

The Dyson-Schwinger equation for a model with instantons – the Schwinger model*

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Abstract:

Using the exact path integral solution of the Schwinger model – a model where instantons are present – the Dyson-Schwinger equation is shown to hold by explicit computation. It turns out that the Dyson-Schwinger equation separately holds for every instanton sector. This is due to Θ invariance of the Schwinger model.

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Introduction

The Schwinger model (SM), which is QED₂ with one massless fermion, is known to be exactly soluble ([1]), and explicit solutions within the operator formalism ([2]) and the path integral (PI) approach ([3], [4], [5], [6], [7], [8]) are wellknown. Further instantons and an instanton vacuum have to be present in the SM in order to obtain a consistent quantization ([7], [8]). So the question of whether there is a modification of the Dyson-Schwinger (DS) equation ([10], [11], [12]) when an instanton vacuum exists may be answered by explicit calculation in the SM. On one hand we will compute that for the trivial sector (instanton number $k = 0$) the DS equation holds as it must be. On the other hand it will turn out that the DS equation even holds for the nontrivial sector $k \neq 0$. More precisely it holds separately for every instanton sector k . So, at least for the model at hand, the DS equation may be extended to the case where the true vacuum is not the perturbative one but an instanton or Θ vacuum. The reason why every instanton sector separately fulfills the DS equation is related to the Θ invariance of the SM, as we will discuss below.

Our computations are done in flat Euclidean space E^2 , and we use conventions like in [7], [8], [9]:

$$\begin{aligned} g_{\mu\nu} &= \delta_{\mu\nu} \quad , \quad \gamma_5 = -i\gamma_E^0\gamma_E^1 =: -i\gamma^0\gamma^1 \quad , \\ \gamma_\mu\gamma_5 &= \epsilon_{\mu\nu}\gamma^\nu \quad , \quad \epsilon_{\mu\nu}\epsilon^{\nu\lambda} = g_\mu^\lambda \quad , \quad \epsilon_{01} = -i. \end{aligned} \quad (1)$$

The abelian gauge field A_μ is parametrized like in [7], [8],

$$A_\mu(x) = \frac{1}{e}(\partial_\mu\alpha(x) + \epsilon_{\mu\nu}\partial^\nu\beta(x)), \quad (2)$$

where e is the dimensionfull charge. The gauge fixing $\alpha = 0$ is chosen in the sequel. β , which is the dynamical part of A_μ , may carry an integer instanton number (Pontryagin index)

$$\nu = -\frac{i}{2\pi} \int dx \square\beta(x) = -\frac{ie}{2\pi} \int dx \tilde{F}(x) = k \in \mathbf{Z}. \quad (3)$$

As representative for a "k-instanton" (wellbehaving at all spacetime but not minimizing the action) we choose

$$eA_\mu(x) = ik\epsilon_{\mu\nu}\frac{x^\nu}{x^2 + \lambda^2} = \epsilon_{\mu\nu}\partial^\nu\frac{ik}{2}\ln\frac{x^2 + \lambda^2}{\lambda^2} =: \epsilon_{\mu\nu}\partial^\nu\beta_k(x). \quad (4)$$

The vacuum functional has to be summed over all k instanton configurations,

$$Z[J_\mu, \eta, \bar{\eta}] = \sum_{k=-\infty}^{\infty} Z_k[J_\mu, \eta, \bar{\eta}], \quad (5)$$

where

$$Z_k[J_\mu, \eta, \bar{\eta}] = N \int (DA_\mu)_k D\bar{\Psi}D\Psi e^{\int dx [\bar{\Psi}(i\cancel{\partial} - eA)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu + \bar{\Psi}\eta + \bar{\eta}\Psi]}. \quad (6)$$

When one separates the zero modes of the Dirac operator and integrates out the fermions, the resulting expression for the vacuum functional leading to the correct quantum theory of the SM is (see [7], [8] for details)

$$Z_k[J_\mu, \eta, \bar{\eta}] = N \int D\beta \prod_{i_0=0}^{k-1} (\bar{\eta}\Psi_{i_0}^{\bar{\beta}})(\bar{\Psi}_{i_0}^{\bar{\beta}}\eta).$$

$$e^{i \int \bar{\eta}(x) G(x, y) \eta(y) dx dy} e^{\int (\frac{1}{2} \bar{\beta} \mathbf{D} \bar{\beta} + \frac{1}{e} \bar{\beta} \lambda) dx}, \quad (7)$$

where $\bar{\beta} = \beta + \beta_k$, β_k is the "instanton" (4) and β has zero instanton number; further

$$A_\mu J^\mu = \beta \lambda,$$

so

$$\frac{\delta}{\delta J_\mu(x)} = \epsilon^{\mu\nu} \partial_\nu \frac{\delta}{\delta \lambda(x)}; \quad (8)$$

