

Application of the Heat-Kernel Method to
the Constituent Quark Model
at
Finite Temperature *

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February 1, 2008

Abstract

Abstract: The Heat Kernel Method is applied to the constituent quark model. We calculate the effect of thermal quark fluctuations on the meson action and the resulting quark condensate and $\pi\pi$ -scattering amplitude at finite temperature. The quarks produce a chiral phase transition only by their effect on the mesonic coupling constants. The s-wave isospin zero $\pi\pi$ -scattering amplitude diverges near the phase transition showing the necessity for a more sophisticated treatment of meson fluctuations.

*Supported by the GSI (Darmstadt)

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1 Introduction

The chiral constituent quark model is a mixture of the potential model and the σ -model which contains spontaneous chiral symmetry breaking in a dynamical way. The quarks get their masses via the interaction with the scalar field, which develops a vacuum expectation value. In a previous paper [1] the thermodynamics of this model has been investigated, in particular the restoration of chiral symmetry with temperature. It is important to understand the possible evolution of this model in QCD in order to handle it correctly at finite temperatures. For high resolution, i.e. short distance processes fundamental QCD with quarks and gluons is the most efficient theory. A typical scale associated with perturbative QCD is $\Lambda \geq 1 - 1.5$ GeV. RHIC and LHC physics for particles with $p_{\perp} \geq \Lambda$ will be dominated by such processes. In nucleus nucleus collisions, however, scattering at smaller momentum scale will be non negligible. Since the strength of the QCD interactions increases with decreasing momentum transfer, characteristic $q\bar{q}$ bound states will form and influence the dynamics at larger distances. A typical resolution where these processes start to become important is $\Lambda_{\chi SB} = 0.5$ GeV [2]. Below this scale the vacuum changes and acquires a quark condensate and/or meson condensate. The spontaneous breaking of chiral symmetry will persist down to $\Lambda \approx \Lambda_{QCD}$. Around this scale the confining gluon configurations make themselves felt via confining forces. In the interval $[\Lambda_{QCD}, \Lambda_{\chi SB}]$ the dynamics is governed by constituent quarks interacting via pions and sigma mesons. One should keep in mind that the effective couplings of these mesonic degrees of freedom to themselves and to the quarks are induced by the underlying physics of the quarks and gluons. When one immerses the quarks and gluons into a heat bath, it is to be expected that the induced couplings will be temperature dependent. In the following paper we calculate the temperature dependence of the couplings in the linear sigma model. We apply the Heat Kernel technique to integrate out the fluctuations of the fermion variables.

We use the constituent quark model as an effective theory in the region between $\Lambda_{QCD} \approx 200$ MeV and $\Lambda_{\chi SB} \approx 0.5$ GeV with constituent quarks and confining gluons, which are neglected in this paper [7]. If the integration over quark and meson fields is done in one step, then one obtains a first order phase transition contrary to our expectations for the sigma model and QCD for zero quark masses. This calculation therefore presents an intermediate step to illustrate the technique of the heat kernel which will be finally applied in the context of the renormalization group equations to get a more accurate description of the second order phase transition. In the course of developing this technique we solve the problem of quark condensate in the constituent quark model. The natural order parameter in the linear sigma model is the vacuum expectation value of the scalar field. We show how we can relate the vacuum expectation value of the scalar field to the quark antiquark condensate. In addition we calculate $\pi\pi$ -scattering as a function of temperature. This calculation shows the slightly different physics concept underlying the constituent quark model and the Nambu–Jona-Lasinio (NJL) model in a physical observable. The $\pi\pi$ -scattering amplitude can be tested via transport properties. We also show that no new renormalization constants or cut offs are necessary in the constituent

model contrary to the NJL model, where the meson loops generate additional singularities which have to be regulated by new phenomenological cut off parameters. The heat kernel technique allows to restrict the effect of quarks of a fixed mass to the bosonic vertices. That means we propose to ignore the free fermion loop contribution of unconfined quarks in the free energy and also to freeze the quark mass in order not to allow artificially long distance meson meson interactions, which would be cut off by the confining forces between quarks and antiquarks below the critical temperature. In the constituent quark model it is assumed that the meson degrees of freedom persist beyond the chiral phase transition. Since the gluonic forces are especially strong in the π and σ channels these mesons may not dissolve but remain active degrees of freedom beyond the chiral transition.

The outline of the paper is as follows: In the next section 2 we derive the heat kernel method at zero temperature and apply it to the constituent quark model at finite temperature in section 3. We calculate the free energy, which is the necessary thermodynamical tool for the analysis of a phase transition and study the chiral transition. In section 4 we evaluate the quark condensate and the $\pi\pi$ -scattering length as functions of temperature explicitly. At the end we conclude and compare our results to other calculations.

