String Representation
of the Abelian Higgs Theory
and Aharonov–Bohm Effect on the Lattice

M.I. Polikarpov

ITEP, Moscow, 117259, Russia,

e-mail: polykarp@vxdsyc.desy.de

U.-J. Wiese

HLRZ Jülich, 5170 Jülich, Germany

and M.A. Zubkov

ITEP, Moscow, 117259, Russia

Abstract

The partition function of the $4D$ lattice Abelian Higgs theory is represented as the sum over world sheets of Nielsen–Olesen strings. The creation and annihilation operators of the strings are constructed. The topological long–range interaction of the strings and charged particles is shown to exist; it is proportional to the linking number of the string world sheet and particle world trajectory.

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1　Introduction

There are several examples in the lattice field theory showing that a change of the variables in the functional integral allows for a formulation of the theory in terms of the physical excitations, *e.g.* the partition function of the $2D$ XY-model is equivalent to the partition function of the Coulomb gas [1, 2], the partition function of the $4D$ compact electrodynamics can be represented as a sum over the monopole–antimonopole world lines [3]. In the present publication we show that the partition function of the four–dimensional Abelian Higgs theory can be represented as the sum over closed surfaces which are the world sheets of the Abrikosov–Nielsen–Olesen strings [4, 5]. We construct the string creation operators, which create the string world sheets spanned on the given loops. If the field which is condensed has the charge Ne, then there is a non–trivial long–range topological interaction of Nielsen–Olesen strings with particles of charge Me, provided that $\frac{M}{N}$ is non–integer. This long–range interaction is described by the term in the action which is proportional to the linking number of the string world sheet and the world line of charge Me. This is the four–dimensional analogue [6, 7, 8] of the Aharonov–Bohm effect: strings correspond to solenoids which scatter charged particles.

2　World Sheets of the Nielsen–Olesen Strings

We consider the model describing interaction of the noncompact gauge field A_μ with the scalar field $\Phi = |\Phi|e^{i\varphi}$, whose charge is Ne. The selfinteraction of the scalar field is described by the potential $V = \lambda(|\Phi|^2 - \eta^2)^2$. For simplicity, we consider the limit as $\lambda \to \infty$, so that the radial part of the scalar field is frozen, and the dynamical variable is compact: $\varphi \in (-\pi, \pi]$. The partition function for the Villain form of the action is given by:

$$\mathcal{Z} = \int\limits_{-\infty}^{+\infty} \mathcal{D}A \int\limits_{-\pi}^{+\pi} \mathcal{D}\varphi \sum_{l(c_1)\in\mathbb{Z}} \exp\left\{-S_l(A, \mathrm{d}\varphi)\right\}, \tag{1}$$

where

$$S_l(A, \mathrm{d}\varphi) = \frac{1}{2e^2}\|\mathrm{d}A\|^2 + \frac{\kappa}{2}\|\mathrm{d}\varphi + 2\pi l - NA\|^2. \tag{2}$$

We use the notations of the calculus of differential forms on the lattice [9], which are briefly described in Appendix A. The symbol $\int \mathcal{D}\varphi$ ($\int \mathcal{D}A$) denotes the integral over all site (link) variables φ (A). Fixing the gauge $\mathrm{d}\varphi = 0$, we get the following expression for the action (2): $S_l = \frac{1}{2e^2}[A, (\delta\mathrm{d} + N^2\kappa e^2)A] +$ (*terms linear in A*); therefore, due to the Higgs mechanism, the gauge field acquires the mass $m = N\kappa^{\frac{1}{2}}e$; there are also soliton sectors of the Hilbert space which contain Abrikosov–Nielsen–Olesen strings, hidden in the summation variable l in (1).

The partition function of the compact electrodynamics can be represented as a sum over closed world lines of monopoles [3]. In the same way the partition function (1) can be rewritten as the sum over closed world sheets of the Nielsen–Olesen strings[1]:

$$\mathcal{Z}^{BKT} = \mathrm{const.} \cdot \sum_{\substack{{}^*\sigma({}^*c_2)\in\mathbb{Z} \\ \delta^*\sigma=0}} \exp\left\{-2\pi^2\kappa\left({}^*\sigma, (\Delta + m^2)^{-1*}\sigma\right)\right\}. \tag{3}$$

