Nuclear Physics B364 (1991) 127–161
North-Holland

VANISHING OF ZERO-MOMENTUM LATTICE GLUON PROPAGATOR AND COLOR CONFINEMENT

Daniel ZWANZIGER*

Physics Department, New York University, New York, NY 10003, USA

Received 21 December 1990
(Revised 12 April 1991)

A minimal Landau gauge is defined by choosing the gauge which minimizes the action $S_L(U) \equiv -\sum \mathrm{Re\,tr}\, U$, where the sum extends over all *links* of the lattice, and a minimal Coulomb gauge is defined analogously. The positivity of the second variation of this action at a minimum determines the lattice Gribov region. It is shown that if an external "magnetic" field H is coupled to the color spins then, in the infinite-volume limit, the color magnetization $M(H)$ vanishes identically for all H. Consequently all gluon correlation functions vanish at zero-momentum. This implies a maximal violation of reflection positivity for gluons in a minimal Landau gauge. A confinement mechanism is hypothesized whereby color-singlet gauge-invariant states are stabilized by reflection positivity which gives them a real mass, whereas color non-singlet objects are unstable because they are not gauge invariant and consequently develop a complex mass, which is observable, in principle, in jet events.

1. Introduction

Wilson's lattice gauge theory provides a gauge-invariant discretization of quantum chromodynamics. It is suitable for numerical calculations of the correlation functions of gauge-invariant quantities, from which the hadron spectrum may be deduced. It also provides a natural explanation for the confinement of external quarks, for Wilson [1] showed that in the strong-coupling limit the Wilson loops satisfy an area law, which implies a linearly rising potential energy between external quarks. The issue of the confinement of gluons is generally disposed of by holding to the philosophy that one should only calculate gauge-invariant quantities. However the prominence of gluon jets [2] in elementary particle scattering requires us to understand both the existence of gluon jets and the confinement of gluons⋆. High-energy gluon jets are well described by perturbative QCD, a gauge-fixed theory of unconfined, massless gluons that are asymptotically free. One would like

* Research supported in part by the National Science Foundation under grant no. PHY 87-15995.
⋆ For the current status of jet physics see the first of ref. [2] and for theoretical calculations see the second of ref. [2].

to see continuum perturbative QCD emerge as a limiting case of lattice gauge theory.

In virtue of the asymptotic freedom relation $g_0^{-2} = 2b \ln(\xi/a)$, the continuum limit $\xi/a \to \infty$ is the weak-coupling limit $g_0 \to 0$. Here ξ is a physical correlation length, a is the lattice spacing, g_0 is the unrenormalized coupling constant, and b is a positive constant. Weak-coupling calculations on the lattice require gauge fixing. In a subsequent article, we shall describe lattice weak-coupling perturbation theory in the Landau gauge. In the present article we shall derive a number of rigorous bounds which hold in the Coulomb and Landau gauges. In particular we shall show that the zero-momentum lattice gluon propagator vanishes in the Landau or Coulomb gauge, and hypothesize a confinement mechanism suggested by this result.

We consider two classes of gauge-fixings on the Wilson lattice, a minimal Landau gauge and a minimal Coulomb gauge. These are defined by making a gauge transformation which minimizes

$$S_U(g) \equiv S_{\rm L}(U^g)\,, \tag{1.1a}$$

where

$$S_{\rm L}(U) \equiv -\sum_L \mathrm{Re}\,\mathrm{tr}\,U_L\,. \tag{1.1b}$$

Here, for a minimal Landau gauge, the sum extends over all links L of the lattice and, for a minimal Coulomb gauge, the sum over all links L that lie within a time-slice. The minimal Landau gauge approaches the renormalizable continuum Landau gauge which is successful in describing gluon jets.

We shall prove: *In either a minimal Landau or Coulomb gauge, and in the infinite-volume limit, the lattice gluon propagator $\mathscr{D}(k)$ vanishes at $k = 0$.* The result is quite general and holds in any number of dimensions and for any gauge-invariant action (after it is gauge fixed). This is in striking contrast to zero-order perturbation theory according to which the gluon propagator diverges like $1/k^2$ at $k = 0$, and in yet stronger contrast to a theory of confinement according to which a linearly rising quark–quark potential comes from a gluon propagator that behaves like $1/k^4$ at $k = 0$ [3].

This result holds for a gauge fixing which is either a local or an absolute minimum of the action (1.1a), or some weighted average of minima. This includes numerical gauge fixing by an algorithm which may lead to a relative rather than an absolute minimum.

It is possible to give an analytic representation of the gauge in which the absolute minimum of the action (1.1a) is chosen [4]. The expectation value of a gauge-invariant observable $F(U)$ is given by

$$\langle F \rangle = N \int \mathrm{d}U \exp[-S(U)]\, F(U)\,, \tag{1.2}$$

where $S(U)$ is the Wilson action. We have

$$\langle F \rangle = N \int \mathrm{d}U \, \mathrm{d}g \exp\left[-S(U) - M^2 S_L(U^g)\right] F(U)/I(U),$$

where

$$I(U) \equiv \int \mathrm{d}g \exp\left[-M^2 S_L(U^g)\right]. \tag{1.3}$$

By the gauge invariance of $\mathrm{d}U = \mathrm{d}U^g$, and of $S(U) = S(U^g)$, and similarly for $F(U)$ and $I(U)$, this may be written

$$\langle F \rangle = N \int \mathrm{d}U \exp\left[-S(U) - M^2 S_L(U) - W(U)\right] F(U). \tag{1.4}$$

Here $W(U) \equiv \ln[I(U)]$ is a gauge-invariant but non-local action. In the limit $M^2 \to \infty$, this gives the absolutely minimal Landau gauge.

The bounds which will be obtained are a consequence of the existence in the lattice Landau and Coulomb gauges of the analog of the Gribov horizon of continuum gauge theory. As in the continuum theory, gauge orbits bunch up to pass within the narrow compass of the Gribov horizon, and one may fix a gauge entirely within it. The horizon is very close for infrared modes which strongly suppresses them. In particular, the constant component of the gauge field on a hypercubic lattice of edge L is bounded by

$$\left| V^{-1} \sum_x A_\mu(x) \right| \leqslant 2 \tan(\pi/L), \tag{1.5}$$

where $V = L^D$ and

$$A^a_\mu(x) \equiv -\mathrm{tr}\left\{ t^a \left[U_\mu(x) - U^\dagger_\mu(x) \right] \right\}. \tag{1.6}$$

(Conventions are stated at the beginning of sect. 2.) Observe that in the infinite-volume limit ($L \to \infty$), the color magnetization $V^{-1}\Sigma_x A(x)$ vanishes in every configuration, so that, if a source $H \Sigma_x A(x)$ is added to the action, the mean color magnetization, $M(H) \equiv \langle V^{-1} \Sigma_x A(x) \rangle$, vanishes identically in H. Consequently, the color susceptibility $\chi(H) = \partial M(H)/\partial H$, and all higher derivatives vanish, which is the statement that all correlation functions vanish at zero momentum in the infinite-volume limit. We shall also obtain bounds on other Fourier components of $A(x)$. Some of the results presented here have been given elsewhere in condensed form [5].

Although we mainly use the language of gauge-fixing, it is possible to give a gauge-invariant formulation of the results obtained here. To do so, one turns formula (1.4) around and uses it to assign a gauge-invariant observable $F(U)$ to any gauge non-invariant quantity $F_{ni}(U) \to F(U)$ according to

$$F(U) \equiv \int \mathrm{d}g \exp\left[-M^2 S_L(U^g)\right] F_{ni}(U^g)/I(U). \tag{1.7}$$

One has $\langle F \rangle = \langle F_{ni} \rangle$, and $\langle F \rangle$ may be evaluated using the gauge-invariant Wilson formula (1.2). For any $F_{ni}(U)$, one may in principle calculate $F(U)$ in a hopping expansion, with hopping parameter M^2, so that $F(U)$ gets expressed as a sum of Wilson loops, $F(U) = \Sigma$(Wilson loops). If this is done for the gluon bilocal,

$$4\,\mathrm{Re\,tr}\left[t^a U_\mu(x)\right]\mathrm{Re\,tr}\left[t^b U_\nu(y)\right] \to \left[A_\mu^a(x) A_\nu^b(y)\right](U) = \sum(\text{Wilson loops}), \tag{1.8}$$

where each Wilson loop passes through the links (x, μ) and (y, ν), then the bound on the propagator in the minimal Landau gauge becomes a bound on the expectation value of the sum of Wilson loops. (Is the destructive interference implied by the vanishing of the zero-momentum gluon propagator, eq. (1.10) below, related to the Wilson area law?) Because the absolute minimum is invariant under a global gauge transformation, a slight modification of formula (1.7) is required to obtain a non-vanishing gauge field at a single link namely

$$A_\mu^a(x, U) \equiv \lim_{M \to \infty}\left\{ I^{-1}(U) \int' \mathrm{d}g \exp\left[-M^2 S_L(U^g)\right](-2)\mathrm{Re\,tr}\left[t^a U^g(x)\right]\right\}, \tag{1.9}$$

where the prime means that $g_0 = g(x_0) = 1$ is not integrated over. This field undergoes a global gauge transformation by g_0 when U undergoes a local gauge transformation.

COLOR CONFINEMENT AND REFLECTION POSITIVITY

The reader who prefers proven results unadorned by speculation is advised to proceed directly to sect. 2 and the succeeding sections, for in the remainder of this section, we shall discuss a speculative but simple hypothesis for gluon and color confinement. It is suggested by the vanishing of the zero-momentum gluon propagator

$$\mathscr{D}(0) = \int \mathrm{d}^D x \langle A(x/2) A(-x/2) \rangle = 0. \tag{1.10}$$

Here, for simplicity, we have written only the (unproven) continuum analog of the correct lattice relation given in sect. 5. If reflection positivity held for the gluon field, then the integrand would be non-negative for each x. Thus either the gluon correlation function vanishes identically in the Landau gauge, or reflection positivity is maximally violated in the sense that the gluon correlation function is positive and negative in equal measure. A gauge non-invariant field need not satisfy reflection positivity, for in lattice gauge theory one proves only that the states created by local gauge-invariant fields satisfy reflection positivity [6]. In formula (1.4), the gauge non-invariant action $S_L(U)$ destroys the proof of reflection positivity, as would the gauge-invariant but highly non-local field defined in eq. (1.9b) because its support extends to both sides of any reflection plane. Thus it is not surprising that reflection positivity is violated by the gluon correlation function, although the maximal violation may be unexpected.

Reflection positivity is a condition which assures that a euclidean or statistical mechanical system has a consistent quantum-mechanical interpretation. Its failure for the lattice gluon field in the Landau gauge implies that it has a complex mass spectrum, or residues that are not positive, or both [6]. Here we have a perfect set-up to accommodate gluon jets without gluons. It is sufficient that the states created by the lattice gluon field in a minimal Landau gauge have a complex mass. This is appropriate for particles that decay. Color-singlet states are immune to infection by a complex mass, and consequent decay and confinement, because the principle of reflection positivity guarantees that local gauge-invariant fields have a real mass spectrum.

