CONFINEMENT, MAGNETIC Z_N SYMMETRY AND LOW ENERGY EFFECTIVE THEORY OF GLUODYNAMICS

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In these notes I explain the idea how one could understand confinement by studying the low energy effective dynamics of non Abelian gauge theories. I argue that under some mild assumptions, the low energy dynamics is determined universally by the spontaneous breaking of the magnetic symmetry introduced by 't Hooft more than 20 years ago. The degrees of freedom in the effective theory are magnetic vortices. They play a similar role in confining dynamics to the role played by pions and sigma in the chiral symmetry breaking dynamics. I give explicit derivation of the effective theory in 2+1 dimensional weekly coupled confining models and give arguments that it remains qualitatively the same in strongly coupled 2+1 dimensional gluodynamics. Confinement in this effective theory is a very simple classical statement about long range interaction between topological solitons, which follows by a simple direct classical calculation from the structure of the effective Lagrangian. I discuss the elements of this picture which generalize to 3+1 dimensions and point to the open questions still remaining.

1 Introduction.

In these notes I am going to discuss an approach to understanding confinement in gluodynamics in terms of universal properties determined by realization of global symmetries.

First, let me be precise in terminology. By gluodynamics I mean a non Abelian gauge theory without dynamical fields in fundamental representation. Reason d'etre being that I believe that confinement is the property of the pure glue sector of a non Abelian theory and it is therefore important to understand the pure glue theory. Presence of fundamental fermions in real life QCD is from this point of view secondary and although it changes various properties of the low energy dynamics in significant ways, it is not driving the confinement phenomenon. This has been traditional point of view among the workers in field. It is however in marked contradiction to Gribovs ideas who maintained that it is the light fundamental fermions that bear ultimate responsibility for confinement. These ideas, although very interesting will not be discussed in these notes.

Confinement is something of a mystery. It is certainly the most striking qualitative phenomenon in QCD. Still we do not even have a satisfactory definition of what exactly is meant by this word. Who is confined? Global color charges? No! States with nonvanishing color charge (be it local or global) are

not part of the Hilbert space of the non Abelian gauge theory. The only physical states are those that satisfy Gauss' law - the color singlets. The global color is thus "confined" by construction, and that's certainly not what we mean. We are after a dynamical effect. Consistently with the vagueness of the definition of the effect itself, most models that have been put forward to explain it suffer from similar blind spots.

Perhaps the most popular simple model of confinement is the dual superconductor model due to 't Hooft and Mandelstam [1]. The idea itself is very simple and as such certainly very appealing. Consider a superconductor. It is described by the Landau - Ginzburg model of the complex order parameter field H (the Higgs field) coupled to the vector potential A_i. The free energy (or Lagrangian of this Abelian Higgs model (AHM) in the field theoretical context) for static field configurations is

$$-L = |D_i H|^2 + \frac{1}{2}B^2 + \lambda(H^* H - v^2)^2 \tag{1}$$

where

$$D_i = \partial_i - igA_i \tag{2}$$

The superconducting phase is characterized by a nonvanishing condensate of the Higgs field

$$< H >= v \tag{3}$$

The only way the magnetic field can penetrate a superconductor is in the form of the flux tubes with the core size of the penetration depth $l = (gv)^{-1}$ - Abrikosov-Nielsen-Olesen (ANO) vortex. The magnetic flux inside a flux tube is quantized in units of $2\pi/g$. The flux tube solution can be found numerically and its properties in various limits are well understood analytically, see for example [2]. Suppose now, that in addition to the charged field H our model also has very heavy magnetic monopoles. Since they are very heavy, they do not affect the dynamics of the order parameter. But the structure of the vacuum does affect strongly the interaction between the monopoles themselves. A monopole and an antimonopole in the superconducting vacuum feel linear confining potential. Since a monopole is a source of magnetic flux, this flux in the superconducting vacuum will form a ANO flux tube, which terminates on an antimonopole. The energy of the flux tube is proportional to its length, hence the linear potential.

So, here is a theory in which certain objects are confined by a linear potential. The dual superconductor hypothesis asserts that the Yang Mills theory is basically an "upside down" version of the Abelian Higgs model, as far as the confinement mechanism is concerned. That is, if we use the following dictionary, the preceding discussion describes confinement in any non Abelian gauge

theory, which is in the confining phase. The magnetic field of the Higgs model should be called the '"color electric field", the magnetic monopoles should be called the "color charges", and by analogy the electrically charged field H should turn into "color magnetic monopole".

This dual superconductor mechanism works in the simplest known confining theory - Abelian compact $U(1)$ model. This theory on the lattice is known to have a phase transition at a finite value of the coupling constant. In the strongly coupled phase it is confining. Note that in the Abelian, as opposed to non Abelian case, the global part of the gauge group is a physical symmetry. Charged states do exist in the Hilbert space a priori, and in this case confinement clearly means linear potential between these charges. In this lattice theory one can perform a duality transformation which transforms the compact $U(1)$ pure gauge theory into a noncompact $U(1)$ theory with charged fields, i.e. Abelian Higgs model [3]. The "Higgs" field in the dual model indeed describes excitations carrying magnetic charge - magnetic monopoles. Those are condensed in the confining phase and dual superconductivity captures perfectly the physics of confinement in this theory.

However as soon as one tries to generalize it to non Abelian theories many new questions pop up. In the Abelian theory the monopole charge is unambiguous and well defined. In non Abelian theories there is no gauge invariant definition of a monopole, or for that matter even of the magnetic charge that it carries. In order to identify monopole - like objects, the common practice is to to gauge fix the non Abelian gauge symmetry is a way that leaves only a $U(1)$ subgroup not fixed. This can be accomplished by gauge rotating any adjoint field R into a canonical form. The original idea of 't Hooft was that the "monopoles" should be identified with the singular obstructions to this gauge fixing, and the physics should not depend on which particular quantity R we use to set up the gauge fixing procedure.

This however turned out not to be the case. Since the definition of monopoles depends on the gauge, it is unavoidable that the correlation between the quantitative properties of these monopoles (e.g. monopole density) and confinement (the value of the string tension) also vary from gauge to gauge. Although some qualitative properties are shared by monopoles in different gauges [5], the gauge dependence has been seen clearly in various lattice simulations. In the so called maximal Abelian gauge these correlations are indeed very strong [6]. This gauge is designed to make the gauge fixed Yang Mills Lagrangian to be as similar as possible to the compact $U(1)$ theory by attempting to maximally suppress the fluctuations of all fields except the color components of the gauge potential which belong to the Cartan subalgebra. In particular it looks like the monopoles contribute a large share of the string ten-

sion [6]. The monopole condensate disappears above the critical temperature in the deconfined phase. In other gauges correlations between the monopole properties and the confining properties of the theory are much weaker, and in some cases are completely absent [4]. Even in the maximal Abelian gauge it is not clear that the monopole dominance is not a lattice artifact. One expects that if the monopoles are indeed physical objects in the continuum limit, their size would be of order of the only dimensional parameter in the theory Λ_{QCD}^{-1}. The monopoles which appear in the context of lattice calculations on the other hand are all pointlike objects.

During the recent years there have been some attempts to relate dual superconductivity to confinement somewhat more directly in continuum limit. The case in point is the exactly solvable $N = 2$ supersymmetric gauge theories [7]. Although $N = 2$ super Yang Mills theory is not confining, Witten and Seiberg showed that at one point in the moduli space it contains massless monopoles. Introducing a particular supersymmetry breaking perturbation the monopoles can be made to condense leading to confinement. Unfortunately this construction does not address the genuine non Abelian dynamical questions. The $N = 2$ super Yang Mills theory close to the monopole point is essentially Abelian. The spectrum contains an Abelian gauge field, while the analogs of the non Abelian gluons are massive due to the Higgs mechanism. Confinement in this model is therefore similar to the Abelian confinement of the compact Abelian theory. As discussed in [8] the properties of the spectrum and various characteristics of confining strings are very different in this model and in nonsupersymmetric QCD. In particular it is not known whether the Abelian monopole charge survives at all in the genuine non Abelian regime and whether the condensation of Abelian monopoles is at all relevant to confinement in the non Abelian limit. It is entirely unclear at this point how this supersymmetry inspired construction can help us understand the real theory of strong interactions.

Thus although the dual superconductor picture is appealing due to its simplicity, it is far from having been substantiated. To my mind there is quite a general objection to it. The dual superconductor hypothesis in effect states that the effective low energy theory of the pure Yang Mills theory when written in terms of appropriate field variables is an Abelian Higgs model. If so, the low lying spectrum of pure Yang Mills should be the same as that of an Abelian Higgs model.