2 The Heat Kernel Method for $T = 0$

We consider the chiral constituent model with quarks, σ and π mesons. At zero temperature in Euclidean space the partition function or the generating functional without external sources Δ is given by

$$Z[\Delta = 0] = \int \mathcal{D}q \mathcal{D}\bar{q} \mathcal{D}\sigma \mathcal{D}\vec{\pi} \exp\left\{-\int d^4x (\mathcal{L}_F + \mathcal{L}_B)\right\} \quad (1)$$

with a fermionic \mathcal{L}_F and a bosonic \mathcal{L}_B part.

$$\mathcal{L}_F = \bar{q}(x) (\gamma_E \partial_E + g (\sigma + i\vec{\tau}\vec{\pi}\gamma_5)) q(x) \quad (2)$$

$$\mathcal{L}_B = \frac{1}{2} \left((\partial_\mu \sigma)^2 + (\partial_\mu \vec{\pi})^2 \right) - \frac{\mu_0^2}{2} (\sigma^2 + \vec{\pi}^2) + \frac{\lambda_0}{4} (\sigma^2 + \vec{\pi}^2)^2. \quad (3)$$

The Yukawa coupling of the constituent quarks to the mesons is denoted by g . The parameters of the linear σ -model at $T = 0$ are chosen in the same way as in ref. [1]. The mass squared μ_0^2 is positive reflecting the spontaneously broken ground state. The minimum of the potential $U(\sigma)$ lies at $\langle \sigma_0 \rangle = 0.093$ GeV. The light constituent quarks have a mass of $m = 300$ MeV, the σ -mass $m_\sigma = 700$ MeV, $\mu_0^2 = (0.495 \text{ GeV})^2$, $\lambda = 28.33$ and $g = 3.23$. These couplings are effective couplings obtained after the integration over high momentum quark and gluon degrees of freedom. Therefore we must not reintegrate these high momentum modes at zero temperature, since their effect is already included in the couplings. In the following we demonstrate the heat kernel technique at zero temperature to set up the notation

we are going to use at finite temperature. Also in a later paper we are investigating the renormalization group flow with the same method [12].

By means of a formal integration over the quarks one obtains a non-local determinant, which is in general complex

$$Z[\Delta = 0] = \int \mathcal{D}\sigma \mathcal{D}\vec{\pi} \det(\gamma_E \partial_E + g(\sigma + i\vec{\tau}\vec{\pi}\gamma_5)) \exp\{-\int d^4x \mathcal{L}_B\}$$

$$\det(\gamma_E \partial_E + g(\sigma + i\vec{\tau}\vec{\pi}\gamma_5)) = \int \mathcal{D}q \mathcal{D}\bar{q} \exp\{-\int d^4x \mathcal{L}_F\}.$$

This determinant can be expressed as a logarithm and in the following we consider the real part of the effective action (the modulus of the determinant)

$$\Re S_F = -\frac{1}{2} Tr \ln DD^+ \quad (4)$$

with the elliptic operator

$$DD^+ = -\partial_E^2 + g^2 MM^+ + g\gamma_\mu(\partial_\mu M^+)$$

and $M(x) = \sigma(x) + i\vec{\tau}\vec{\pi}(x)\gamma_5$. The symbol 'Tr' is defined by a functional trace over an operator \mathcal{O}

$$Tr \mathcal{O} \equiv \int d^d x tr \langle x | \mathcal{O} | x \rangle$$

and tr is reserved for the remaining trace over all inner spaces i.e. spin, positive and negative energy solutions associated with the gamma matrices, isospin and color [4].

Since DD^+ is a positive definite operator it is possible to rewrite the logarithm as an Schwinger proper time integral. Then we find for the real part of the effective action in $d = 4$ dimensions

$$\begin{aligned} \Re S_F &= \lim_{\Lambda \rightarrow \infty} \frac{1}{2} \int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau} Tr e^{-\tau DD^+} \\ &= \lim_{\Lambda \rightarrow \infty} \frac{1}{2} \int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau} \int d^4x H_0(x, x; \tau) \end{aligned}$$

where the second line defines the diagonal part of the zero temperature heat kernel $H_0(x, x; \tau)$. In general the proper time integration diverges for vanishing temperature therefore we introduce a proper time regulator Λ . The order of magnitude for Λ is estimated in ref. [3] from the meson spectrum and lies in the range of about 1 GeV. Physically one can interpret Λ as a scale at which chiral symmetry breaking starts.

The non-diagonal part of the heat kernel $H_0(x, y; \tau)$ can further be evaluated by including a plane wave basis [5]. We get

$$\begin{aligned} H_0(x, y; \tau) &= tr e^{-\tau DD_x^+} \langle x | y \rangle \\ &= tr \int \frac{d^d k}{(2\pi)^d} e^{-\tau DD_x^+} \langle x | k \rangle \langle k | y \rangle \\ &= tr \int \frac{d^d k}{(2\pi)^d} e^{-iky} e^{-\tau DD_x^+} e^{-ikx} \end{aligned} \quad (5)$$

which yields the diagonal part ($y \rightarrow x$) of the heat kernel

$$H_0(x, x; \tau) = \text{tr} \int \frac{d^d k}{(2\pi)^d} e^{-\tau(k^2 - 2ik_\mu \partial_\mu + DD_x^+)} \mathbf{1} .$$

Note, the unit operator on the right is to be associated with other possible functions on which the derivatives with respect to x can act. The product rule of differentiation can be implemented by the substitution $\partial_\mu \rightarrow \partial_\mu + ik_\mu$ in the operator DD_x^+ . After a subsequent rescaling of the momenta $k_\mu \rightarrow k_\mu/\sqrt{\tau}$ the following expansion in powers of the proper time τ is found

$$H_0(x, x; \tau) = \frac{\text{tr}}{(4\pi\tau)^{d/2}} \int \frac{d^d k}{\pi^{d/2}} e^{-k^2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sqrt{\tau} 2ik_\mu \partial_\mu - \tau DD_x^+ \right)^n \mathbf{1} . \quad (6)$$