Concrete calculations are required to establish whether the hypothesis of a complex gluon mass is realized. In appendix C, a simple lattice model with an approximate lattice Gribov horizon is defined and solved. One finds a pair of complex conjugate masses. To understand how this comes about, notice that with $\partial \cdot A = 0$, we may write $A_\nu = \partial_\mu \Pi_{\mu\nu}$, where $\Pi_{\mu\nu} = -\Pi_{\nu\mu}$, is a relativistic Hertz potential that satisfies $\partial_\lambda \Pi_{\mu\nu} + \partial_\mu \Pi_{\nu\lambda} + \partial_\nu \Pi_{\lambda\mu} = 0$. (Again, for simplicity, we have written only the continuum analog of the correct lattice relations.) In terms of Π, the free action $V^{-1}\Sigma_k k^2 |a(k)|^2$ is given by $V^{-1}\Sigma_k (k^2)^2 |\pi(k)|^2$ and the ellipsoidal bound, $V^{-1}\Sigma_k |a(k)|^2/k^2 \leqslant cV$, proven in appendix B, is given by $V^{-1}\Sigma_k |\pi(k)|^2 \leqslant cV$. This bound effectively introduces a mass term for the Π field, so that Π has the propagator $[(k^2)^2 + \gamma]^{-1}$. This gives the gluon propagator

$$k^2\left[(k^2)^2 + \gamma\right]^{-1}, \tag{1.11}$$

which corresponds to imaginary $m^2 = \pm i\gamma^{1/2}$, and mass $m = (1 \pm i)2^{-1/2}\gamma^{1/4}$. In sect. 6 we suggest numerical investigations to verify the hypothesis that in the minimal Landau gauge the gluon has a complex mass.

The violation of reflection positivity by the gluon field in the Landau gauge should be viewed less as the consequence of a specific dynamics, than as a general feature of statistical mechanical spin systems which will occur unless some specific property such as gauge invariance prevents it. Indeed for gluon fields, the vanishing of $\mathscr{D}(0)$ holds for any gauge-invariant dynamics, once it is fixed in the Landau gauge. By contrast, the area law for Wilson loops is a specific property of the plaquette action that does not hold when dynamical quarks are present. It is to be expected that other color non-singlet fields such as the quark field also acquire a complex mass and get confined because they are not protected by gauge invariance.

The first person who obtained a gluon propagator with a complex mass was Gribov [7], in his famous paper of 1978. In an approximate calculation in continuum gauge theory he obtained the gluon propagator (1.11). In 1986, Stingl [8], without explicitly introducing the Gribov horizon, adopted the same gluon propagator, and also a more general one, as an ansatz for a non-perturbative solution to the Schwinger–Dyson equations. He pointed out that the complex mass implies an unstable particle with energy and lifetime given, at large 3-momentum $\boldsymbol{k}$, by

$$\epsilon(\boldsymbol{k}) = |\boldsymbol{k}| \left[1 + 8^{-1}\gamma \boldsymbol{k}^{-4} + \dots\right],$$

$$\tau(\boldsymbol{k}) = |\boldsymbol{k}|\gamma^{-1/2}\left[1 + 8^{-1}\gamma \boldsymbol{k}^{-4} + \dots\right].$$

The lifetime grows linearly with $\boldsymbol{k}$, and the leading correction to the energy is of order $\gamma/\boldsymbol{k}^4$ instead of $m^2/\boldsymbol{k}^2$, as it would be for a massive particle. Stingl observed, "These relations, which reflect 'naive' asymptotic freedom, are in qualitative agreement with observation: when endowed with larger momenta the gluonic excitations grow increasingly particlelike, and 'jets get jettier'." He added, "...at gluon energies admittedly exorbitant, $\epsilon \approx 10^{12}$ GeV, these excitations would travel over distance of millimeters during their intrinsic lifetime, and could in principle be intercepted by macroscopic detectors *before* hadronization." In 1987, Mandula and Ogilvie [9] performed a Monte Carlo calculation of the gluon propagator in SU(3) lattice gauge theory on $4^3 \times 10$ and $4^3 \times 8$ lattices for three values of β in the scaling region. They used a minimal Landau gauge (for which the result derived here holds), calculated the gluon propagator in position space, fitted it with a decaying exponential as if to determine the mass of an ordinary stable particle, and observed, "A striking aspect...is that for each β the effective mass grows with separation. Such behavior is only possible if the spectral function describing the gauge potential propagator is not positive definite." This behavior is also possible if the gluon mass is complex. More recently a perturbative scheme in continuum gauge theory propagator has been proposed [10], with zeroth order gluon as given in eq. (1.11).

For the Coulomb gauge, which is a physical gauge in the continuum theory, it is shown in sect. 5 that the vanishing of the equal-time gluon correlation function at zero spatial momentum implies that the zero spatial momentum gluon field annihilates the vacuum. This is a literal statement of infrared slavery. It is as if the gluon has an infinite mass in the Coulomb gauge. This gives a clear and simple picture of gluon confinement, but it does not appear to be of help for understanding gluon jets, or the confinement of other color non-singlet states.

The organization of this article is as follows. In sect. 2 we consider the first and second derivatives of the action (1.1a). At a stationary point of the action, the condition $\Delta \cdot A = 0$ is satisfied, where $\Delta \cdot A$ is the lattice divergence of A. The matrix of second derivatives defines the lattice Faddeev–Popov matrix, $M(U)$, which is positive at a minimum of the action, so $M(U)$ is positive in a minimal Landau or Coulomb gauge. It is shown that $M(U)$ has the following special structure, $M(U) = K(A) + M_1(U)$, where $M_1(U)$ is negative for all U, and $K(A)$ depends only on $A(x)$, defined in eq. (1.6). Consequently $K(A)$ is a positive matrix in a minimal Landau or Coulomb gauge. (This is theorem 2.1.) Moreover $K(A)$ is a naive lattice analog of the continuum Faddeev–Popov operator $(-\partial^2 - A \cdot \partial)$, with derivatives replaced by lattice differences. Here A acts in the adjoint representation. Methods of proof developed for the continuum theory can be extended to the finite hypercubic lattice [11–16]. The region Θ in A-space, defined by the condition that $K(A)$ be positive, is convex and bounded in every direction [11]. A bound on the Fourier components $a(k)$ of $A(x)$ of the form $|a(k)| \leqslant \sigma(k)V$, where $\sigma(k)$ vanishes with k is established in appendix A (theorem A.1). An ellipsoidal bound on Θ is proven by the method of ref. [12]* in appendix B (theorem B.1). In sect. 3 it is shown that these bounds imply bounds on $W(J)$ the generating function of connected correlation functions (theorem 3.1). In sect. 4, a constant source $J(x) = H$, is considered, so $W(H)$ is the free energy of a system of color spins coupled to an external magnetic field. It is proven, as a special case of a more general bound (theorem 4.1) that the free energy per unit volume vanishes identically in the infinite-volume limit

$$w(H) \equiv \lim_{V \to \infty} W(H)/V = 0. \tag{1.12}$$

This implies that the color magnetization $M(H) = \partial w(H)/\partial H$, the susceptibility $\chi = \partial M/\partial H$, and all higher derivatives vanish identically, which is the statement that all correlation functions vanish at zero momentum (theorem 5.1). In sect. 5, it is proven that the zero-momentum component of the gluon field in the Coulomb gauge annihilates the vacuum (theorem 5.2a), and that either the zero spatial momentum Landau-gauge gluon propagator vanishes or reflection positivity is

*The paradox pointed out in ref. [12] that the continuum ellipsoidal bound contradicts the perturbative renormalisation group is resolved by the vanishing of the renormalisation constant.

maximally violated in the Landau gauge (theorem 5.2b). In appendix C, a simple lattice model is defined and solved. Its two-point function is a lattice analog of the propagator (1.11). In sect. 6 we propose some numerical studies to test the hypothesis that gluons have a complex mass in a minimal Landau gauge.

2. Linear matrix bound on Landau and Coulomb lattice gauge fields

Let the sites of a D-dimensional periodic hypercubic lattice of integer edge L and volume $V = L^D$, be labelled by D-dimensional vectors x^μ, $\mu = 1, \dots, D$ with integer components, $x^\mu = 1, \dots, L$. Denote by $U_\mu(x)$ the element of the SU(N) Lie group associated with the link from x to $x + e_\mu$, where e_μ is a unit vector in the positive μ direction, and $U_\mu(x) = U_\mu(x + Le_\nu)$, for periodicity. The local gauge transform U^g of a gauge field configuration $U = \{U_\mu(x)\}$ is defined by

$$U_\mu^g(x) = g^{-1}(x) U_\mu(X) g(x + e_\mu), \tag{2.1}$$

where $g(x)$ is an element of the unitary group associated to the site x. The action

$$S(U) \equiv \sum_{x,\mu} \left[1 - N^{-1} \operatorname{Re} \operatorname{tr} U_\mu(x)\right], \tag{2.2}$$

where the sum extends over all *links*, is a gauge-dependent quantity which approaches half the Hilbert norm $\|A\|^2$ of the gauge potential A in the naive continuum limit. For a given gauge configuration U, the gauge is fixed by making a local gauge transformation $g = \{g(x)\}$, where g is either a local or absolute minimum of the action on the orbit through U defined by

$$S_U(g) = S(U^g). \tag{2.3}$$

We call the resulting gauge choice "a minimal Landau gauge" if D is the space-time dimension of a euclidean lattice gauge theory, and "a minimal Coulomb gauge" if D is the spatial dimension of a $(D+1)$ euclidean lattice gauge theory, and a minimum of the action (2.3) is chosen on each time-slice.