The derivation of this representation is given in Appendix B. The sum here is over the integer variables ${}^*\sigma$, which are attached to the plaquettes ${}^*c_2$ of the dual lattice. The condition $\delta^*\sigma = 0$ means that for each link of the dual lattice the "conservation law" is satisfied: $\sigma_1 + \sigma_2 + \sigma_3 = \sigma_4 + \sigma_5 + \sigma_6$, where σ_i are integers corresponding to plaquettes connected to the considered link. The signs of σ_i's in this "conservation law" are dictated by the definition of δ (by the orientation of the plaquettes 1,...,6). If $\sigma = 0, 1$, then the condition $\delta^*\sigma = 0$ means that we consider closed surfaces made of plaquettes with $\sigma = 1$. In (3) we have $\sigma \in \mathbb{Z}$, which means that one plaquette may "decay" into several ones, but still the surfaces, made of plaquettes with $\sigma \neq 0$, are

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[1]We use the superscript BKT, since a similar transformation of the partition function was first found by Berezinskii [1], Kostrlitz and Thouless [2], who showed that the XY model is equivalent to the Coulomb gas in two dimensions

closed. It follows from (3) that the strings interact with each other via the Yukawa forces[2] $- (\Delta + m^2)^{-1}$.

3   String Creation Operators

The creation of a string (as a nonlocal object) involves nonlocal operators. Strings are surrounded by a cloud of bosons, just as charged particles are surrounded by their photon cloud. Creation operators for charged particles were first constructed by Dirac [10], whose idea was to compensate the gauge variation of a charged field $\Phi(x)' = \Phi(x) \exp(i\alpha(x))$ by a contribution of the gauge field representing the photon cloud:

$$\Phi_c(x) = \Phi(x) \exp\left\{i \int d^3y\, B_i(x-y) A_i(y)\right\}, \tag{4}$$

where $\partial_i B_i(x) = \delta(x)$, and $A_i(x)' = A_i(x) + \partial_i\alpha(x)$ is the photon field. The gauge invariant operator $\Phi_c(x)$ creates a scalar charged particle at point x, together with the photon cloud surrounding it. Our construction of string creation operators [11] is based on the same idea, and is quite similar to the construction of soliton creation operators suggested by Fröhlich and Marchetti [12]. It is convenient to consider the model dual to the original one (1). As shown in Appendix C, its partition function has the form:

$$\mathcal{Z}^d = \text{const.} \cdot \sum_{*p(*c_2)\in\mathbb{Z}} \int\limits_{-\infty}^{+\infty} \mathcal{D}^*C \exp\left\{-\frac{1}{2\kappa}\|\mathrm{d}^*p\|^2 - \frac{N^2 e^2}{2}\|\mathrm{d}^*C + {}^*p\|^2\right\}. \tag{5}$$

The dual model describes the interaction of the integer valued hypergauge field ${}^*p({}^*c_2)$ (antisymmetric rank 2 tensor) with the real valued gauge field ${}^*C({}^*c_3)$; the action is invariant under the hypergauge transformations:

$$^*p' = {}^*p + \mathrm{d}^*r; \quad {}^*C' = {}^*C - {}^*r. \tag{6}$$

Thus we have three equivalent representations of the partition function: the original one (1), the BKT–representation (3), and the dual representation (5).

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[2]Due to the definition of the integration by parts $(\varphi, \delta\psi) = (\mathrm{d}\varphi, \psi)$, the operator $(\Delta + m^2)^{-1}$ (and not $(-\Delta + m^2)^{-1}$) is positively defined on the Euclidean lattice

Consider now the Wilson loop $W(\mathcal{C}) = \exp\{i({}^*C, {}^*j_{\mathcal{C}})\}$, where the current ${}^*j_C({}^*c_3)$ is equal to unity on the links of the dual lattice which belong to the loop $\mathcal{C}$, and vanishes on the other links. $W(\mathcal{C})$ is gauge invariant, but not hypergauge invariant. In order to make $W(\mathcal{C})$ hypergauge invariant, we use the analogue of the Dirac procedure, namely we surround the loop by the cloud of the hypergauge bosons:

$$U_{\mathcal{C}} = W(\mathcal{C}) \cdot \exp\{i({}^*D_{\mathcal{C}}, {}^*p)\}, \tag{7}$$

