf{x}) = \Omega_\infty^n = (-1)^n \,. \tag{7}$$

Following the standard procedure [22] one derives the following functional integral representation of the Yang-Mills transition amplitude:

$$\langle C | e^{-HT} | C' \rangle = \int_{C'}^{C} \mathcal{D}A_i \, \exp(-S_{\mathrm{YM}}[A_0 = 0, \mathbf{A}]) \tag{8}$$

Here, the functional integration runs over all field configurations $A_i(x)$ satisfying the boundary conditions $A_i(x_0 = 0, \mathbf{x}) = C_i'(\mathbf{x})$, $A_i(x_0 = T, \mathbf{x}) = C_i(\mathbf{x})$. Furthermore

$$S_{\mathrm{YM}}[A_0, \mathbf{A}] = -\frac{1}{2g^2}\int d^4x \mathrm{Tr} F_{\mu\nu}(x) F_{\mu\nu}(x) \tag{9}$$

is the standard Yang-Mills action with

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c \tag{10}$$

denoting the field strength. Inserting eq. (8) into eq. (3) we obtain for the partition function

$$Z = \sum_n e^{-in\Theta} \int_G \mathcal{D}\mu(\Omega_n) \int_{C^\Omega}^{C} \mathcal{D}A_i \quad e^{-S_{\mathrm{YM}}[A_0=0,\mathbf{A}]} \,, \tag{11}$$

where the functional integration is performed with boundary conditions

$$A_i(x_0 = 0, \mathbf{x}) = C_i^\Omega(\mathbf{x}) \,, \quad A_i(x_0 = T, \mathbf{x}) = C_i(\mathbf{x}) \,. \tag{12}$$

[2]We are using antihermitian fields $A_\mu = A_\mu^a T^a$ with generators T^a satisfying $[T^a, T^b] = f^{abc} T^c$, $\mathrm{Tr}(T^a T^b) = -\frac{1}{2}\delta^{ab}$.

Note that equation (11) is almost the standard Yang-Mills functional integral representation except that, instead of an integral over the time component of the gauge field A_0, we have the integration over the gauge group with the Haar measure.[3] The ultimate relation between the gauge transformation and the time component of the gauge field can be easily established by performing the time-dependent gauge transformation $A \to A^U$ with

$$U = \Omega_n^{\frac{x_0}{T}-1} , \tag{13}$$

which removes the gauge transformation Ω_n from the initial values of the spatial gauge fields (12) but introduces at the same time a time-independent temporal gauge potential

$$A_0 = -\frac{1}{T} \ln \Omega_n . \tag{14}$$

The partition function then becomes (for simplicity A^U_μ is replaced by A_μ)

$$Z = \sum_n e^{-in\Theta} \int_G \mathcal{D}\mu(e^{-TA_0}) \int \mathcal{D}A_i \, e^{-S_{\rm YM}[A_0,\mathbf{A}]} , \tag{15}$$

where the functional integration now runs over temporally periodic spatial gauge fields $A_i(T, \mathbf{x}) = A_i(0, \mathbf{x})$. This is almost the standard Yang-Mills functional integral except for the presence of the Haar measure for the temporal gauge field (14) and the absence of the gauge fixing by the Faddeev-Popov method. It has been shown [2], however, that the representation (15) is completely equivalent to the standard Yang-Mills functional integral, where the integration is performed with a flat measure over all four components of the gauge field $A_\mu(x)$ but with the gauge fixed by the Faddeev-Popov method. In this case the Haar measure arises from the Faddeev-Popov determinant. In the equivalence proof given in ref. [2] it was tacitly assumed that the gauge transformation (13) does not change the topological properties of the gauge field. To be more precise, for $\Theta \neq 0$ the equivalence of eq. (15) with the standard Yang-Mills functional integral requires that the winding number n of the gauge function Ω coincides with the (negative) Pontryagin index

$$\nu[A] = -\frac{1}{16\pi^2} \int d^4x {\rm Tr} F_{\mu\nu} \tilde{F}_{\mu\nu} , \quad \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\kappa\lambda} F_{\kappa\lambda} \tag{16}$$

of the gauge rotated field A^U_μ. This will be shown in the next section.

3 Topology of the Gauge Field

The gauge fields are topologically classified by the Pontryagin index (16). It is well known that the integrand of eq. (16) is a total derivative

$$-\frac{1}{16\pi^2} {\rm Tr} F_{\mu\nu} \tilde{F}_{\mu\nu} = \partial_\mu X_\mu , \tag{17}$$

[3]The importance of the Haar measure has been emphasized in ref. [19]. Let me also mention that eq. (11) was previously obtained in ref. [20].

where

$$X_\mu[A] = -\frac{1}{16\pi^2}\epsilon_{\mu\nu\kappa\lambda}\mathrm{Tr}\left\{\frac{1}{2}A_\nu\partial_\kappa A_\lambda + \frac{1}{3}A_\nu A_\kappa A_\lambda\right\} \tag{18}$$

is the topological current. For smooth gauge fields[4] we can apply Gauss' theorem to obtain

$$\nu[A] = \int_{\partial M} d\Sigma_\mu X_\mu[A]\,. \tag{19}$$

Under a gauge transformation the topological current transforms as

$$X_\mu[A^U] = X_\mu[A] + \frac{1}{16\pi^2}\epsilon_{\mu\alpha\beta\gamma}\mathrm{Tr}\Big\{2\partial_\beta(L_\alpha A_\gamma) + 2A_\alpha\ell_{\beta\gamma} + L_\alpha\ell_{\beta\gamma} - \frac{2}{3}L_\alpha L_\beta L_\gamma\Big\} \tag{20}$$

with the current

$$L_\alpha = U\partial_\alpha U^\dagger \tag{21}$$

and

$$\ell_{\mu\nu} = U(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)U^\dagger\,. \tag{22}$$

The last quantity vanishes for non-singular gauge functions $U(x)$, which we will assume. Furthermore, for smooth $U(x)$ the surface term in (20) will not contribute and we obtain

$$\nu[A^U] = \nu[A] + \bar{n}[U]\,, \tag{23}$$

where

$$\bar{n}[U] \equiv \nu[U\partial U^\dagger] = -\frac{1}{24\pi^2}\int\limits_{\partial M} d^3\Sigma_\mu\epsilon_{\mu\alpha\beta\gamma}\mathrm{Tr}(L_\alpha L_\beta L_\gamma) \tag{24}$$

is the extension of the winding number (6) to time-dependent gauge functions $U(\mathbf{x}, t)$ (see below).

For sake of completeness let us first determine the Pontryagin index (19) in the Weyl gauge $A_0 = 0$. The integration in eq. (19) is over the surface of four-dimensional Euclidean space, which, for a finite time interval T, is given by a four-dimensional cylinder, $M = [0, 1]\times B^3_\infty$, see fig. 1a. The faces of the cylinder are given by our ordinary 3-dimensional space, which we assume here to be a 3-dimensional ball B^3_∞ of infinite radius.[5] The mantle of the cylinder does not contribute to the Pontryagin index (19) in the Weyl gauge provided

[4]We will assume that the gauge fields $A_i(x)$ are smooth in the Weyl gauge.

[5]In view of the boundary condition (7) the 3-dimensional space B^3_∞ can be topologically compactified to a sphere S^3_∞ of infinite radius. Let us emphasize that compactification is understood here only in the topological sense but not in the metrical sense.

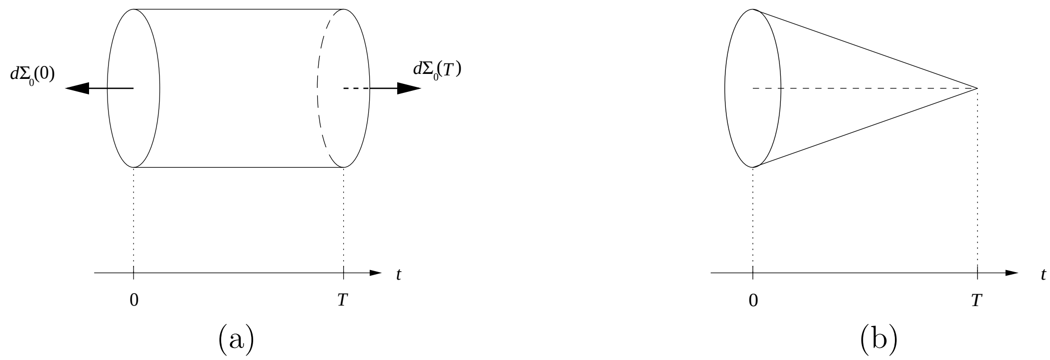

Figure 1: Compactification of four-dimensional Euclidean space from a cylinder (a) to a cone (b) by contracting the face of the cylinder at $t = T$ to a single point.