The momentum integration can be done using the relation

$$\begin{aligned} \int \frac{d^d k}{\pi^{d/2}} e^{-k^2} k_{\mu_1} k_{\mu_2} \cdots k_{\mu_{2m}} &= \frac{1}{2^m} \delta_{\mu_1 \mu_2 \cdots \mu_{2m}} \\ &= \frac{1}{2^m} \sum_{\sigma \in S(2m)} \delta_{\mu_{\sigma(1)} \mu_{\sigma(2)}} \delta_{\mu_{\sigma(3)} \mu_{\sigma(4)}} \cdots \delta_{\mu_{\sigma(2m-1)} \mu_{\sigma(2m)}} \end{aligned}$$

with $\sigma(1) < \sigma(2)$, $\sigma(3) < \sigma(4)$, \cdots and $\sigma(1) < \sigma(3) < \sigma(5) \cdots$. All in all there are $(2m-1)!!$ permutations. For the constituent quark model the corresponding expression for the heat kernel in $d=4$ dimensions looks like

$$H_0(x, x, \tau) = \frac{\text{tr}}{(4\pi\tau)^2} \int \frac{d^4 k}{\pi^2} e^{-k^2} e^{-\tau m^2} e^{-\tau(-\partial^2 - \frac{2i}{\sqrt{\tau}} k \cdot \partial + \Omega + V - m^2)} \mathbf{1}$$

with $\Omega := g^2 MM^+$ and $V := g\gamma \cdot (\partial M^+)$. A mass term m^2 in the exponent has been introduced to guarantee the convergence of the proper time integration. It is chosen equal to the quark constituent mass in the vacuum. Quantum fluctuations of the theory for short distances enter in the $\tau \rightarrow 0$ region of the τ integration while the behaviour of quantum fluctuations at long distances contribute in the limit $\tau \rightarrow \infty$. Therefore through the introduction of a mass term m^2 no infrared divergences for $\tau \rightarrow \infty$ emerge. The mass term m^2 is an IR regulator and is not uniquely determined. We checked the sensitivity of our later results with respect to variations in m^2 and found that in a reasonable interval there is no strong dependence.

The heat kernel method makes the fermion functional integration doable by expanding the exponential in the proper time variable up to the second order. Performance of the inner traces yields the following result for the effective action

$$\begin{aligned} \Re S_F &= 4N_c \lim_{\Lambda \rightarrow \infty} \int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau} \frac{1}{(4\pi\tau)^2} e^{-\tau m^2} \\ &\left[1 - \tau \left\{ g^2(\sigma^2 + \vec{\pi}^2) - m^2 \right\} + \frac{\tau^2}{2} \left\{ g^4(\sigma^2 + \vec{\pi}^2)^2 \right. \right. \\ &\left. \left. + g^2((\partial_\mu \sigma)^2 + (\partial_\mu \vec{\pi})^2) + m^4 - 2m^2 g^2(\sigma^2 + \vec{\pi}^2) + g^2 \partial^2(\sigma^2 + \vec{\pi}^2) \right\} \right] . \end{aligned}$$

The proper time integration leads to an incomplete gamma functions in the fermionic action

$$\Re S_F = \frac{m^4}{4\pi^2} \sum_{k=0}^{\infty} \frac{\Gamma(k-2, \frac{m^2}{\Lambda^2})}{m^{2k}} Tr h_k$$

where $Tr h_k$ symbolizes a short hand notation for the integrals resulting from different powers in τ containing the fields and their derivatives [3]. One observes that the proper time expansion leads to a gradient expansion, since the proper time has dimension $mass^{-2}$. The first three coefficients of this expansion are always divergent for $\Lambda \rightarrow \infty$, therefore a regularization scheme should be applied at this stage.

3 The Heat Kernel Method at finite Temperature

Before we apply the heat kernel method to the constituent quark model we want to show how to extend it to finite temperature.

The main difference in finite temperature field theory compared to zero temperature field theory is the change of the continuous time integration of the effective action to a finite integration in Euclidean space from zero to $\beta = 1/T$. We can generalize the definition of the heat kernel at zero temperature cf. Eq. (5) to finite temperature

$$\begin{aligned} H_T(x, x', \tau) &= e^{-\tau DD^+} \langle x|x' \rangle_{\beta} \\ &= e^{-\tau DD^+} \delta_{\beta}^4(x - x') \end{aligned} \quad (7)$$

where $'\beta'$ stands for the effect of finite temperature in the Fourier representation of $\delta_{\beta}^4(x - x')$. This means one has to insert the correct boundary condition of the fermions (or bosons) into the Dirac delta function. For the quarks we have to take anti-periodic boundary conditions in Euclidean time, i.e. the Fourier representation has only half integer Matsubara frequencies. With the help of a generalized theta-transformation one can find a connection between the zero temperature and the finite temperature heat kernel for fermions [6]

$$H_T(x, x, \tau) = H_0(x, x, \tau) \left[1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{-n^2 \beta^2 / 4\tau} \right]. \quad (8)$$

This result is remarkable, because the finite temperature heat kernel separates into a sum of the zero temperature heat kernel and a temperature dependent piece, which again is a product of the zero temperature heat kernel times a temperature dependent function.