A statement about the form of the effective Lagrangian is not easily verified, since it necessitates an appropriate choice of variables. On the other hand a statement about the spectrum does not depend on the choice of variables and is in this sense universal and easily verifiable. The spectrum of the Abelian

Higgs model is well known. The two lowest mass excitations are a massive vector particle and a massive scalar particle. The role of the vector particle is crucial in the framework of the dual superconductor model. Indeed in the Seiberg-Witten model the light vector is present in the spectrum. However this is not the case in gluodynamics. The spectrum of the Yang Mills theory is not known from analytical calculations. However, in recent years a rather clear picture of it emerged from the lattice simulations [9]. The lowest lying particle in the spectrum is a scalar glueball with the mass $1.5 - 1.7$ Gev. The second excitation is a spin 2 tensor glueball with a mass of around 2.2 Gev. Vector (and pseudo vector) glueballs are conspicuously missing in the lowest lying part of the spectrum. The simulations indicate that they are relatively heavy [a] with masses above 2.8 Gev [9]. The pattern of the Yang Mills spectrum therefore seems to be rather different from the one suggested by the Abelian Higgs model. Since the vector glueball is so much heavier than the scalar and tensor ones, it seems quite unlikely that it plays so prominent a role in the confinement mechanism as the one reserved for the massive photon in the dual superconductor scenario.

Of course, it's not over until it's over. Since no dynamical solution to the problem exist we cannot completely rule out the possibility that it is the properties of the high energy, rather than of low energy part of the spectrum, that are directly related to confinement. To me however this sounds like a far cry. It seems much more likely that the key is at low energies. And so rather than looking for it under the lamp post of the dual superconductor model, we may, while we are at it, ask a more general question. Can we learn about the mechanism of confinement by studying the effective low energy theory? This does not sound like such a stupid idea. If confinement (whatever it is) is a low energy phenomenon, it is very likely that inside the low energy effective theory we would be able to understand it in a simple way. The catch is of course obvious: we don't know what the low energy theory is and what principles determine it. And so it could be that determining the low energy theory is itself even more difficult an undertaking than just understanding confinement.

In these notes I will try to argue that this is not necessarily the case, and that such an undertaking is not as hopeless as it may seem. There are instances when we can understand main features of the low energy dynamics without being able to solve in detail the "microscopic" theory. This happens whenever we are lucky enough to have a spontaneously broken global continuous symmetry. The Goldstone theorem assures us that such a spontaneous symmetry breaking results in appearance of massless particles. Those of course are

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[a]Qualitatively, precisely this pattern was predicted 20 years ago [10] on the basis of QCD low-energy theorems.

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natural low energy degrees of freedom. Moreover the original symmetry must be manifest in the effective theory that governs the interactions between the Goldstone bosons, albeit its implementation is nonlinear. This severely constrains the interactions of the Goldstone bosons and in fact lends considerable predictive power to the effective theory. The classic example of this type is the spontaneous breaking of chiral symmetry which has the effect that the low energy physics is dominated by pions and is described by the chiral effective Lagrangian [11]. In real life the chiral symmetry is broken explicitly and the pion is massive. But effects of small explicit breaking are easily accommodated in the chiral Lagrangian. The great thing about the chiral Lagrangian of course is that it is universal. It does not care what exact dynamics is responsible for the symmetry breaking, what where the degrees of freedom of "microscopic" theory or any other fine details. All you need to know is that there was a symmetry, and this symmetry was spontaneously broken.

Can some universal considerations of a similar kind determine the structure of low energy theory in pure gluodynamics? The chiral symmetry here is certainly of no relevance. Nevertheless I will argue that the symmetry path is a very fruitful one. The main thesis is this. There is a global discreet symmetry in gluodynamics, which is spontaneously broken in the vacuum. In the following this symmetry will be called the magnetic Z_N. Although the symmetry is discreet and therefore does not have all the bliss of the Goldstone theorem, under some natural assumptions it does indeed determine the low energy dynamics. In physical 3+1 dimensional theory this symmetry is of a somewhat unusual type - its charge is not a volume integral, but rather a surface integral. The implications of such a symmetry for the low energy Lagrangian have to be studied in more detail and this has not been done so far. However one can go much further in 2+1 dimensions, where this symmetry is of a familiar garden variety. The larger part of these notes will be devoted therefore to 2+1 dimensional non Abelian theories. The structure of these notes is the following. Sections 2-5 are devoted to 2+1 dimensional physics. In section 2 I will discuss in detail the nature of the magnetic Z_N symmetry and the explicit construction of both, the generator of the group and the local order parameter. In Section 3 I will show how this symmetry is realized in different phases of Abelian gauge theories. I will show that in the Coulomb phase this symmetry is spontaneously broken and in the Higgs phase it is respected by the vacuum state. In the Abelian theories the magnetic symmetry is continuous $U(1)$ group, and so its spontaneous breaking leads to the appearance of a massless particle - the photon. I will derive the low energy effective Lagrangian that describes this symmetry breaking pattern and show that it exhibits logarithmic confinement in the Coulomb phase. In Section 4 I will

discuss weakly interacting confining theories, like the Georgi-Glashow model. Here the magnetic symmetry is discreet, but it is also spontaneously broken like in the Coulomb phase of QED. Again it is possible to derive the effective Lagrangian which follows from this symmetry breaking pattern. I will demonstrate that due to the fact that the magnetic symmetry is discreet the effective theory exhibits linear confinement, and that this confinement mechanism in the effective theory appears very simply on the classical level. In Section 5 I will give arguments that the basic structure of the effective Lagrangian as well as the physics of confinement stays the same in the pure gluodynamics limit. I will discuss the similarities and differences between the confining properties of the weakly interacting non Abelian theories and gluodynamics. In section 6 I will discuss the generalization of these ideas to the physical dimensionality of 3+1, including the symmetry structure, the order parameter and the realization of the symmetry. The explicit construction of the low energy effective theory in 3+1 dimensions has not been completed, although I will discuss some modest steps in this direction. Finally section 7 is devoted to the discussion of the lessons, past and future that can be hopefully learned from this approach.

2   The magnetic symmetry in 2+1 dimensions.

A while back 't Hooft gave an argument establishing that a non Abelian $SU(N)$ gauge theory without charged fields in fundamental representation possesses a global Z_N symmetry [12].

Consider first a theory with several adjoint Higgs fields so that varying parameters in the Higgs sector the $SU(N)$ gauge symmetry can be broken completely. In this phase the perturbative spectrum will contain the usual massive "gluons" and Higgs particles. However in addition to those there will be heavy stable magnetic vortices. Those are the analogs of Abrikosov-Nielsen-Olesen vortices in the superconductours and they must be stable by virtue of the following topological argument. The vortex configuration away from the vortex core has all the fields in the pure gauge configuration

$$H^\alpha(x) = U(x)h^\alpha, \quad A^\mu = iU\partial^\mu U^\dagger \tag{4}$$

Here the index α labels the scalar fields in the theory, h^α are the constant vacuum expectation values of these fields, and $U(x)$ is a unitary matrix. As one goes around the location of the vortex in space, the matrix U winds nontrivially in the gauge group. This is possible, since the gauge group in the theory without fundamental fields is $SU(N)/Z_N$ and it has a nonvanishing first homotopy group $\Pi_1(SU(N)/Z_N) = Z_N$. Practically it means that when going full circle around the vortex location, U does not return to the same $SU(N)$

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group element U_0, but rather ends up at $\exp\{i\frac{2\pi}{N}\}U_0$. Adjoint fields do not feel this type of discontinuity in U and therefore the energy of such a configuration is finite. Since such a configuration can not be smoothly deformed into a trivial one, a single vortex is stable. Processes involving annihilation of N such vortices into vacuum are allowed since N-vortex configurations are topologically trivial. One can of course find explicit vortex solutions once the Higgs potential is specified. As any other semi classical solution in the weak coupling limit the energy of such a vortex is inversely proportional to the gauge coupling constant and therefore very large. One is therefore faced with the situation where the spectrum of the theory contains a stable particle even though its mass is much higher than masses of many other particles (gauge and Higgs bosons) and the phase space for its decay into these particles is enormous. The only possible reason for the existence of such a heavy stable particle is that it carries a conserved quantum number. The theory therefore must possess a global symmetry which is unbroken in the completely Higgsed phase. The symmetry group must be Z_N since the number of vortices is only conserved modulo N. 't Hooft dubbed this symmetry "topological", but I prefer to call it "magnetic" for reasons that will become apparent in a short while.

Now imagine changing smoothly the parameters in the Higgs sector so that the expectation values of the Higgs fields become smaller and smaller, and finally the theory undergoes a phase transition into the confining phase. One can further change the parameters so that the adjoint scalars become heavy and eventually decouple completely from the glue. This limiting process does not change the topology of the gauge group and therefore does not change the symmetry content of the theory. One concludes that the pure Yang-Mills theory also possesses a Z_N symmetry. Of course since the confining phase is separated from the completely Higgsed phase by a phase transition one may expect that the Z_N symmetry in the confining phase is realized differently in the confining vacuum than in the completely "Higgsed" phase. In fact the original paper of 't Hooft as well as subsequent work [13] convincingly argued that in the confining phase the Z_N symmetry is spontaneously broken and this breaking is related to the confinement phenomenon. We will have more to say about this later.