the field configurations $A_i(x)$ drop off faster than $1/|\mathbf{x}|$ at spatial infinity $|\mathbf{x}| \to \infty$.[6] Furthermore, since

$$d^3\Sigma_0(t = T) = -d^3\Sigma_0(t = 0) = d^3x \tag{25}$$

the contribution from the faces of the cylinder reads

$$\nu[A] = W[A(t = T)] - W[A(t = 0)]\,, \tag{26}$$

where

$$W[A] = \int d^3x\, X_0[A] \tag{27}$$

is the charge of the topological current (20), which is nothing but the Chern-Simons action. Therefore, for gauge fields which contribute to the partition function, i.e. satisfy the boundary conditions (12), we obtain for the Pontryagin index

$$\nu[A] = W[C] - W[C^\Omega]\,. \tag{28}$$

From the transformation property of the topological current (20) we obtain for smooth $\Omega(\mathbf{x})$

$$W[A^\Omega] = W[A] + n[\Omega] \tag{29}$$

and from eq. (28), it follows that

$$\nu[A] = -n[\Omega]\,, \tag{30}$$

[6] This is the case in the absence of magnetic monopoles. We will assume here that magnetic monopoles are absent in the Weyl gauge. Magnetic monopoles will later on occur as gauge artifacts after performing singular gauge transformations, i.e. by going from the smooth Weyl gauge to singular gauges.

where $n[\Omega]$ (6) is the reduction of the winding number $\bar{n}$ (24) to time-independent $\Omega(\mathbf{x})$. It is this quantity $n[\Omega] \in \Pi_3(S^3)$ which is usually referred to as "winding number".

It remains to be shown how the Pontryagin index changes under the time-dependent gauge transformation $U(x)$ (13), see eq. (23). In this case the integration in the expression for the winding number (24) is over the four-dimensional cylinder illustrated in fig. 1a. With eq. (25) we can rewrite the winding number as

$$\bar{n}[U] = \int d^3x X_0(t=T) - \int d^3x X_0(t=0) + \int_{\mathcal{M}} d^3\Sigma_i X_i \tag{31}$$

where $\mathcal{M}$ denotes the mantle of the cylinder. Since $U(t=T) = 1$, the integrand of the first term vanishes. Furthermore, since $U(t=0) = \Omega^{-1}$, the second term yields the contribution

$$-\, n[\Omega^{-1}] = n[\Omega]\,. \tag{32}$$

Let us now calculate the contribution from the mantle of the cylinder (the last term in eq. (31)). Exploiting the properties of the anti-symmetric tensor $\epsilon_{\mu\alpha\beta\gamma}$, the mantle contribution becomes ($\epsilon_{0ijk} = \epsilon_{ijk}$)

$$\bar{n}^{(\mathcal{M})}[U] = \frac{1}{16\pi^2} \int_{\mathcal{M}} d^3\Sigma_i \epsilon_{ijk} \,\mathrm{Tr}(L_0[L_j, L_k])\,. \tag{33}$$

Using $F_{ij}[L] = 0$, i.e.

$$[L_i, L_j] = -(\partial_i L_j - \partial_j L_i)\,, \tag{34}$$

we can rewrite the above expression as

$$\bar{n}^{(\mathcal{M})}[U] = -\frac{1}{8\pi^2} \int_{\mathcal{M}} d^3\Sigma_i \epsilon_{ijk} \mathrm{Tr}(L_0 \partial_j L_k)\,. \tag{35}$$

To simplify the following calculations I will restrict myself to the gauge group $G = SU(2)$. Parameterizing the group element by ($T^a = -\frac{i}{2}\tau^a$, where τ^a are the Pauli matrices)

$$\Omega = \exp(i\boldsymbol{\chi\tau}) = \cos\chi + i\boldsymbol{\tau}\hat{\boldsymbol{\chi}} \sin\chi\,, \quad \chi = |\boldsymbol{\chi}|\,, \quad \hat{\boldsymbol{\chi}} = \boldsymbol{\chi}/\chi \tag{36}$$

we find

$$U = \Omega^{\frac{t}{T}-1} = \cos\bar{\chi} - i\hat{\boldsymbol{\chi}}\boldsymbol{\tau} \sin\bar{\chi}\,, \quad \bar{\chi} = \chi\left(1 - \frac{t}{T}\right) \tag{37}$$

and therefore

$$L_t = U\partial_t U^\dagger = -\frac{i}{T}\boldsymbol{\chi\tau}\,. \tag{38}$$

For the spatial currents straightforward evaluation yields

$$\begin{aligned} L_k &= U\partial_k U^\dagger \\ &= i\hat{\boldsymbol{\chi}}\boldsymbol{\tau}\partial_k\bar{\chi} + i\sin\bar{\chi}\cos\bar{\chi}\partial_k\hat{\boldsymbol{\chi}}\cdot\boldsymbol{\tau} + i\sin^2\bar{\chi}(\hat{\boldsymbol{\chi}}\times\partial_k\hat{\boldsymbol{\chi}})\boldsymbol{\tau}\,. \end{aligned} \tag{39}$$

Hence we find

$$\epsilon_{ijk}\mathrm{Tr}(L_0\partial_j L_k) = 2\epsilon_{ijk}\frac{\chi}{T}\sin^2\bar{\chi}(\partial_j\hat{\boldsymbol{\chi}}\times\partial_k\hat{\boldsymbol{\chi}})\cdot\hat{\boldsymbol{\chi}}\,. \tag{40}$$

Inserting the last expression into eq. (35), we obtain for the mantle contribution

$$\bar{n}^{(\mathcal{M})}[U] = -\frac{1}{4\pi^2}\int\limits_{\mathcal{M}} d^3\Sigma_i\frac{\chi}{T}\sin^2\bar{\chi}\epsilon_{ijk}(\partial_j\hat{\boldsymbol{\chi}}\times\partial_k\hat{\boldsymbol{\chi}})\hat{\boldsymbol{\chi}}\,. \tag{41}$$

The surface element of the mantle is given here by

$$d^3\Sigma_i = dt d^2\sigma_i\,, \tag{42}$$

where $d^2\sigma_i$ denotes the surface element of three-dimensional space, i.e. the integration runs over $\mathcal{M} = S^2_\infty \times [0, T]$ where S^2_∞ is the surface of our 3-space B^3_∞. In eq. (33) the integrand has to be taken at spatial infinity, where, in view of the boundary condition on Ω (7),

$$\lim_{r\to\infty}\chi(r,\hat{\mathbf{x}}) = n\pi\,. \tag{43}$$

The time integral can then be readily performed, since $\hat{\boldsymbol{\chi}}$ is time-independent, yielding

$$\frac{1}{T}\int\limits_0^T dt\,\chi\sin^2\bar{\chi} = \frac{n\pi}{T}\int\limits_0^T dt\sin^2 n\pi\left(1-\frac{t}{T}\right) = \int_0^{n\pi} dz\sin^2 z = \frac{n\pi}{2}\,. \tag{44}$$

We therefore obtain for the contribution from the mantle of the cylinder

$$\bar{n}^{(\mathcal{M})}[U] = -nm[\hat{\boldsymbol{\chi}}]\,, \tag{45}$$

where

$$m[\hat{\boldsymbol{\chi}}] = \frac{1}{8\pi}\int\limits_{S^2_\infty} d\sigma_i\,\epsilon_{ijk}(\partial_j\hat{\boldsymbol{\chi}}\times\partial_k\hat{\boldsymbol{\chi}})\hat{\boldsymbol{\chi}} \tag{46}$$

defines the winding number $m[\hat{\boldsymbol{\chi}}] \in \Pi_2(S^2)$ of the mapping $\hat{\boldsymbol{\chi}}(x)$ of the surface S^2_∞ of R^3 into the equator of the gauge group $\Omega\left(\chi = \frac{\pi}{2}\right) = i\hat{\boldsymbol{\chi}}\boldsymbol{\tau}$, which is also S^2 (since $\hat{\boldsymbol{\chi}}^2 = 1$).

By definition of homotopy classes mappings with the same winding number can be smoothly deformed into each other. Therefore, we can smoothly map any gauge function $\Omega(\mathbf{x})$ onto the corresponding "hedgehog" map $\hat{\boldsymbol{\chi}}(\mathbf{x}) = \hat{\mathbf{x}}$ with the same winding number. Since $m[\hat{\mathbf{x}}] = 1$ we find from (45)

$$\bar{n}^{(\mathcal{M})}[U] = -n[\Omega] \,, \tag{47}$$

which together with eqs. (31), (32) implies

$$\bar{n}[U] = 0 \,. \tag{48}$$

This result can be easily understood from fig. 1. Since $U(t = T) = 1$, all points of the face at $t = T$ can be identified. This deforms the cylinder to the cone shown in fig. 1b. This cone can be thought of as a deformation of the face at $t = 0$ (with opposite orientation), since both manifolds have the same boundary, namely the boundary S^2_∞ of 3-space. Furthermore, by our choice of boundary conditions, $\Omega(\mathbf{x})$ traces out the same group space on both manifolds. However, the two manifolds have opposite orientations and hence give opposite contributions to the winding number.