Since we are interested in the pure temperature contribution of the effective action we consider only the difference between the finite temperature and zero-temperature heat kernel. The temperature dependent effective action is then given by

$$\Re S_F^T = \frac{1}{2} \int d^4x \int_0^{\infty} \frac{d\tau}{\tau} \{ H_T(x, x; \tau) - H_0(x, x; \tau) \} .$$

The vacuum divergences in both kernels cancel each other and no new divergences are generated at finite temperatures. Therefore we do not need the regulator Λ anymore and take the proper time integration over all positive real τ . With the result of Eq. (8) we find

$$\Re S_F^T = \frac{1}{2} \int d^4x \int_0^\infty \frac{d\tau}{\tau} 2H_0(x, x; \tau) \sum_{n=1}^{\infty} (-1)^n e^{-n^2 \beta^2 / 4\tau} .$$

In the previous section the expansion of $H_0(x, x; \tau)$ was shown and by using the formula

$$\int_0^\infty d\tau \tau^{\nu-1} e^{-\frac{\alpha}{\tau} - \gamma\tau} = 2 \left(\frac{\alpha}{\gamma} \right)^{\nu/2} K_\nu(2\sqrt{\alpha\gamma}) \quad (9)$$

where $K_\nu(x)$ denotes modified Bessel functions, we finally obtain

$$\begin{aligned} \Re S_F^T = & \int d^4x \left[\frac{N_c}{2\pi^2} \frac{8m^2}{\beta^2} F_2(\beta m) - \frac{N_c}{2\pi^2} \frac{4m}{\beta} F_1(\beta m) \{ g^2(\sigma^2 + \vec{\pi}^2) - m^2 \} \right. \\ & + \frac{N_c}{2\pi^2} F_0(\beta m) \left\{ m^4 + g^2((\partial_\mu \sigma)^2 + (\partial_\mu \vec{\pi})^2) + g^4(\sigma^2 + \vec{\pi}^2)^2 \right. \\ & \left. \left. - 2m^2 g^2(\sigma^2 + \vec{\pi}^2) + \frac{1}{3} g^2 \partial^2(\sigma^2 + \vec{\pi}^2) \right\} \right] . \end{aligned}$$

Here we define

$$F_i(\beta m) := \sum_{n=1}^{\infty} (-1)^n \frac{K_i(n\beta m)}{n^i} \quad \text{for } i = 0, 1, 2 .$$

This result is substituted into the partition function Eq. (1), which then becomes only a functional of the mesonic degrees of freedom alone.

$$\begin{aligned} Z = & \int D\sigma D\vec{\pi} \exp \left(-\frac{8N_c}{(4\pi)^2} \left\{ \frac{8m^2}{\beta^2} F_2(\beta m) + \frac{4m^3}{\beta} F_1(\beta m) + m^4 F_0(\beta m) \right\} \right) \times \quad (10) \\ & \exp \left(-\int d^4x \left[\frac{1}{2} z(T) \{ (\partial_\mu \sigma)^2 + (\partial_\mu \vec{\pi})^2 \} - \frac{\mu'^2(T)}{2} (\sigma^2 + \vec{\pi}^2) + \frac{\lambda'(T)}{4} (\sigma^2 + \vec{\pi}^2)^2 \right] \right) \end{aligned}$$

The fermionic integration has given a prefactor which comes from the quark loop at finite temperature. This prefactor is due to the free motion of quarks, which we think is inadequately described by the linear sigma model without confinement. Therefore we will not include it in the equation of state. We take, however, the fermionic effects on the mesonic couplings seriously. Due to the freezing of the infrared parameter m , the fermion loop contributions arise from short time fluctuations in the thermal system, where quark confinement is not relevant. After sorting one encounters the following temperature dependent wave function renormalization constant and couplings up to second order in the proper time expansion:

$$z(T) = 1 + \frac{8N_c}{(4\pi)^2} 2g^2 F_0(\beta m) \quad (11)$$

$$\mu'^2(T) = \mu_0^2 + \frac{8N_c}{(4\pi)^2} 4m^2 g^2 F_0(\beta m) + \frac{8N_c}{(4\pi)^2} \frac{8m}{\beta} g^2 F_1(\beta m) \quad (12)$$

$$\lambda'(T) = \lambda_0 + \frac{8N_c}{(4\pi)^2} 4g^4 F_0(\beta m) . \quad (13)$$

The first terms on the right hand sides in Eqns. (11-13) are the $T = 0$ terms coming from the bosonic action Eq. (3). The finite temperature corrections are generated from the heat kernel. Since the heat kernel expansion is an one loop integration, its result corresponds to the results in perturbation theory which gives temperature dependent modifications of the couplings. The wave function renormalization comes from the graph where a quark radiates a meson. The mass of the mesons is changed by the meson polarization operator in quark antiquark pairs to order g^2 . The meson meson scattering amplitude gets modifications from the box graph in fourth order g^4 of the coupling. The numerical values of the couplings are shown in Fig. 1 as a function of temperature. One sees that μ' decreases and becomes zero at $T \approx 185$ MeV. The coupling λ' and the wave function renormalization z also become smaller. Note the wave function renormalization z does not vanish at the chiral transition, which means the pion has not dissolved at this temperature. The probability to find a free pion at $T \approx 185$ MeV, however, has decreased to 50%.