$G(x, y)$ is the exact fermion propagator,

$$G(x, y) = e^{i(\bar{\beta}(x) - \bar{\beta}(y)) \gamma_5} G_0(x - y), \quad (9)$$

and G_0 is the free fermion propagator. The brackets mean

$$(\bar{\eta} \Psi) \equiv \int dz \bar{\eta}^\alpha(z) \Psi^\alpha(z). \quad (10)$$

\mathbf{D} is the operator of the effective photon action after the integration of the fermions (see [9], [8])

$$\mathbf{D} = \frac{\square}{e^2} (\square - \mu^2) \quad , \quad \mu^2 := \frac{e^2}{\pi}, \quad (11)$$

which has the Green's function ([7], [8])

$$\begin{aligned} \mathbf{G}(x - y) &= \pi(D(\mu, x - y) - D(0, x - y)), \\ \mathbf{D}_x \mathbf{G}(x - y) &= \delta(x - y), \end{aligned} \quad (12)$$

$D(\mu, x)$ and $D(0, x)$ being the massive and massless scalar field propagators in two dimensions.

$\Psi_{i_0}^{\bar{\beta}}$ in (7) is a zero mode of the Dirac operator with respect to the gauge field $\bar{\beta}$. It may be related to a zero mode with respect to the "instanton" β_k of (4) by

$$\Psi_{i_0}^{\bar{\beta}}(x) = e^{\sigma i \beta(x)} \Psi_{i_0}^{\beta_k}(x), \quad (13)$$

where $\sigma = \pm 1$ is the eigenvalue of γ_5 , $\gamma_5 \Psi_{i_0}^{\beta_k} = \pm \Psi_{i_0}^{\beta_k}$ for $k \geq 0$. The zero modes $\Psi_{i_0}^{\beta_k}$ may be computed ([7], [8]),

$$\begin{aligned} \Psi_{i_0}^{\beta_k} &= \frac{1}{\sqrt{2\pi}} (x^-)^{i_0} \left(\frac{x^2 + \lambda^2}{\lambda^2} \right)^{-\frac{k}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad k > 0 \quad , \quad i_0 = 0 \dots k - 1, \\ \Psi_{i_0}^{\beta_k} &= \frac{1}{\sqrt{2\pi}} (x^+)^{i_0} \left(\frac{x^2 + \lambda^2}{\lambda^2} \right)^{\frac{k}{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \quad k < 0 \quad , \quad i_0 = 0 \dots |k| - 1, \\ x^+ &= x_1 + i x_0 \quad , \quad x^- = x_1 - i x_0. \end{aligned} \quad (14)$$

The Dyson-Schwinger equations

The general derivation of the Dyson-Schwinger (DS) equations for QED starts with the observation that a field derivative of the vacuum functional vanishes ([10]):

$$\frac{\delta Z[J_\mu, \eta, \bar{\eta}]}{\delta \bar{\Psi}_\alpha(x)} = \left(\frac{\delta S}{\delta \bar{\Psi}_\alpha(x)} \left[\frac{\delta}{\delta J_\mu}, \frac{\delta}{\delta \eta}, \frac{\delta}{\delta \bar{\eta}} \right] + \eta_\alpha(x) \right) Z[J_\mu, \eta, \bar{\eta}] = 0 \quad (15)$$

and

$$\frac{\delta Z[J_\mu, \eta, \bar{\eta}]}{\delta A^\nu(x)} = 0. \quad (16)$$

Inserting the Dirac equation in (15) or the Maxwell equation in (16), performing derivatives with respect to the sources and setting the sources equal to zero at the end results in equations for the exact n -point functions of the theory - the DS equations.

Now the only nontrivial vacuum expectation values (VEVs) of the SM involving instanton contributions are VEVs of products of (pseudo)scalar currents ([3], [7], [8]). The Maxwell equations however generate only gauge field and vector current correlators, so they are not interesting for the problem which we want to discuss - the DS equations in an instanton vacuum. Therefore it suffices to work with equation (15), which is the quantum analog of the Dirac equation.

The simplest nontrivial DS equation is obtained from (15) by performing three fermion source derivatives, resulting in an equation between fermionic two- and four-point functions. In the explicit calculation of this DS equation we will observe a cancellation mechanism between different Green's functions that may be easily generalized to higher n -point functions. So it is enough to deal with this DS equation (another reason for doing the computation for the four-point function is its importance for a Bethe-Salpeter analysis of the SM, which will be done in a forthcoming paper). It reads

$$\frac{\delta}{\delta \eta_\epsilon(y_2)} \frac{\delta}{\delta \eta_\delta(x_2)} \frac{\delta}{\delta \bar{\eta}_\gamma(y_1)} \left(\eta^\alpha(x_1) + \gamma_\mu^{\alpha\beta} (i\partial_{x_1}^\mu - e \frac{\delta}{\delta J_\mu(x_1)}) \frac{\delta}{\delta \bar{\eta}_\beta(x_1)} \right) Z|_0 = 0, \quad (17)$$

