TOWARDS A STRING REPRESENTATION OF INFRARED SU(2) YANG-MILLS THEORY

Edwin Langmann* and Antti J. Niemi**

*Department of Theoretical Physics, Royal Institute of Technology, Stockholm, Sweden

**Department of Theoretical Physics, Uppsala University, Box 803, S-75108, Uppsala, Sweden

**Helsinki Institute of Physics, P.O. Box 9, FIN-00014 University of Helsinki, Finland

** The Mittag-Leffler Institute, Auravägen 17, S-182 62 Djursholm, Sweden

LANGMANN@THEOPHYS.KTH.SE, NIEMI@TEORFYS.UU.SE

We employ a heat kernel expansion to derive an effective action that describes four dimensional SU(2) Yang-Mills theory in the infrared limit. Our result supports the proposal that at large distances the theory is approximated by the dynamics of knotted string-like fluxtubes which appear as solitons in the effective theory.

Confinement remains an important unsolved problem in four dimensional Yang-Mills theory, even though several approaches have been developed for explaining it [1]. From the present point of view an interesting one is the construction of string variables that describe the theory at large distances [2]. In [3] an explicit string-like decomposition of four dimensional SU(2) gauge field has been introduced. Wilsonian renormalization group arguments suggest that in these variables the infrared SU(2) Yang-Mills theory is in the universality class of

$$S(\mathbf{n}) \;=\; \int d^4x \;\left[(\partial_\mu \mathbf{n})^2 \;+\; \frac{1}{e^2}(\mathbf{n}\cdot\partial_\mu\mathbf{n}\times\partial_\nu\mathbf{n})^2 \right] \quad (1)$$

where $\mathbf{n}$ is a three component unit vector. The action (1) supports string-like knotted solitons [4], [5], which suggests that at large distances SU(2) Yang-Mills theory describes the dynamics of knotted fluxtubes. The first term in (1) is relevant in the infrared limit, and it is uniquely defined. The second term is marginal, and it stabilizes the solitons. Even though Lorentz invariance permits additional marginal contributions, the second term in (1) is special in the sense that time derivatives appear at most bilinearly. This ensures that (1) admits a Hamiltonian interpretation and as such it is unique.

In this Letter we present a first-principles heat-kernel derivation of (1). We start from the SU(2) Yang-Mills Lagrangian

$$L_{YM} \;=\; \frac{1}{4g^2} Tr F_{\mu\nu}^2 \quad (2)$$

where we decompose the gauge field as [3]

$$A_\mu^a \;=\; C_\mu n^a \;+\; \epsilon_{abc}\partial_\mu n^b n^c \;+\; \rho\partial_\mu n^a \;+\; \sigma\epsilon_{abc}\partial_\mu n^b n^c \quad (3)$$

with C_μ a vector field and $\phi = \rho + i\sigma$ a complex scalar. Together they form an abelian Higgs multiplet, covariant under U(1) gauge transformations in the direction n^a of the Lie algebra SU(2). We show that if one integrates over this multiplet in the path integral, the ensuing effective action for the unit vector $\mathbf{n}$ is in the universality class of (1). Indeed, if we substitute (3) to (2), we get

$$L_{YM} \;=\; \frac{1}{4g^2}\big(\;[G_{\mu\nu} - H_{\mu\nu}(1-|\phi|^2)]^2$$

$$+\,[(v_{\mu\nu} \;+\; iH_{\mu\nu})(D_\mu\phi)^* D_\nu\phi + h.c\;]\;\big) \quad (4)$$

where we have defined the U(1) covariant derivative $D_\mu\phi = \partial_\mu\phi + iC_\mu\phi$, and we have defined $G_{\mu\nu} = \partial_\nu C_\mu - \partial_\mu C_\nu$, and $H_{\mu\nu} = \mathbf{n}\cdot\partial_\mu\mathbf{n}\times\partial_\nu\mathbf{n}$, and $v_{\mu\nu} = \delta_{\mu\nu}(\partial_\rho\mathbf{n})^2 - \partial_\mu\mathbf{n}\cdot\partial_\nu\mathbf{n}$. In particular, $H_{\mu\nu}^2$ is bilinear in $v_{\mu\nu}$,

$$H_{\mu\nu}^2 \;=\; \frac{1}{3}(v_{\mu\mu})^2 - v_{\mu\nu}v_{\nu\mu}$$

If we then average (4) over (C_μ, ϕ) with $< |\partial_\lambda\phi|^2\delta_{\mu\nu} - \partial_\mu\phi^*\partial_\nu\phi >\propto\; \delta_{\mu\nu}$ we get the Lagrangian in (1). Furthermore, if we average (4) over $\mathbf{n}$ with $< H_{\mu\nu} >= 0$ and define $< v_{\mu\nu} >= m^2\delta_{\mu\nu}$ and set $< H_{\mu\nu}^2 >= \lambda$ we get the abelian Higgs model

$$S \;=\; \frac{1}{4g^2}\int d^4x \left\{ G_{\mu\nu}^2 + m^2|D_\mu\phi|^2 + \lambda(|\phi|^2-1)^2 \;\right\} \quad (5)$$