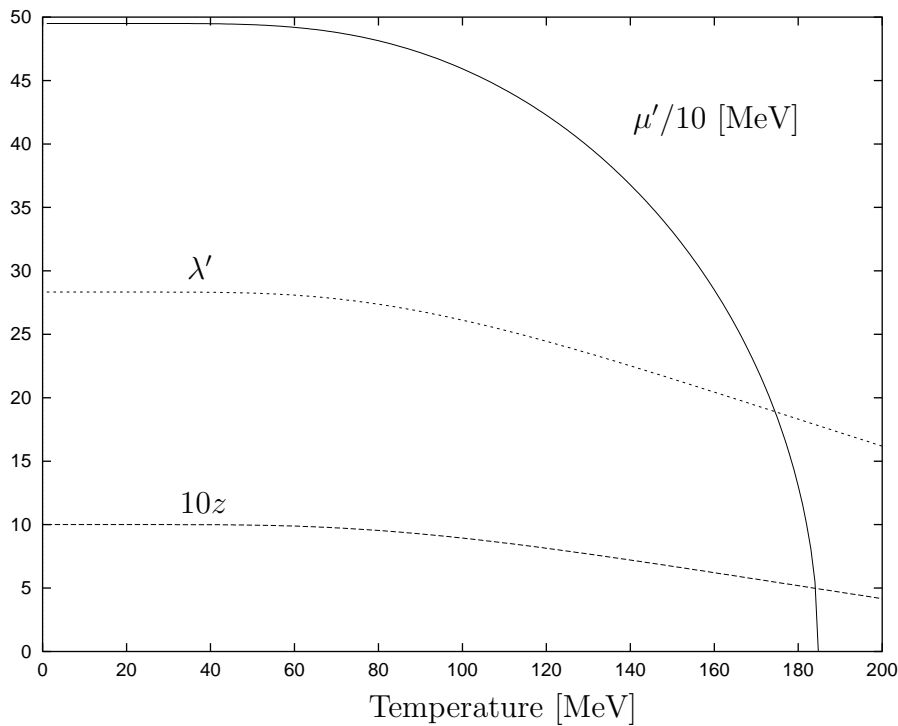


Figure 1: couplings λ' , μ' and z as function of the temperature.

We also apply the presented heat kernel method to evaluate the quark condensate $\langle \bar{q}q \rangle$ at finite temperature in the chiral limit and to calculate $\pi\pi$ -scattering for finite pion masses. These calculations are possible by adding an additional source term

$$2\Delta\bar{q}q = \Delta \sum_{i=1}^2 \bar{q}_i q_i$$

to the fermionic Lagrangian Eq. (2) and then differentiating with respect to Δ . Thus the parameter Δ plays the role of a current quark mass and breaks chiral symmetry explicitly. We again calculate only the difference between the finite temperature

heat kernel H_T and the zero temperature H_0 . Therefore we have to include in our bosonic action at zero temperature an explicit symmetry breaking term, arising from the integration over the quarks with finite masses at zero temperature which we do not execute explicitly. This term will receive a temperature dependent modification in our calculation. In addition another term cubic in meson fields will arise at finite temperature through the fermion integral. The symmetry breaking term in the linear sigma model at $T = 0$ is given by $\mathcal{L}_c = -c\sigma$ with a value of c which reproduces the physical pion mass.

$$c\sigma = -m_\pi^2 f_\pi \sigma.$$

Using the relation $m_\pi^2 f_\pi^2 = -2\Delta \langle \bar{q}q \rangle_0$ we get

$$c = \Delta \left(\frac{2\langle \bar{q}q \rangle_0}{f_\pi} \right) =: \Delta \tilde{c}$$

Employing the heat kernel technique we obtain the final effective bosonic action at $T \neq 0$ including explicit symmetry breaking due to finite quark masses Δ . The meson wave function renormalization constant has been scaled out.

$$S[\sigma, \vec{\pi}] = \int d^4x \left\{ const + \frac{1}{2} \left((\partial_\mu \sigma)^2 + (\partial_\mu \vec{\pi})^2 \right) - \frac{\mu''(T)^2}{2} (\sigma^2 + \vec{\pi}^2) + t_3(T) \Delta \sigma (\sigma^2 + \vec{\pi}^2) + \frac{\lambda''(T)}{4} (\sigma^2 + \vec{\pi}^2)^2 + t_1(T) \Delta \sigma \right\}$$

with additional temperature dependent functions entering the symmetry breaking terms.

$$\begin{aligned} t_1(T) &= \frac{-2(\mu'(T)^2 - \mu_0^2) + \tilde{c}}{z(T)^{1/2}} \\ t_3(T) &= \frac{4(\lambda'(T) - \lambda_0)}{z(T)^{3/2}} \\ \mu''(T)^2 &= \frac{\mu'(T)^2}{z(T)} \\ \lambda''(T) &= \frac{\lambda'(T)}{z(T)^2} \end{aligned} \tag{14}$$

There are two new temperature dependent loop corrections in this formula proportional to the external source Δ . The sigma tadpole term $t_1 \Delta \sigma$ gets a temperature dependent fermion loop correction. A temperature dependent Yukawa vertex between two mesons and the sigma meson is generated due to the quark loop. The term $t_3 \Delta$ adds to the Yukawa vertex produced after spontaneous symmetry breaking, when σ develops a vacuum expectation value. In Table 1, we give the numerical values of μ'' , λ'' , $t_1 \Delta / (f_\pi m_\pi^2)$ and $t_3 \Delta$. Both terms depending on the explicit symmetry breaking Δ influence the order of the phase transition. Previously [10] we did not allow such temperature dependent terms in calculations of the order of the phase transition. Especially a third order term would add to the determinant term in Flavour- $SU(3)$ from instantons which is nine times larger at T_c [11].