Consider the one-dimensional subgroup of the local gauge group defined by

$$g(\tau) = \{g(\tau, x)\} = \{\exp[\tau\omega(x)]\}. \tag{2.4}$$

Here $\omega(x)$ is an element of the local Lie-algebra,

$$\omega(x) = t^a \omega^a(x), \qquad \omega(x) = -\omega^\dagger(x), \tag{2.5}$$

and the N-dimensional anti-hermitian matrices t^a satisfy the Lie algebra commutation relations in the fundamental representation,

$$[t^a, t^b] = f^{abc} t^c, \qquad t^a = -t^{a\dagger}, \tag{2.6}$$

normalized to $\mathrm{tr}(t^a t^b) = -\delta^{a,b}/2$. A summation over repeated color indices is understood. For a fixed gauge-field configuration V define the function

$$S(\tau) = S_V(g(\tau)). \tag{2.7}$$

We have

$$\partial S(\tau)/\partial\tau = -N^{-1} \sum_{x,\mu} \mathrm{Re\,tr}\{[\omega(x+e_\mu) - \omega(x)] U_\mu(x)\}. \tag{2.8}$$

$$\partial^2 S(\tau)/\partial\tau^2 = -N^{-1} \sum_{x,\mu} \mathrm{Re\,tr}\{[\omega^2(x+e_\mu) - 2\omega(x+e_\mu)\omega(x) + \omega^2(x)] U_\mu(x)\}. \tag{2.9}$$

where $U = V^g$, and we have used the cyclicity of the trace. At a stationary point of the action $S_V(g)$, we have $\partial S/\partial\tau = 0$,

$$\sum_x \mathrm{Re\,tr}\Big\{\omega(x) \sum_\mu [U_\mu(x-e_\mu) - U_\mu(x)]\Big\} = 0. \tag{2.10}$$

Since this must vanish for any $\omega^a(x)$, we conclude that at a stationary point of the action

$$\sum_\mu [A^a_\mu(x) - A^a_\mu(x-e_\mu)] = 0. \tag{2.11}$$

Here we have introduced the real variables

$$A^a_\mu(x) \equiv -\mathrm{tr}\{t^a[U_\mu(x) - U^\dagger_\mu(x)]\}, \tag{2.12}$$

which will be used in the rest of this article. In the continuum limit, $A(x)$ approaches the unrenormalized gauge connection $g_0 A_0$, where g_0 is the unrenormalized coupling constant and A_0 is the unrenormalized canonical field. Eq. (2.11) expresses the vanishing of the lattice divergence of A, $\Delta \cdot A = 0$. This condition defines a hyperplane in A-space which we call Γ.

The second derivative of the action defines the quadratic form

$$(\omega, M(U)\omega) = \sum_{x,\mu} \mathrm{tr}\Big\{-[\omega(x+e_\mu) - \omega(x)]^2 [U_\mu(x) + U^\dagger_\mu(x)] + [\omega(x+e_\mu), \omega(x)][U_\mu(x) - U^\dagger_\mu(x)]\Big\}, \tag{2.13}$$

where the real symmetric matrix $M(U)$ is the lattice Faddeev–Popov operator. Let Ω be the subspace of Γ where $M(U) \geqslant 0$. (This condition means that $(\omega, M\omega)$ is non-negative for all ω.) This set includes all $U = V^g$ which are relative or absolute minima of the action $S_V(g)$ on all gauge orbits, and thus all minimal Landau or Coulomb gauge fields. By analogy with the continuum theory, we call Ω the lattice Gribov region. On a finite lattice, each gauge orbit is compact, and the action $S_U(g)$ is a bounded function, so there is at least one minimum on each orbit*. Therefore Ω contains a fundamental modular region or physical configuration space (i.e. a representative of each gauge orbit) Λ, $\Lambda \subset \Omega$.

It is convenient to decompose M as

$$M(U) = K(A) + M_1(U), \tag{2.14}$$

where the real symmetric matrix $K(A)$ is defined by

$$(\omega, K\omega) \equiv \sum_{x,\mu} \mathrm{Re}\,\mathrm{tr}\Big\{-2\big[\omega(x+e_\mu) - \omega(x)\big]^2 + \big[\omega(x+e_\mu), \omega(x)\big]\big[U_\mu(x) - U_\mu^\dagger(x)\big]\Big\},$$

$$(\omega, K\omega) = \sum_{x,\mu}\Big\{\sum_a \big[\omega^a(x+e_\mu) - \omega^a(x)\big]^2 + f^{abc}\omega^a(x+e_\mu)A_\mu^b(x)\omega^c(x)\Big\}. \tag{2.15}$$

and depends explicitly only on the variables A. In fact $K(A)$ is linear in A. The difference $M_1 \equiv M - K$, given by

$$(\omega, M_1\omega) = -\sum_{x,\mu} \mathrm{tr}\Big\{\big[\omega(x+e_\mu) - \omega(x)\big] \times \big[1 - U_\mu(x)\big]\big[1 - U_\mu^\dagger(x)\big]\big[\omega^\dagger(x+e_\mu) - \omega^\dagger(x)\big]\Big\}, \tag{2.16}$$

is manifestly negative for all U, $M_1(U) \leqslant 0$. Because the Gribov region Ω is defined by the condition $M(U) \geqslant 0$, it follows that $K(A) = M - M_1$ is a positive matrix, $K(A) \geqslant 0$, for all A in Ω. Call Θ the subset of Γ (the hyperplane $\Delta \cdot A = 0$) where $K(A) \geqslant 0$. We have established:

Theorem 2.1. Let Λ be a fundamental modular region, let Ω be the set of relative and absolute minima of the action (2.3) on each orbit (which includes the

* See ref. [13] for a proof of the corresponding continuum statement.

minimal Landau or Coulomb gauge fields), let $K(A)$ be the real symmetric matrix defined by the quadratic form (2.15), and let Θ be the region defined by the two conditions

$$\Delta \cdot A = 0 \tag{2.17a}$$

$$K(A) \geqslant 0, \tag{2.17b}$$

where $A_\mu^b(x)$ is defined in eq. (2.5). Then the sets so defined are related by the inclusions

$$\Lambda \subset \Omega \subset \Theta. \tag{2.18}$$

We do not have an explicit description of the fundamental modular region Λ, and the characterization of the lattice Gribov region Ω by the condition $M(U) \geqslant 0$ is not transparent. On the other hand it is a simple matter to obtain bounds on Θ, because $K(A)$ is linear in A. In other sections of this article we shall obtain bounds on Θ, that is to say, bounds on all A in Θ, which come from these two conditions alone, although there are further restrictions on $A_\mu^a(x)$ which come from the compactness of the configuration space. (Recall that by its definition, eq. (2.12), $A_\mu^a(x)$ is a component of an element of the Lie group, not of the Lie algebra.) The advantages of proceeding in this way are: (i) the intricacies of the non-linear configuration space are avoided, (ii) it turns out that these conditions are very restrictive, and (iii) in the continuum limit, Θ approaches Ω, so if one is interested in the continuum limit only, nothing is given up by considering Θ instead of Ω.

The two condition (2.17) are isomorphic to the characterization of the continuum Gribov region, with continuum derivatives replaced by lattice differences. Consequently the proofs of familiar results on the continuum Gribov region may be extended to proofs of the same results for the lattice region Θ in terms of the variable A. In particular, Θ possesses the following three properties [12]: (1) Θ includes the origin, $A = 0$. (2) Θ is convex, or in other words, if A_1 and A_2 are in Θ, then $\alpha A_1 + (1-\alpha)A_2$ is in Θ, where $0 < \alpha < 1$. (3a) Θ is bounded in every direction, or in other words, for any $A \neq 0$, there is a positive number λ such that λA lies outside of Θ. It is possible to replace (3a) by a stronger property namely: (3b) Θ is included in an ellipsoid E, defined in eq. (B.20), and we have

$$\Lambda \subset \Omega \subset \Theta \subset \mathrm{E}. \tag{2.19}$$

The main technique to bound Θ is variational calculation. Choose any vector ω. If $(\omega, K(A)\omega)$ is negative for some A, then that A lies outside Θ, and moreover, so does λA for all $\lambda > 1$ because Θ is convex. This technique is used in appendix

A to obtain a bound on the Fourier components of all A in Θ, and in appendix B an ellipsoidal bound on Θ is established.

We conclude this section with an example of a lattice configuration which, though far from the continuum, has a gauge copy within the Gribov horizon that approaches a typical continuum configuration as the size of the lattice grows. A well-known symmetry of the Wilson action is the transformation $U \to U'$ where U' is obtained from U by multiplying all links that cross a fixed time-slice by an element of the center Z of the group. This lattice symmetry has no continuum counterpart. Take a hypercubic lattice of edge L, gauge group SU(2) whose center is $Z = \{\pm 1\}$, and label coordinates by $x = (t, \boldsymbol{x})$, where $t = x^0$. Let the configuration $U' = U'(t, \boldsymbol{x})$ be obtained from the vacuum configuration $U_\mu(t, \boldsymbol{x}) = 1$ by the above-mentioned symmetry transformation applied to the $t = 0$ hyperplane, so

$$U'_\mu(t, \boldsymbol{x}) = -1 \quad \text{for } t = \mu = 0,$$

$$U'_\mu(t, \boldsymbol{x}) = 1 \qquad \text{otherwise}.$$

All Wilson loops have the value 1, except Polyakov loops (those that wind around the torus in the zero-direction an odd number of times) which have the value -1. This configuration may be shown to be gauge equivalent to U'' defined by

$$U''_\mu(t, \boldsymbol{x}) = \exp(2\pi t_3/L) \quad \text{for } \mu = 0$$

$$U''_\mu(t, \boldsymbol{x}) = 1 \qquad \text{otherwise},$$

as one would expect because all Wilson loops have the same value in configurations U' and U''. It is not hard to show that $U'' = U'^g$ is an absolute minimum of $S_{U'}(g)$. Moreover

$$A^{b\prime\prime}_\mu(t, \boldsymbol{x}) = 2\sin(\pi/L)\delta^0_\mu\,\delta^3_b, \tag{2.20}$$

approaches a typical continuum configuration on a large lattice. The lattice Faddeev–Popov operator $M(U'')$, defined in (2.13), is given by

$$(M\omega(x))^a = \cos(\pi/L)\left[2\omega^a(x) - \omega^a(x + e_\mu) - \omega^a(x - e_\mu)\right]$$
$$- \sin(\pi/L)\epsilon^{a3c}\left[\omega^c(x + e_1) - \omega^c(x - e_1)\right],$$

where ϵ^{abc} is the anti-symmetric symbol. Because $M(U'')$ is independent of x, all eigenvectors may be found by lattice Fourier transform. The wave function

$$\omega^a(x) = \exp(2\pi i x_1/L)(\delta_{a,1} + i\delta_{a,2})$$

is an eigenfunction of $M(U'')$ belonging to the eigenvalue 0, and all other eigenvalues of $M(U'')$ are positive. Thus the configuration U'' is actually a gauge copy of U' that lies on the boundary of the Gribov region Ω. On large lattices, the configuration (2.20) approaches a bound, obtained in appendix A, on all A in Θ, namely

$$V^{-1}\left|\sum_x A_\mu(x)\right| \leqslant 2\tan(\pi/L)\,. \tag{2.21}$$

3. Two bounds on the generating function of connected correlation functions

In this section we shall derive two bounds on the generating function of connected correlation functions $W(J)$, which follow from the two variational bounds on Θ which are derived in appendices A and B.