These Wilsonian arguments [3] are suggestive, but leave a number of important issues unexplained. For example, (2) is (classically) scale invariant but (1) is not, and these arguments fail to explain how the mass scale appears. For a justification of (1) a first principles computation is needed. For this we redefine $C_\mu \to gC_\mu$ and $\phi \to g\phi$ and re-write (4) as

$$L_{YM} \;=\; \frac{1}{4}G_{\mu\nu}^2 \;+\; \frac{1}{4}(\frac{1}{g} - g|\phi|^2)^2 H_{\mu\nu}^2 \;+\; \frac{1}{2}v_{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi$$

$$+\frac{i}{2}g\; v_{\mu\nu}[(\partial_\mu\phi^*)\phi - \phi^*\partial_\mu\phi]C_\nu \;+\; \frac{1}{2}v_{\mu\nu}g^2|\phi|^2 C_\mu C_\nu$$

$$- C_\mu \left((\frac{1}{g} - g|\phi|^2)\partial_\nu H_{\mu\nu} - 3H_{\mu\nu} g \partial_\nu |\phi|^2 \right) \qquad (6)$$

Here it is obvious that the combination $v_{\mu\nu}$ is the natural order parameter in the effective action that follows when we integrate over (C_μ, ϕ). Furthermore, since the ground state is both rotation and translation invariant, we conclude that the free energy *i.e.* effective potential can only depend on the constant part of the trace of $v_{\mu\nu}(x)$. This prompts us to redefine

$$v_{\mu\nu}(x) \ \rightarrow \ v_{\mu\nu}(x) \ + \ m^2 \delta_{\mu\nu} \qquad (7)$$

where m is a constant with the dimensions of a mass. The effective action then admits a derivative expansion in powers of $v_{\mu\nu}(x)$ around $v_{\mu\nu} = m^2\delta_{\mu\nu}$ and the leading term in this expansion, the effective potential, depends only on the rotation and translation invariant m^2.

We note that the second term in (1) already appears in (6) as the second term in its *r.h.s.* In order to justify (1) it is then sufficient to show that the first term in (1) is induced when we integrate over (C_μ, ϕ). We also note that the second term in (6) is reminiscent of a Higgs mechanism, see also (5). This suggests the shift $\phi \rightarrow \phi + a$ with a some *a priori* arbitrary complex parameter, the *v.e.v.* of the complex scalar ϕ. Consequently we introduce the parameter $\Delta^2 = (\frac{1}{g} - g|a|^2)^2$ which measures the deviation of $|a|$ from $1/g$, the value of the complex scalar $|\phi|$ for which the second term on the *r.h.s.* of (6) vanishes.

We eliminate the U(1) gauge invariance from the abelian Higgs multiplet by selecting the gauge fixing term

$$L_{gf} \ = \ \frac{1}{2\xi} \left(\ M^{-2}\partial_\mu (v_{\mu\nu} C_\nu) \ + \ \xi M^2 g \frac{i}{2}(\phi^* a - a^* \phi) \ \right)^2 \qquad (8)$$

This is a variant of the conventional R_ξ-gauge in spontaneously broken gauge theories, with M an arbitrary mass scale which we need to introduce since in our units the field ϕ is dimensionless. The corresponding ghost action is

$$L_{ghost} \ = \ \bar\eta \ (-M^{-2}\frac{1}{g}\partial_\mu v_{\mu\nu}\partial_\nu \ + \ \xi M^2 g|a|^2) \ \eta \qquad (9)$$

The integration measure over (C_μ, ϕ) is determined as follows: We recall that geometrically $\mathcal{A}_4$, the space of all four dimensional Yang-Mills connections A^a_μ, is an infinite dimensional Euclidean space with Cartesian flat metric which we write as

$$ds^2 \ \sim \ \mathcal{G}(A,A) \ = \ \int d^4x d^4y \ \delta^{\mu\nu}_{ab}(x-y) dA^a_\mu(x) dA^b_\mu(x)$$