T [MeV]	0	40.	80.	120.	160.	$T_c \approx 184.8$
μ'' [MeV]	495.	495.	493.	469.	362.	0.
λ'' [1]	28.3	28.4	30.1	36.9	53.0	72.9
$\Delta t_1/f_\pi m_\pi^2$ [1]	-1.	-1.	-0.97	-0.81	-0.42	0.
Δt_3 [MeV]	0.	-0.1	-7.4	-38.2	-116.4	-219.

Table 1: Couplings in the effective Boson action $S[\sigma, \vec{\pi}]$ for different temperatures.

4 Quark Condensate and $\pi\pi$ -Scattering

The next task is to track the behaviour of the vacuum expectation value of the σ field. When we expand the effective action around the spontaneously broken ground state $\sigma_0 = \text{const}$, we get up to a normalization factor \mathcal{N} the partition function with the tree contribution to the effective action. This action contains all the effects of the temperature fluctuations of the fermions besides the free energy of free quarks and antiquarks. These effects suffice to produce a chiral phase transition.

$$S[\sigma_0] = (\beta V)(t_1 \Delta \sigma_0 - \frac{\mu''^2}{2} \sigma_0^2 + t_3 \Delta \sigma_0^3 + \frac{\lambda''}{4} \sigma_0^4)$$

In mean field approximation we determine from this tree action the quark condensate in the chiral limit as a function of temperature.

$$2\langle \bar{q}q \rangle_{MF} = -\frac{1}{\beta V} \left. \frac{\partial \ln Z_{MF}(\Delta)}{\partial \Delta} \right|_{\Delta=0} = t_1 \sigma_0 + t_3 \sigma_0^3. \quad (15)$$

Minimizing the action with respect to σ one finds that σ_0 and the resulting quark condensate are smooth functions of temperature (cf. Fig. 2), which vanish at the chiral transition temperature where μ''^2 vanishes. The phase transition is at a higher temperature compared to the calculation in ref. [1], because the effects of the quark fields enter only via the intermediate coupling of the meson fields. If one would set $m = g\sigma_0$ in the prefactor in Eq. (10), a variation of this part would give a faster decrease of the vacuum expectation value of σ_0 . But this procedure is questionable since the quarks are not free below the transition temperature.

In order to estimate the mesonic effects on the quark condensate we include their thermal fluctuations in the partition functions. In the linear sigma model the integration over the bosonic degrees of freedom is much simpler than in the NJL model [8]. The zero temperature bose fluctuations are already included in the meson coupling constants at zero temperature. Therefore we use a simple one loop approximation for the boson integration. Including mesonic terms we obtain the following partition function, where the integrals are to be understood at finite temperature with the mean field masses.

$$Z = \mathcal{N} e^{-S[\sigma_0]} - \frac{1}{2} Tr \ln(-\partial^2 + M_\sigma^2) - \frac{3}{2} Tr \ln(-\partial^2 + M_\pi^2)$$

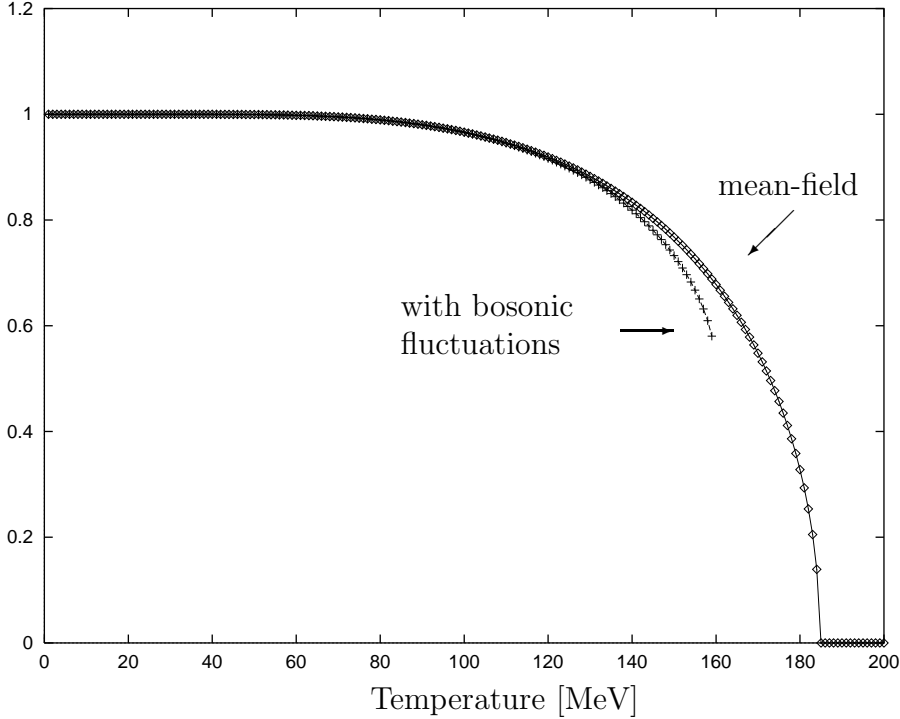


Figure 2: Normalized condensate $\langle \bar{q}q \rangle_{MF}$ and $\langle \bar{q}q \rangle$ with bosonic fluctuations as function of the temperature.