The gluon propagator $\mathscr{D}(k)$ in momentum space is defined by

$$\begin{aligned}\langle A^b_\mu(x)A^c_\nu(0)\rangle &\equiv V^{-1}\sum_k \mathscr{D}_{\mu\nu}(k)\,\delta^{bc}\exp(ik\cdot x)\\ &\equiv V^{-1}\delta^{bc}\left[\mathscr{D}(0)\delta_{\mu\nu} + \sum_{k\neq 0}\mathscr{D}(k)P_{\mu\nu}(k)\exp(ik\cdot x)\right],\end{aligned} \tag{3.1a}$$

where $P_{\mu\nu}(k)$ is the transverse projector

$$P_{\mu\nu}(k) \equiv \delta_{\mu\nu} - \sin(k_\mu/2)\sin(k_\nu/2)\Big/\sum_\lambda \sin^2(k_\lambda/2)\,.$$

The bound (A.7), $|a^b_\mu(k)| \leqslant \sigma_\mu(k)V$, on the Fourier components $a(k)$ of $A(x)$, defined by the expansion (A.2), which holds in a minimal Landau or Coulomb gauge, gives a bound on

$$\mathscr{D}_{\mu\nu}(k)\delta^{bc} = V^{-1}\big\langle a^b_\mu(k)a^c_\nu(-k)\big\rangle \tag{3.1b}$$

namely

$$\mathscr{D}_{\mu\mu}(k) \leqslant \sigma^2_\mu(k)V \qquad (\text{no sum on }\mu)\,. \tag{3.2}$$

Similarly the ellipsoidal bound (B.20c) gives the bound

$$V^{-1}\sum_{k,\mu}\mathscr{D}_{\mu\mu}(k)/\nu^2_\mu(k) \leqslant 4DN, \tag{3.3}$$

where the semi-major axis $\nu_\mu(k)$ is given in eq. (B.20a). The quantities $\nu_\mu(k)$ and

$\sigma_\mu(k)$ have the infinite-volume limit

$$\lim_{L\to\infty} \nu_\mu^2(k) = \lambda_k \cos^{-2}(k_\mu/2), \tag{3.4a}$$

$$\lim_{L\to\infty} \sigma_\mu^2(k) = 2\lambda_k \cos^{-2}(k_\mu/2), \tag{3.4b}$$

where

$$\lambda_k \equiv 4\sum_\mu \sin^2(k_\mu/2), \tag{3.4c}$$

so that in the infinite-volume limit the bound (3.3) becomes

$$(2\pi)^{-D}\int_{-\pi}^{\pi} d^D k \mathscr{D}(k)\tau_k/\lambda_k \leqslant 4DN, \tag{3.5a}$$

where the momentum k_μ is a continuous angular variable in the Brillouin zone $-\pi \leqslant k_\mu \leqslant \pi$, and

$$\tau_k \equiv \sum_\mu \left[1 - 4(\lambda_k)^{-1}\sin^2(k_\mu/2)\right]\cos^2(k_\mu/2). \tag{3.5b}$$

The fact that $\sigma(k)$ and $\nu(k)$ are both proportional to $|k|$ at low k in the infinite-volume limit suggests that there may be a strong bound on $\mathscr{D}(k)$ at low k, although the bound (3.2) is of no use in the infinite-volume limit because of the explicit power of the volume V. However we shall gain a power of V by deriving a bound on the generating function of connected correlation functions given by

$$Z(J) \equiv \int dU \rho(U) \exp(J, A(U)). \tag{3.6}$$

Here $A(U)$ is transverse and lies in Θ, $\rho(U)$ is any positive normalized probability distribution, $J(x)$ is a source for $A(U)$,

$$\begin{aligned}(J, A) &= \sum_{x,\mu,b} J_\mu^b(x) A_\mu^b(x) \\ &= V^{-1}\sum_{k,\mu,b} j_\mu^b(k) a_\mu^b(-k).\end{aligned} \tag{3.7}$$

where $j(k)$ is defined as in eq. (A.2). This expression for $Z(J)$ is quite general and includes possible coupling to quark or other fields which are integrated out. In the Coloumb gauge, one is calculating spatial correlations within a given time-slice, and all link variables that do not lie within the time-slice are also integrated out. We take the $A_\mu^a(x)$ as the variables of integration and sum over possible additional variables which may be required to label gauge-fixed configurations U, so the generating function takes the form

$$Z(J) = \int \mathrm{d}A\, \rho'(A) \exp\left\{V^{-1} \sum_{k,\mu,b} \left[j_\mu^b(k) a_\mu^b(-k)\right]\right\}. \tag{3.8}$$

where $\mathrm{d}A$ is the Cartesian measure, and $\rho'(A)$ is some positive normalized probability distribution, whose support is contained in Θ. The bound (A.7) on the Fourier coefficients $a(k)$ gives

$$Z(J) \leqslant \int \mathrm{d}A\, \rho'(A) \exp\left[\sum_{k,\mu,b} |j_\mu^b(k)| \sigma_\mu(k)\right],$$

$$Z(J) \leqslant \exp\left[\sum_{k,\mu,b} |j_\mu^b(k)| \sigma_\mu(k)\right], \tag{3.9}$$

because the probability is normalized to unity.

By the Schwartz inequality and the ellipsoidal bound (B.20), we have

$$(J, A) = (\nu J, \nu^{-1} A) \leqslant \|\nu J\|\, \|\nu^{-1} A\| \leqslant \|\nu J\| \left[4DN(N^2-1)V\right]^{1/2}. \tag{3.10}$$

The operator ν, defined in eq. (B.20) in a momentum basis, is not invertible because it vanishes for $k = 0$, so this inequality holds only for all sources J with vanishing zero-momentum component. From the definition (3.6) this bound implies

$$Z(J) \leqslant \exp\left\{\|\nu J\| \left[4DN(N^2-1)V\right]^{1/2}\right\} \qquad [j(0) = 0]. \tag{3.11}$$

We have established:

Theorem 3.1. In a minimal Coulomb or Landau gauge, the generating function of connected correlations $W(J) \equiv \ln Z(J)$ satisfies the bounds

$$W(J) \leqslant \sum_{k,\mu,b} |j_\mu^b(k)| \sigma_\mu(k), \tag{3.12}$$

$$W(J) \leqslant \left[4DN(N^2-1)V\right]^{1/2} \|\nu J\| \qquad [j(0) = 0], \tag{3.13}$$

where $\sigma_\mu(k)$ is defined in eq. (A.7) and ν is defined in a momentum basis in eq. (B.20). The Fourier components $j(k)$ of the source $J(x)$ are defined as in eq. (A.2).

4. Vanishing of color magnetization on the infinite lattice

In this section we shall prove that the lattice gluon propagator vanishes at $k = 0$. Take the source

$$J_\mu^b(x) = H_\mu^b \cos(k \cdot x), \tag{4.1}$$

which may be interpreted as a spatially modulated "magnetic" field which is coupled to the color spins $A_\mu^b(x)$. For this source we write

$$W(H, k) \equiv W(J), \tag{4.2}$$

so $W(H, k)$ is the analog of the free energy of the spin system in a magnetic field.

On a finite lattice $W(H, k)$ is an analytic function of H. It satisfies

$$W(0, k) = 0, \tag{4.3}$$

because the probability is normalized, and

$$W''(H, k) \geqslant 0, \tag{4.4}$$

where W'' is the matrix of second derivatives of W, because this quantity represents the connected correlation function in the presence of the source. These relations imply that $W(H, k)$ is a positive convex function with a unique minimum at $H = 0$. Thus, from the bound (3.12), with $j_\mu^b(k) = j_\mu^b(-k) = H_\mu^b V/2$, one has

$$0 \leqslant W(H, k) \leqslant \sum_{\mu, b} |H_\mu^b| \sigma_\mu(k) V, \tag{4.5}$$

or

$$0 \leqslant w(H, k) \leqslant \sum_{\mu, b} |H_\mu^b| \sigma_\mu(k), \tag{4.6}$$

where

$$w(H, k) \equiv W(H, k)/V, \tag{4.7}$$

is the analog of the free energy per unit volume. The last inequality holds at all volumes V and hence it holds also in the infinite-volume limit where $\sigma_\mu(k)$ has the form given in eq. (A.8). We have proven:

Theorem 4.1. In the infinite-volume limit the free energy per unit volume satisfies the bound

$$0 \leqslant w_\infty(H, k) \leqslant (2\lambda_k)^{1/2} \sum_{\mu, b} |H_\mu^b| \cos^{-1}(k_\mu/2), \tag{4.8}$$

where $\lambda_k = 4\Sigma_\nu \sin^2(k_\nu/2)$.

We arrive at the remarkable result that for a constant magnetic field, $k=0$, the free energy per unit volume $w_\infty(H) \equiv w_\infty(H,0)$ vanishes identically for all H,

$$w_\infty(H) = 0. \tag{4.9}$$

The system does not respond at all to a constant magnetic field coupled to the color spins. The magnetization $M(H) = \partial w_\infty(H)/\partial H$, and susceptibility, $\chi(H) = \partial M(H)/\partial H$, vanish, together with all higher derivatives. It is as if the constant color degree of freedom does not exist.

To see that this is the correct interpretation, we return to the bound $|a(k)| \leqslant \sigma(k)V$ for all A in Θ. For $k=0$ this reads

$$\left|V^{-1}\sum_x A_\mu(x)\right| \leqslant \sigma(0) = 2\tan(\pi/L), \tag{4.10}$$

or, in words, the average color spin is less than $2\tan(\pi/L)$ for all configurations in Θ. In the infinite-volume limit $L \to \infty$, and the mean color spin vanishes for all configurations in Θ.

5. Lattice gluon propagator

The magnetic susceptibility is the two-point function at zero momentum, so the vanishing of the susceptibility is equivalent to the vanishing of the gluon propagator at $k=0$. [To see this, observe that the contribution to $W(J)$ which is quadratic in $J_\mu^b(x) = H\cos(k\cdot x)$, with μ and b fixed, is given by

$$W^{(2)}(H,k) = (H^2/2)\sum_{x,y}\cos(k\cdot x)\cos(k\cdot y)W_{\mu\mu}^{(2)bb}(x-y)$$

$$= (H^2V/4)\sum_x \cos(k\cdot x)W_{\mu\mu}^{(2)bb}(x) \qquad \text{(no sum on } b \text{ and } \mu)$$

$$w^{(2)}(H,k) = W^{(2)}(H,k)/V = (H^2/4)\mathscr{D}_{\mu\mu}(k),$$

where $\mathscr{D}_{\mu\nu}(k) = \mathscr{D}_{\mu\nu}(-k)$ is the gluon propagator in momentum space defined by

$$\mathscr{D}_{\mu\nu}(k)\,\delta^{ab} = \sum_x W_{\mu\nu}^{(2)ab}(x)\exp(-ik\cdot x).$$

In fact, all higher point correlation functions vanish at all external momenta $k=0$.]

Theorem 5.1. In the infinite-volume limit, the zero-momentum lattice gluon propagator vanishes in a minimal Landau or Coulomb gauge,

$$\lim_{L\to\infty} \mathscr{D}(0) = 0, \tag{5.1}$$

together with all higher point correlation functions at zero momentum.