Similarly, the *naive* path integral measure over $\mathcal{A}_4$ is an infinite dimensional analog of the Cartesian measure $[dA] \sim \prod \ dA^a_\mu(x)$. Geometrically the decomposition (3) defines an embedding of a surface in $\mathcal{A}_4$, and integration over (C_μ, ϕ) is with respect to the measure that is induced by this embedding. In order to determine this induced measure we first define $\mathbf{n} = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$. The natural intrinsic measure for the components in (3) is then $[dC_\mu][\sin\theta d\theta][d\varphi][d\rho][d\sigma]$. But since (3) is also a parametrization of a surface which is embedded in $\mathcal{A}_4$, in addition we need to account for a Jacobian $|\mathrm{G}|^{\frac{1}{2}}$ factor that emerges from the embedding. This we determine as follows: We first evaluate the variation of (3) *w.r.t.* its components

$$dA^a_\mu(x) \ = \ \int d^4y \{ \ \frac{\delta A^a_\mu(x)}{\delta C_\nu(y)} dC_\nu(y) + \frac{\delta A^a_\mu(x)}{\delta\rho(y)} d\rho(y)$$

$$+ \frac{\delta A^a_\mu(x)}{\delta\sigma(y)} d\sigma(y) + \frac{\delta A^a_\mu(x)}{\delta\theta(y)} d\theta(y) + \frac{\delta A^a_\mu(x)}{\delta\varphi(y)} d\varphi(y) \ \}$$

and then infer the Jacobian $|\mathrm{G}|^{\frac{1}{2}}$ factor by evaluating the determinant of the metric that we read from the corresponding line element ds^2. We interpret the result as an additional contribution to our Lagrangian. In a derivative expansion in powers of $v_{\mu\nu}$ this gives us the contribution $L_{measure} = -ln\left[v^2_{\mu\mu}(x)\right] + \ldots$ where we have ignored terms that do not contribute at the level of the Gaussian approximation. The Lagrangian that we use in our computation of the effective action is then

$$L_{eff} = L_{YM} + L_{gf} + L_{ghost} \ + \ L_{measure}$$

We compute the path integral over the fields $(C_\mu, \phi, \eta, \bar\eta)$ using standard heat kernel expansion. Details of the computation are straightforward but somewhat tedious, and we have used Maple to perform symbolic manipulations. After combining the various functional determinants that emerge from the Gaussian integrations we conclude that to leading order in a gradient expansion in $v_{\mu\nu}$ the effective action is

$$L_{eff} \ = \ \frac{\Delta^2}{4} H^2_{\mu\nu}(x)$$

$$+ \int\limits_{|p|\leq\Lambda} \frac{d^4p}{(2\pi)^4} \left(\ \frac{1}{2} ln det[\hat Q^C_{\mu\nu}] \ + \ \frac{1}{2} ln[\hat Q] \ - \ ln[v_{\mu\mu}] \ \right) \qquad (10)$$

where $\hat Q^C_{\mu\nu}(x) = p^2\delta_{\mu\nu} - p_\mu p_\nu + g^2|a|^2 v_{\mu\nu}(x)$ and $\hat Q = v_{\mu\nu}(x) p_\mu p_\nu + 2g^2 H^2_{\mu\nu}$ and where we also need to perform the shift (7). Here Λ is an ultraviolet cut-off that we introduce in order to regulate the momentum integral.

We note that the result is *manifestly* U(1) gauge invariant *i.e.* it is independent of the parameters M^2 and ξ in (9), as it should.

The evaluation of the momentum integral in (10) is straightforward. We describe the final result as

$$L_{eff} = \frac{\Delta^2}{4} H_{\mu\nu}^2(x) + \left\{ \ L^{(0)}(m) + L^{(2)}(v) + L^{(4)}(v) \right\}$$

where $L^{(0)}(m)$, $L^{(2)}(v)$ and $L^{(4)}(v)$ are of zeroth, first and second order in powers of $v_{\mu\nu}$, in a derivative expansion. These contributions can all be presented using the function

$$\begin{aligned} f(u) \ &= \ \frac{1}{8} + \int_0^1 d\xi \ \xi^3 \ ln(\xi^2 + u) \\ &= \ \frac{1}{4} u \left[1 - u \cdot ln(\frac{1+u}{u}) \right] \ + \ \frac{1}{4} ln(1+u) \end{aligned}$$

and its derivatives, where $u = (\frac{m}{\Lambda})^2$ is a dimensionless parameter. We note that the function $f(u)$ is non-negative and monotonically increasing, with $f(u \to 0) = \mathcal{O}(u)$ and $f(u \to \infty) = \mathcal{O}\left(ln(u)\right)$.