where the vertical line indicates that the sources have to be set equal to zero after the differentiation. We rewrite (17) in three terms that we will compute separately, and they have to sum up to zero:

$$\left[-\delta_\delta^\alpha \delta(x_2 - x_1) \frac{\delta}{\delta \eta_\epsilon(y_2)} \frac{\delta}{\delta \bar{\eta}_\gamma(y_1)} + \delta_\epsilon^\alpha \delta(y_2 - x_1) \frac{\delta}{\delta \eta_\delta(x_2)} \frac{\delta}{\delta \bar{\eta}_\gamma(y_1)} \right] Z|_0 \equiv A \quad (18)$$

$$i\gamma_\mu^{\alpha\beta} \partial_{x_1}^\mu \frac{\delta}{\delta \eta_\epsilon(y_2)} \frac{\delta}{\delta \eta_\delta(x_2)} \frac{\delta}{\delta \bar{\eta}_\gamma(y_1)} \frac{\delta}{\delta \bar{\eta}_\beta(x_1)} Z|_0 \equiv B \quad (19)$$

$$-e\gamma_\mu^{\alpha\beta} \frac{\delta}{\delta J_\mu(x_1)} \frac{\delta}{\delta \eta_\epsilon(y_2)} \frac{\delta}{\delta \eta_\delta(x_2)} \frac{\delta}{\delta \bar{\eta}_\gamma(y_1)} \frac{\delta}{\delta \bar{\eta}_\beta(x_1)} Z|_0 \equiv C. \quad (20)$$

Next we must remember that for a VEV of n fermions and n antifermions instanton sectors up to $k = \pm n$ contribute ([3], [7], [8], see formula (7)). So for the term A $k = 0, \pm 1$ will be important, and for B, C $k = 0, \pm 1, \pm 2$.

Further the general spinor structure is important in the DS equation only for the indices α, β, δ belonging to the x_i variables, on which the DS equation acts. We therefore may focus on

the scalar component $\gamma = \epsilon$ for the y_i variables without loss of information (remember that only scalar components are important for instanton effects). This will simplify the computations and is fixed in the sequel.

Now we are ready to start the computations. We will do them separately for $k = 0, +1, +2$ (-1 and -2 are analogous to $+1, +2$) and find that the DS equation (17) separately holds for every instanton sector.

Computation for $k = 0$

Using (7) and (8) we get for the term C^0 (20) (the zero index indicates zero instanton sector)

$$\begin{aligned} & -e\gamma_\mu^{\alpha\beta} \frac{\delta}{\delta J_\mu(x_1)} \frac{\delta}{\delta \eta_\gamma(y_2)} \frac{\delta}{\delta \eta_\delta(x_2)} \frac{\delta}{\delta \bar{\eta}_\gamma(y_1)} \frac{\delta}{\delta \bar{\eta}_\beta(x_1)} Z|_0 = \\ & = \gamma_\mu^{\alpha\beta} \epsilon^{\mu\nu} N \int D\beta(\partial_\nu^{x_1} \beta(x_1)) [G^{\gamma\delta}(y_1, x_2) G^{\beta\gamma}(x_1, y_2) - \\ & \quad - G^{\beta\delta}(x_1, x_2) G^{\gamma\gamma}(y_1, y_2)] e^{\frac{1}{2} \int dz \beta \mathbf{D}\beta}, \end{aligned} \quad (21)$$

where the rules of Grassmann differentiation have been used. With $G^{\gamma\gamma} = \text{tr}G = 0$ we further obtain (see (9))

$$\begin{aligned} & \epsilon^{\mu\nu} \gamma_\mu^{\alpha\beta} N \int D\beta(\partial_\nu^{x_1} \beta(x_1)) [e^{i\gamma_5(\beta(x_1) + \beta(x_2) - \beta(y_1) - \beta(y_2))}]^{\beta\gamma} \cdot \\ & \quad \cdot G_0^{\gamma\epsilon}(x_1 - y_2) G_0^{\epsilon\delta}(y_1 - x_2) e^{\frac{1}{2} \int dz \beta \mathbf{D}\beta} \end{aligned} \quad (22)$$

and, introducing the real field $\beta' = i\beta$ (we omit the prime) and using the short notation

$$D\mu[\beta] := N \int D\beta e^{-\frac{1}{2} \int dz \beta \mathbf{D}\beta} \quad (23)$$