The vanishing of the gluon propagator at $k=0$ in the Coulomb and Landau gauges means that there are no massless gauge quanta in these gauges. However for this it is sufficient that the propagator does not have a pole at $k=0$. The vanishing of the propagator at $k=0$ is a stronger condition which has drastic consequences.

Theorem 5.2a. In a minimal Coulomb gauge the strong (norm) limit holds.

$$\lim_{L\to\infty} \left\| L^{-D/2} \sum_x A(t,x)\Omega \right\| = 0. \tag{5.2}$$

Here Ω is the quantum mechanical vacuum state, $A(t,x)$ is the minkowskian field operator in the Coulomb gauge, euclidean space-time is of $D+1$ dimensions, the previously suppressed time variable t has been resurrected, and x is the D-dimensional spatial variable of the previous section. Roughly speaking, this theorem states that in a minimal Coulomb gauge the zero-spatial-momentum component of the gluon field annihilates the vacuum.

Proof. The two-point correlation function in the Coulomb gauge which we have been considering is in a single time slice, so it is identical to the equal-time minkowskian vacuum expectation value

$$\mathscr{D}(k) = \sum_x (\Omega, A(t,x)A(t,0)\Omega)\exp(ik\cdot x), \tag{5.3}$$

where indices are suppressed. In particular, for $k=0$, one has, by translation invariance,

$$\mathscr{D}(0) = \left(\Omega, \sum_x A(t,x)A(t,0)\Omega\right) = L^{-D}\left(\Omega, \left[\sum_x A(t,x)\right]^2 \Omega\right).$$

Thus, from $\lim_{L\to\infty} \mathscr{D}(0) = 0$ the assertion follows. ∎

Note that the vanishing of a zero-momentum state is not a relativistically invariant concept, and if it holds also in the continuum limit, it would imply that the vacuum is annihilated by all momentum components of the gluon field in a minimal Coulomb gauge. For the equal-time propagator of a free field of mass M

at zero spatial momentum, one has

$$\mathscr{D}(0) = L^{-1} \sum_{k_0} \left[k_0^2 + M^2 \right]^{-1} \to (2M)^{-1},$$

so the vanishing of this quantity in a Coulomb gauge means that Coulomb-gauge gluons have an infinite effective mass.

Theorem 5.2b. In a minimal Landau gauge, either reflection positivity is violated or the zero-spatial momentum component of the gluon field annihilates the vacuum.

Proof. Consider $\mathscr{D}(0)$ at finite volume,

$$\mathscr{D}(0) = \sum_{t,x} \langle A(t,x) A(0,0) \rangle = \sum_{t} \langle B(t) A(0,0) \rangle$$

$$\mathscr{D}(0) = L^{-D+1} \sum_{t} \langle B(t) B(0) \rangle, \tag{5.4}$$

where we have written $x = (t, x)$, used translation invariance, and introduced $B(t) \equiv \sum_x A(t, x)$, which is a finite sum for a finite lattice. To check reflection positivity, it is convenient to label the reflection plane by $t = 0$, and time-slices containing sites of the lattice by $t = \pm 1/2, \pm 3/2 \ldots$. Use translation invariance again to rewrite $\mathscr{D}(0)$ as

$$\mathscr{D}(0) = 2^{-1} L^{-D+1} \langle \{ 2B^2(1/2) + B(-1/2)B(1/2) + B(-3/2)B(3/2) + [B(-1/2) + B(-3/2)][B(1/2) + B(3/2)] + \ldots \} \rangle. \tag{5.5}$$

The first term is positive and every other term in this sum is positive if reflection positivity holds. Thus, if reflection positivity holds, each term must vanish in the infinite-volume limit, and we have,

$$\lim_{L \to \infty} L^{-D+1} \langle B(t) B(0) \rangle = 0. \tag{5.6}$$

This is the Coulomb case with $D \to D - 1$. ■

Of the two alternatives, one naturally takes reflection positivity to be violated because it is established only for gauge-invariant objects [6], and the other alternative is too strong. The surprise is that the violation of reflection positivity is maximal, since the vanishing of $\sum_x \langle A(x) A(0) \rangle$ means that $\langle A(x) A(0) \rangle$ is positive and negative in equal measure.

Reflection positivity implies both that there is a positive quantum mechanical inner product and that the spectrum of the hamiltonian is real and positive. Since it is violated in the minimal Landau gauge, either or both of these consequences may fail to hold. The methods of proof used here do not establish which of these alternatives holds. In appendix C we consider a crude model with support inside the Gribov horizon, which may approximate lattice gauge theory in the Landau gauge. It is found that the mass is complex with positive real part. Thus the gluon is an unstable particle in this model.

[We would like to comment on whether the propagator vanishes at $k=0$ in the continuum limit. All the above bounds hold for all values of the lattice spacing, so they are satisfied in the continuum limit. However it may happen that they are trivially satisfied by $W(J)=0$ in the continuum limit, because $W(J)$ is the generating function of the unrenormalized Green functions. To obtain a bound on the generating function of the renormalized Green functions one replaces J by $J/Z^{1/2}$, where $Z^{1/2}$ is the renormalization constant of the connection

$$g_0 A_0 = Z^{1/2} g_r A_r , \tag{5.7}$$

and the subscripts 0 and r refer to unrenormalized and renormalized quantities. The bound (3.12) becomes a bound on the generating function of renormalized connected correlation functions given by

$$W_r(J) \leqslant \sum \left| j_\mu^b(k) \right| \sigma_\mu(k) Z^{-1/2} . \tag{5.8}$$

Thus, if Z approaches zero in the continuum limit, obviously no conclusion concerning the renormalized correlation functions can be drawn without further consideration. In space-time with dimension D less than 4, the renormalization constant is believed to be finite, which implies that the renormalized continuum propagator vanishes at $k=0$. In 4-dimensional QCD, if Z is calculated according to the perturbative renormalization group, one does indeed find $Z=0$, and so the bound (5.8) is trivially satisfied in continuum QCD in 4 dimensions, and the behavior of the renormalized propagator cannot be determined by the present method. The issue of renormalization in 4 dimensions will be addressed in the future [17]*.

Although the derivation presented here relied on periodic boundary conditions, the proofs may be extended without change to any boundary conditions which preserve translational invariance such as, for example, anti-periodic boundary conditions.

Physical observables are gauge invariant. On the lattice, gauge-invariant quantities are Wilson loops which are products of the link variables U. Recall that the lattice variables $A(x)$, defined in eq. (2.12), are components of the link variables

* It appears that the low-energy bounds obtained in the present article hold for the renormalized propagator. However high-energy bounds implied by the ellipsoidal bound are modified by renormalisation.

$U(x)$ which are elements of the Lie group and not of the Lie algebra. As such the $A(x)$ are not elements of a linear space (although we have been considering linear bounds on them) and consequently they cannot be renormalized multiplicatively. It should be possible to take the continuum limit of expectation values of the lattice gauge-invariant objects directly, without ever introducing renormalized fields.]

6. Conclusion

We have learned that the lattice gluon propagator vanishes at zero momentum in a minimal Landau or Coulomb gauge, but the degree of its zero has not been established. Does it vanish like k^2 at low k, as suggested by the model of appendix C, namely $\mathscr{D}(k) \approx k^2/\gamma$? This could be investigated by numerical evaluation of the Fourier components of the gluon propagator. If it does, then, in the continuum limit, the coefficient of k^2 determines a constant of dimension (mass)^4. From the discussion in sect. 5 one would expect that in the Coulomb gauge this constant would approach infinity, whereas in the Landau gauge it could be finite, and scale according to the perturbative renormalization group. If so, one has a mechanism for mass generation in QCD which may be understood as originating in the suppression of infrared components by the lattice Gribov horizon.

By simulations on lattices of different volumes, one could also determine for a fixed mode, for example $k=0$, the dependence on the volume. On a lattice of edge L, does $\mathscr{D}_L(0)$ vanish like $(2\pi L^{-1})^2$?

One may attempt to verify the hypothesis, proposed in the introduction, that confinement of gluons may be understood in the Landau gauge if the gluon has a complex mass, as suggested by the maximum violation of reflection positivity. To verify this, one may calculate numerically the gluon propagator at large separations. The quantity $\mathscr{D}(t) \equiv \sum_x \mathscr{D}(t, x)$ may behave as in the model of appendix C, namely, asymptotically at large t like

$$\mathscr{D}(t) \sim \text{const.} \times \exp(-mt)\cos(mt + \pi/4)\,, \tag{6.1}$$

or more generally like

$$\mathscr{D}(t) \sim \text{const.} \times \exp(-m_1 t)\cos(m_2 t + \varphi)\,. \tag{6.2}$$

To be relevant for physics, mass parameters must scale in accordance with the perturbative renormalization group.

It is possible that such behavior is easier to observe numerically in the gauge that is fixed at the absolute minimum of the action $S_U(g)$, which is represented analytically in eq. (14). For this purpose one may numerically obtain the required gauge transformation g for any given configuration U by defining an ensemble on the gauge orbit of U by means of the Boltzmann factor

$$\exp\left[-M^2 S_U(g)\right] = \exp\left[-M^2 S_L(U^g)\right], \tag{6.3}$$

where $S_L(U)$ is the link action defined in eq. (2.2). The absolute minimum on the orbit of U is achieved by equilibrating the site variables g numerically, and adiabatically cooling the system to zero temperature, which corresponds to letting the parameter $\beta = M^2$ approach infinity.

One may also evaluate the quark propagator numerically in a minimal Landau gauge, to see if it also has a complex mass which scales according to the perturbative renormalization group. If both the quark and gluon have this behavior, then the hypothesis proposed in the introduction will have been verified, and the confinement of colored particles can then be attributed to the instability described by a complex physical mass which arises because colored particles are created by gauge non-invariant colored fields which violate reflection positivity. On the other hand, color singlet particles, created by local gauge-invariant fields, are stabilized by reflection positivity which insures that they have a real mass.

The author recalls with pleasure many stimulating discussions with Richard Brandt, Frank Brown, Gianfausto Dell'Antonio, Stefano Fachin, Sergio Fanchiotti, Claudio Parrinello and Alan Sokal.