With (48) we finally obtain

$$\nu[A^U] = \nu[A] \,, \tag{49}$$

showing that the Pontryagin index does not change under the gauge transformation $U(x)$ (13). Therefore we can replace $(-n)$ in eq. (15) by $\nu[A]$ and obtain the desired result.

4 Cartan Decomposition of the Gauge Transformation

It is convenient to perform a diagonalization of the map $\Omega_n(x)$ provided by the gauge function,

$$\Omega(x) = V^\dagger \omega V \,, \tag{50}$$

where ω is an element of the Cartan subgroup (maximal torus) $H = U(1)^{N-1}$ and V lives in the coset G/H. By gauge invariance of the Yang-Mills Hamiltonian, we have

$$\langle C | e^{-HT} | C^{V^\dagger \omega V} \rangle = \langle C^V | e^{-HT} | (C^V)^\omega \rangle \,. \tag{51}$$

By shifting the integration invariable $C(x)^V \to C(x)$ it is seen that the integrand in the partition function (3), (4) does not depend on the coset V.

The integration over the coset G/H can then be explicitly performed by using the Weyl integration formula [23]

$$\int d\mu(\Omega_n) f(\Omega_n) = \frac{1}{|\mathcal{W}|} \int\limits_H d\bar{\mu}(\omega_n) \int\limits_{G/H} dV_n f(V_n^\dagger \omega_n V_n) \,, \tag{52}$$

where $|\mathcal{W}|$ is the order of the Weyl group ($|\mathcal{W}| = N!$ for $G = SU(N)$) and the reduced Haar measure $\bar{\mu}(\omega)$ is defined by

$$d\bar{\mu}(\omega) = \prod_k d\lambda_k \sum_p \delta\left(\sum_i \lambda_i - 2\pi p\right) \prod_{i<j} \sin^2 \frac{\lambda_i - \lambda_j}{2} \,. \tag{53}$$

Here $i\lambda_k$ denotes the eigenvalues of $\ln \omega_n$ (λ_k real).[7] The partition function then becomes (cf. eqs. (11) and (15))

$$Z = \sum_n \quad e^{-in\Theta} \int\limits_H \mathcal{D}\bar{\mu}(e^{-Ta_0}) \int \mathcal{D}A_i'(x) \exp(-S_{\mathrm{YM}}[a_0, \mathbf{A}']) \,, \tag{54}$$

where

$$a_0 = -\frac{1}{T} \ln \omega_n \tag{55}$$

and an irrelevant constant, which arises from the integration over the coset space V_n, has been dropped. The spatial gauge fields $A_i'(x)$ in eq. (54) are related to the ones in eq. (15) by the time-independent gauge transformation $V_n \in G/H$,

$$A_i' = A_i^V = V A_i V^\dagger + V \partial_i V^\dagger \,, \tag{56}$$

and hence the $A_i'(x)$ also fulfill temporally periodic boundary conditions (we will omit the prime in the following).

Since a specific Lorentz component of the gauge field $A_\mu(x)$, μ fixed, enters the Yang-Mills action $S_{\mathrm{YM}}[A_0, A_i]$ at most quadratically, we can explicitly integrate out one spatial component of the gauge field, say $A_3(x)$, and obtain

$$Z = \sum_n e^{-in\Theta} \int\limits_H \mathcal{D}\mu(e^{-Ta_0}) \int \mathcal{D}A_1 \mathcal{D}A_2 \mathrm{Det}^{-1/2}(-\hat{D}_{\bar{\mu}} \hat{D}_{\bar{\mu}}) \exp(-\tilde{S}(a_0, A_1, A_2)) \,. \tag{57}$$

Here

$$\tilde{S}(A_0, A_1, A_2,) = \frac{1}{2g^2} \int d^4x \, \partial_3 A_{\bar{\mu}} \mathcal{P}_{\bar{\mu}\bar{\nu}} \partial_3 A_{\bar{\nu}} + \frac{1}{4g^2} \int d^4x (F_{\bar{\mu}\bar{\nu}})^2 \tag{58}$$

[7] When the Weyl formula is extended to functional integrals over two-dimensional compact manifolds, topological obstructions occur, as discussed recently in ref. [24]. In higher dimensions these obstructions are absent.

with

$$\mathcal{P}_{\bar{\mu}\bar{\nu}} = \delta_{\bar{\mu}\bar{\nu}} - \hat{D}_{\bar{\mu}} \frac{1}{\hat{D}_{\bar{\lambda}}\hat{D}_{\bar{\lambda}}} \hat{D}_{\bar{\nu}} \,, \quad \hat{D}^{ab}_{\mu} = \delta^{ab}\partial_{\mu} + f^{acb}A^{c}_{\mu} \tag{59}$$

is the effective action of the remaining gauge degrees of freedom. Note that the indices $\bar{\mu}, \bar{\nu}, \ldots$ run only from 0 to 2, and for notational simplicity we have replaced a_0 by A_0 in eq. (58). The quantity (59) represents a generalized transverse projector [25].

For $\Theta = 0$ eq. (57) is precisely the path integral expression derived for the Yang-Mills partition function in ref. [6] by explicitly resolving Gauss' law in the modified axial gauge

$$A^{(\mathrm{ch})}_0 = 0 \,, \quad \partial_0 A^{(\mathrm{n})}_0 = 0 \,, \tag{60}$$

where $A^{(\mathrm{n})}_0$ and $A^{(\mathrm{ch})}_0$ denote the "neutral" and "charged" parts of A_0, which live in the Cartan algebra $\mathcal{H}$ and in the coset $\mathcal{G}/\mathcal{H}$, respectively.[8] (See also ref. [5] where Gauss' law was resolved in the same gauge in the canonical operator approach.[9]) The gauge (60) is equivalent to the so-called Polyakov gauge defined by diagonalizing the Polyakov line $\mathcal{P}\exp(-\int_0^T dx_0 A_0)$.

In the standard Yang-Mills functional integral with gauge (60) the reduced Haar measure $\mathcal{D}\bar{\mu}(\exp(-Ta_0))$ of eq. (57) arises from the corresponding Faddeev-Popov determinant [2]. This connection has been recently also established in 1+1 dimensions [26]. The derivation of the gauge fixed Yang-Mills partition function (57) given above, starting from the gauge invariant projection, is more concise than the explicit resolution of Gauss' law in either the operator approach [5] or in the standard functional integral approach [6]. Of course both methods yield the same result.

5 Magnetic monopoles and strings

As we have seen in the previous section, the gauge invariant projection combined with the diagonalization of the gauge function (50) is equivalent to a resolution of Gauss' law in the gauge (60). This gauge represents a variant of 't Hooft's Abelian gauges [7] in which magnetic monopoles are known to occur. For later purposes it is instructive to briefly demonstrate the emergence of monopoles in the diagonalization (50), cf. also refs. [7], [8], [16], [28]. For simplicity I consider again the gauge group $SU(2)$.

Adopting the parameterization (36) chosen above for an element of the gauge group, the coset matrix V (50) is defined by

$$\hat{\boldsymbol{\chi}}\boldsymbol{\tau} = V^{\dagger}\tau_3 V \,. \tag{61}$$

[8] More precisely, the expression derived in reference [6] follows from eq. (57) by a permutation of the Lorentz indices. In reference [6], A_0 was integrated out to yield Gauss' law and the gauge $A^{(\mathrm{ch})}_3 = 0$ and $\partial_3 A^{(\mathrm{n})}_3 = 0$ was used to resolve Gauss' law.

[9] Similar gauges have been considered recently in refs. [26], [27].

Obviously the coset matrix V depends only on the unit vector $\hat{\boldsymbol{\chi}}(\theta,\phi)$, which can be parameterized in the usual fashion by polar and azimuthal angles θ and ϕ. In the parameterization (36) the element of the invariant torus has the representation

$$\omega = e^{i\chi\tau_3} = \cos\chi + i\tau_3 \sin\chi\,. \tag{62}$$

Let us also quote the explicit representation of the coset matrix V which diagonalizes the general group element. In fact, this matrix is defined only up to an element of the Cartan subgroup $V \to gV$, $g \in H$, which does not change the decomposition (50). Since this matrix has to rotate an arbitrary vector in group space $\hat{\boldsymbol{\chi}}$ into the 3-direction, this matrix can be chosen as the rotational matrix

$$V = e^{i\frac{\theta}{2}\mathbf{e}_\phi \boldsymbol{\tau}}\,, \quad \mathbf{e}_\phi = -\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2\,. \tag{63}$$

For $\theta = \pi$ there is an ambiguity in the choice of the rotational axis $\mathbf{e}_\phi$, when rotating $\hat{\boldsymbol{\chi}}(\theta = \pi) = -\mathbf{e}_3$ to $\mathbf{e}_3$ (any axis in the 1-2-plane can be chosen). This ambiguity is reflected by the coordinate singularity of ϕ for $\theta = \pi$, which gives rise to a line singularity of V at $\theta = \pi$, where $V(\theta = \pi) = i\mathbf{e}_\phi\boldsymbol{\tau}$. For the identification of the singularities, the polar coordinate representation (63) is not very convenient since the polar coordinates themselves have singularities. To exhibit the singular structure of V it is more convenient to use Cartesian coordinates, in which the matrix V can be chosen as

$$V = -i\hat{\boldsymbol{\alpha}}\boldsymbol{\tau}\,, \quad \hat{\alpha}^a = \frac{\delta^{a3} + \hat{\chi}^a}{\sqrt{2(\hat{\chi}^3 + 1)}}\,, \quad \hat{\boldsymbol{\alpha}}^2 = 1\,. \tag{64}$$