For the zeroth order contribution $L^{(0)}(m)$ we have the following u dependence,

$$L^{(0)} \ = \ \frac{\Lambda^4}{16\pi^2} \left\{ \frac{16\pi^2}{3} \Delta^2 u^2 \ + \ 3f(g^2|a|^2 u) \ + \ f(\frac{8}{3} g^2 u) \ \right\} \tag{11}$$

This is the vacuum (Casimir) energy in the Gaussian approximation. It has the expected $\mathcal{O}(\Lambda^4)$ divergence multiplying the (uniform) vacuum energy density. The ground state values of the various parameters in the background of a constant $\mathbf{n}$ can be determined by minimizing (11). This suggests that we set $u = 0$ so that (11) vanishes. However, we note that for a fully reliable minimization one needs to account for certain additional terms. In particular one needs to include the full ghost contribution that comes from fixing the full SU(2) gauge invariance. For this we need to understand how (3) emerges from the conventional gauge fixing procedure, which is beyond the scope of the present letter.

The first-order contribution $L^{(2)}(v)$ yields in the leading order our desired first term in (1),

$$\begin{aligned} L^{(2)}(u) \ = \ & \frac{\Lambda^2}{64\pi^2} \{ \frac{32}{3} \pi^2 \Delta^2 u \ + \ 3g^2 |a|^2 f'(g^2|a|^2 u) \\ & + \ \frac{8}{3} g^2 f'(\frac{8}{3} g^2 u) \} \cdot (\partial_\mu \mathbf{n})^2 \end{aligned} \tag{12}$$

The derivative of $f(u)$ is monotonically decreasing, with $f'(0) = \frac{1}{2}$ and $f'(u \to \infty) \to 0$. In particular, when $u = 0$ and the Casimir energy (11) vanishes, the coefficient in (12) remains non-zero. Indeed, since $f'(u)$ is non-vanishing and positive, the coefficient in (12) is always positive and its minimum value occurs for a *non-vanishing* value of u. This suggests that in order to locate the ground state in the background of a nontrivial $\mathbf{n}$ we should select $u \neq 0$. This means that $m \neq 0$ and it scales nontrivially in Λ. But for our present purposes the exact determination of the ground state value of u, or the scaling of m in Λ is inessential. Here it is sufficient to note that for all u (12) is positive and non-vanishing. This means we have a mass gap and the first term in (1) is present, irrespectively of the exact value of u.

Since (12) is infrared relevant and the second, infrared marginal term in (1) appears in (6), we may already at this point conclude that (1) is indeed an effective action that approximates the infrared limit of the SU(2) Yang-Mills theory. But for completeness we also evaluate the Gaussian correction to the coefficient of the second term in (1). At this point it becomes apparent that there is certain latitude in the definition of the coefficient multiplying $H_{\mu\nu}^2$: The natural order parameter for the effective action is $v_{\mu\nu}$, and $H_{\mu\nu}$ is but a particular bilinear combination of this order parameter. There are also various additional bilinear combinations of $v_{\mu\nu}$ that can be present, and here we select a particular basis of such bilinears that allows us to write the leading contribution as

$$L^{(4)}(u) \ = \ \frac{1}{16\pi^2}(4\pi^2\Delta^2 \ - \ A) H_{\mu\nu}^2 \ + \ \frac{1}{16\pi^2}(\frac{A}{3} - B) v_{\mu\mu}^2 \tag{13}$$

where

$$\begin{aligned} A \ = \ & \frac{1}{192u^2} \ + \ \frac{4}{9}\frac{g^2}{u} f'(\frac{8}{3}g^2 u) \ + \ \frac{4}{27} g^4 f''(\frac{8}{3} g^2 u) \\ & + \ \frac{g^2|a|^2}{24u} f'(g^2|a|^2 u) \ + \ \frac{1}{48} g^4 |a|^4 f''(g^2|a|^2 u) \end{aligned}$$

and

$$\begin{aligned} B \ = \ & \frac{1}{48u^2} \ + \ \frac{16}{9}\frac{g^2}{u} f'(\frac{8}{3}g^2 u) \ - \ \frac{8}{27} g^4 f''(\frac{8}{3} g^2 u) \\ & + \ \frac{g^2|a|^2}{6u} f'(g^2|a|^2 u) \ - \ \frac{7}{24} g^4 |a|^4 f''(g^2|a|^2 u) \end{aligned}$$

For the second derivative we have $f''(0) = -\infty$ and $f''(u \to \infty) \to 0$. Depending on the relative value of parameters, the coefficient of $H_{\mu\nu}^2$ is typically negative when $u < u_0 < 1$ and positive when $u_0 < u < 1$ for some $u_0 > 0$, while the coefficient of the second term in (13) is negative. Since the energy density should be positive this yields restrictions on the possible values of the parameters, and suggests that we should set $u > u_0$. We note that this resembles the situation in [6]. In analogy we then conclude that the negative value of the energy density (13) for some range of parameters is an indication that higher order terms in the derivative expansion become important. Indeed, one can show that the *full* energy density is necessarily positive for all values of the parameters.