The physical considerations given above can be put on a firmer formal basis. In particular one can construct explicitly the generator of the Z_N as well as the order parameter associated with it - the operator that creates the magnetic vortex [15]. We start this discussion by considering Abelian theories, where things are simpler and are completely under control.

In the $U(1)$ case the homotopy group is Z and so the magnetic symmetry is $U(1)$ rather than Z_N. It is in fact absolutely straightforward to identify the relevant charge. It is none other than the magnetic flux through the equal time plane, with the associated conserved current being the dual of the electromagnetic field strength

$$\Phi = \int d^2x B(x), \qquad \partial^\mu \tilde{F}_\mu = 0 \qquad (5)$$

The current conservation is insured by the Bianchi identity. It may come as a surprise that we are considering seriously a current whose conservation equation is an "identity". However "identity" is a relative thing. The conservation equation is trivial only because we have written the components of the field strength tensor in terms of the vector potential. But the introduction of vector potential is nothing but potentiation of the conserved current $\tilde{F}_\mu$, that is explicit solution of the conservation equation. In exactly the same way one can potentiate any conserved current, and such a potentiation will turn the pertinent conservation equation into identity. Thus $\tilde{F}_\mu$ has exactly the same status as any other local conserved gauge invariant current, and should be treated as such.

Once we have the current and the charge, we also know the elements of the symmetry group. A group element of the $U(1)$ magnetic symmetry group is

$$W_\alpha(\infty) = \exp\{i\alpha\Phi\} \qquad (6)$$

for arbitrary value of α. The notation W is not accidental here, since the group element is indeed a large spatial Wilson loop defined on a contour that encloses the whole system.

The question that might bother us is whether this group acts at all nontrivially on any local physical observable in our theory. The obvious gauge invariant observables like B and E commute with W. There is however another set of local gauge invariant observables in the theory which do indeed transform nontrivially under the action of W. Consider, following 't Hooft the operator of the "singular gauge transformation"

$$V(x) = \exp \frac{i}{g} \int d^2y \; \left[\epsilon_{ij} \frac{(x-y)_j}{(x-y)^2} E_i(y) + \Theta(x-y) J_0(y) \right] \qquad (7)$$

where $\Theta(x-y)$ is the polar angle function and J_0 is the electric charge density of whatever matter fields are present in the theory. The cut discontinuity in

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the function Θ looks bothersome, but is in fact not physical and completely harmless. The gauge function jumps across the discontinuity by $2\pi/g$, but since the only dynamical fields in the theory have charges that are integer multiples of g the discontinuity is not observable. The cut can be chosen parallel to the horizontal axis. Using the Gauss' law constraint this can be cast in a different form, which we will find more convenient for our discussion

$$V(x) = \exp\frac{2\pi i}{g}\int_C dy^i \ \epsilon_{ij}E_i(y) \tag{8}$$

where the integration goes along the cut of the function Θ which starts at the point x and goes to spatial infinity. In this form it is clear that the operator does not depend on where precisely one chooses the cut to lie. To see this, note that changing the position of the cut C to C' adds to the phase $\frac{2\pi}{g}\int_S d^2x\partial_i E^i$ where S is the area bounded by $C-C'$. In the theory we consider only charged particles with charges multiples of g are present. Therefore the charge within any closed area is a multiple integer of the gauge coupling $\int_S d^2x\partial_i E^i = gn$ and the extra phase factor is always unity. The only point in space where the action of $V(x)$ on any physical state is nontrivial is the point x. The field $V(x)$ therefore acts like any other local field. With a little more work one can prove not only that $V(x)$ is a local field, in the sense that it commutes with any other local gauge invariant operator $O(y)$, $x \neq y$ but also that it is a bona fide Lorentz scalar. [14].

The physical meaning of the operator V is very simple. Calculating its commutator with the magnetic field B we find

$$V(x)B(y)V^\dagger(x) = B(y) + \frac{2\pi}{g}\delta^2(x-y) \tag{9}$$

Thus V creates a pointlike magnetic vortex of flux $2\pi/g$. This commutator also tells us that

$$W_\alpha^\dagger V(x)W_\alpha = e^{i\frac{2\pi\alpha}{g}}V(x) \tag{10}$$

and so V is indeed the local eigenoperator of the magnetic $U(1)$ symmetry group.

Eqs.(5,8) formalize the physical arguments of 't Hooft in the Abelian case. We have explicitly constructed the generator and the local order parameter of the magnetic symmetry.

2.2 Non-Abelian theories at weak coupling.

Let us now move onto the analogous construction for non Abelian theories. Ultimately we are interested in the pure Yang - Mills theory. It is however

illuminating to start with the theory with an adjoint Higgs field and take the decoupling limit explicitly later. For simplicity we discuss the $SU(2)$ gauge theory. Consider the Georgi-Glashow model - $SU(2)$ gauge theory with an adjoint Higgs field.

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}(\mathcal{D}^{ab}_\mu H^b)^2 + \tilde{\mu}^2 H^2 - \tilde{\lambda}(H^2)^2 \tag{11}$$

where

$$\mathcal{D}^{ab}_\mu H^b = \partial_\mu H^a - gf^{abc}A^b_\mu H^c \tag{12}$$

At large and positive $\tilde{\mu}^2$ the model is weakly coupled. The $SU(2)$ gauge symmetry is broken down to $U(1)$ and the Higgs mechanism takes place. Two gauge bosons, $W^\pm$, acquire a mass, while the third one, the "photon", remains massless to all orders in perturbation theory. The theory in this region of parameter space resembles very much electrodynamics with vector charged fields. The Abelian construction can therefore be repeated. The $SU(2)$ gauge invariant analog of the conserved dual field strength is

$$\tilde{F}^\mu = \frac{1}{2}[\epsilon_{\mu\nu\lambda}F^a_{\nu\lambda}n^a - \frac{1}{g}\epsilon^{\mu\nu\lambda}\epsilon^{abc}n_a(\mathcal{D}_\nu n)^b(\mathcal{D}_\lambda n)^c] \tag{13}$$

where $n^a \equiv \frac{H^a}{|H|}$ is the unit vector in the direction of the Higgs field. Classically this current satisfies the conservation equation

$$\partial^\mu \tilde{F}_\mu = 0 \tag{14}$$

The easiest way to see this is to choose a unitary gauge of the form $H^a(x) = H(x)\delta^{a3}$. In this gauge $\tilde{F}$ is equal to the Abelian part of the dual field strength in the third direction in color space. Its conservation then follows by the Bianchi identity. Thus classically the theory has a conserved $U(1)$ magnetic charge $\Phi = \int d^2x \tilde{F}_0$ just like QED. However the unitary gauge can not be imposed at the points where H vanishes, which necessarily happens in the core of an 't Hooft-Polyakov monopole. It is well known of course [17] that the monopoles are the most important nonperturbative configurations in this model. Their presence leads to a nonvanishing small mass for the photon as well as to confinement of the charged gauge bosons with a tiny nonperturbative string tension. As far as the monopole effects on the magnetic flux, their presence leads to a quantum anomaly in the conservation equation (14). As a result only the discrete Z_2 subgroup of the transformation group generated by Φ remains unbroken in the quantum theory. The detailed discussion of this anomaly, the residual Z_2 symmetry and their relation to monopoles is given in

[15]. The nonanomalous Z_2 magnetic symmetry transformation is generated by the operator

$$U = \exp\{i\frac{g}{2}\Phi\} \tag{15}$$

The order parameter for the magnetic Z_2 symmetry is constructed analogously to QED as a singular gauge transformation generated by the gauge invariant electric charge operator

$$J^\mu = \epsilon^{\mu\nu\lambda}\partial_\nu(\tilde{F}^a_\lambda n^a), \quad Q = \int d^2x J_0(x) \tag{16}$$

Explicitly

$$
\begin{aligned}
V(x) &= \exp\frac{i}{g}\int d^2y \left[\epsilon_{ij}\frac{(x-y)_j}{(x-y)^2}n^a(y)E^a_i(y) + \Theta(x-y)J_0(y)\right] \\
&= \exp\frac{2\pi i}{g}\int_C dy^i\epsilon_{ij}n^a E^a_j(y)
\end{aligned} \tag{17}
$$

One can think of it as a singular $SU(2)$ gauge transformation with the field dependent gauge function

$$\lambda^a(y) = \frac{1}{g}\Theta(x-y)n^a(\vec{y}) \tag{18}$$

This field dependence of the gauge function ensures the gauge invariance of the operator V. Just like in QED it can be shown [15], [14] that the operator V is a local scalar field. Again like in QED, the vortex operator V is a local eigenoperator of the Abelian magnetic field $B(x) = \tilde{F}_0$.

$$[V(x), B(y)] = -\frac{2\pi}{g}V(x)\delta^2(x-y) \tag{19}$$

That is to say, when acting on a state it creates a pointlike magnetic vortex which carries a quantized unit of magnetic flux.