Obviously this matrix has a string singularity at the negative 3-axis, $\hat{\chi}^3 = -1$. The end points of this line singularity correspond to the irregular group elements (50) $\Omega = 1$ ($\chi = 2k\pi$) and $\Omega = -1$ ($\chi = (2k{+}1)\pi$). At these points in group space, the corresponding induced gauge potential

$$\mathcal{A}_\alpha = V\partial_\alpha V^\dagger \tag{65}$$

develops a magnetic monopole. The string singularity observed in the coset matrix (64) is nothing but the Dirac string (here in group space), which interpolates between two monopoles with opposite magnetic charges or runs from a monopole to infinity. Note that due to the imposed boundary condition (7) $\lim_{r\to\infty}\chi(r) = n\pi$, spatial infinity is mapped onto an irregular group element $\Omega = (-1)^n$. This implies that we may have a magnetic monopole at spatial infinity and hence also a Dirac string which extends to spatial infinity.

With the representation (63), straightforward evaluation yields for the induced gauge potential (65)

$$\mathcal{A}_\alpha = -i\frac{1}{2}\mathbf{e}_\phi\boldsymbol{\tau}\partial_\alpha\theta + \frac{i}{2}\sin\theta\mathbf{e}_\rho\cdot\boldsymbol{\tau}\,\partial_\alpha\phi + i\frac{1}{2}(1-\cos\theta)(\partial_\alpha\phi)\tau_3\,, \tag{66}$$

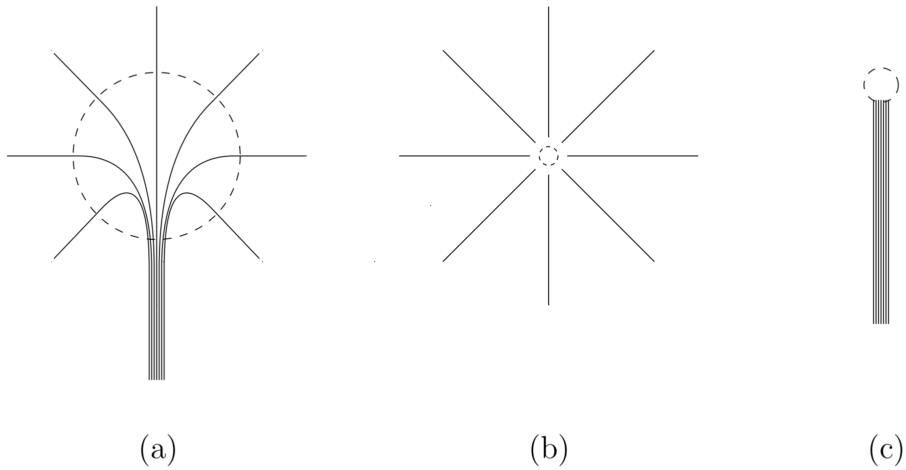

Figure 2: Illustration of the various magnetic fields introduced in sect. 5 for a monopole configuration: (a) the Abelian magnetic field $\boldsymbol{\mathcal{B}}$ (67), (b) the non-Abelian contribution $(-\mathbf{B})$ (71) and (c) the total magnetic field $\mathbf{B}^3$ (70). For illustrative purposes the monopole fields have been regularized. The true fields are obtained by contracting the dashed circle to a point.

where we have used $\frac{\partial}{\partial\phi}\mathbf{e}_\phi = -\mathbf{e}_\rho$ and $\mathbf{e}_\rho \times \mathbf{e}_\phi = \mathbf{e}_3$. Here the last term in front of τ_3 represents the gauge potential of a magnetic monopole with Dirac string at $\theta = \pi$.

In fact, the Abelian magnetic field

$$\boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} \times \boldsymbol{\mathcal{A}}^3 \,, \quad \boldsymbol{\mathcal{A}}^3 = -2\mathrm{Tr}(T^3 \boldsymbol{\mathcal{A}}) \tag{67}$$

is that of a Dirac monopole shown in fig. 2a. The magnetic flux, flowing outward from the center of the monopole, is oppositely the same as the magnetic flux of the Dirac string, flowing inside the center, and the net magnetic flux of the monopole and the Dirac string vanishes,

$$\int_{S^2_\infty} d\boldsymbol{\Sigma}\boldsymbol{\mathcal{B}} = 0 \,, \tag{68}$$

where S^2_∞ denotes the surface of R^3.

Since the induced gauge potential (65) is pure gauge, its total (non-Abelian) field strength

$$F_{\mu\nu}[\mathcal{A}] = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \tag{69}$$

vanishes except at the singularities of V, where the r.h.s. can be non-zero. In the presence of a magnetic monopole, V has a string singularity and we expect a non-vanishing total magnetic field

$$B_i[\mathcal{A}] = \frac{1}{2}\epsilon_{ijk} F_{jk}[\mathcal{A}] \tag{70}$$

at the Dirac string. These are the magnetic strings found in ref. [16], where it was observed that unlike the Abelian field (67), the total non-Abelian field (69) does not contain monopole type fields, but only strings of magnetic flux, which are the remnants of the Dirac string of the Abelian magnetic field. What happens is that, in the total field strength (69), the non-Abelian commutator term $[\mathcal{A}_\mu, \mathcal{A}_\nu]$ cancels the monopole part of the Abelian magnetic field (67), leaving only the Dirac string. Therefore, the magnetic field defined by

$$\mathsf{B}_i = \mathcal{B}_i - B_i^3 = -\frac{1}{2}\epsilon_{ijk}[\mathcal{A}_j, \mathcal{A}_k]^3 = \epsilon_{ijk}\mathrm{Tr}([\mathcal{A}_j, \mathcal{A}_k]T_3) \tag{71}$$

represents the magnetic field of a monopole without the Dirac string. This field then obviously satisfies

$$\boldsymbol{\nabla}\mathsf{B} = 4\pi \sum_i m_i \delta^{(3)}(\mathbf{x} - \bar{\mathbf{x}}_i)\,, \tag{72}$$

where $\bar{\mathbf{x}}_i$ are the positions of the monopoles and m_i their magnetic charges, which agree with the total magnetic flux of the monopole field,

$$m_i = \frac{1}{4\pi}\int_{S^2_\varepsilon(i)} d\boldsymbol{\sigma}\mathsf{B} = \frac{1}{4\pi}\int d\sigma_i \epsilon_{ijk}\mathrm{Tr}([\mathcal{A}_j, \mathcal{A}_k]T_3)\,, \tag{73}$$

where $S^2_\varepsilon(i)$ is a sphere of infinitesimal radius around the center of the monopole, $\bar{\mathbf{x}}_i$. It is straightforward to show that the magnetic flux (73) agrees with the winding number $m[\hat{\boldsymbol{\chi}}] \in \Pi_2(SU(2)/U(1)) = \Pi_2(S_2)$ of the mapping $\hat{\boldsymbol{\chi}}(\hat{\mathbf{x}})$ given in eq. (46), cf. also refs. [8], [16]. This ensures that the charge of the magnetic monopole is quantized

$$m_i = 0, \pm 1, \pm 2, \ldots\,. \tag{74}$$

The B field obviously agrees with the Abelian magnetic field $\boldsymbol{\mathcal{B}}$ everywhere except at the Dirac string. Therefore, the magnetic flux (73) can be alternatively evaluated from the Abelian field $\boldsymbol{\mathcal{B}}$ by leaving out that point $\bar{\bar{\mathbf{x}}}$ of S^2_∞ where the Dirac string pierces the surface,

$$m_i = \frac{1}{4\pi}\int\limits_{\tilde{S}^2(i)} d\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{B}}\,. \tag{75}$$

The integration is then over a punctured sphere $\tilde{S}^2(i) = S^2_\varepsilon(i)\setminus\{\bar{\bar{\mathbf{x}}}_i\}$ and can be performed by applying Stoke's theorem,

$$m_i = \frac{1}{4\pi}\int\limits_{C_i} d\mathbf{x}\,\boldsymbol{\mathcal{A}}^3\,, \tag{76}$$

where C_i is an infinitesimal circle enclosing the Dirac string of the monopole. Inserting here the explicit form of $\boldsymbol{\mathcal{A}}^3$ (66) and taking into account that on the infinitesimal circle C_i around the Dirac string we have $\theta \simeq \pi$ we obtain for the magnetic flux (76)

$$m_i = -\frac{1}{2\pi} \int\limits_{C_i} d\mathbf{x}\, \boldsymbol{\nabla}\phi = -m[\phi]\,. \tag{77}$$

This is the winding number $m[\phi] \in \Pi_1(U(1))$ of the mapping

$$\phi(x) : \mathbf{x} \in C_i \simeq S_1 \to \phi \in S_1 \simeq U(1)\,. \tag{78}$$

As discussed above the magnetic flux m_i is also given by the winding number (46) $m[\hat{\boldsymbol{\chi}}] \in \Pi_2(SU(2)/U(1))$. The equality of both winding numbers, $m[\phi]$ and $m[\hat{\boldsymbol{\chi}}]$, is guaranteed by the relation

$$\Pi_2(SU(2)/U(1)) = \Pi_1(U(1))\,. \tag{79}$$

In the next section we will see that it is the flux (73) of the monopole field **B** which also determines the winding number $n[\Omega] \in \Pi_3(SU(2))$.