Appendix A. Variational bound on Fourier components

We shall use a trial wave function and exploit the positivity of the operator $K(A)$ defined in eq. (2.15) to prove a simple bound on the Fourier components of the lattice gluon field $A(x)$ in a minimal Landau or Coulomb gauge. We write

$$K = K_0 + K_1, \tag{A.1a}$$

where

$$(K_0\omega)^a(x) = -\Delta^2\omega^a(x) = 2\omega^a(x) - \omega^a(x+e_\mu) - \omega^a(x-e_\mu), \tag{A.1b}$$

$$(K_1\omega)^a(x) = (1/2)f^{abc}\sum_\mu\left\{-A_\mu^b(x)\omega^c(x+e_\mu) + A_\mu^b(x-e_\mu)\omega^c(x-e_\mu)\right\}. \tag{A.1c}$$

We extend K to an operator on complex wave functions in order to use plane-wave states. We shall also make use of the lattice Fourier transform of A

$$A_\mu^b(x) = V^{-1}\sum_k a_\mu^b(k)\exp\left[ik\cdot(x+e_\mu/2)\right], \tag{A.2}$$

where $k_\mu = 2\pi n_\mu/L$, and n_μ is an integer in the interval $-L/2 < n_\mu \leqslant L/2$. In terms of the Fourier components, transversality is simply

$$\sum_\mu \sin(k_\mu/2) a_\mu^c(k) = 0. \tag{A.3}$$

The matrix elements between normalized plane-wave states are given by

$$\langle k_1, a | K_0 | k_2, c \rangle = \delta^{ac} \delta_{k_1, k_2} \lambda_{k_1}, \tag{A.4a}$$

$$\langle k_1, a | K_1 | k_2, c \rangle = -iV^{-1} f^{abc} \sum_\mu \sin\left[(k_{1\mu} + k_{2\mu})/2\right] a_\mu^b(k_1 - k_2), \tag{A.4b}$$

where $\lambda_k = 4\Sigma_\mu \sin^2(k_\mu/2)$.

We will use some simple trial functions to bound the Fourier components of $A(x)$. First take

$$|\omega\rangle = |k_0, \chi\rangle = |k_0\rangle|\chi\rangle.$$

Here $|\chi\rangle$ is a normalized color vector, $|k\rangle$ is a normalized plane-wave state of momentum k, and k_0 is one of the $2D$ vectors

$$k_0 = \pm 2\pi e_\mu/L, \tag{A.5a}$$

where e_μ is a unit vector in the μ-direction. We have

$$(\omega, K\omega) = \lambda_0 - iV^{-1} \chi^{a*} f^{abc} \sum_\mu \sin(k_0{}^\mu) a_\mu^b(0) \chi^c,$$

where

$$\lambda_0 \equiv \lambda_{k_0} = 4 \sin^2(\pi/L). \tag{A.5b}$$

For the SU(2) group, $(S^b)_{ac} \equiv if^{abc}$ are the angular momentum operators in the spin-one representation and so, for any color vector a^b, the eigenvalues of the operator $S^b a^b$ are 0 and $\pm|a|$. Hence by choosing k_0 to lie along the μ-axis, and χ to be an eigenfunction of highest or lowest weight, we obtain from the positivity of $K(A)$

$$\lambda_0 - V^{-1} \sin(2\pi/L) \left| a_\mu(0) \right| \geqslant 0,$$

which holds for each μ. This gives the bound

$$|a_\mu(0)| \leqslant 2 \tan(\pi/L) V \equiv \sigma(0) V. \tag{A.6}$$

In fact, this bound holds for every SU(N) group because the highest weights are the same in the adjoint representation of any SU(N) group.

To obtain a bound on other Fourier components, choose a trial wave function of the form

$$|\omega\rangle = |k_0, \chi_0\rangle + \alpha |k, \chi\rangle,$$

where α is a complex variational parameter. We have

$$\begin{aligned} 0 \leqslant (\omega, K\omega) = \lambda_0 + |\alpha|^2 \lambda_k \\ - V^{-1} \sum_\nu \Big\{ \sin(k_{0\nu})(\chi_0, S \cdot a_\nu(0)\chi_0) + |\alpha|^2 \sin(k_\nu)(\chi, S \cdot a_\nu(0)\chi) \\ + 2\sin[(k_{0\nu} + k_\nu)/2] \operatorname{Re}[\alpha(\chi_0, S \cdot a_\nu(k_0 - k)\chi)] \Big\}. \end{aligned}$$

By the bound on $a(0)$ and transversality, we have

$$\begin{aligned} 0 \leqslant 2\lambda_0 + |\alpha|^2 \Big[\lambda_k + \sum_\nu |\sin(k_\nu)| 2\tan(\pi/L) \Big] \\ - 4V^{-1} \sum_\nu \sin(k_{0\nu}/2)\cos(k_\nu/2) \operatorname{Re}[\alpha(\chi_0, S \cdot a_\nu(k_0 - k)\chi)]. \end{aligned}$$

The second sum on ν has only one non-zero term because $k_{0\nu}$ has only one non-zero component, say along the μ-axis. Choose χ and χ_0 to be eigenvectors with minimal or maximal weight for $S \cdot a_\mu(k_0 - k)$, and obtain

$$\begin{aligned} 0 \leqslant 2\lambda_0 + |\alpha|^2 \Big[\lambda_k + \sum_\nu |\sin(k_\nu)| 2\tan(\pi/L) \Big] \\ \pm 4V^{-1} \sin(\pi/L)\cos(k_\mu/2)(\operatorname{Re}\alpha) \big| a_\mu(k_0 - k) \big|. \end{aligned}$$

By appropriate choice of the sign of α we obtain

$$\begin{aligned} 0 \leqslant 2\lambda_0 + \beta^2 \Big[\lambda_k + \sum_\nu |\sin(k_\nu)| 2\tan(\pi/L) \Big] \\ - 4V^{-1}\beta \sin(\pi/L)\cos(k_\mu/2) \big| a_\mu(k_0 - k) \big|, \end{aligned}$$

which holds for all real β. Upon minimization with respect to β, one has

$$\big| a_\mu(k - k_0) \big|^2 \leqslant 2\cos^{-2}(k_\mu/2) \Big[\lambda_k + \sum_\nu |\sin(k_\nu)| 2\tan(\pi/L) \Big] V^2.$$

For the SU(N) group this holds for each color index, and we have proven:

Theorem A.1. In a minimal Landau or Coulomb gauge, the Fourier components $a(k)$ of the lattice field $A(x)$, defined by eqs. (2.12) and (A.2), satisfy the bound

$$|a_\mu(k)| \leqslant \sigma_\mu(k) V, \tag{A.7a}$$

$$\left[\sigma_\mu(k)\right]^2 \equiv 2\cos^{-2}[(k+k_0)_\mu/2]\left[\lambda_{k+k_0} + \sum_\nu |\sin[(k+k_0)_\nu]| 2\tan(\pi/L)\right]. \tag{A.7b}$$

where k_0 may be any one of the vectors $\pm 2\pi e_\mu/L$.

By comparison with the bound (A.6), we see that this also holds for $k=0$. In the infinite-volume limit with fixed k, $\sigma_\mu(k)$ is given by

$$\left[\sigma_\mu(k)\right]^2 = 2\lambda_k \cos^{-2}(k_\mu/2). \tag{A.8}$$

These bounds are weak when the $\cos(k_\mu/2)$ is small. However we may obtain another bound on $a(k)$ from the unitarity of U. We write

$$U_\mu(x) = \sum_a t^a C^a + R,$$

where $\mathrm{tr}(t^a R) = \mathrm{tr}(t^a R^\dagger) = 0$. From the definition of A, eq. (2.12), we have

$$A_\mu^a(x) = \mathrm{Re}\, C^a,$$

and from

$$\mathrm{tr}\left[U_\mu(x) U_\mu^\dagger(x)\right] = N = (1/2)\sum_a |C^a|^2 + \mathrm{tr}[R^\dagger R],$$

one obtains, $\Sigma_b [A_\mu^b(x)]^2 \leqslant 2N$,

$$|A_\mu(x)| \leqslant (2N)^{1/2}. \tag{A.9}$$

Thus unitarity of $U_\mu(x)$ implies the bound on the Fourier coefficients

$$|a_\mu(k)| \leqslant (2N)^{1/2} V. \tag{A.10}$$

We may combine these two bounds as

$$\left|a_\mu^b(k)\right| \leqslant \sigma'_\mu(k) V, \tag{A.11a}$$

where

$$\sigma'_\mu(k) \equiv \min\left[\sigma_\mu(k), (2N)^{1/2}\right]. \tag{A.11b}$$

Appendix B. Ellipsoidal bound

An ellipsoidal bound on A, which holds for all minimal lattice Landau and Coulomb gauges, will be proven from the positivity of $K(A)$. The result is stated as theorem B.1 below.

B.1. VARIATIONAL INEQUALITY

Let P be any real positive symmetric matrix. The positivity of $K(A)$ implies

$$\operatorname{tr}(KP) \geqslant 0. \tag{B.1}$$

For P we take

$$P = \{1 - \alpha[(1-P_0)/(K_0 - \lambda_0)]K_1\} P_0 [1 - \alpha K_1 (1-P_0)/(K_0-\lambda_0)]. \tag{B.2}$$

Here K_0 and K_1 are defined in eq. (A.1). P_0 is the projector onto the eigenspace of K_0 belonging to one of its eigenvalues λ_0, and α is a variational parameter. This form for P is inspired by first-order perturbation theory, according to which the first-order change in an eigenvector ψ_0 is given by $\psi_1 = -[(1-P_0)/(K_0-\lambda_0)]K_1\psi_0$; but we shall of course obtain an exact, non-perturbative variational bound. Substitute eq. (B.2) into the inequality (B.1) and obtain

$$\operatorname{tr}(KP) = I(\alpha) \equiv X - 2\alpha Y + \alpha^2 Z \geqslant 0, \tag{B.3}$$

where

$$X \equiv \operatorname{tr}[(K_0 + K_1)P_0], \tag{B.4a}$$

$$Y \equiv \operatorname{tr}\{K_1[(1-P_0)/(K_0-\lambda_0)]K_1 P_0\}, \tag{B.4b}$$

$$Z \equiv \operatorname{tr}\{(K_0+K_1)[(1-P_0)/(K_0-\lambda_0)]K_1 P_0 K_1[(1-P_0)/(K_0-\lambda_0)]\}. \tag{B.4c}$$

Here we have used $P_0K_0(1-P_0)=0$ to simplify Y. Positivity of $I(\alpha)$ for all α implies

$$X \geqslant 0, \qquad Z \geqslant 0, \tag{B.5}$$

$$Y^2 \leqslant XZ. \tag{B.6}$$

B.2. BOUND ON K

The following *lemma* will be useful: For the SU(N) gauge group and for all A in Θ, i.e. A such that $K(A)$ is a positive operator, $K(A)$ is bounded from above by

$$(\omega, K\omega) = (\omega, (K_0+K_1)\omega) \leqslant N^2(\omega, K_0\omega). \tag{B.7}$$