The magnetic Z_2 acts on the vortex field V as a phase rotation by π

$$e^{i\frac{g}{2}\Phi}V(x)e^{-i\frac{g}{2}\Phi} = -V(x) \tag{20}$$

This is the explicit realization of the magnetic Z_2 symmetry in the Georgi-Glashow model.

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2.3   Gluodynamics.

From the Georgi Glashow model we can get easily to the pure Yang Mills theory. This is achieved by smoothly varying the $\tilde{\mu}^2$ coefficient in the Lagrangian so that it becomes negative and eventually arbitrarily large. In this limit the Higgs field has a large mass and therefore decouples leaving the pure gluodynamics behind. It is well known that in this model the weakly coupled Higgs regime and strongly coupled confining regime are not separated by a phase transition [16]. The pure Yang Mills limit in this model is therefore smooth.

In the pure Yang Mills limit the expressions Eq. 13,17,15 have to be taken with care. When the mass of the Higgs field is very large, the configurations that dominate the path integral of the theory are those with very small value of the modulus of the Higgs field $|H| \propto 1/M$. The modulus of the Higgs field in turn controls the fluctuations of the unit vector n^a, since the kinetic term for n in the Lagrangian is $|H|^2(D_\mu n)^2$. Thus as the mass of the Higgs field increases the fluctuations of n grow in both, amplitude and frequency and the magnetic field operator B as defined in Eq. 13 fluctuates wildly. This situation is of course not unusual. It happens whenever one wants to consider in the effective low energy theory an operator which explicitly depends on fast, high energy variables. The standard way to deal with it is to integrate over the fast variables. There could be two possible outcomes of this integrating out procedure. Either the operator in question becomes trivial (if it depends strongly on the fast variables), or its reduced version is well defined and regular on the low energy Hilbert space. The "magnetic field" operator B in Eq. 13 is obviously of the first type. Since in the pure Yang Mills limit all the orientations of n^a are equally probable, integrating over the Higgs field at fixed A_μ will lead to vanishing of B. However what interests us is not so much the magnetic field but rather the generator of the magnetic Z_2 transformation. It is actually instructive to consider the operator that performs the Z_2 transformation not everywhere in space, but only inside a contour C

$$U_C = \exp\{i\frac{g}{2}\int_S d^2x B(x)\} \tag{21}$$

with the area S being bounded by C. In the limit of gluodynamics we are lead to consider the operator

$$U_C = lim_{H\to 0} \quad \int Dn^a \exp\left\{-|H|^2(\vec{D}n_a)^2\right\} \tag{22}$$

$$\exp\left\{i\frac{g}{4}\int_C d^2x\left(\epsilon_{ij}F_{ij}^a n^a - \frac{1}{g}\epsilon^{ij}\epsilon^{abc}n_a(\mathcal{D}_i n)^b(\mathcal{D}_j n)^c\right)\right\}$$

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The weight for the integration over n is the kinetic term for the isovector n_a. As was noted before the action does not depend on n^a in the YM limit since $H^2 \to 0$. The first term in Eq. 23 however regulates the path integral and we keep it for this reason. This operator may look somewhat unfamiliar at first sight. However in a remarkable paper [18] Diakonov and Petrov showed that Eq. 23 is equal to the trace of the fundamental Wilson loop along the contour C^b.

$$U_C = W_C \equiv \mathrm{Tr}\mathcal{P}\exp\left\{ig\int_C dl^i A^i\right\} \qquad (23)$$

Taking the contour C to run at infinity, we see that in gluodynamics the generator of the magnetic Z_2 symmetry is the fundamental spatial Wilson loop along the boundary of the spatial plane^c.

Of course one does not have to go through the exercise with the Georgi Glashow model in order to show that the fundamental Wilson loop generates a symmetry. Instead one can directly consider the commutator

$$[W, H] = \lim_{C\to\infty}\oint_C dx_i \mathrm{Tr}\mathcal{P}E_i(x)\exp\{i\oint_{C(x,x)} dy_i A^i(y)\} \to_{C\to\infty} 0 \qquad (24)$$

Here the integral in the exponential on the right hand side starts and ends at the point of insertion of the electric field. For a finite contour C the commutator does not vanish only along the contour itself, but it does not contain any bulk terms. Making the contour C go to infinity and assuming as usual that in a theory with finite mass gap at infinity no physical modes are excited we conclude that the commutator of H with infinitely large Wilson loop vanishes^d.

^{b}We note that Diakonov and Petrov had to introduce a regulator to define the path integral over n. The regulator they required was precisely of the same form as in Eq. 23.

^{c}There is a slight subtlety here that may be worth mentioning. The generator of a unitary transformation should be a unitary operator. The trace of the fundamental Wilson loop on the other hand is not unitary. One should therefore strictly speaking consider instead a unitarized Wilson loop $\tilde{W} = \frac{W}{\sqrt{WW^\dagger}}$. However the factor between the two operators $\sqrt{WW^\dagger}$ is an operator that is only sensitive to behavior of the fields at infinity. It commutes with all physical local operators $O(x)$ unless $x \to \infty$. In this it is very different from the Wilson loop itself, which has a nontrivial commutator with vortex operators $V(x)$ at all values of x. Since the correlators of all gauge invariant local fields in the pure Yang Mills theory are massive and therefore short range, the operator $\sqrt{W(C)W^\dagger(C)}$ at $C \to \infty$ must be a constant operator on all finite energy states. The difference between W and $\tilde{W}$ is therefore a trivial constant factor and we will not bother with it in the following.

^{d}Note that the nonvanishing of the commutator at finite C is precisely of the same nature as for any other "conserved charge" which is defined as an integral of local charge density $Q = \lim_{C\to\infty}\int_{|x|\in C} d^2x\rho(x)$. The commutator of such a charge with a Hamiltonian also contains surface terms, since the charge density ρ itself never commutes with the Hamiltonian. The

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Next we consider the vortex operator Eq. 17. Again in order to find the pure Yang-Mills limit of it we have to integrate this expression over the orientations of the unit vector n^a. This integration in fact is equivalent to averaging over the gauge group. Following [18] one can write n_a in terms of the SU(2) gauge transformation matrix Ω.

$$\vec{n} = \frac{1}{2}\mathrm{Tr}\Omega\tau\Omega^\dagger\tau_3 \tag{25}$$

The vortex operator in the pure gluodynamics limit then becomes

$$\tilde{V}(x) = \int D\Omega \exp\frac{2\pi i}{g}\int_C dy_i\epsilon_{ij}\mathrm{Tr}\Omega E_j\Omega^\dagger\tau_3 \tag{26}$$

This form makes it explicit that $\tilde{V}(x)$ is defined as the gauge singlet part of the following, apparently non gauge invariant operator

$$V(x) = \exp\frac{2\pi i}{g}\int_C dy^i\epsilon_{ij}E_i^3(y) \tag{27}$$

The integration over Ω obviously projects out the gauge singlet part of V. In the present case however this projection is redundant. This is because even though V itself is not gauge invariant, when acting on a physical state it transforms it into another physical state[e]. By physical states we mean the states which satisfy the Gauss' constraint in the pure Yang-Mills theory. This property of V was noticed by 't Hooft [12]. To show this let us consider $V(x)$ as defined in Eq. 27 and its gauge transform $V_\Omega = \Omega^\dagger V\Omega$ where Ω is an arbitrary nonsingular gauge transformation operator. The wave functional of any physical state depends only on gauge invariant characteristics of the vector potential, i.e. only on the values of Wilson loops over all possible contours.

$$\Psi[A_i] = \Psi[\{W(C)\}] \tag{28}$$

commutator is rather a total derivative. For a conserved charge, due to the continuity equation this surface term is equal to the circulation of the spatial component of the current

$$[Q,H] = \to_{C\to\infty}\oint dx^i j_i$$

. The vanishing of this term again is the consequence of the vanishing of the physical fields at infinity in a theory with a mass gap. When the charge is not conserved, the commutator in addition to the surface term contains also a bulk term. It is the absence of such bulk terms that is the unique property of a conserved charge. The same conclusion is reached if rather than considering the generator of the algebra, one considers the commutator of the group element for either continuous or discreet symmetry groups. The commutator in Eq. 24 therefore indeed tells us that W is a conserved operator.

[e]This is not a trivial statement, since a generic non gauge invariant operator has nonvanishing matrix elements between the physical and an unphysical sectors.