Let us explicitly quote the various magnetic fields introduced above for a generic mapping $\Omega(\mathbf{x})$ given by the generalized hedgehog field[10]

$$\begin{aligned} &\chi = \chi(r) \quad \text{with} \quad \chi(0) = 0\,, \quad \chi(\infty) = n\pi \\ &\theta = p\vartheta\,, \quad \phi = q\varphi\,. \end{aligned} \tag{80}$$

To get a handle on the singularities of $\mathcal{A}_\alpha$ (65) we use the representation (64) for V and introduce a regularization [16]

$$\mathcal{A}_k^a = -\lim_{\varepsilon\to 0} \frac{2}{\alpha^2+\varepsilon^2}\, \epsilon^{abc} \alpha^b \partial_k \alpha^c \tag{81}$$

where

$$\alpha^a = \chi^a + \chi\, \delta^{a3}\,, \quad \alpha^2 = 2\,\chi^2(1+\hat{\chi}^3)\,. \tag{82}$$

For $\chi = \chi(r)$ independent of ϑ and φ one finds in spherical coordinates

$$\begin{aligned} \mathcal{A}_r^3 &= 0 \\ \mathcal{A}_\vartheta^3 &= -\lim_{\varepsilon\to 0} \frac{2\chi^2}{\alpha^2+\varepsilon^2}\, \epsilon^{3bc}\, \hat{\chi}^b\, \frac{1}{r}\frac{\partial}{\partial\vartheta}\hat{\chi}^c \\ \mathcal{A}_\varphi^3 &= -\lim_{\varepsilon\to 0} \frac{2\chi^2}{\alpha^2+\varepsilon^2}\, \epsilon^{3bc}\, \hat{\chi}^b\, \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}\hat{\chi}^c \end{aligned} \tag{83}$$

[10]Recall, while (ϑ, φ) are polar and azimuthal angle in ordinary space, (θ, ϕ) denote the corresponding quantities in group space.

which gives rise to the Abelian magnetic field (67)

$$\boldsymbol{\mathcal{B}} = -q\,\frac{\sin p\vartheta}{r^2\sin\vartheta}\,\lim_{\varepsilon\to 0}\left\{p\,\frac{\alpha^4}{(\alpha^2+\varepsilon^2)^2}\,\mathbf{e}_r + \frac{\varepsilon^2}{(\alpha^2+\varepsilon^2)^2}\,4\chi^2\left[p\cos p\vartheta\,\mathbf{e}_r - \frac{r\chi'}{\chi}\sin p\vartheta\,\mathbf{e}_\vartheta\right]\right\} \quad (84)$$

with $\chi' = d\chi/dr$. The first term is regular for $\varepsilon \to 0$. Using

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^2}{(\alpha^2+\varepsilon^2)^2} = \pi\delta(\chi^1)\delta(\chi^2)\Theta(-\chi^3) \quad (85)$$

we obtain for $\varepsilon \to 0$

$$\boldsymbol{\mathcal{B}} = -q\,\frac{\sin p\vartheta}{r^2\sin\vartheta}\left\{p\,\mathbf{e}_r + \pi\delta(\chi^1)\delta(\chi^2)\Theta(-\chi^3)\,4\chi^2\left[p\cos p\vartheta\,\mathbf{e}_r - \frac{r\chi'}{\chi}\sin p\vartheta\,\mathbf{e}_\vartheta\right]\right\}. \quad (86)$$

For simplicity let us consider the case $p = q = 1$. Near the singularity (monopole position) $\chi = 0$ we have $\chi(r) \simeq r\chi'(0)$ so that $r\chi'/\chi \simeq 1$ and the magnetic field (86) reduces to ($\mathbf{e}_3 = \cos\vartheta\mathbf{e}_r - \sin\vartheta\mathbf{e}_\vartheta$)

$$\boldsymbol{\mathcal{B}} = -\frac{\hat{\mathbf{x}}}{r^2} - 4\pi\Theta(-x_3)\delta(x_1)\delta(x_2)\mathbf{e}_3\,, \quad (87)$$

which is the magnetic field of the familiar Dirac monopole with magnetic charge $m = 1$, in agreement with the topological quantization of the magnetic flux. For the field $\mathbf{B}$ (71) one finds near the singularity

$$\mathbf{B} = -\frac{\hat{\mathbf{x}}}{r^2}\,, \quad (88)$$

which is a monopole field without the Dirac string.

Before concluding this section let us comment on the physical meaning of the induced gauge potential $\mathcal{A}_\alpha$ (65) and, in particular, of the monopole singularities: The (time-independent) gauge function $\Omega(\mathbf{x}) = \exp(-TA_0(\mathbf{x}))$ is nothing but the Polyakov line in the gauge $\partial_0 A_0 = 0$ and diagonalization of this map $\Omega(\mathbf{x}) = V^\dagger\omega V$ is equivalent to going to the Polyakov gauge. Lattice calculations performed in the Polyakov gauge show not only Abelian dominance but also dominance of magnetic monopoles [10], i.e. most of the string tension comes from the magnetic monopole configurations alone. In this respect the magnetic monopoles arising in the induced gauge potential $\mathcal{A}_\alpha = V\partial_\alpha V^\dagger$ from singular gauge transformations V (63) represent the dominant infrared degrees of freedom. This is very analogous to Yang-Mills theory in the maximum Abelian gauge defined by

$$\left[\partial_\mu + A_\mu^{(\mathrm{n})}, A_\mu^{(\mathrm{ch})}\right] = 0 \quad (89)$$

where the infrared dominance of magnetic monopoles is even more pronounced [9].

6 Relation between winding number and monopole charges

In what follows, we consider how the winding number (6) transforms under the Cartan decomposition (50). Thereby we will derive a relation between the winding number $n[\Omega = V^\dagger \omega V]$ and the magnetic charges of the monopoles induced by the (coset) gauge transformation $V(x)$.[11]

The relevant current can be rewritten as

$$\begin{aligned} L_\alpha = \Omega \partial_\alpha \Omega^\dagger &= V^\dagger \omega \mathcal{A}_\alpha \omega^\dagger V + V^\dagger s_\alpha V + V^\dagger \partial_\alpha V \\ &= V^\dagger (\tilde{\mathcal{A}}_\alpha - \mathcal{A}_\alpha + s_\alpha) V \,. \end{aligned} \tag{90}$$

Here we have used the definition of $\mathcal{A}$ (65) and the abbreviations

$$\tilde{\mathcal{A}}_\alpha = \omega \mathcal{A}_\alpha \omega^\dagger \,, \quad s_\alpha = \omega \partial_\alpha \omega^\dagger \,. \tag{91}$$

Using the cyclic properties of the trace and the fact that two elements of the Cartan algebra commute, i.e. $[s_\alpha, s_\beta] = 0$, the expression for the winding number (6) can be reduced to

$$\begin{aligned} n[\Omega] = \frac{1}{24\pi^2} \int d^3x \, \epsilon^{\alpha\beta\gamma} \mathrm{Tr}\Big\{ & (\tilde{\mathcal{A}}_\alpha - \mathcal{A}_\alpha)(\tilde{\mathcal{A}}_\beta - \mathcal{A}_\beta)(\tilde{\mathcal{A}}_\gamma - \mathcal{A}_\gamma) \\ + & \; 3(\tilde{\mathcal{A}}_\alpha - \mathcal{A}_\alpha)(\tilde{\mathcal{A}}_\beta - \mathcal{A}_\beta) s_\gamma \Big\} \,. \end{aligned} \tag{92}$$