we get

$$\begin{aligned} & -i\epsilon^{\mu\nu} \gamma_\mu^{\alpha\beta} \int D\mu[\beta] (\partial_\nu^{x_1} \beta(x_1)) \{[\mathbf{1} \cosh(\beta(x_1) + \beta(x_2) - \beta(y_1) - \beta(y_2)) + \\ & \quad + \gamma_5 \sinh(\beta(x_1) + \beta(x_2) - \beta(y_1) - \beta(y_2))] G_0(x_1 - y_2) G_0(y_1 - x_2)\}^{\beta\delta} = \\ & = -i\gamma_\mu^{\alpha\beta} [\gamma_5 G_0(x_1 - y_2) G_0(y_1 - x_2)]^{\beta\delta} \epsilon^{\mu\nu} \partial_\nu^{x_1} \int D\mu[\beta] \cdot \\ & \quad \cdot \cosh(\beta(x_1) + \beta(x_2) - \beta(y_1) - \beta(y_2)), \end{aligned} \quad (24)$$

where we used the Leibnitz rule for the cosh and the fact that in the Gaussian PI only even powers of the variable β contribute. With the identity $\gamma_\mu \gamma_5 \epsilon^{\mu\nu} = -\gamma^\nu$ (see (1)) we finally arrive at

$$\begin{aligned} C^0 & = i\gamma_\mu^{\alpha\beta} G_0^{\beta\gamma}(x_1 - y_2) G_0^{\gamma\delta}(y_1 - x_2) \partial_{x_1}^\mu \int D\mu[\beta] \cdot \\ & \quad \cdot \cosh(\beta(x_1) + \beta(x_2) - \beta(y_1) - \beta(y_2)). \end{aligned} \quad (26)$$

The computation of B^0 is nearly identical up to formula (22), just instead of $-\epsilon^{\mu\nu}(\partial_\nu^{x_1} \beta(x_1))$ we have to insert $i\partial_{x_1}^\mu$ acting on the whole four-point function (see (19), (20)). Introducing again a real β like in (24), this time only the $\mathbf{1}$ cosh part of the PI contributes and we finally get

$$B^0 = -i\gamma_\mu^{\alpha\beta} \partial_{x_1}^\mu G_0^{\beta\gamma}(x_1 - y_2) G_0^{\gamma\delta}(y_1 - x_2) \int D\mu[\beta].$$

$$\cdot \cosh(\beta(x_1) + \beta(x_2) - \beta(y_1) - \beta(y_2)) = \quad (27)$$

$$= -C^0 - i[\partial_{x_1} G_0(x_1 - y_2)]^{\alpha\gamma} G_0^{\gamma\delta}(y_1 - x_2) \int D\mu[\beta].$$

$$\cdot \cosh(\beta(x_1) + \beta(x_2) - \beta(y_1) - \beta(y_2)) = \quad (28)$$

$$= -C^0 - i\delta(x_1 - y_2) G_0^{\alpha\delta}(y_1 - x_2) \int D\mu[\beta] \cosh(\beta(x_2) - \beta(y_1)). \quad (29)$$

So the first term in B^0 just cancels C^0 , whereas the second one results in a two-point function that will cancel A^0 of (18) as we now shortly compute.

Keeping β real from the very beginning we find (because we fixed $\gamma = \epsilon$ again the first term of (18) is zero!)

$$\begin{aligned} A^0 &= \delta(y_2 - x_1) \frac{\delta}{\delta\eta_\delta(x_2)} \frac{\delta}{\delta\bar{\eta}_\alpha(y_1)} Z|_0 = \\ &= i\delta(y_2 - x_1) \int D\mu[\beta] G^{\alpha\delta}(y_1 - x_2) = \\ &= i\delta(y_2 - x_1) G_0^{\alpha\delta}(y_1 - x_2) \int D\mu[\beta] \cosh(\beta(y_1) - \beta(x_2)), \end{aligned} \quad (30)$$

which indeed cancels the rest of B^0 in (29).

So the DS equation is separately fulfilled for $k = 0$, which however is the perturbative vacuum. Therefore this result is not too surprising.

Computation for $k = 1$

We compute the case $k = +1$, $k = -1$ is nearly identical. In formula (7) now one zero mode is present and we obtain for C^1 (20)