Acting on this state by the operators V and V_Ω respectively we obtain

$$
\begin{aligned}
V|\Psi> &= \Psi_V[A_i] = \Psi[\{VW(C)V^\dagger\}] \\
V_\Omega|\Psi> &= \Psi_V^\Omega[A_i] = \Psi[\{V_\Omega W(C)V_\Omega^\dagger\}]
\end{aligned} \tag{29}
$$

It is however easy to see that the action of $V(x)$ and $V_\Omega(x)$ on the Wilson loop is identical - they both multiply it by the phase belonging to the center of the group if x is inside C and do nothing otherwise. Therefore

$$
V|\Psi> = V_\Omega|\Psi> \tag{30}
$$

for any physical state Ψ. Thus we have

$$
\Omega V|\Psi> = \Omega V \Omega^\dagger |\Psi> = V|\Psi> \tag{31}
$$

where the first equality follows from the fact that a physical state is invariant under action of any gauge transformation Ω and the second equality follows from Eq. 30. But this equation is nothing but the statement that the state $V|\Psi>$ is physical, i.e. invariant under any nonsingular gauge transformation.

We have therefore proved that when acting on a physical state the vortex operator creates another physical state. For an operator of this type the gauge invariant projection only affects its matrix elements between unphysical states. Since we are only interested in calculating correlators of V between physical states, the gauge projection is redundant and we can freely use V rather than $\tilde{V}$ to represent the vortex operator.

The formulae of this section can be straightforwardly generalized to $SU(N)$ gauge theories. Once again one can start with the Georgi-Glashow like model, where the $SU(N)$ is Higgsed to $U(1)^{(N-1)f}$. The construction of the vortex operator and the generator of Z_N in this case is very similar and the details are given in [15]. Taking the mass of the Higgs field to infinity again projects the generator onto the trace of the fundamental Wilson loop. The vortex operator can be taken as

$$
V(x) = \exp\{\frac{4\pi i}{gN} \int_C dy^i \epsilon_{ij} \mathrm{Tr}(YE_i(y)) \tag{32}
$$

where the hyper charge generator Y is defined as

$$
Y = \mathrm{diag}\,(1, 1, ..., -(N-1)) \tag{33}
$$

---

[f]In $SU(N)$ theories with $N > 2$ there in principle can be phases separated from each other due to spontaneous breaking of some global symmetries. For instance the $SU(3)$ gauge theory with adjoint matter has a phase with spontaneously broken charge conjugation invariance [22]. Still even in this phase the confining properties are the same as in the strongly coupled pure Yang-Mills theory, with the Wilson loop having an area law.

and the electric field is taken in the matrix notation $E_i = \lambda^a E_i^a$ with λ^a - the $SU(N)$ generator matrices in the fundamental representation.

To summarize this section, we have established two important facts. First, $SU(N)$ gauge theories in 2+1 dimensions have global Z_N magnetic symmetry. The generator of this magnetic symmetry group is the fundamental Wilson loop around the spatial boundary of the system. Second, this symmetry has a local order parameter. This order parameter is a local gauge invariant scalar field which creates a magnetic vortex of fundamental flux.

The next question we should ask is whether this global symmetry is at all relevant for low energy dynamics. In the next section we will show that this is indeed the case. We will calculate the expectation value of V in confining and nonconfining situations and will show that confinement is rigidly related to the spontaneous breaking of the magnetic symmetry.

3   The vacuum realization of the magnetic symmetry, the effective Lagrangian and confinement. Abelian theory.

We again start our discussion with Abelian theory. Consider a $U(1)$ gauge theory with scalar matter field

$$\mathcal{L} = -\frac{1}{4}F^2 + |D_\mu\phi|^2 - M^2|\phi|^2 - \lambda(\phi^*\phi)^2 \tag{34}$$

Depending on the values of the coupling constant this theory can be either in the Coulomb phase with massless photon and logarithmically confined charges, or in the Higgs phase which is massive with screened electric charges.

3.1   Realizations of the magnetic symmetry.

Let us start by calculating $< V >$ in the Coulomb phase. This can be done using the standard weak coupling perturbation theory. The expectation value of V is given by the following expression

$$< V(x) >= N^{-1} \lim_{T\to\infty} \int dA_0 < 0| e^{iTH} e^{\frac{2\pi i}{g} \int_C dy^i \epsilon_{ij} E_i(y)} e^{i \int A_0[\partial_i E_i - J_0]} e^{iTH} |0 > \tag{35}$$

Here N^{-1} is the normalization factor - the usual vacuum-to-vacuum amplitude, $|0 >$ is the perturbative Fock vacuum and the integral over A_0 is the standard representation of the projection operator which projects the Fock vacuum $|0 >$ onto the gauge invariant subspace which satisfies the Gauss' law. As usual, discretising time, introducing resolution of identity at every time slice and integrating over E_i in the phase space path integral this expression

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can be rewritten as path integral in the field space. The result is easy to understand - it is almost the same as for the vacuum-to-vacuum amplitude, except that at time $t = 0$ the spatial derivative of the scalar potential A_0 is shifted by the c-number field due to the presence of the vortex operator:

$$< V(x) >= N^{-1} \int \mathcal{D}A_\mu \exp\left[-\frac{1}{4}\int d^3y[\tilde{F}_\mu(y) - \tilde{f}_\mu(y-x)]^2 + L_{Higgs}\right] \tag{36}$$

The c-number field $\tilde{f}_\mu$ is the magnetic field of an infinitely thin magnetic vortex which terminates at point x. One can view it as the Dirac string of a (three dimensional Euclidean) magnetic monopole.

$$\tilde{f}_0 = \tilde{f}_2 = 0, \quad \tilde{f}_1(y) = \frac{2\pi}{g}\theta(y_1)\delta(y_2)\delta(y_3) \tag{37}$$

Thus at weak coupling we have to find the solution of the classical equations of motion following from the action with the external source Eq. 36. The nature of this solution is clear: it is just a Dirac monopole. The action of this solution is IR finite, since the contribution of the Dirac string (which normally would be linearly IR divergent) is canceled by the external source.

$$< V >= \exp\{-S_{cl}\} \tag{38}$$

with

$$S_{cl} = \frac{\Lambda}{g^2} \tag{39}$$

Here Λ is the ultraviolet cutoff which has to be introduced since the action of a pointlike monopole diverges in the ultraviolet. This ultraviolet divergence is benign since it can be eliminated by the multiplicative renormalization of the vortex operator [19]. The important point is that since there is no divergence in the infrared, the expectation value of V is nonvanishing. Thus we conclude that in the Coulomb phase of QED the magnetic symmetry is spontaneously broken.

The spontaneous breaking of a continuous symmetry must be accompanied by appearance of a massless Goldstone particle. Indeed in QED such a particle exists - it is the massless photon. The matrix element of the magnetic current between the vacuum and the one photon state is

$$< 0|\tilde{F}_\mu|k_i >= Z^{1/2}(0)k_\mu \tag{40}$$

where $Z(0)$ is the on shell photon wave function renormalization. This is the standard form of a matrix element of a spontaneously broken current, with $Z(0)$ playing the role of f_π^2.

Let us now perform the same calculation in the Higgs phase. The path integral representation Eq. 36 is still valid. However the classical solution that dominates this path integral is now very different. Since in the Higgs phase the photon has nonzero mass μ the classical action of the three dimensional monopole in the superconducting medium is linearly divergent in the infrared [2]. Essentially the magnetic flux that emanates from the monopole can not spread out in space(time) but rather is concentrated inside a flux tube of the thickness $1/\mu$ that starts at the location of the monopole and goes to the spatial boundary at infinity. The action of such a field configuration is proportional to the linear size of the system and diverges in the thermodynamic limit. As a result the expectation value of V in the Higgs phase vanishes.

$$< V >= e^{-L} \to_{L \to \infty} 0 \tag{41}$$

The vortex field correlator similarly is given in terms of the classical energy of a monopole-antimonopole pair in the superconductor. The Euclidean action of this configuration is proportional to the distance between the monopole and the antimonopole and so the correlator of the V decays exponentially

$$< V^*(x)V(y) >\sim e^{-M_V |x-y|} \tag{42}$$

with $M_V \propto 1/g^2$ being the mass of the Nielsen-Olesen vortex.

This simple calculation can be improved perturbatively. In the next to leading order one has to calculate the determinant of the Schrödinger operator of a particle in the field of a monopole and this corrects the value of the mass M_V. Higher orders in perturbation theory can be calculated, but we will not pursue this calculation here.

The main lesson is that the expectation value of the order parameter vanishes in the Higgs phase, and thus the magnetic symmetry is unbroken.