The two terms that are present in (13) are both local and Lorentz-invariant. Both are also marginal in the infrared, but since the second term in (13) has a fourth order contribution from time derivatives it does not admit a canonical Hamiltonian interpretation. Consequently (1) is unique in the sense that it admits a Lorentz invariant Hamiltonian interpretation. In particular we may view (1) as a Hamiltonian that represents the universality class of infrared SU(2) Yang-Mills theory. But we have also found that there is some latitude in the definition of the coefficient multiplying $H^2_{\mu\nu}$. In modelling the dynamical details of large distance SU(2) Yang-Mills theory, it is then necessary to include additional infrared marginal terms. We note that this is parallel to the situation in the Skyrme model, where one should account for all three terms with four derivatives.

Finally, we verify that (1) together with its knotted string-like solitons is indeed consistent with certain familiar aspects of non-perturbative Yang-Mills theory. For this we remind that in the temporal $A_0 = 0$ gauge Yang-Mills instantons interpolate between asymptotic $t = \pm\infty$ vacua, classified by $\pi_3(S^3) \sim Z$ homotopy classes. The difference in the $t = \pm\infty$ homotopy classes coincides with the second Chern character of the interpolating instanton configuration. For consistency, this picture should be reflected in the topological properties of the knotted solitons: If we substitute the decomposition (3) in the three dimensional Chern-Simons action that counts the $t = \pm\infty$ homotopy classes, we find that for a flat $A_0 = 0$ connection the Chern-Simons action yields the Hopf invariant of the configuration $\mathbf{n}$,

$$\omega_3(A) \ = \ \epsilon_{ijk} Tr\{A_i \partial_j A_k + \frac{2}{3} A_i A_j A_k\} \ = \ \frac{1}{4}\epsilon_{ijk} C_i H_{jk}$$

with $H_{ij} = \partial_j C_j - \partial_j C_i = \mathbf{n} \cdot \partial_i \mathbf{n} \times \partial_j \mathbf{n}$. Consequently there is a relationship between the $A_0 = 0$ instanton structure of Yang-Mills theory and the (self-)linking number of knotted solitons in the effective infrared theory. In particular, instantons can be viewed as configurations that interpolate between different knotted vacuum configurations in $\mathbf{n}$. We suggest that this relationship is additional evidence supporting the consistency of the present interpretation of the infrared structure of SU(2) Yang-Mills theory.

In conclusion, we have studied the large distance structure of four dimensional SU(2) Yang-Mills theory. By employing a heat-kernel expansion of the effective action we have shown that the Lagrangian (1) indeed follows from the decomposition (3) of the gauge field. Our result supports the proposal made in [3], that at large distances SU(2) Yang-Mills theory becomes an effective string theory, where the strings appear as solitons that are stabilized against shrinkage by their nontrivial, knotted topology. We have also verified that this interpretation is consistent with the familiar instanton θ-vacuum structure of SU(2) Yang-Mills theory.

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[1] see *e.g.* Proceedings of the Confinement and Topology Session in *Lattice '98*, Nucl. Phys. (Proc. Suppl.) (1999) (to appear)

[2] A. Polyakov, Nucl. Phys. Proc. Suppl. **68** (1998) 1

[3] L. Faddeev and A.J. Niemi, Phys. Rev. Lett. **82** (1999) 1624; L. Faddeev and A.J. Niemi, Phys. Lett. **B449** (1999) 214

[4] L.D. Faddeev and A.J. Niemi, Nature **387** 58 (1997)

[5] J. Gladikowski and M. Hellmund, Phys. Rev. **D56** (1997) 5194; R. Battye and P. Sutcliffe, hep-th/9811077, hep-th/9808129; J. Hietarinta and P. Salo, hep-th/9811053

[6] M.K. Volkov and D. Ebert, Sov. J. Nucl. Phys. **36** (1982) 186; A. Dhar and S. Wadia, Phys. Rev. Lett. **52** (1984) 959; D. Diakonov and M. Eides (unpublished) (1983)