where $D_{\mathcal{C}}$ satisfies the equation: $\delta^{(3)}{}^*D_{\mathcal{C}} = {}^*j_{\mathcal{C}}$. Since the operator of the creation of the string should act at a definite time slice, we use the three-dimensional operator of the codifferentiation $\delta^{(3)}$, and the loop $\mathcal{C}$ belongs to the considered time slice. It is easy to see that the operator (7) is invariant under the hypergauge transformations (6): $U'_{\mathcal{C}} = U_{\mathcal{C}}\exp\{-i({}^*r, {}^*j_{\mathcal{C}}) + i(\delta^*D_{\mathcal{C}}, {}^*r)\} = U_{\mathcal{C}}$. The quantum average of this operator in the BKT representation,

$$< U_{\mathcal{C}} > = \frac{1}{\mathcal{Z}^{BKT}} \sum_{\substack{{}^*\sigma({}^*c_2)\in\mathbb{Z} \\ \delta^*\sigma={}^*j_{\mathcal{C}}}} \exp\left\{-2\pi^2\kappa\left(({}^*\sigma - {}^*D_{\mathcal{C}}), (\Delta + m^2)^{-1}({}^*\sigma - {}^*D_{\mathcal{C}})\right)\right\}$$
$$\tag{8}$$

shows, that this is indeed string creation operator, since the above sum is taken over all closed world sheets of the strings, and over all world sheets spanned on the contour C.

Performing the inverse duality transformation of the average value of the operator (7), we get the expectation value of the string creation operator in terms of the original fields:

$$< U_{\mathcal{C}} > = \frac{1}{\mathcal{Z}} \sum_{l(\mathrm{c}_1)\in\mathbb{Z}} \int_{-\infty}^{+\infty} \mathcal{D}A \int_{-\pi}^{+\pi} \mathcal{D}\varphi \exp\left\{S_l(A, \mathrm{d}\varphi - 2\pi\delta\Delta^{-1}(D_{\mathcal{C}} - \rho_{\mathcal{C}}))\right\}. \tag{9}$$

Here the integer valued field $\rho_{\mathcal{C}}$ satisfies the equation $\delta^{(3)}({}^*D_{\mathcal{C}} - {}^*\rho_{\mathcal{C}}) = 0$; ${}^*\rho_{\mathcal{C}}$ is the analog of the (invisible) Dirac string. The Dirac string connected to the monopole is a one-dimensional object, while ${}^*\rho_{\mathcal{C}}$, being defined on the plaquettes, is a two-dimensional one. The invisibility of ρ follows from the invariance of the $< U_{\mathcal{C}} >$ given by (9) under the deformations of the "Dirac sheet": $\rho' = \rho + \mathrm{d}\xi$.

4 Linking of Strings World Sheets and Particle World Trajectories

The approach considered here allows us to understand more clearly the four-dimensional analogue of the Aharonov – Bohm effect, discussed in [6, 7, 8]. Let us calculate the quantum average of the Wilson loop for the charge Me, $W_M(\mathcal{C}) = \exp\{iM(A, j_\mathcal{C})\}$, in the BKT representation. Repeating all steps which transform (1) into (3) we get:

$$< W_M(\mathcal{C}) >= \frac{1}{\mathcal{Z}^{BKT}} \sum_{\substack{*\sigma(^*c_2)\in\mathbb{Z} \\ \delta^*\sigma=0}} \exp\left\{-2\pi^2\kappa(^*\sigma, (\Delta + m^2)^{-1*}\sigma)\right. \tag{10}$$

$$\left. -\frac{M^2e^2}{2}(j_\mathcal{C}, (\Delta + m^2)^{-1}j_\mathcal{C}) - 2\pi i\frac{M}{N}(j_\mathcal{C}, (\Delta + m^2)^{-1}\delta\sigma) + 2\pi i\frac{M}{N}\mathbb{L}(\sigma, j_\mathcal{C})\right\}.$$

The first three terms in the exponent describe the short–range (Yukawa) interactions: surface – surface, current – current and current – surface. In spite of the gauge field acquiring the mass $m = N\kappa^{\frac{1}{2}}e$, there is *long–range* interaction of geometrical nature, described by the last term in the exponent $\mathbb{L}(\sigma, j_\mathcal{C})$, $\mathbb{L}$ being the four–dimensional analogue of the Gauss linking number for loops in three dimensions, *i.e.* the linking number of surfaces defined by $\{\sigma\}$ and loop defined by $j_\mathcal{C}$. The explicit expression for $\mathbb{L}$ is:

$$\mathbb{L} = (^*j_\mathcal{C}, \Delta^{-1}\mathrm{d}^*\sigma) = (^*j_\mathcal{C}, ^*n) \tag{11}$$