3.2 The low energy effective Lagrangian and the logarithmic confinement of electric charges.

Thus we see that the Coulomb - Higgs phase transition can be described as due to restoration of the magnetic $U(1)$ symmetry, the pertinent local order parameter being the vortex operator V. It should be true then that the low energy dynamics in the vicinity of the phase transition is described by the effective low energy Lagrangian. For the $U(1)$ symmetry breaking such an effective Lagrangian can be immediately written down

$$\mathcal{L} = \partial_\mu V^* \partial^\mu V - \lambda(V^*V - \mu^2)^2 \tag{43}$$

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Although this Lagrangian may seem a bit unfamiliar in the context of QED, a little thought convinces one that it indeed describes all relevant light degrees of freedom of the theory. In the Coulomb phase, where $<V>= \mu \neq 0$, the physical particles are interpolated by the phase and the radial part of V

$$V(x) = \rho e^{i\chi} \tag{44}$$

The phase χ is of course the massless Goldstone boson field, i.e. the photon. The fluctuation of the radial component $\rho - \mu$, is the lightest scalar particle, which in this case is the lightest meson, or scalar positronium. In the Higgs phase, the field V itself interpolates physical excitations - the ANO vortices. Of course far from the phase transition the vortices are heavy and there are other, lighter excitations in the spectrum. The validity of this effective Lagrangian on the Higgs phase side is therefore limited to the narrow critical region where vortices are indeed the lightest particles.

One type of objects that have not appeared in our discussion so far are charged particles. Indeed in 2+1 dimensions electrical charges are confined, and therefore we do not expect the charged fields to appear as basic degrees of freedom in the effective low energy Lagrangian. However our original purpose was precisely to understand the mechanism of confinement through studying the effective Lagrangian. Thus if we are unable to identify the charged objects in this framework our program is doomed to failure. Fortunately it is not difficult to understand how charged states are represented in the Lagrangian Eq. 43. The easiest way to do this is to identify the electric charge through the Maxwell equation

$$J_\mu = \frac{1}{4}\epsilon_{\mu\nu\lambda}\partial_\nu \tilde{F}_\lambda \tag{45}$$

The dual field strength $\tilde{F}_\mu$ is obviously proportional to the conserved $U(1)$ current

$$\tilde{F}_\mu = \frac{2\pi}{g}i(V^*\partial V - h.c.) \tag{46}$$

The proportionality constant in this relation is dictated by the fact that in the Higgs phase the magnetic vortices that carry one unit of the $U(1)$ charge, carry the magnetic flux of $2\pi/g$. The above two relations give

$$\frac{g}{\pi}J_\mu = i\epsilon_{\mu\nu\lambda}\partial_\nu(V^*\partial_\lambda V) \tag{47}$$

To calculate the electric charge we integrate the zeroth component of the current over the two dimensional plane

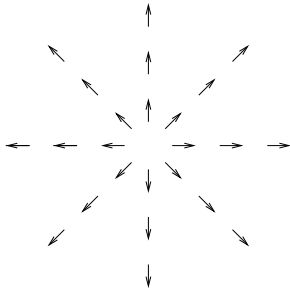$$\frac{g}{\pi}Q = \mu^2 \oint_{C\to\infty} dx_i \partial_i \chi \tag{48}$$

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Figure 1: The hedgehog configuration of the field V in the state of unit charge in an Abelian theory.

The electric charge is therefore proportional to the winding number of the phase of the field V.

So the charged states do appear in the low energy description in a very natural way. A charged state is a soliton of V with a nonzero winding number.

This identification tells us immediately that the charged particles are logarithmically confined. Consider for example the minimal energy configuration in the sector with the unit winding number. This is a rotationally invariant hedgehog, Fig. 1, which far from the soliton core has the form

$$V(x) = \mu e^{i\theta(x)}. \tag{49}$$

Here $\theta(x)$ is an angle between the vector x and one of the axes.

The self energy of this configuration is logarithmically divergent in the infrared due to the contribution of the kinetic term

$$E = \pi\mu^2 \ln(\lambda\mu^2 L^2) \tag{50}$$

This is nothing but the electromagnetic self energy of an electrically charged state associated with the logarithmic Coulomb potential in two spatial dimensions. The logarithmic confinement of QED is therefore indeed very easily and transparently seen on the level of the low energy effective Lagrangian. This in itself perhaps is not such big a deal, since confinement in this model is a linear phenomenon: it is the direct consequence of the logarithmic behavior of Coulomb potential. We will see later however that this low energy picture generalizes naturally also to non Abelian theories and easily accommodates linear confinement.

But before moving on to non Abelian theories, let me make two comments. First, that in perturbative regime the couplings of the effective Lagrangian can

be determined in terms of the couplings of the fundamental QED Lagrangian. To determine the two constants in Eq. 43 one needs two matching conditions. One of them can be naturally taken as the coefficient of the infrared logarithm in the self energy of a charged state. Matching Eq. 50 with the Coulomb self energy gives

$$\mu^2 = \frac{g^2}{8\pi^2} \tag{51}$$

The other coefficient is determined by requiring that the mass of the radial excitation ρ matches the mass of the scalar positronium, which to leading order in g is just $2M$. This condition gives

$$\lambda = \frac{2\pi^2 M}{g^2} \tag{52}$$

The second comment is about the relation between the vortex operator as defined in Eq. 17 and the field V that enters the effective Lagrangian Eq. 43.

The vortex operator as defined in Eq. 17 has a fixed length whereas the field V which enters the Lagrangian Eq. 43 is a conventional complex field. How should one understand that? First of all at weak gauge coupling the quartic coupling in the dual Lagrangian is large $\lambda \to \infty$. This condition freezes the radius of V dynamically. In fact even at finite value of

λ if one is interested in the low energy physics, the radial component is irrelevant as long as it is much heavier than the phase. Indeed at weak gauge coupling the phase of V which interpolates the photon is much lighter than all the other excitations in the theory. Effectively therefore at low energies Eq. 43 reduces to a nonlinear sigma model and one can identify the field V entering Eq. 43 directly with the vortex operator of Eq. 17. However it is well known that quantum mechanically the radial degree of freedom of a sigma model field is always resurrected. The spectrum of such a theory always contains a scalar particle which can be combined with the phase into a variable length complex field. The question is only quantitative - how heavy is this scalar field relative to the phase.

Another way of expressing this is the following. The fixed length field V is defined at the scale of the UV cutoff in the original theory. To arrive at the low energy effective Lagrangian one has to integrate over all quantum fluctuations down to some much lower energy scale. In the process of this integrating out the field is "renormalized" and it acquires a dynamical radial part. The mass of this radial part then is just equal to the mass of the lowest particle with the same quantum numbers in the original theory. This is in fact why the parameters in Eq. 43 must be such that the mass of the radial part of V is equal to the mass of the lightest scalar positronium.

4 Non Abelian theories.

In this section I want to extend the construction of low energy effective Lagrangian to non Abelian theories and see how the realization of magnetic symmetry in this Lagrangian is related to confinement. Before discussing in detail specific models let me present a general argument that establishes that in a non Abelian theory spontaneous breaking of magnetic Z_N implies the area law behavior of the fundamental Wilson loop.

4.1 Broken Z_N means confinement.

As we have discussed in the previous section, the generator of magnetic Z_N in pure gluodynamics is the fundamental Wilson loop around the spatial boundary of the system. By the same token the Wilson loop around a closed spatial contour C generates the Z_N transformation at points inside the contour C. Let us imagine that the Z_N symmetry is spontaneously broken in the vacuum and consider in such a state $|0>$ the expectation value of $W(C)$. The expectation value $<0|W(C)|0>$ is nothing but the overlap of the vacuum state $<0|$ and the state $|S>$ which is obtained by acting with $W(C)$ on the vacuum: $|S>=W(C)|0>$. If the symmetry is broken, the wave function $|0>$ depends explicitly on the degrees of freedom which are non invariant under the symmetry transformation, and is peaked around some specific orientation of these variables in the group space. For simplicity let us think about all these non invariant variables as being represented by the vortex field V. The field V in the vacuum state has nonvanishing VEV and is pointing in some fixed direction in the internal space. In the state $|S>$ on the other hand its direction in the internal space is different - rotated by $2\pi/N$ - at points inside the area S bounded by C, since at these points the field V has been rotated by the action of $W(C)$. In the local theory with finite correlation length the overlap between the two states approximately factorizes into the product of the overlaps taken over the regions of space of linear dimension of order of the correlation length l

$$< 0|S >= \Pi_x < 0_x|S_x >$$ (53)

where the label x is the coordinate of the point in the center of a given small region of space. For x outside the area S the two states $|0_x>$ and $|S_x>$ are identical and therefore the overlap is unity. However for x inside S the states are different and the overlap is therefore some number $e^{-\gamma}$ smaller than unity. The number of such regions inside the area is obviously of order S/l^2 and thus

$$< W(C) >= \exp\{-\gamma\frac{S}{l^2}\}$$ (54)

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In the broken phase the spatial Wilson loop therefore has an area law behavior.

Now consider the unbroken phase. Again the average of $W(C)$ has the form of the overlap of two states which factorizes as in Eq. 53. Now however all observables non invariant under Z_N vanish in the vacuum. The action of the symmetry generator does not affect the state $|0>$. The state $|S>$ is therefore locally exactly the same as the state $|0>$ except along the boundary C. Therefore the only regions of space which contribute to the overlap are those which lay within one correlation length from the boundary. Thus

$$< W(C) >= \exp\{-\gamma P(C)\} \tag{55}$$

where $P(C)$ is the perimeter of the curve C.