We first derive the lemma for the SU(2) group. We write

$$K_1^b = S^b H^b, \tag{B.8}$$

where $(S^b)_{ac} \equiv if^{abc}$ are the angular momentum operators in the spin-one representation, and H^b is a hermitian operator that acts only on spatial variables, but not on color variables. We shall first bound the operator

$$(K_1^3)_{ac} \equiv (S^3)_{ac} H^3. \tag{B.9}$$

The spin-one operator S^3 has eigenvalues 0 and ± 1. Let $\omega_\pm^a(x)$ be the product wave function

$$\omega_\pm^a(x) \equiv e_\pm^a \varphi(x), \tag{B.10a}$$

where $\varphi(x)$ is any function of x, and $e_\pm$ are the normalized highest and lowest weight eigenvectors of S_3 satisfying

$$S_3 e_\pm = \pm e_\pm. \tag{B.10b}$$

We have

$$0 \leqslant (\omega_\pm, (K_0+K_1)\omega_\pm) = (\varphi, K_0\varphi) \pm (\varphi, H^3\varphi),$$

where we have use the fact that $\langle \pm|S^1|\pm\rangle = \langle \pm|S^2|\pm\rangle = 0$ in the highest weight states because S^1 and S^2 may be expressed in terms of raising and lowering operators, and the inequality holds because $K=K(A)$ is a positive operator for A in Θ. This gives

$$|(\varphi, H^3\varphi)| \leqslant (\varphi, K_0\varphi),$$

which holds for any spatial wave function φ. It follows that for any wave function

ω which is a color vector

$$|(\omega, S^3 H^3 \omega)| \leqslant (\omega, K_0 \omega)$$

since we may expand ω into eigenstates of S^3. This inequality holds for $S^1 H^1$ and $S^2 H^2$ also, from which we conclude, for the SU(2) group

$$|(\omega, K_1 \omega)| \leqslant 3(\omega, K_0 \omega) .$$

For the SU(N) group, the proof is identical except that there are $N^2 - 1$ terms in K_1. This gives

$$|(\omega, K_1 \omega)| (N^2 - 1)(\omega, K_0 \omega) .$$

and the lemma is established.

B.3. EVALUATION OF THE INEQUALITY

Note that Z is of the form

$$Z = \operatorname{tr}(KQ) = \operatorname{tr}[(K_0 + K_1)Q] , \tag{B.11}$$

where Q is a positive operator. From inequality (B.7) we have

$$Z = \operatorname{tr}(KQ) \leqslant N^2 \operatorname{tr}(K_0 Q) \equiv N^2 Z_0 , \tag{B.12}$$

which gives the simpler bound

$$Y^2 \leqslant N^2 X Z_0 . \tag{B.13}$$

Let us now evaluate this expression for the operators at hand. Note that both K_0 and K_1 have a trivial null space $\mathscr{H}_0$ consisting of constant wave functions. $\Delta_\mu \omega^a(x) = 0$, where $\Delta_\mu \omega^a(x) \equiv \omega^a(x + e_\mu) - \omega^a(x)$. Since $\mathscr{H}_0$ is annihilated by both K_0 and K_1, it shall be understood henceforth, without loss of generality, that K_0 and K_1 act on the subspace orthogonal to $\mathscr{H}_0$. To obtain an interesting bound, we choose P_0 to be the projector onto the next-lowest-lying eigenspace of $K_0 = -\Delta^2$, which belongs to the eigenvalue $\lambda_0 = 4\sin^2(\pi/L)$. It consists of wave functions of the form

$$u^a(x) = v^a \exp(ik_0 \cdot x) , \tag{B.14a}$$

where v^a is an x-independent color vector, and k_0 is one of $2D$ momentum vectors (pointing in positive and negative directions along the D principal axes) that satisfies

$$\lambda_{k_0} = \lambda_0 = 4\sin^2(\pi/L) . \tag{B.14b}$$

Because the dimension of the adjoint representation is N^2-1, one has

$$\operatorname{tr} P_0 = 2D(N^2-1). \tag{B.15}$$

It is convenient to express X, Y and Z by saturating with momentum eigenstates, which gives

$$X = \sum_{k_0} \langle k_0, b | K_0 + K_1 | k_0, b \rangle,$$

$$Y = \sum_{k, k_0} \langle k_0, b | K_1 | k, c \rangle \left[1/(\lambda_k - \lambda_0) \right] \langle k, c | K_1 | k_0, b \rangle,$$

$$Z_0 = \sum_{k, k_0} \langle k_0, b | K_1 | k, c \rangle \left[\lambda_k/(\lambda_k - \lambda_0)^2 \right] \langle k, c | K_1 | k_0, b \rangle.$$

Here repeated color indices b, c are summed over, k_0 is summed only over values satisfying $\lambda_{k_0} = \lambda_0$, and k is summed subject to the restriction $\lambda_k > \lambda_0$. One obtains from eqs. (A.2)–(A.4),

$$X = \sum_{k_0, b} \lambda_0 \delta^{bb} = 2D(N^2-1)\lambda_0,$$

$$Y = NV^{-2} \sum_{k, k_0, \mu, b} 4\sin^2(k_{0\mu}/2)\left[a_\mu^b(k_0 - k) a_\mu^b(k - k_0) \cos^2(k_\mu/2)/(\lambda_k - \lambda_0) \right],$$

$$Z_0 = NV^{-2} \sum_{k, k_0, \mu, b} 4\sin^2(k_{0\mu}/2)\left[a_\mu^b(k_0 - k) a_\mu^b(k - k_0) \cos^2(k_\mu/2) \lambda_k/(\lambda_k - \lambda_0)^2 \right]$$

(B.16)

We have used $\sum_{b,d} f^{bcd} f^{deb} = -N\delta^{ce}$, where N is the value of the Casimir operator in the adjoint representation of SU(N). The proof may be extended to any semi-simple group since for such groups the Casimir operator is strictly positive.

Observe that if the edge of the lattice has length

$$L \geqslant 4, \tag{B.17}$$

then $\lambda_k \geqslant 2\lambda_0$, so the positivity of the summands of Y and Z_0 yields

$$Z_0 \leqslant 2Y, \tag{B.18}$$

and the inequality (B.13) implies $Z_0^2 \leqslant 4N^2 X Z_0$ or

$$Z_0 \leqslant 8DN^2(N^2-1)\lambda_0. \tag{B.19}$$

By a change of summation variables, Z_0 may be written

$$Z_0 = 2\lambda_0 N V^{-2} \sum_{k,\mu,b} a_\mu^b(-k) a_\mu^b(k)/\nu_\mu^2(k),$$

where

$$\nu_\mu^{-2}(k) \equiv 2^{-1} {\sum_{k_0}}' \cos^2[(k+k_0)_\mu/2] \lambda_{k+k_0}/(\lambda_{k+k_0} - \lambda_0)^2, \tag{B.20a}$$

and we have used the fact that k_0 has only one non-zero component. In this expression, k never vanishes, k_0 is summed only over the two values

$$k_0 = \pm(2\pi/L) e_\mu, \tag{B.20b}$$

and the prime on summation sign means that a term is dropped if each denominator is not strictly positive. The ellipsoidal bound is established:

Theorem B.1. In a minimal Landau or Coulomb gauge, the Fourier components $a(k)$ of the lattice gluon field $A(x)$, defined by eqs. (2.12) and (A.2), lie in the ellipsoid defined by

$$\sum_{k,\mu,b} a_\mu^b(-k) a_\mu^b(k)/\nu_\mu^2(k) \leqslant 4DN(N^2-1)V^2. \tag{B.20c}$$

The semi-major axis of the ellipsoid is very large when $\cos(k_\mu/2)$ is small, however, $a(k)$ is independently bounded by eq. (A.11).

Appendix C. A simple model

In this appendix we consider a simple lattice model which is defined by the free-field action on a euclidean lattice,

$$S_0(A) = (2V)^{-1} \sum_k a^2(k) \lambda_k. \tag{C.1}$$

By analogy with the rigorous ellipsoidal bound obtained in appendix B, we suppose that the classical configuration space Λ is the ellipsoid in A-space given by

$$S_1(A) \equiv (2V)^{-1} \sum_k a^2(k) \nu_k^{-2} \leqslant cV. \tag{C.2}$$

Consider the generating function

$$Z(J) \equiv N \int \prod_k \mathrm{d}a(k) \theta(cV - S_1(A)) \exp\left[-\beta S_0(A) + V^{-1} \sum_k j(k) a(k)\right], \tag{C.3}$$

where the normalization constant N is chosen such that $Z[0]=1$, and

$$\beta \equiv g_0^{-2} a^{D-4}, \tag{C.4}$$

where a is the lattice spacing. We shall show by direct evaluation that in the infinite-volume limit, it is given by

$$Z(J) = \exp\left\{(2\beta)^{-1}(2\pi)^{-D}\int_{-\pi}^{\pi} \mathrm{d}^D k\, |j(k)|^2 \left[\lambda_k + \gamma\nu_k^{-2}\right]^{-1}\right\}. \tag{C.5}$$

One recognizes that $Z(J)$ is the generating functional of a free field with kinetic energy $\lambda_k + \gamma\nu_k^{-2}$. Here γ is a thermodynamic parameter whose value is uniquely fixed by

$$c = (2\beta)^{-1}(2\pi)^{-D}\int_{-\pi}^{\pi} \mathrm{d}^D k \left[\nu_k^2\lambda_k + \gamma\right]^{-1}. \tag{C.6}$$

To evaluate $Z(J)$ we write

$$Z(J) = N(2\pi i)^{-1}\int \mathrm{d}\omega\,(\omega - i\epsilon)^{-1}\exp(i\omega\beta c V)$$

$$\times \int \prod_k \mathrm{d}a(k) \exp\left[-\beta\{S_0(A) - i\omega S_1(A)\} + V^{-1}\sum_k j(k)a(k)\right]. \tag{C.7}$$

With

$$S_0(A) + i\omega S_1(A) = (2V)^{-1}\sum_k a^2(k)\left[\lambda_k + i\omega\nu_k^{-2}\right],$$

integration over the $a(k)$ is easily evaluated by gaussian quadrature,

$$Z(J) = N(2\pi i)^{-1}\int \mathrm{d}\omega\,(\omega - i\epsilon)^{-1}\exp(i\omega\beta c V)\prod_k\left[\lambda_k + i\omega\nu_k^{-2}\right]^{-1/2}$$

$$\times \exp\left\{(2\beta V)^{-1}\sum_k |j(k)|^2\left[\lambda_k + i\omega\nu_k^{-2}\right]^{-1}\right\}. \tag{C.8}$$

(The normalization constant N is redefined, to maintain $Z(0)=1$, as shall be done again without further comment.)