$$\begin{aligned}
 M_\sigma^2 &= -\mu''^2 + 3\lambda''\sigma_0^2 + 6t_3\Delta\sigma_0 \\
 M_\pi^2 &= -\mu''^2 + \lambda''\sigma_0^2 + 2t_3\Delta\sigma_0
 \end{aligned}
 \tag{16}$$

The value σ_0 entering these equations is determined in mean field from the minimum of the bosonic tree action. Consequently, σ_0 depends on the explicit symmetry breaking parameter Δ . The quark condensate in this approximation contains fermionic and mesonic effects. If the couplings t_1 and t_3 would not depend on temperature, the quark dynamics would be invisible in the variation of the quark condensate with temperature.

$$2\langle \bar{q}q \rangle_T = t_1(T)\sigma_0 + t_3(T)\sigma_0^3 + I_\sigma(\sigma_0, T) + I_\pi(\sigma_0, T)
 \tag{17}$$

where the bosonic integrals are defined as follows:

$$\begin{aligned}
 I_\sigma(\sigma_0, T) &= -\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{p^2 + M_\sigma^2}} \frac{1}{e^{\beta\sqrt{p^2 + M_\sigma^2}} - 1} \frac{\partial M_\sigma^2}{\partial \Delta} \Big|_{\Delta=0} \\
 &= -\frac{M_\sigma T}{4\pi^2} \frac{\partial M_\sigma^2}{\partial \Delta} \Big|_{\Delta=0} \sum_{n=1}^{\infty} \frac{K_1(nM_\sigma/T)}{n}
 \end{aligned}$$

and

$$\begin{aligned}
 I_\pi(\sigma_0, T) &= -\frac{3}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{p^2 + M_\pi^2}} \frac{1}{e^{\beta\sqrt{p^2 + M_\pi^2}} - 1} \frac{\partial M_\pi^2}{\partial \Delta} \Big|_{\Delta=0} \\
 &= -\frac{T^2}{8} \frac{\partial M_\pi^2}{\partial \Delta} \Big|_{\Delta=0},
 \end{aligned}$$

In the last equation we have to take into account the pion with zero mass. To specify the result in the chiral limit we list the masses and σ_0 field derivatives

$$\begin{aligned}\frac{\partial M_\pi^2}{\partial \Delta}\Big|_{\Delta=0} &= 2t_3\sigma_0 + (2\lambda''\sigma_0)\frac{\partial \sigma_0}{\partial \Delta}\Big|_{\Delta=0} \\ \frac{\partial M_\sigma^2}{\partial \Delta}\Big|_{\Delta=0} &= 3\frac{\partial M_\pi^2}{\partial \Delta}\Big|_{\Delta=0} \\ \frac{\partial \sigma_0}{\partial \Delta}\Big|_{\Delta=0} &= -\frac{t_1 + 3t_3\sigma_0^2}{M_\sigma^2}\Big|_{\Delta=0}.\end{aligned}\tag{18}$$

In Fig. 2 we show the resulting quark condensate including the bosonic thermal fluctuations. It decreases faster and leads to a first order transition. We are aware that the meson one loop approximation is not sufficient to generate a second order phase transition for a pure mesonic $O(4)$ -theory. In the present paper we are mainly interested in the effect of quarks on the mesonic action, in a further paper we are going to improve our calculation by combining renormalization group methods with the heat kernel technique [12].

The $\pi\pi$ s-wave scattering amplitude vanishes in the zero momentum limit for massless pions. Therefore we have to use the masses of σ - and π -mesons in the form with finite symmetry breaking Δ (cf. Eq. (16)). In addition we need the Yukawa coupling $g_{\sigma\pi\pi}$ from the bosonic action after symmetry breaking:

$$g_{\sigma\pi\pi} = \lambda''\sigma_0 + t_3\Delta\tag{19}$$

The s-wave projection of the $\pi\pi$ -scattering amplitude in the isospin zero channel then has the following form

$$T_{00}(s = 4M_\pi^2) = \frac{1}{3}\left((30\lambda'' - 4g_{\sigma\pi\pi}^2)\left(\frac{9}{M_\sigma^2 - 4M_\pi^2} + \frac{6}{M_\sigma^2}\right)\right)\tag{20}$$

Inserting the temperature dependent masses Eq.(16) and coupling constants Eq.(14) we find a divergence of the s-wave isospin zero scattering length (cf. ref. [9]) shortly before the chiral phase transition. As long as one treats the bosons very naively in mean field or one loop approximation there comes a point where the σ mass becomes degenerate with twice the pion mass, which leads to an exploding scattering length a_{00}

$$a_{00} = -\frac{T_{00}}{32\pi M_\pi}\tag{21}$$

In Fig. 3 we plot the result of our calculation near T_c and compare the result with a simple extrapolation of the Weinberg scattering length

$$a_{00} = -\frac{7m_\pi}{32\pi f_\pi^2}$$

to finite temperature.