where $^*n$ is an integer valued 3-form which is the solution of the equation: $\delta^*n = {}^*\sigma$. It is clear now that $\mathbb{L}$ is equal to the number of points at which the loop $j_\mathcal{C}$ intersects the three–dimensional volume $^*n$ bounded by the closed surface defined by $^*\sigma(^*c_2)$. The elements of the surface $^*\sigma$ may carry any integer number, so that any intersection point may contribute an integer into $\mathbb{L}$. Therefore $\mathbb{L}$ is the linking number of the world sheet of the strings and the current $j_\mathcal{C}$ which define Wilson's loop $W_M(\mathcal{C})$. The reason for the long–range interaction is that the charges e, $2e$, $\ldots(N-1)e$ cannot be completely screened by the condensate of the field of charge Ne; if M/N is integer, then the screening is complete and there are no long–range forces. The long–range particle–particle interaction may appear in that phase of the theory where the condensate of strings exists, and $\mathbb{L}(\sigma, j_\mathcal{C})$ does not vanish

for large Wilson loops. The dynamical properties of quantum Nielsen–Olesen strings are discussed in [13].

Another interesting operator, which can be calculated *exactly*, was suggested in [8]; this operator is the product of the Wilson loop $W_M(\mathcal{C})$ and the operator suggested in [14], which creates the world sheet of the string on the closed surface Σ:

$$F_N(\Sigma) = \sum_{l(c_1) \in \mathbb{Z}} \exp\left\{-S_l\left(A - \frac{2\pi k}{N}, \mathrm{d}\varphi\right) + S_l(A, \mathrm{d}\varphi)\right\}, \qquad (12)$$

where k defines the surface Σ on the dual lattice: $\delta^* k = \delta_\Sigma$; δ_Σ is the lattice δ–function which is equal to unity on the plaquettes of the dual lattice belonging to the surface Σ, and δ_Σ vanishes on all other plaquettes. We can change the integration variable: $A \to A + \frac{2\pi k}{N}$ therefore $< F_N(\Sigma) >= 1$. The operator which has a nontrivial expectation value has the form [8]:

$$A_{NM}(\Sigma, \mathcal{C}) = F_N(\Sigma) \cdot \frac{W_M(\mathcal{C})}{< W_M(\mathcal{C}) >}. \qquad (13)$$

Performing the same steps which lead to (10) we get:

$$< A_{NM}(\Sigma, \mathcal{C}) >= e^{2\pi i \frac{M}{N} \mathbb{L}(\Sigma, \mathcal{C})}. \qquad (14)$$

The meaning of this result is very simple. If the surface Σ lies in a given time slice, then $F(\Sigma) = \exp\left\{\frac{2\pi i}{Ne} Q_\Sigma\right\}$ (see [14, 8]), where Q_Σ is the total charge inside the volume bounded by the surface Σ; if $\mathbb{L}(\Sigma, \mathcal{C}) = n$ then there is charge Mne in the volume bounded by Σ.

5   Conclusions; Acknowledgments

We have shown that the partition function of Abelian Higgs theory can be represented as the sum over world sheets of the Nielsen–Olesen strings; the dynamics of these strings (*e.g.* scattering and decay properties) can be studied by means of string creation and annihilation operators. Similar analysis can be carried out in the case of a compact gauge field [13]. In this case, the monopoles are present in the theory, and strings can be open carrying monopole and antimonopole on their ends. String–like excitations exist also

in the scalar theory without gauge field (global strings). Our formulas (1)–(9) are valid in this case ($e = 0$, $m = 0$). Now the topological interaction is absent, but there exists the long–range interaction Δ^{-1} which is due to the cloud of Goldstone bosons surrounding the string. Our formulas are also valid for the three–dimensional case. For example for $D = 3$, $e = 0$, $m = 0$, we have a theory of vortices in the two–dimensional superfluid. The dual formulation of the continuum Abelian–Higgs theory has been recently discussed in [15].