Using the Cartan decomposition of the Lie algebra of the gauge group, it is straightforward to show that the quantity $\tilde{\mathcal{A}}_\alpha - \mathcal{A}_\alpha$ lives entirely in the coset space $\mathcal{G}/\mathcal{H}$. Therefore, the first term in eq. (92) vanishes for $SU(2)$. Indeed, with the above adopted parameterization of the gauge group (92), (62), one finds

$$s_\alpha = i\tau_3 \partial_\alpha \chi \,, \tag{93}$$

$$\tilde{\mathcal{A}}_\alpha - \mathcal{A}_\alpha = -2\sin^2\chi \mathcal{A}_\alpha^{(\mathrm{ch})} - \sin 2\chi \epsilon_{3\bar{a}\bar{b}} \mathcal{A}_\alpha^{\bar{a}} T^{\bar{b}} \,, \tag{94}$$

where $\mathcal{A}_\alpha^{(\mathrm{ch})} = \mathcal{A}_\alpha^{\bar{a}} T^{\bar{a}}$ denotes the charged part of $\mathcal{A}_\alpha$. (The indices $\bar{a}, \bar{b} = 1, 2$ are restricted to the generators of the coset $\mathcal{G}/\mathcal{H}$.) The winding number (92) then becomes

$$n[\Omega] = \frac{i}{4\pi^2} \int d^3x \, \sin^2\chi \epsilon^{\alpha\beta\gamma} \partial_\gamma \chi \mathrm{Tr}([\mathcal{A}_\alpha, \mathcal{A}_\beta]\tau_3) \,. \tag{95}$$

Using

$$\sin^2\chi \partial_k \chi = \frac{1}{2} \partial_k (\chi - \sin\chi \cos\chi) \tag{96}$$

[11] A naive application of eq. (29) would yield the result $n[\Omega] = n[V^\dagger] + n[\omega] + n[V] = 0$ since $n[V^\dagger] = -n[V]$ and $n[\omega] = 0$. However, unlike Ω, V does not approach an angle-independent limit for $r \to \infty$, which is required for a gauge function in order for its winding number to be well defined.

and the definition of the monopole field B (71), the topological charge can be expressed as

$$n[\Omega] = -\frac{1}{4\pi^2} \int d^3\Sigma_0 \mathsf{B}_k \partial_k (\chi - \sin\chi\cos\chi)\,. \tag{97}$$

Performing a partial integration and using Gauss' theorem, we obtain

$$\begin{aligned} n[\Omega] &= -\frac{1}{4\pi^2} \int d^2\sigma_k \mathsf{B}_k (\chi - \sin\chi\cos\chi) \\ &\quad + \frac{1}{4\pi^2} \int d^3\Sigma_0 (\chi - \sin\chi\cos\chi) \boldsymbol{\nabla}\mathsf{B} \\ &= n^{(1)} + n^{(2)}\,. \end{aligned} \tag{98}$$

Due to our chosen boundary condition $\lim_{r\to\infty} \chi(r) = n\pi$, the first term yields

$$n^{(1)} = -\frac{n}{4\pi} \int d\boldsymbol{\sigma} \cdot \mathsf{B} = -nm[\hat{\boldsymbol{\chi}}]\,, \tag{99}$$

where

$$m[\hat{\boldsymbol{\chi}}] = \sum_i m_i[\hat{\boldsymbol{\chi}}] \tag{100}$$

is the magnetic flux of the monopoles (73), (46). The second term $n^{(2)}$ receives contributions from the monopoles, which are the sources of the field B. Using eq. (72), the integration in $n^{(2)}$ can performed and we obtain

$$n^{(2)} = \frac{1}{\pi} \sum_k (\chi(\bar{x}_k) - \sin\chi(\bar{\mathbf{x}}_k)\cos\chi(\bar{\mathbf{x}}_k)) m_k\,, \tag{101}$$

where $\bar{\mathbf{x}}_k$ denotes the positions of the magnetic monopoles. At a monopole position, where the gauge function $\Omega(\mathbf{x})$ is irregular, we have (cf. eq. (36))

$$\chi(\bar{\mathbf{x}}_k) = n_k\pi\,, \tag{102}$$

where n_k is integer. Hence we find

$$n^{(2)} = \sum_k n_k m_k \tag{103}$$

and the total winding number (98) becomes

$$n[\Omega] = -\sum_k (n - n_k) m_k = \sum_k \ell_k m_k\,, \tag{104}$$

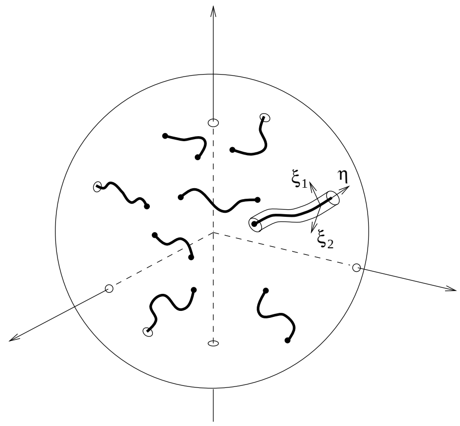

Figure 3: Illustration of a gas of magnetic strings. For one of the strings the local coordinate system (η, ξ_1, ξ_2) (see text) has been indicated by the tube.

where

$$\ell_k = n_k - n \tag{105}$$

times π are the (signed) lengths of the Dirac strings of the monopoles in group space, cf. fig. 3 and the appendix. Note that for gauge functions $\Omega(\mathbf{x})$ with well-defined winding number $n[\Omega]$ (6), the n_k and hence the ℓ_k are integer.

Eq. (104) expresses the winding number of the mapping $n[\Omega = V\omega V^\dagger]$ in terms of the charges m_k of the magnetic monopoles of the induced gauge potential $\mathcal{A}_i = V\partial_i V^\dagger$ and the properties ℓ_k of the Cartan element ω. Let us emphasize that ℓ_k are the (integer valued) lengths of the string singularities of $\mathcal{A}_i$ in group space. In ordinary space the mapping of such a string singularity (which is usually referred to as the Dirac string) can have arbitrary length and furthermore the string singularity can even be split into several strings or deformed to conic sheet singularities. In the appendix we will illustrate this by means of a few generic mappings $\Omega(\mathbf{x})$ which illustrate what happens in the general case.

A more physical derivation of the above result can be obtained in the following way. The Pontryagin index of the gauge field entering the Yang-Mills partition function (54) is given by

$$\nu[A'] \equiv \nu[a_0, A_i^V] = -n\,. \tag{106}$$

This can be easily seen by noticing that the gauge field in eq. (54) differs from the one in (15), A_μ^U, by the time-independent gauge transformation $V \in G/H$, which in view of $\Pi_3(SU(2)/U(1)) = 1$ does not change the Pontryagin index. The relation (106) follows then from eqs. (30) and (49). On the other hand, the Pontryagin index can be expressed in terms of the colour electric and magnetic fields by

$$\nu[A] = \frac{1}{4\pi^2}\int dt \int d^3x\, \mathrm{Tr}(\mathbf{E}\mathbf{B})\,. \tag{107}$$

Obviously only long range fields of the monopole type give here non-zero contributions, cf. eq. (19). For the calculation of the Pontryagin index it is therefore sufficient to keep from the spatial gauge field only the induced gauge field (65), which contains all the monopoles and by construction is time-independent. The electric field is then given by

$$E_i = -\partial_i a_0 + [a_0, \mathcal{A}_i]\,, \tag{108}$$

where the Abelian field a_0 is defined by eq. (55). From the induced gauge field $\mathcal{A}_i = V\partial_i V^\dagger$ only the Abelian monopole part is long range, cf. eq. (66), and has hence to be included. Then the commutator term in eq. (108) vanishes and we obtain with eqs. (55), (62)

$$E_i = \frac{1}{T}\partial_i \ln \omega_n = -\frac{2}{T}T_3\partial_i\chi\,. \tag{109}$$

Obviously the quantity χ figures as the scalar potential for the Abelian electric field. Inserting eq. (109) into eq. (107), the Pontryagin index becomes

$$\nu[A'] = \nu[a_0, \mathcal{A}_i] = -\frac{1}{4\pi^2}\cdot\frac{1}{T}\int dt \int d^3x\,(\partial_k\chi)B^3_k\,, \tag{110}$$

where $B^3_k = -2\mathrm{Tr}(T_3 B_k[\mathcal{A}])$ is the Abelian component of the full (non-Abelian) magnetic field. As shown in sect. 5, this field vanishes, since $\mathcal{A}_i$ is pure gauge, except for the string singularities. Therefore the Pontryagin index receives contributions only from the Dirac strings of the magnetic monopoles. It becomes hence a sum of the contributions from the individual monopoles,

$$\nu[A'] = \sum_i \nu_{(i)}[A']\,, \tag{111}$$

where $\nu_{(i)}$ is the contribution of the ith monopole. To calculate this contribution it is convenient to introduce local coordinates in the neighborhood of the string, see fig. 3. Let η and ξ_1, ξ_2 denote the coordinates parallel and orthogonal, respectively, to the Dirac string. The magnetic flux is conserved (in its magnitude) along the Dirac string and hence independent of η. On the other hand, χ depends only on η. Hence the integration orthogonal to the string can be explicitly performed and yields ($\int d\xi_1\, d\xi_2\, (\mathbf{B}^3\cdot\mathbf{e}_\eta) = 4\pi m_i$)

$$\nu_{(i)}[A'] = m_i\frac{1}{\pi}\int d\eta \frac{\partial\chi}{\partial\eta}\,, \tag{112}$$

where $(-m_i)$ is the magnetic flux of the Dirac string, which is equal in magnitude but opposite in sign to the magnetic flux of the spherically symmetric monopole field $\mathbf{B}$, defined by eq. (71). In view of our boundary condition (43), the Dirac string runs between the singular field configuration $\chi(\eta = 0) = n_i\pi$ ($\eta = 0$ corresponds here to the monopole position while $\eta \to \infty$ corresponds to spatial infinity) and $\chi(\infty) = n\pi$. Hence, with eq. (105), we obtain

$$\nu_{(i)}[A'] = -\ell_i m_i\,, \tag{113}$$

which together with eqs. (111) and (106) agrees with our previous result (104).