$$\begin{aligned} C^1 &= -\epsilon^{\mu\nu} \gamma_\mu^{\alpha\beta} \frac{\delta}{\delta\eta_\gamma(y_2)} \frac{\delta}{\delta\eta_\delta(x_2)} \frac{\delta}{\delta\bar{\eta}_\gamma(y_1)} \frac{\delta}{\delta\bar{\eta}_\beta(x_1)} N \int D\beta(\partial_\nu^{x_1} \bar{\beta}(x_1)) \cdot \\ &\quad \cdot (\bar{\eta}\Psi_0)(\bar{\Psi}_0\eta) e^{i \int dz dz' \bar{\eta}(z) G(z, z') \eta(z')} e^{\frac{1}{2} \int dz \bar{\beta} \mathbf{D} \bar{\beta}} = \\ &= -i\epsilon^{\mu\nu} \gamma_\mu^{\alpha\beta} N \int D\beta(\partial_\nu^{x_1} \bar{\beta}(x_1)) [-\Psi_0^\beta(x_1) \bar{\Psi}_0^\delta(x_2) G^{\gamma\gamma}(y_1, y_2) + \Psi_0^\beta(x_1) \bar{\Psi}_0^\gamma(y_2) G^{\gamma\delta}(y_1, x_2) + \\ &\quad + \Psi_0^\gamma(y_1) \bar{\Psi}_0^\delta(x_2) G^{\beta\gamma}(x_1, y_2) - \Psi_0^\gamma(y_1) \bar{\Psi}_0^\gamma(y_2) G^{\beta\delta}(x_1, x_2)] e^{\frac{1}{2} \int dz \bar{\beta} \mathbf{D} \bar{\beta}}, \end{aligned} \quad (31)$$

where again $G^{\gamma\gamma} = 0$ in the first term is zero. Inserting for the G formula (9), transforming the zero modes like in (13) and introducing the zero mode projectors (see (14))

$$S^{\alpha\beta}(x, y) := (\Psi_0^{(\beta_1)}(x))^\alpha (\bar{\Psi}_0^{(\beta_1)}(y))^\beta = \frac{1}{2\pi} P_+^{\alpha\beta} e^{i(\beta_1(x) + \beta_1(y))} \quad (32)$$

we get

$$\begin{aligned} C^1 &= -i\epsilon^{\mu\nu} \gamma_\mu^{\alpha\beta} N \int D\beta(\partial_\nu^{x_1} \bar{\beta}(x_1)) [e^{i\beta(x_1) + i\beta(x_2)} S^{\beta\gamma}(x_1, y_2) \cdot \\ &\quad \cdot (e^{i\gamma_5(\bar{\beta}(y_1) - \bar{\beta}(x_2))})^{\gamma\epsilon} G_0^{\epsilon\delta}(y_1 - x_2) + e^{i\beta(y_1) + i\beta(x_2)} S^{\gamma\delta}(y_1, x_2) \cdot \\ &\quad \cdot (e^{i\gamma_5(\bar{\beta}(x_1) - \bar{\beta}(y_2))})^{\beta\epsilon} G_0^{\epsilon\gamma}(x_1 - y_2) - e^{i\beta(y_1) + i\beta(y_2)} S^{\gamma\gamma}(y_1, y_2) \cdot \end{aligned}$$

$$\cdot (e^{i\gamma_5(\bar{\beta}(x_1)-\bar{\beta}(x_2))})^{\beta\epsilon} G_0^{\epsilon\delta}(x_1-x_2)] e^{\frac{1}{2} \int dz \bar{\beta} \mathbf{D} \bar{\beta}}, \quad (33)$$

and, turning to real β and using $\gamma_5 P_+ = P_+ \gamma_5 = P_+$, we further get

$$\begin{aligned} C^1 = & -\frac{1}{2\pi} \epsilon^{\mu\nu} \gamma_\mu^{\alpha\beta} N \int D\beta (\partial_\nu^{x_1} \bar{\beta}(x_1)) e^{-\frac{1}{2} \int dz \bar{\beta} \mathbf{D} \bar{\beta}} \cdot \\ & \cdot [\sinh(\bar{\beta}(x_1) - \bar{\beta}(x_2) + \bar{\beta}(y_1) + \bar{\beta}(y_2)) P_+^{\beta\gamma} G_0^{\gamma\delta}(y_1 - x_2) + \\ & + \sinh(-\bar{\beta}(x_1) + \bar{\beta}(x_2) + \bar{\beta}(y_1) + \bar{\beta}(y_2)) P_+^{\gamma\delta} G_0^{\beta\gamma}(x_1 - y_2) - \\ & e^{\bar{\beta}(y_1) + \bar{\beta}(y_2)} \sinh(\bar{\beta}(x_1) - \bar{\beta}(x_2)) \gamma_5^{\beta\epsilon} G_0^{\epsilon\delta}(x_1 - x_2) - \\ & - e^{\bar{\beta}(y_1) + \bar{\beta}(y_2)} \cosh(\bar{\beta}(x_1) - \bar{\beta}(x_2)) \delta^{\beta\epsilon} G_0^{\epsilon\delta}(x_1 - x_2)]. \end{aligned} \quad (34)$$