The crucial requirement for this argument to hold is the existence of a mass gap in the theory. If the theory contains massless excitations the factorization of the overlap does not hold, and so in principle even in the broken phase the Wilson loop can have perimeter behavior. This indeed is the case in the Abelian theories.

We now turn to the discussion of low energy effective theories. This will allow us to see the explicit realization of this general argument.

4.2 The Georgi-Glashow model.

Let us again start with the Georgi-Glashow model. For simplicity all explicit calculations in this section will be performed for the $SU(2)$ gauge theory. Generalization to the $SU(N)$ group is not difficult and is discussed in [15].

Not much has to be done here to parallel the calculation of $<V>$ of the previous section. The theory is weakly interacting, and all calculations are explicit. Choosing unitary gauge $n^a = \delta^{a3}$ the perturbative calculation becomes essentially identical to that in the Coulomb phase of QED. The only difference is that the charged matter fields are vectors $(W_\mu^\pm)$ rather than scalars (ϕ), but this only enters at the level of the loop corrections. The nonperturbative monopole contributions are there, but they affect the value of $<V>$ very little, since $<V>\neq 0$ already in perturbation theory. Thus just like in the Coulomb phase of QED, $<V>\neq 0$ and the magnetic symmetry is spontaneously broken. The real difference comes only when we ask what is the effective Lagrangian that describes the low energy physics. Here the monopole contributions are crucial, since as we have seen before the $U(1)$ magnetic symmetry of QED is explicitly (anomalously) broken by these contributions to Z_2. The effective Lagrangian therefore must have an extra terms which reduce the symmetry of the Eq. 43. The relevant effective Lagrangian is

$$\mathcal{L} = \partial_\mu V^* \partial^\mu V - \lambda (V^* V - \mu^2)^2 + \zeta (V^2 + (V^*)^2) \tag{56}$$

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The addition of this extra symmetry breaking term has an immediate effect on the mass of the "would be" photon - the phase of V. Expanding around $< V >= \mu$ we see that the phase field now has a mass $m_{ph}^2 = 4\zeta$. This is consistent with the classical analysis by Polyakov [17]- the monopole contributions turn the massless photon of QED into a massive (pseudo)scalar with the exponentially small mass $m_{ph} \propto \exp\{-M_W/g^2\}$. As a matter of fact for very weak coupling, when the modulus of V can be considered as frozen, the Lagrangian Eq. 56 in terms of the phase χ reduces to Polyakov's dual Lagrangian. The exact correspondence between the two is discussed in [15].

The explicit symmetry breaking causes a dramatic change in the topologically charged (soliton) sector. We know from Polyakov's analysis that the charges in this model are confined by linear potential. This as opposed to QED where confinement is logarithmic. In the effective Lagrangian description this is due to the explicit symmetry breaking term. The crucial point is that the vacuum of the theory is not infinitely degenerate $< V >= e^{i\chi}\mu$ with arbitrary constant χ, as in the case of QED but only doubly degenerate $< V >= \pm\mu$. Thus the lowest energy state in the nontrivial winding sector can not be a hedgehog. In a hedgehog configuration the field V at each point in spatial infinity points in a different direction in the internal space. This is OK if all these directions are minima of the potential. Then the total energy of the configuration comes from the kinetic term, and as we saw is logarithmic. However now the potential has only two minima. Thus the hedgehog field is far from the vacuum everywhere in space. The energy of such a state therefore diverges as the volume of the system: $E \propto g^2 m^2 L^2$. Clearly to minimize the energy in a state with a nonzero winding, the system must be as in one of the two vacuum states in as large a region of space as possible. However since the field has to wind when one goes around the position of a soliton even at arbitrarily large distance, V can not be aligned with the vacuum everywhere at infinity. The best bet for a system is therefore to choose a string like configuration Fig. 2. The phase of $V(x)$ deviates from 0 (or π) only inside a strip of width $d \sim 1/m$ stretching from the location of the soliton (charge) to infinity. The energy of such a configuration diverges only linearly with the dimension L. In fact a back-of-the-envelope estimate with the effective Lagrangian Eq. 56 gives the energy of such a confining string as $E \propto g^2 m_{ph} L$. Clearly the energy of a soliton and an anti soliton separated by a large distance R is $E = \sigma R$ with the string tension $\sigma \propto g^2 m_{ph}$.

This is the simple picture of confinement in the effective Lagrangian approach in the weakly coupled regime.

The preceding discussion pertains to confinement of "adjoint" color charges. So far we have been considering topological solitons with unit winding, which
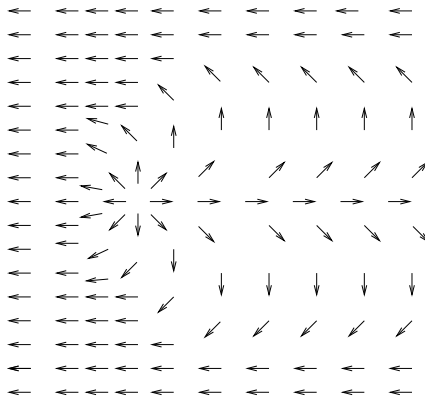
Figure 2: The string like configuration of the field V in the state of unit charge in the presence of the symmetry breaking terms in the effective Lagrangian.

corresponds to the charge of the massive $W^{\pm}$ bosons, or "massive gluons". It should be noted that the notion of adjoint string tension is not an absolute one. Our discussion so far neglected the fact that the solitons have a finite core energy and therefore in principle can be created in pairs from the vacuum. Thus the soliton - anti soliton interaction at distance R can be screened by creating such a pair, if R is big enough. The distance at which the string breaking occurs can be estimated from the energy balance between the energy stored in the string $E_S = \sigma_{Adj} R$ and the core energy of the soliton-anti soliton pair, which in our model is twice the mass of the W-boson $2M_W$.

$$g^2 m_{ph} R = 2M_W \tag{57}$$

The distance at which the string breaks is therefore

$$R_{breaking} \propto \frac{M_W}{g^2} \frac{1}{M_{ph}} \tag{58}$$

Since the width of the string is of order $1/m_{ph}$ and in the weak coupling $M_W \gg g^2$, the length of the string is indeed much greater than its width. One can therefore sensibly talk about a well formed adjoint confining string.

As opposed to adjoint string tension, the concept of the fundamental string tension is sharply defined even in principle. This is because the theory does not contain particles with fundamental charge and thus an external fundamental charge can not be screened. To discuss confinement of external fundamental charges we have to learn how to deal with half integer windings. Imagine adding

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to the Georgi-Glashow model some extra very heavy fields in the fundamental representation. The quanta of these fields will carry half integer "electric" charge Eq. 48 and will be confined with a different string tension than $W^{\pm}$. To calculate this string tension we should consider the Abelian Wilson loop with half integer charge. We will now do it in the effective theory framework.

Let us first consider a space like Wilson loop. As discussed in the previous section, this operator is closely related to the generator of the magnetic Z_2. In fact $W(C)$ is nothing but the operator that performs the Z_2 transformation inside the area bounded by the contour C.

It is straightforward to write down an operator in the effective theory in terms of the field V that has the same property

$$W(C) = e^{i\pi \int_S d^2x P(x)} \tag{59}$$

Here S is the surface bounded by the contour C, and P is the operator of momentum conjugate to the phase of V. In terms of the radius and the phase of V the path integral representation for calculating the vacuum average of this Wilson loop is

$$< W(C) > = \int DV \exp\left\{ i \int d^3x \rho^2 (\partial_\mu \chi - j^S_\mu)^2 + (\partial_\mu \rho)^2 - U(V) \right\} \tag{60}$$

where $U(V)$ is the Z_2 invariant potential of Eq. 56. The external current $j^S_\mu(x)$ does not vanish only at points x which belong to the surface S and is proportional to the unit normal n_μ to the surface S. Its magnitude is such that when integrated in the direction of n it is equal to π. These properties are conveniently encoded in the following expression

$$\int_T dx_\mu j^S_\mu(x) = \pi n(T, C) \tag{61}$$

Here T is an arbitrary closed contour, and $n(T, C)$ is the linking number between two closed curves T and C^g.

This path integral representation follows immediately if we notice that the conjugate momentum P is $P(x) = 2\rho^2(x)\partial_0\chi(x)$ at some fixed time t. This accounts for the linear in $\partial_\mu \chi$ in the exponential Eq. 60. The constant term j^2 arises due to standard integration over the conjugate momenta in arriving to the path integral representation.