A more careful investigation of pion scattering at finite energies shows that the effect of the singularity is diminished when one averages over the cm-momentum of

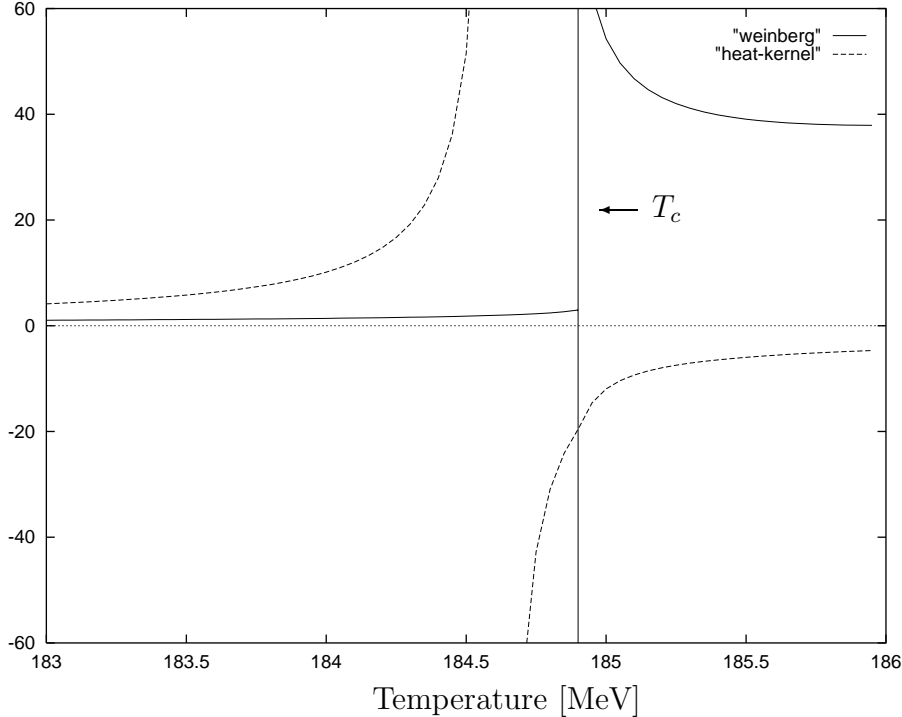


Figure 3: Scattering lengths a_{00} as functions of the temperature.

the pions, as the heat bath only contains finite momentum pions. The total effect is almost completely washed out, if one includes an average over the unitarized s-wave amplitude, which contains $\pi\pi$ -rescattering effects. It is therefore very important to include the ring sum or an equivalent damping mechanism to control the pion interactions near T_c . The scattering length is a good illustration for the inadequacy of the mean field or one loop approximation in the presence of Goldstone Bosons. It remains to be seen how far away from T_c these effects are important in the order parameter, since in the physical mass case the second order phase transition is smeared out. Also a quantitative comparison of the ring or large N_f approximation with the renormalization group equations is interesting. The heat kernel technique allows without any additional assumption about the form of the infrared regulator to calculate the evolution equations. At lower temperatures the presented coupling constants can be included in the constituent quark model calculation with confinement in ref. [1] to take into account the effects of thermal quark fluctuations on the couplings of the sigma model. The simplicity of NJL model estimates remains unchallenged by the linear sigma model. If one wants to tackle higher order effects, however, our calculation shows that the constituent quark model is by far superior in its ability to include meson dynamics. The physical question whether mesonic degrees of freedom are modeled adequately by this model near T_c has to be answered by calculating susceptibilities not only in the scalar and pseudoscalar, but also in the vector channels. These results can be compared with numerical lattice calculations but also with stunning experimental data from dilepton spectra [13].

Acknowledgments

We would like to thank J. Hüfner, D.-U. Jungnickel and S.P. Klevansky for useful discussions.

References

- [1] H.-J. Pirner and M. Wachs, Nucl. Phys. **A617** (1997) 395.
- [2] D.-U. Jungnickel and C. Wetterich, Phys. Rev. **D53** (1996) 5142.
- [3] D. Ebert and H. Reinhardt, Nucl. Phys. **B271** (1986) 188.
- [4] R. D. Ball, Phys. Rep. **182** (1989) 1; J.A. Zuk, J. Phys. A18 (1985) 1795.
- [5] R. I. Nepomechie, Phys. Rev. **D31** (1985) 3291.
- [6] H. Boschi-Filho and C. P. Natividade, Phys. Rev. **D45** (1992) 586.
- [7] A. Manohar and H. Georgi, Nucl. Phys. **B234** (1984) 189.
- [8] W. Florkowski and W. Broniowski, Phys. Lett. **B386** (1996) 62; W. Broniowski et al., Preprint RUB-TPII-19-95, (1995) hep-ph/9509363.
- [9] E. Quack, P. Zhuang, Y. Kalinovsky, S.P. Klevansky and J. Hüfner, Phys. Lett. **B348** (1995) 1.
- [10] H. Meyer-Ortmanns, A. Patkos and H.-J. Pirner, Phys. Lett. **B295** (1992) 255.
- [11] H. Meyer-Ortmanns and B.-J. Schaefer, Phys. Rev. **D53** (1996) 6586.
- [12] B.-J. Schaefer and H.-J. Pirner, work in progress.
- [13] A. Drees, Nucl. Phys. **A610** (1996) 536c.