The result obtained above shows that magnetic monopoles with Dirac strings running to infinity are sufficient to account for the topologically non-trivial sectors of Yang-Mills theory. While magnetic monopoles connected by finite strings do not contribute to the topology, they are presumably the relevant infrared degrees of freedom which are responsible for confinement [9], [11].

Finally, let me comment on previous related work. In ref. [29] the Pontryagin index of dyon configurations, treated as periodic in time, was shown to coincide with the magnetic charge. This corresponds to the case $\ell_i = 1$ in eq. (113). There have been also recent investigations of Yang-Mills theory in the modified axial gauge (60) on the torus [30], where large gauge transformations have been found to induce magnetic flux.

7 Concluding remarks

In this paper I have demonstrated that gauge invariant projection provides an efficient way of resolving Gauss' law in Yang-Mills theory. For the partition function, the projection onto gauge invariant orbits amounts to integration over the Cartan subgroup. The irregular group elements give rise to magnetic monopoles which are responsible for the non-trivial topological structure of the gauge fields. I have explicitly shown how these magnetic charges build up the Pontryagin index. The latter is determined by the magnetic charges and the lengths of the associated Dirac strings in group space (cf. eq. (104)).

An important conclusion of the present paper is that the magnetic monopoles are entirely sufficient to account for the non-trivial topological structure of Yang-Mills theory. This result is perhaps not surprising since intuitively one may expect that the global, i.e. topological properties of gauge fields are related to the long range monopole fields. Furthermore this result is also consistent with the monopole dominance seen in lattice calculations [9], [10].

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A Generic mappings

By definition of homotopy classes, any mapping $\Omega(\mathbf{x}) \in SU(2)$ can be smoothly deformed into a field of the hedgehog type (with the same winding number). In this sense, the hedgehog is a generic topologically non-trivial mapping.

As a first illustrative example, we consider the ordinary hedgehog with winding number $n[\Omega] = 1$, defined by eq. (36) with

$$\hat{\boldsymbol{\chi}} = \hat{\mathbf{x}}\,, \quad \chi(\mathbf{x}) = \chi(|\mathbf{x}|) \tag{114}$$

and with the boundary conditions

$$\chi(0) = 0\,, \quad \chi(\infty) = \pi\,. \tag{115}$$

The field obviously has the right asymptotics (7) and yields a conformal mapping of ordinary space onto group space. Since $\hat{\boldsymbol{\chi}} = \hat{\mathbf{x}}$, the Dirac string is obviously now also along the 3-axis in ordinary space and from (115) it follows that its length (in group space) is $\ell = 1$, cf. eq. (104).

From the hedgehog with winding number one, we can construct the hedgehog with winding number n by taking the hedgehog field to the nth power. The new mapping still satisfies eq. (114) and is hence diagonalized by the same $V(\mathbf{x})$, but the boundary condition (115) is changed to

$$\chi(0) = 0\,, \qquad \chi(\infty) = n\pi\,, \tag{116}$$

which increases the length ℓ of the Dirac string in group space by a factor n but leaves its position unchanged. Also the induced monopole at $\mathbf{x} = 0$ has the same magnetic charge $m = 1$ as in the case $n = 1$. Thus the increase in the winding number by going from Ω to Ω^n is entirely due to the increase of the length of the Dirac string in group space.

There are, however, alternative modifications of the hedgehog which lead to higher winding numbers. Consider the mapping

$$\theta = \vartheta\,, \quad \phi = n\varphi\,, \tag{117}$$

where θ, ϕ are the polar and azimuthal angle in group space, while ϑ, φ are the corresponding quantities in ordinary space. Furthermore, we assume that $\chi(r)$ depends only on the radius and satisfies the boundary conditions (115). Obviously this field has again winding number n. Since this map represents the identity map for the polar angle $\theta = \vartheta$, there is only one Dirac string along the negative 3-axis, like in group space. However, this string now carries n times the magnetic flux of the hedgehog field with winding number one, as can be easily shown by using Stoke's theorem. Obviously, in this field configuration n monopoles are sitting on top of each other in the origin of the coordinate system.

From these n monopoles, magnetic flux lines run to infinity, where they are contracted by oppositely charged monopoles.

Finally let us consider the mapping

$$\theta = p\vartheta\,, \quad \phi = \varphi\,, \tag{118}$$

Naively one would expect that this mapping $\Omega(\mathbf{x})$ has also winding number $n[\Omega] = p$. This is, however, not true. To see this let us explicitly calculate the winding number for the generalized hedgehog mapping defined by eq. (80). The above considered examples are special cases of this mapping.

Using eq. (34) the winding number (6) is expressed as

$$n[\Omega] = \frac{1}{24\pi^2} \int d^3x\, \epsilon_{ijk} \mathrm{Tr}\,(L_i \partial_j L_k) \tag{119}$$

where in the representation (36)

$$L_k = -i\hat{\boldsymbol{\chi}}\boldsymbol{\tau}\partial_k\chi - \frac{i}{2}\sin 2\chi\,(\partial_k\hat{\boldsymbol{\chi}})\boldsymbol{\tau} + i\sin^2\chi\,(\hat{\boldsymbol{\chi}} \times \partial_k\hat{\boldsymbol{\chi}}) \cdot \boldsymbol{\tau} \tag{120}$$

so that

$$\begin{aligned} \epsilon_{ijk}\mathrm{Tr}\,(L_i\partial_j L_k) = 2\epsilon_{ijk} \Big\{ & 3\sin^2\chi\,(\partial_i\chi)(\partial_j\hat{\boldsymbol{\chi}} \times \partial_k\hat{\boldsymbol{\chi}}) \cdot \hat{\boldsymbol{\chi}} + \\ & + \cos\chi\sin^3\chi\,\partial_i\hat{\boldsymbol{\chi}} \cdot (\partial_j\hat{\boldsymbol{\chi}} \times \partial_k\hat{\boldsymbol{\chi}}) \Big\}\,. \end{aligned} \tag{121}$$

For the mapping (80) it is convenient to switch to spherical coordinates. Then it is seen that the last term on the r.h.s. of eq. (121) vanishes and the winding number becomes

$$n[\Omega] = n\, m[\hat{\boldsymbol{\chi}}]\,. \tag{122}$$

Here we have used

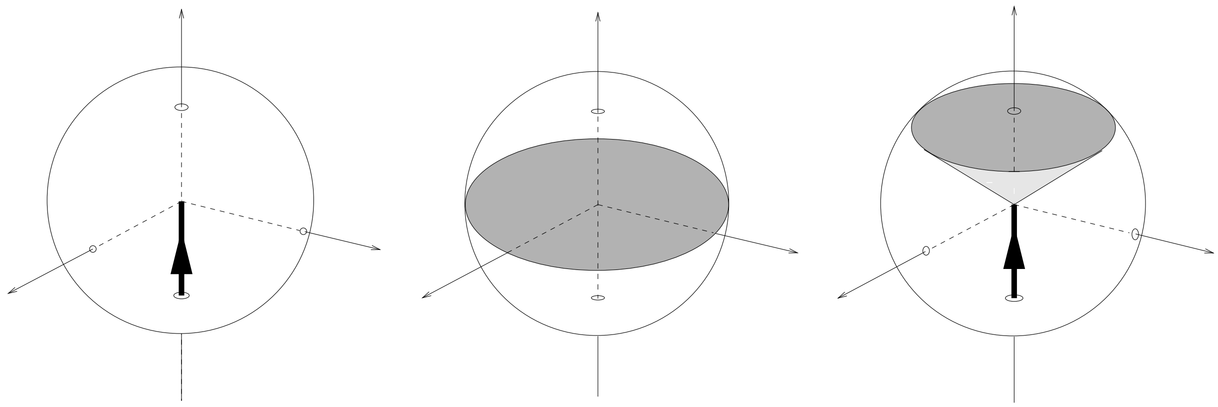
$$\int\limits_0^\infty dr\,\chi'(r)\sin^2\chi(r) = \int\limits_{\chi(0)}^{\chi(\infty)} d\chi\,\sin^2\chi = \frac{n\pi}{2} \tag{123}$$

and

$$m[\hat{\boldsymbol{\chi}}] = \frac{1}{4\pi}\int\limits_0^\pi d\vartheta \int\limits_0^{2\pi} d\varphi \left(\frac{\partial\hat{\boldsymbol{\chi}}}{\partial\vartheta} \times \frac{\partial\hat{\boldsymbol{\chi}}}{\partial\varphi}\right) \cdot \hat{\boldsymbol{\chi}} \tag{124}$$

is the $\Pi_2(S_2)$ winding number (46) of the mapping $\hat{\boldsymbol{\chi}}(\vartheta,\varphi)$ which, as shown in section 5, coincides with the magnetic charge. For the mapping (80) straightforward evaluation yields

$$m[\hat{\boldsymbol{\chi}}] = -q\frac{1}{2}(1-(-1)^p)\,. \tag{125}$$



(a) (b) (c)

Figure 4: Illustration of the Dirac type of singularities of the gauge functions $V(x)$ which diagonalize the hedgehog type of mapping Ω (118) for various numbers p: (a) $p = 1$, which is the usual hedgehog, where $V(x)$ has a string singularity at $\vartheta = \pi$, which is the familiar Dirac string. (b) $p = 2$, where $V(x)$ is singular in the equatorial plane $\vartheta = \frac{\pi}{2}$. However, in this plane the (Abelian) magnetic field vanishes and there is no magnetic monopole in the center of the plane. (c) $p = 3$, in this case $V(x)$ is singular on the familiar string at $\vartheta = \pi$ and on the cone defined by $\vartheta = \frac{\pi}{3}$.