Here the first two terms in (34) may be treated in a way identical to that for the $k = 0$ case (leading from formula (24) to (26)). The third and fourth term, after applying Leibnitz rule for the derivative $\partial_\nu^{x_1}$ and keeping track of even β powers only, look like $\cosh(\dots) \cdot \cosh(\dots)$ and $\sinh(\dots) \cdot \sinh(\dots)$ respectively. However by using the formulae

$$\begin{aligned} \cosh a \cosh b &= \frac{1}{2} (\cosh(a+b) + \cosh(a-b)), \\ \sinh a \sinh b &= \frac{1}{2} (\cosh(a+b) - \cosh(a-b)), \end{aligned} \quad (35)$$

they may be brought into standard form. Besides, these two terms, being proportional to $\mathbf{1}$ and γ_5 , combine into P_+ , P_- via formulae (35), and after some gamma matrix algebra we find the final result

$$\begin{aligned} & \frac{1}{2\pi} \{ \gamma_\nu^{\alpha\beta} G_0^{\beta\gamma}(y_1 - x_2) P_-^{\gamma\delta} \partial_{x_1}^\nu \int D\mu[\beta] \cosh(\beta(x_1) - \beta(x_2) + \beta(y_1) + \beta(y_2)) + \\ & + \gamma_\nu^{\alpha\beta} G_0^{\beta\gamma}(x_1 - y_2) P_+^{\gamma\delta} \partial_{x_1}^\nu \int D\mu[\beta] \cosh(-\beta(x_1) + \beta(x_2) + \beta(y_1) + \beta(y_2)) - \\ & - \gamma_\nu^{\alpha\beta} G_0^{\beta\gamma}(x_1 - x_2) P_-^{\gamma\delta} \partial_{x_1}^\nu \int D\mu[\beta] \cosh(\beta(x_1) - \beta(x_2) + \beta(y_1) + \beta(y_2)) - \\ & - \gamma_\nu^{\alpha\beta} G_0^{\beta\gamma}(x_1 - x_2) P_+^{\gamma\delta} \partial_{x_1}^\nu \int D\mu[\beta] \cosh(-\beta(x_1) + \beta(x_2) + \beta(y_1) + \beta(y_2)) \}. \end{aligned} \quad (36)$$

For the computation of B^1 we may again use an intermediate result for C^1 , (33), substitute $i\epsilon^{\mu\nu}(\partial_\nu^{x_1} \bar{\beta}(x_1))$ by $\partial_\mu^{x_1}$, and find

$$\begin{aligned} B^1 = & \frac{1}{2\pi} \gamma_\mu^{\alpha\beta} \partial_{x_1}^\mu \{ -P_+^{\beta\gamma} G_0^{\gamma\delta}(y_1 - x_2) \int D\mu[\beta] \cosh(\beta(x_1) - \beta(x_2) + \beta(y_1) + \beta(y_2)) - \\ & - G_0^{\beta\gamma}(x_1 - y_2) P_+^{\gamma\delta} \int D\mu[\beta] \cosh(-\beta(x_1) + \beta(x_2) + \beta(y_1) + \beta(y_2)) + \\ & + G_0^{\beta\delta}(x_1 - x_2) \int D\mu[\beta] \cosh(\beta(x_1) - \beta(x_2)) \cosh(\beta(y_1) + \beta(y_2)) + \\ & + \gamma_5^{\beta\gamma} G_0^{\gamma\delta}(x_1 - x_2) \int D\mu[\beta] \sinh(\beta(x_1) - \beta(x_2)) \sinh(\beta(y_1) + \beta(y_2)) \} = \end{aligned} \quad (37)$$

$$\begin{aligned}
&= -C^1 - \frac{1}{2\pi} (\not{\partial}_{x_1} G_0(x_1 - y_2))^{\alpha\gamma} P_+^{\gamma\delta} \int D\mu[\beta] \cosh(-\beta(x_1) + \beta(x_2) + \beta(y_1) + \beta(y_2)) + \\
&\quad + (\not{\partial}_{x_1} G_0(x_1 - x_2))^{\alpha\delta} \int D\mu[\beta] \cosh(\beta(x_1) - \beta(x_2)) \cosh(\beta(y_1) + \beta(y_2)) - \\
&\quad - (\not{\partial}_{x_1} G_0(x_1 - x_2))^{\alpha\gamma} \gamma_5^{\gamma\delta} \int D\mu[\beta] \sinh(\beta(x_1) - \beta(x_2)) \sinh(\beta(y_1) + \beta(y_2)) \} = \quad (38)
\end{aligned}$$

$$\begin{aligned}
&= -C^1 + \frac{1}{2\pi} \{ -\delta(x_1 - y_2) P_+^{\alpha\delta} \int D\mu[\beta] \cosh(\beta(x_2) + \beta(y_1)) + \\
&\quad + \delta(x_1 - x_2) \delta^{\alpha\delta} \int D\mu[\beta] \cosh(\beta(y_1) + \beta(y_2)) \}, \quad (39)
\end{aligned}$$

where the remaining two-point functions are again cancelled by A^1 :