The contour of integration may be deformed into the lower half complex ω-plane, where the integrand is an analytic function. To find a saddle point there, we write, with $z = i\omega$,

$$Z(J) = N(2\pi i)^{-1}\int \mathrm{d}z \exp F(z) \exp\left\{(2V)^{-1}\sum_k |j(k)|^2\left[\lambda_k + z\nu_k^{-2}\right]^{-1}\right\}, \tag{C.9}$$

where z satisfies $\mathrm{Re}(z) > 0$, the integration is parallel to the imaginary axis, and

$$F(z) = -\ln z + \beta c V z - 2^{-1} \sum_k \ln\left[\lambda_k + z \nu_k^{-2}\right]. \tag{C.10}$$

The critical points $z = \gamma$ of $F(z)$ are determined from

$$F'(\gamma) = -\gamma^{-1} + \beta c V - 2^{-1} \sum_k \left[\nu_k^2 \lambda_k + \gamma\right]^{-1} = 0. \tag{C.11}$$

or

$$c = (2\beta V)^{-1} \sum_k \left[\nu_k^2 \lambda_k + \gamma\right]^{-1} + (\beta \gamma V)^{-1}, \tag{C.12}$$

(We have ignored the term containing j^2 in the search for the critical point because Green functions are obtained by setting $j = 0$, after differentiating with respect to j.) There are no roots for γ complex. For γ real and positive, the right-hand side is a monotonically decreasing function of γ which ranges from ∞ to 0, which gives a unique value for γ. As V grows, the first term approaches a finite integral. If c is less than the critical value

$$c_0 \equiv (2\beta)^{-1} (2\pi)^{-D} \int_{-\pi}^{\pi} \mathrm{d}^D k \left(\nu_k^2 \lambda_k\right)^{-1},$$

then γ approaches a finite value as $V \to \infty$, the last term in eq. (C.12) becomes completely negligible, and eq. (C.6) follows. In the cases of interest, namely $D \leqslant 4$ with $\lambda_k = \nu_k^2 = 4\Sigma \sin^2(k_\mu/2)$, c_0 is infinite, so obviously $c < c_0$. (If c exceeds c_0, then γ approaches zero in the infinite-volume limit and the restriction to the interior of the ellipsoid becomes vacuous.)

To prove that contributions away from the saddle point are negligible, we pose $z = \gamma + iy$. The path of integration is $-\infty < y < +\infty$. Observe that in eq. (C.9), the dominant factor in the integrand $\exp F(z)$ may be written

$$\exp\left\{V\left[z\beta c - 2^{-1}(2\pi)^{-D} \int_{-\pi}^{\pi} \mathrm{d}^D k \ln\left(\lambda_k + z\nu_k^{-2}\right)\right]\right\},$$

with $z = \gamma + iy$. When y is different from zero the real part of the quantity in brackets is less than at $y = 0$. Because this factor is multiplied by the enormous number V which appears in the exponent, the contribution away from $y = 0$ is negligible.

The second derivative at the critical point is given by

$$F''(\gamma) = \gamma^{-2} + 2^{-1} \sum_k \left[\nu_k^2 \lambda_k + \gamma\right]^{-2}. \tag{C.13}$$

At large volumes it is given by

$$F''(\gamma) = VE + \mathrm{O}(1), \tag{C.14a}$$

where

$$E \equiv 2^{-1}(2\pi)^{-D}\int_{-\pi}^{\pi} \mathrm{d}^D k\left(\nu_k^2\lambda_k + \gamma\right)^{-2}. \tag{C.14b}$$

To second order in the Taylor series expansion for $F(\gamma + iy)$ we have

$$F(\gamma + iy) = F(\gamma) - \tfrac{1}{2}VEy^2. \tag{C.15}$$

The peak becomes very narrow, at large volume, and the saddle-point approximation becomes exact in the infinite-volume limit. By integrating over z in eq. (C.9) along the contour $z = \gamma + iy$, where $-\infty < y < \infty$, the stated result, eq. (C.5) follows, and γ is the unique solution of eq. (C.6). This completes the evaluation, and we now discuss the result.

The generating function (C.5) corresponds to the gaussian measure

$$\mathrm{d}\mu\gamma \equiv N\prod_k \mathrm{d}a(k)\exp(-\beta S(A)), \tag{C.16a}$$

where

$$S(A) = S_0(A) + \gamma S_1(A). \tag{C.16b}$$

Note that eq. (C.12), which determines γ, may be written (with neglect of the last term which vanishes with V^{-1})

$$\langle S_1(A)\rangle = cV, \tag{C.16c}$$

where the expectation value refers to the measure (C.16a). This relation expresses the fact that the support of the measure which was restricted to the interior of the ellipsoid $S_1(A)/V < c$, in fact approaches the boundary of the ellipsoid $S_1(A)/V = c$, as $V \to \infty$. This agrees with the elementary fact that the density in radius goes like r^{N-1} for an N-dimensional ball, so the volume of a very high dimensional sphere or ellipsoid gets concentrated as its surface.

The gluon propagator in position space corresponding to the generating functional (C.5) is given by

$$\mathscr{D}(x) = g_0^2(2\pi)^{-D}\int \mathrm{d}^D k\left[\lambda_k + \gamma\nu_k^{-2}\right]^{-1}\exp(ik\cdot x), \tag{C.17}$$

where $-\pi \leqslant k_\mu \leqslant \pi$. To see the mass spectrum, we sum this over a time-slice. With $x = (t, \boldsymbol{x})$, $t = x^0$, and $\mathscr{D}(t) \equiv \Sigma_{\boldsymbol{x}}\mathscr{D}(t, \boldsymbol{x})$, we have

$$\mathscr{D}(t) = g_0^2(2\pi)^{-1}\int \mathrm{d}\theta\,\lambda(\theta)\left[\lambda^2(\theta) + \gamma\right]^{-1}\exp(i\theta t),$$

where we have specialized the model to $\nu_k^2 = \lambda_k$, and $\lambda(\theta) = 4\sin^2(\theta/2)$. This integral may be evaluated by contour integration around the unit circle, with the result

$$\mathscr{D}(t) = g_0^2 \,\mathrm{Re}\left\{\exp(|t|\ln z_-)[2(z_+ - z_-)]^{-1}\right\},$$

$$z_\pm = 1 + i2^{-1}\gamma^{1/2} \pm \left[(1 + i2^{-1}\gamma^{1/2})^2 - 1\right]^{1/2}. \tag{C.18}$$

These roots satisfy $z_+ z_- = 1$, with z_- lying inside the unit circle, so $\rho \equiv |z_-| < 1$. With

$$\exp(|t|\ln z_-) = \exp[(\ln\rho + i\varphi)|t|] \equiv \exp(-m|t|),$$

we see that in this model the lattice gluon has a pair of complex masses $m_\pm = -\ln\rho \pm i\varphi$. The real part of the mass is positive, $\mathrm{Re}(m_\pm) > 0$, (because $\rho < 1$) as it must be for an exponential decay in the euclidean domain. However, because of the imaginary part, the gluon also decays in the minkowskian domain and is thus an unstable particle. For γ to be finite in the continuum limit, one must have $\gamma \ll 1$, in which case one obtains in the continuum limit

$$\mathscr{D}(t) = g_0^2 2^{-1/2}(4m)^{-1}\exp(-mt)\cos(mt + \pi/4), \tag{C.19}$$

where $m = 2^{-1/2}\gamma^{1/4}$.

Finally we consider the continuum limit of eq. (C.6) which determines γ,

$$c g_0^{-2} = (1/2)(2\pi)^{-D}\int \mathrm{d}^D k\left[(k^2)^2 + \gamma\right]^{-1}. \tag{C.20}$$

For $D < 4$, this integral converges, and the coupling constant g is dimensional, as is γ. In this case, condition (C.20) may be used to solve for either $\gamma(g_0)$ or $g_0(\gamma)$, so either may be used as the independent parameter. For $D \geqslant 4$, the integral diverges as the ultraviolet cut-off is removed, and there is no cut-off independent meaning for the original measure. In $D = 4$ euclidean dimensions, the integral (C.20) diverges logarithmically, and the theory may be defined as the limit of the theory for $D < 4$. If one keeps γ fixed at a finite value as the cut-off $\varepsilon \equiv 4 - D$ approaches zero, one obtains [10],

$$g_0^2 \sim \mathrm{const.} \times \epsilon. \tag{C.21}$$

The dependence on the cut-off ϵ agrees with asymptotic freedom and the perturbative renormalization group.

Finally we observe that if a mass M were introduced into S_0, to model, however crudely, a mass of Higgs origin, then one would obtain the continuum propagator

$$k^2\left[(k^2+M^2)k^2+\gamma\right]^{-1},$$

with poles at

$$k^2=-(M^2/2)\pm\left[(M^2/2)^2-\gamma\right]^{1/2}.$$

Depending on the value of M, there are either a pair of complex conjugate poles or a pair of real poles. In the latter case the heavier mass has a positive residue and could represent a weak boson. The lighter mass has negative residue and must decouple from the physical space.

References

[1] K.G. Wilson, Phys. Rev. D10 (1974) 2445

[2] B. Bellettini and A. Scribano, ed., 8th Topical Workshop on Proton–anti-proton collider physics (World Scientific, Singapore 1990) p. 120;
R.K. Ellis and J.C. Sexton, Nucl. Phys. B269 (1986) 445

[3] S. Mandelstam, Phys. Rev. D20 (1979) 3223;
M. Baker, J.S. Ball and F. Zachariason, Nucl. Phys. B186 (1981) 531, 560

[4] D. Zwanziger, Nucl. Phys. B345 (1990) 461;
C. Parrinello and G. Jona-Lasinio, Phys. Lett., to be published

[5] D. Zwanziger, Phys. Lett. B257 (1991) 168

[6] E. Seiler, Gauge theories as a problem of constructive quantum field theory and statistical mechanics (Springer, Berlin, 1982) p. 12 and other references therein

[7] V.N. Gribov, Nucl. Phys. B139 (1978) 1

[8] M. Stingl, Phys. Rev. D34 (1986) 3863

[9] J.E. Mandula and M. Ogilvie, Phys. Lett. B185 (1987) 127

[10] D. Zwanziger, Nucl. Phys. B323 (1989) 513

[11] D. Zwanziger, Nucl. Phys. B209 (1982) 336

[12] G. Dell'Antonio and D. Zwanziger, Nucl. Phys. B326 (1989) 333

[13] G. Dell'Antonio and D. Zwanziger, Every gauge orbit passes inside the Gribov horizon, Commun. Math. Phys., to be published

[14] M.A. Semenov-Tyan-Shanskii and V.A. Franke, Proc. Semin. Leningrad Math. Inst. (1982), English translation (Plenum, New York, 1986)

[15] G.F. Dell'Antonio and D. Zwanziger, *in* Probabilistic methods in quantum field theory and quantum gravity, ed. P. Damgaard, H. Hueffel and A. Rosenblum (Plenum, New York, 1990)

[16] G.F. Dell'Antonio, Gauge fixing: Geometric and probabilistic aspects of Yang–Mills gauge theories, Proc. XIX Int. Conf. on Differential geometric methods in theoretical physics, Rapallo, 1990 (Springer, Berlin, to be published)

[17] D. Zwanziger, manuscript in preparation