Hence the magnetic charge and consequently also the winding number $n[\Omega]$ vanishes for even p. This result can be easily understood by noticing that $\theta = \pi + \alpha$ is equivalent to $\theta = \pi - \alpha$. Increasing θ from $\theta = 0$ to $\theta = \pi$ the $SU(2)$ group is covered once, by increasing θ further to $\theta = 2\pi$ the group is again uncovered.

Let us now return to the discussion of the mapping (118), i.e. $n = 1$, $q = 1$ in eq. (80), and let us also determine the Dirac string in ordinary space, situated still on the negative 3-axis in group space, which occurs for $\theta = (2k+1)\pi$, $k = 0, 1, 2, \ldots$, cf. eqs. (63), (66). Since $0 \leq \vartheta \leq \pi$ the integer k is restricted to

$$k \leq \frac{p-1}{2} \, . \tag{126}$$

Fig. 4 shows the distribution of the (string) singularities of $V(x)$ for various values of p. The case $p = 1$ is the familiar hedgehog, which has been treated above. In the case of $p = 2$ the Dirac string is spread out over the equatorial plane. However, from the explicit expression of the Abelian magnetic field (86) one reads off that the magnetix flux vanishes on this plane. Indeed for $p = 2$, $n = 1$, $q = 1$ we obtain from (86) $\boldsymbol{\mathcal{B}}(\vartheta = \pi/2) = 0$. Thus this singularity of V does not give rise to a singularity in the magnetic field. This is because the singular one-dimensional (Dirac) string (in group space) has been spread out over the whole two-dimensional (equatorial) plane in ordinary space. In the case of $p = 3$ we obtain two types of singularities in ordinary space, corresponding to $k = 0, 1$. For $k = 0$, the area in ordinary space which is mapped onto the singular string in colour space $\theta = \pi$ is given by a cone defined by $\vartheta = \frac{1}{3}\pi$. For $k = 1$, another string singularity occurs at $\vartheta = \pi$, which is the Dirac string we have already found in the field with winding number one. The cone $\vartheta = \pi/3$ can be considered as a deformation of the equatorial singular plane of the $p = 2$ case. Again this cone carries no magnetic flux.

References

[1] G. C. Rossi, M. Testa, Nucl. phys. **B163** (1980) 109

[2] H. Reinhardt, Mod. Phys. Lett. **A11** (1996) 2451

[3] M. B. Halpern, Phys. Rev. **D16** (1977) 3515
K. Johnson, The Yang-Mills Ground State in QCD – 20 years later, p. 795, eds. P. M. Zerwas and H. A. Kastrup
D. Z. Freedman, P. E. Haagensen, K. Johnson and J.-I. Latorre, hep-th/9309045
M. Bauer, D. Z. Freedman, P. E. Haagensen, Nucl. Phys. **B428** (1994) 147
P. E. Haagensen and K. Johnson, Nucl. Phys. **B439** (1995) 597
O. Ganor and J. Sonnenschein, The "dual" variables of Yang-Mills theory and local gauge invariant variables, hep-th/9507036
H. Reinhardt, Dual description of QCD, talk given at the Conference "Quark Confinement and the Hadron spectrum II" at Villa Olmo, Como, Italy, June 26 - 29, 1996, hep-th/9608191

[4] J. Goldstone, R. Jackiw, Phys. Lett. **B74** (1978) 81
V. Baluni, B. Grossman, Phys. Lett. **B78** (1978) 226
A. Izergin et al., Teor. Mat. Fiz. **38** (1979) 3

[5] F. Lenz, H. W. L. Naus, M. Thies, Ann. Phys. **233** (1994) 317
F. Lenz, E. J. Moniz and M. Thies, Ann. Phys. **242** (1995) 429

[6] H. Reinhardt, Phys. Rev. **D55** (1997) 2331

[7] G. 't Hooft, Nucl. Phys. **B190** (1981) 455; Phys. Scr. **25** (1981) 133

[8] A. S. Kronfeld, G. Schierholz, U.-J. Wiese, Nucl. Phys. **B293** (1987) 461
A. S. Kronfeld, M. L. Laursen, G. Schierholz, U.-J. Wiese, Phys. Lett. **B198** (1987) 516

[9] T. Suzuki and I. Yotsuyanagi, Phys. Rev. **D42** (1990) 4257
S. Hioki et al., Phys. Lett. **B272** (1991) 326
T. Suzuki, Proceedings Symposium Lattice 92, Nucl. Phys. B (Proc. Suppl.) **30** (1993) 176
J. D. Stack, S. D. Neimann and R. J. Wensley, Phys. Rev. **D50** (1994) 3399
M. N. Chernodub, M. I. Polikarpov and V. I. Veslov, Phys. Lett. **B342** (1995) 303
S. Bali, V. Bornyakov, M. Mueller-Preussker and K. Schilling, hep-lat/9603012
for a recent review see: M. I. Polikarpov, hep-lat/960902 and references therein

[10] L. Del Debbio, A. Di Giacomo, G. Pafitti and P. Pier, Phys. Lett. **B355**(1995) 255
S. Ejiri et al., Nucl. Phys. B (Proc. Suppl.) **47** (1996) 322
K. Bernstein, G. Di Cecio and R. W. Haymaker, hep-lat/9606018

[11] L. Del Debbio, M. Faber, J. Greensite and S. Olejnik, hep-lat/9610005

[12] V. Bornyakov, G. Schierholz, Phys. Lett. **B384** (1996) 190

[13] A. Hart and M. Teper, Phys. Lett. **B371** (1996) 261
R. C. Brower, K. N. Orginos, C.-I. Tan, hep-th/9610101

[14] M. N. Chernodub, F. V. Gubarev, JETP Lett. **62** (1995) 100

[15] H. Suganuwa, K. Itakura, H. Toki, hep-ph/9512141

[16] K. Langfeld, H. Reinhardt, M. Quandt, Monopoles and Strings in Yang-Mills theory, hep-th/96102213

[17] H. Suganuwa, A. Tanaka, S. Sasaki and O. Miyamura, hep-lat/9512024

[18] M. Lüscher, R. Narayanan, P. Weisz, U. Wolff, Nucl. Phys. **B384** (1992) 168

[19] K. Johnson, L. Lellouch, J. Polonyi, Nucl. Phys. **B367** (1991) 675

[20] G. C. Rossi and M. Testa, Nucl. Phys. **B176** (1980) 477, ibid. **B237** (1984) 442

[21] R. Jackiw, Rev. Mod. Phys. **52** (1980) 661
D. Gross, R. Pisarski, L. Yaffe, Rev. Mod. Phys. **53** (1981) 43

[22] R. P. Feynman, A. R. Hibbs, Quantum Mechanics and Path Integrals, Mc Graw-Hill, New York, 1965

[23] T. Bröcker and T. tom Dieck, Representations of Compact Lie Groups, Springer, Graduate Texts in Mathematics, N. Y. 1985

[24] M. Blau and G. Thompson, Commun. Math. Phys. **171** (1995) 639

[25] O. Babelon, C. M. Viallet, Phys. Lett. **85B** (1979) 296

[26] O. Jahn, T. Kraus and M. Seeger, preprint hep-th/9610056
A. Schaefke, diploma thesis, Universität Tübingen, 1997

[27] E. Langmann, M. Salmhofer and A. Kovner, Mod. Phys. Lett. **A9** (1994) 2913
U. G. Mitreuter, J. M. Pawlowski and A. Wipf, prepint FSUJ-TPI-17/96, hep-th/9611105

[28] H. Suganuma, S. Sasaki and H. Toki, Nucl. Phys. **B 435** (1995) 207

[29] N. Christ and R. Jackiw, Phys. Lett. **B91** (1980) 228

[30] H. Griesshammer, preprint hep-ph/9608390