Appendix A

Here we briefly summarize the main notions from the theory of differential forms on the lattice [9]. The advantages of the calculus of differential forms consists in the general character of the expressions obtained. Most of the transformations depend neither on the space–time dimension, nor on the rank of the fields. With minor modifications the transformations are valid for lattices of any form (triangular, hypercubic, random, *etc.*). A differential form of rank k on the lattice is a function ϕ_k defined on k-dimensional cells c_k of the lattice, *e.g.* the scalar (gauge) field is a 0–form (1–form). The exterior differential operator d is defined as follows:

$$(\mathrm{d}\phi)(c_{k+1}) = \sum_{c_k \in \partial c_{k+1}} \phi(c_k). \tag{A.1}$$

Here ∂c_k is the oriented boundary of the k-cell c_k. Thus the operator d increases the rank of the form by unity; $\mathrm{d}\varphi$ is the link variable constructed, as usual, in terms of the site angles φ, and $\mathrm{d}A$ is the plaquette variable constructed from the link variables A. The scalar product is defined in the standard way: if φ and ψ are k-forms, then $(\varphi, \psi) = \sum_{c_k} \varphi(c_k)\psi(c_k)$, where $\sum_{c_k}$ is the sum over all sells c_k. To any k–form on the D–dimensional lattice there corresponds a $(D - k)$–form $^*\Phi(^*c_k)$ on the dual lattice, $^*c_k$ being the $(D - k)$–dimensional cell on the dual lattice. The codifferential $\delta = {^*}\mathrm{d}{^*}$ satisfies the partial integration rule: $(\varphi, \delta\psi) = (\mathrm{d}\varphi, \psi)$. Note that $\delta\Phi(c_k)$ is a

$(k-1)$–form and $\delta\Phi(c_0) = 0$. The norm is defined by: $\|a\|^2 = (a, a)$; therefore, $\|d\varphi + 2\pi l\|^2$ in (2) implies summation over all links. $\sum_{l(c_1)\in\mathbb{Z}}$ denotes the sum over all configurations of the integers l attached to the links c_1. The action (2) is invariant under the gauge transformations $A' = A + d\alpha$, $\varphi' = \varphi + \alpha$ due to the well known property $d^2 = \delta^2 = 0$. The lattice Laplacian is defined by: $\Delta = d\delta + \delta d$.

Appendix B

To derive eq.(3) we first change the summation variable in (1) (see [16], Part 1, Chapter 4): $\displaystyle\sum_{l(c_1)\in\mathbb{Z}} = \sum_{\substack{\sigma(c_2)\in\mathbb{Z}\\ d\sigma=0}} \sum_{q(c_0)\in\mathbb{Z}}$, here $l = m[\sigma] + dq$ and $m[\sigma]$ is a particular solution of the equation $dm[\sigma] = \sigma$. Using the Hodge decomposition $m[\sigma] = \delta\Delta^{-1}\sigma + d\Delta^{-1}\delta m[\sigma]$ we introduce the noncompact field

$$\Phi = \varphi + 2\pi(\Delta^{-1}\delta m[\sigma] + q), \quad \sum_{q(c_0)\in\mathbb{Z}}\int_{-\pi}^{+\pi}\mathcal{D}\varphi = \int_{-\infty}^{+\infty}\mathcal{D}\Phi, \text{ and we get:}$$

$$\mathcal{Z} = \int_{-\infty}^{+\infty}\mathcal{D}A\mathcal{D}\Phi \sum_{\substack{\sigma(c_2)\in\mathbb{Z}\\ d\sigma=0}} \exp\left\{-\frac{1}{2e^2}\|dA\|^2 - \frac{\kappa}{2}\|d\Phi + 2\pi\Delta^{-1}\delta\sigma - NA\|^2\right\}.$$

$$\text{(B.1)}$$

After fixing the gauge $d\Phi = 0$, the Gaussian integral over A can be easily calculated, and thus we get (3).

Appendix C

To perform the duality transformation of the original theory defined by the partition function (1), we change the dynamical variables, introducing the unity: $1 = \int_{-\infty}^{+\infty}\mathcal{D}F\mathcal{D}B\,\delta(B - d\varphi - 2\pi l + NA)\,\delta(F - dA)$ into the integral in (1). On application of the Poisson summation formula $2\pi\sum_l\delta(x-2\pi l) = \sum_l\exp\{ilx\}$ and the standard representation $\delta(F-dA) = \text{const}\int\mathcal{D}G\exp\{i(G, (F - dA))\}$, the integrals over F and B become Gaussian, the integrals over φ and A give the restrictions $\delta l = 0$ and $\delta G = lN$,

which are solved by the introduction of new variables p and C: $l = {}^*\mathrm{d}^*p({}^*c_2)$, $G = N^*\mathrm{d}^*C({}^*c_3) + Np$. Integrating over F and B we finally get (5).

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