$$\begin{aligned}
A^1 &= -\delta^{\alpha\delta} \delta(x_2 - x_1) \frac{\delta}{\delta\eta_\gamma(y_2)} \frac{\delta}{\delta\bar{\eta}_\gamma(y_1)} Z|_0 + \delta^{\alpha\gamma} \delta(y_2 - x_1) \frac{\delta}{\delta\eta_\delta(x_2)} \frac{\delta}{\delta\bar{\eta}_\gamma(y_1)} Z|_0 = \\
&= -\delta^{\alpha\delta} \delta(x_2 - x_1) \int D\beta \Psi_0^\gamma(y_1) \bar{\Psi}_0^\gamma(y_2) e^{\frac{1}{2} \int dz \bar{\beta} \mathbf{D} \bar{\beta}} + \\
&\quad + \delta^{\alpha\gamma} \delta(y_2 - x_1) \int D\beta \Psi_0^\gamma(y_1) \bar{\Psi}_0^\delta(x_2) e^{\frac{1}{2} \int dz \bar{\beta} \mathbf{D} \bar{\beta}} = \\
&= \frac{1}{2\pi} \{ -\delta^{\alpha\delta} \delta(x_2 - x_1) \int D\mu[\beta] \cosh(\beta(y_1) + \beta(y_2)) + \\
&\quad + P_+^{\alpha\delta} \delta(y_2 - x_1) \int D\mu[\beta] \cosh(\beta(x_2) + \beta(y_1)) \}, \quad (40)
\end{aligned}$$

which we wanted to prove.

The computation for $k = -1$ is analogous.

Computation for $k=2$

For $k = +2$ we find for C^2

$$\begin{aligned}
C^2 &= -\gamma_\mu^{\alpha\beta} \epsilon^{\mu\nu} \frac{\delta}{\delta\eta_\gamma(y_2)} \frac{\delta}{\delta\eta_\delta(x_2)} \frac{\delta}{\delta\bar{\eta}_\gamma(y_1)} \frac{\delta}{\delta\bar{\eta}_\beta(x_1)} N \int D\beta (\partial_\nu^{x_1} \bar{\beta}(x_1)) \cdot \\
&\quad \cdot (\bar{\eta} \Psi_0) (\bar{\Psi}_0 \eta) (\bar{\eta} \Psi_1) (\bar{\Psi}_1 \eta) e^{\frac{1}{2} \int dz \bar{\beta} \mathbf{D} \bar{\beta}} = \\
&= -\gamma_\mu^{\alpha\beta} \epsilon^{\mu\nu} \int D\beta (\partial_\nu^{x_1} \bar{\beta}(x_1)) e^{\frac{1}{2} \int dz \bar{\beta} \mathbf{D} \bar{\beta}} \cdot \\
&\quad \cdot [\Psi_0^\gamma(y_1) \bar{\Psi}_0^\delta(x_2) \Psi_1^\beta(x_1) \bar{\Psi}_1^\gamma(y_2) + \Psi_0^\beta(x_1) \bar{\Psi}_0^\gamma(y_2) \Psi_1^\gamma(y_1) \bar{\Psi}_1^\delta(x_2) - \\
&\quad - \Psi_0^\gamma(y_1) \bar{\Psi}_0^\gamma(y_2) \Psi_1^\beta(x_1) \bar{\Psi}_1^\delta(x_2) - \Psi_0^\beta(x_1) \bar{\Psi}_0^\delta(x_2) \Psi_1^\gamma(y_1) \bar{\Psi}_1^\gamma(y_2)] = \quad (41)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{4\pi^2} \gamma_\mu^{\alpha\beta} \epsilon^{\mu\nu} N \int D\beta (\partial_\nu^{x_1} \bar{\beta}(x_1)) e^{\frac{1}{2} \int dz \bar{\beta} \mathbf{D} \bar{\beta}} \cdot \\
&\quad \cdot e^{i(\bar{\beta}(x_1) + \bar{\beta}(x_2) + \bar{\beta}(y_1) + \bar{\beta}(y_2))} [P_+^{\gamma\delta} P_+^{\beta\gamma} x_1^- y_2^+ + \\
&\quad + P_+^{\beta\gamma} P_+^{\gamma\delta} y_1^- x_2^+ - P_+^{\beta\delta} x_1^- x_2^+ - P_+^{\beta\delta} y_1^- y_2^+], \quad (42)
\end{aligned}$$

where we used the zero modes (14) and the transformation (13). Introducing real β we get ($\epsilon^{\mu\nu}\gamma_\mu P_+ = -\gamma^\nu P_+$)

$$C^2 = -\frac{i}{4\pi^2}\gamma_\nu^{\alpha\beta}P_+^{\beta\delta}(x_1^- - y_1^-)(y_2^+ - x_2^+) \cdot \partial_{x_1}^\nu \int D\mu[\beta] \cosh(\beta(x_1) + \beta(x_2) + \beta(y_1) + \beta(y_2)). \quad (43)$$

For the B^2 part we may, as usual, starting from (42), read off

$$B^2 = \frac{i}{4\pi^2}\gamma_\mu^{\alpha\beta}\partial_{x_1}^\mu P_+^{\beta\delta}(x_1^- - y_1^-)(y_2^+ - x_2^+) \cdot \int D\mu[\beta] \cosh(\beta(x_1) + \beta(x_2) + \beta(y_1) + \beta(y_2)). \quad (44)$$

Further there is no contribution from $k = 2$ to the two-point function in A , (18), so (43) and (44) must cancel completely, or the derivative

$$\gamma_\mu^{\alpha\beta}\partial_{x_1}^\mu P_+^{\beta\delta}x_1^- \quad (45)$$

has to vanish. This is indeed true and is in fact used to construct the zero modes (14) (see [7], [8] for details):

$$\gamma^\mu\partial_\mu^x P_+ x^- = \begin{pmatrix} 0 & \partial_{x^-} \\ \partial_{x^+} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^- = \begin{pmatrix} 0 & 0 \\ \partial_{x^+} & 0 \end{pmatrix} x^- = 0. \quad (46)$$

So the DS equation holds for the sector $k = 2$, too.

Using

$$x^- y^+ = x_\mu y^\mu + \epsilon_{\mu\nu} x^\mu y^\nu \quad (47)$$

we may obtain the final result (which we display for $k = \pm 2$)

$$B^{\pm 2} = \frac{i}{4\pi^2}\gamma_\mu^{\alpha\beta}\partial_{x_1}^\mu P_\pm^{\beta\delta}[(x_1 - y_1)(y_2 - x_2) \pm \epsilon_{\mu\nu}(x_1 - y_1)^\mu(y_2 - x_2)^\nu] \cdot \int D\mu[\beta] \cosh(\beta(x_1) + \beta(x_2) + \beta(y_1) + \beta(y_2)). \quad (48)$$

General discussion

We showed the DS equation to hold for the simplest nontrivial case, the fermion four-point function. For the model at hand it is however easy to generalize the result to higher n -point functions.

To study the DS equation in an instanton vacuum it remains enough to consider scalar (or pseudoscalar) bilocals of the fermion fields, with the exception of the two fields with variables x_1, x_2 on which the DS equation acts. Therefore the spinor structure reduces to simple gamma matrix products. Further the $(\partial_{x_1}^\nu \beta(x_1))$ term enters the computation of the C term in a way completely analogous to our computation, and the PI remains Gaussian. So the partial cancellation between B and C persists to hold.

For a n -point function in the $\cosh(\dots)$ there are n $\beta(x_i)$. Whenever two of them stem from a zero mode projector, both β have the same sign; whenever they stem from a fermion propagator

$G(x_i, x_j)$ their signs are opposite. So when a free propagator $G_0(x_i, x_j)$ is reduced to a δ -function by a derivative the corresponding $\beta(x_i), \beta(x_j)$ cancel in the $\cosh(\dots)$ and lead to the $n-2$ -point function that cancels with the term A . So indeed it remains true for higher n -point functions that the DS equations separately hold for every instanton sector k .

The reason for this separate cancellation is easy to understand. The Schwinger model (SM) is wellknown to be independent of the vacuum angle Θ ([2], [7], [13], [14], [15]). Let us remember how this feature may be seen quickly within the PI formalism. There the Θ vacuum may be taken into account by an additional term in the action

$$S \rightarrow S + ik\Theta = S + \frac{\Theta}{2\pi} \int dx \tilde{F}(x) \quad (49)$$

and by integrating over all instanton configurations (summing over all k) in the vacuum functional. But the additional term on the r.h.s. of (49) is nothing else than the index density. An axial transformation of the fermion fields creates a similar term in the action via the chiral anomaly:

$$\mathcal{A} = \frac{1}{\pi} \int dx \beta \tilde{F}. \quad (50)$$

Usually formula (50) is derived for an infinitesimal axial transformation β , for a constant β formula (50) remains true for finite β . Therefore by choosing

$$\beta = -\frac{\Theta}{2} \quad (51)$$

the Θ term may be absorbed by a simple change of the fermionic integration variables in the PI via an axial transformation.

Our instanton vacuum corresponds to the choice $\Theta = 0$. For general Θ a VEV from the sector k acquires an additional phase $e^{ik\Theta}$. Now when the SM is invariant with respect to Θ the DS equations have to be invariant, too. When the DS equations enforced cancellations between different instanton sectors, those cancellation conditions would acquire a Θ dependence. On the other hand, when the DS equations are fulfilled separately for every instanton sector, they automatically are Θ invariant.

Of course, the interesting question remains to be answered if these simple features of the DS equations in an instanton vacuum may be generalized to more complicated theories.

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