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^{g}Note that here we are dealing with the path integral representation, and thus the contour C and the surface S are embedded into a three dimensional Euclidean space. The linking number between two curves is also defined in three dimensions.
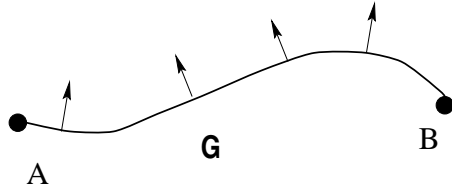
Figure 3: The external current j_μ which creates the pair of static fundamental charges in the effective Lagrangian description.

The path integral representation was constructed for the spatial Wilson loops. However the expression Eq. 60 is completely covariant, and in this form is valid for time like Wilson loops as well. It is important to note that although the expression for the current depends on the surface S, the Wilson loop operator in fact depends only on the contour C that bounds this surface. A simple way to see this is to observe that a change of variables $\chi \rightarrow \chi + \pi$ in the volume bounded by $S + S'$ leads to the change $j_\mu^S \rightarrow j_\mu^{S'}$ in eq.(60). The potential is not affected by this change since it is globally Z_2 invariant. Therefore the operators defined with S and S' are completely equivalent.

To calculate the energy of a pair of static fundamental charges at points A and B we have to consider a time like fundamental Wilson loop of infinite time dimension. This corresponds to time independent j_μ which does not vanish only along a spatial curve G (in equal time cross section) connecting the two points and pointing in the direction normal to this curve Fig.3. The shape of the curve itself does not matter, since changing the curve without changing its endpoints is equivalent to changing the surface S in Eq. 60.

In the classical approximation the path integral Eq. 60 is dominated by a static configuration of V. To determine it we have to minimize the energy on static configurations in the presence of the external current j_μ. The qualitative features of the minimal energy solution are quite clear. The effect of the external current is to flip the phase of V by π across the curve G, as is expressed in Eq. 61. Any configuration that does not have this behavior will have the energy proportional to the length of G and to the UV cutoff scale. Recall that the vacuum in the theory is doubly degenerate. The sign change of V transforms one vacuum configuration into the other one. The presence of j_μ therefore requires that on opposite sides of the curve G, immediately adjacent to G there should be different vacuum states. It is clear however that far away from G in either direction the field should approach the same vacuum state, otherwise the energy of a configuration diverges linearly in the infrared. The phase of V therefore has to make half a wind somewhere in space to return to
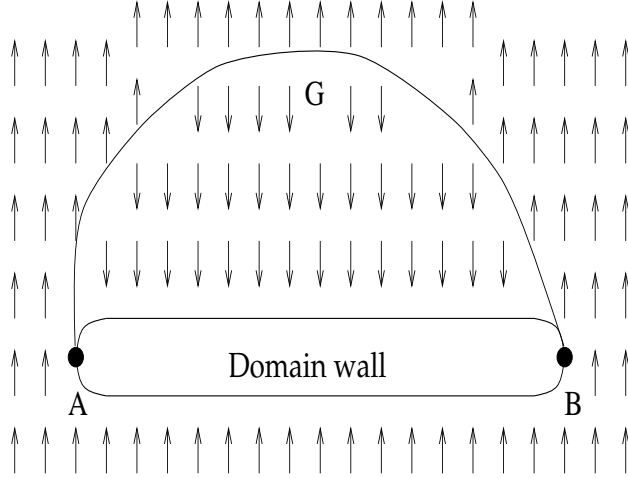
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Figure 4: The minimal energy configuration of V in the presence of a pair of fundamental charges.

the same vacuum state far below G as the one that exists far above G. If the distance between A and B is much larger than the mass of the lightest particle in the theory, this is achieved by having a segment of a domain wall between the two vacua connecting the points A and B. Clearly to minimize the energy the domain wall must connect A and B along a straight line. The energy of such a domain wall is proportional to its length, and therefore the Wilson loop has an area law behavior. The minimal energy solution is schematically depicted on Fig. 4. We see that the string tension for the fundamental string is equal to the tension of the domain wall which separates the two vacua in the theory. This relation has been discussed a long time ago by 't Hooft [12]. Parametrically this string tension is clearly the same as the adjoint one,

$$\sigma_f \propto g^2 m_{ph} \tag{62}$$

although the proportionality constant is different. We will briefly discuss the relation between the adjoint and the fundamental string tensions in the next subsection.

Note that the fundamental string is an absolutely stable topological object in the Z_2 invariant theory: the domain wall. It can not break, if one makes the distance between the two charges larger. In the effective theory it is also obvious since there is no pointlike (particle like) object in the theory on which a domain wall can terminate since there are no dynamical objects with half an

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integer winding number.

We interpreted this calculation as the calculation of the potential between the fundamental adjoint charges - the time like Wilson loop. However in the Euclidean formulation there is no difference between time like and space like Wilson loops. Interpreted in this way the calculation becomes the technical illustration to the argument given in the previous subsection: space like Wilson loop has an area law if the magnetic Z_2 is spontaneously broke.

4.3 Gluodynamics.

In the weakly interacting case the effective low energy Lagrangian can be derived as explained above in perturbation theory plus dilute monopole gas approximation. The more interesting regime is of course that of strong coupling, which is essentially the pure Yang-Mills theory. The luxury one has in 2+1 dimensions is that the weak and the strong coupling regimes are not separated by a phase transition [16]. This means that whatever global symmetries the theory has, their realization must be the same in the weakly coupled and the strongly coupled vacua. The existence of the Z_N symmetry is an exact statement which is not related to the weak coupling limit. It is therefore natural to expect that this symmetry must be nontrivially represented in the effective low energy Lagrangian. It is plausible then that the low energy dynamics at strong coupling is described by the same effective Lagrangian which encodes spontaneous breaking of the magnetic symmetry. The values of the coupling constants will be of course different in the two regimes, but the qualitative behavior should be similar.

Strictly speaking this is an assumption and not a theorem. That is where the difference between continuous and discreet symmetries comes in. If the magnetic symmetry were continuous (like in the Abelian case) its spontaneous breaking would unambiguously determine the structure of the effective Lagrangian, be the theory weakly or strongly coupled. With discrete symmetry this is not necessarily the case. It could happen that even though the symmetry is broken, the "pseudo Goldstone" particle is so heavy that it decouples from the low energy dynamics. For this to happen though, the symmetry breaking would have to occur on a very high energy scale. In gluodynamics this is very unlikely, since the theory has only one dynamical scale. In fact as we have seen in the beginning of this section, the fundamental string tension determines the scale at which Z_N is broken. Wilson loops of linear size $l \leq (\sigma)^{-1/2}$ do not distinguish between confining and nonconfining behavior, and thus between the broken and the unbroken Z_N. The scale of Z_N breaking therefore is $(\sigma)^{1/2}$ which is precisely the natural dynamical scale of QCD.

Generically therefore we expect that the pseudo Goldstone stays among the low energy excitations not only in weakly coupled limit but also in pure gluodynamics. If this is the case the degrees of freedom that enter the effective Lagrangian in weakly coupled phase also interpolate real low energy physical states of the strong coupling regime. That is to say the radial and phase components of the vortex field V must correspond to lightest glueballs of pure $SU(N)$ Yang Mills theory. We can check whether this is the case by considering the lattice gauge theory data on the spectrum [21]. The radial part of V is obviously a scalar and has quantum numbers 0^{++}. The quantum numbers of the phase are easily determined from the definition Eq. 17. Those are 0^{--}. The spectrum of pure $SU(N)$ Yang Mills theory in 2+1 dimensions was extensively studied recently on the lattice [21]. The two lightest glueballs for any N are found to have exactly those quantum numbers. The lightest excitation is the scalar while the next one is a charge conjugation odd pseudo scalar with the ratio of the masses roughly $m_p/m_s = 1.5$ for any N^h.

The situation therefore likely is the following. The low energy physics of the $SU(2)$ gauge theory is always described by the effective Lagrangian Eq. 56. In the weak coupling regime the parameters are given in Eqs. 51,52. Here the pseudo scalar particle is the lightest state in the spectrum and the scalar is the first excitation. The pseudo scalar is the almost massless "photon" and the scalar is the massive Higgs particle. Moving towards the strong coupling regime (decreasing the Higgs VEV in the language of the Georgi-Glashow model) leads to increasing the pseudo scalar mass while reducing the scalar mass and the parameters of the effective Lagrangian change accordingly. The crossover between the weak and the strong coupling regimes occurs roughly where the scalar and the pseudo scalar become degenerate. At strong coupling the degrees of freedom in the effective Lagrangian are the two lightest glueballs. They are however still collected in one complex field which represents nontrivially the exact Z_2 symmetry of the theory (or Z_N for $SU(N)$).

Of course the spectrum of pure Yang Mills theory apart from the scalar and the pseudo scalar glueballs contains many other massive glueball states and those are not separated by a large gap from the two lowest ones. Application of the effective Lagrangian in the strong coupling regime therefore has to be taken in a qualitative sense. On this qualitative level though, as we have seen linear confinement is an immediate property of this type of effective Lagrangian.

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^{h}Actually this state of matters is firmly established only for $N > 2$. At $N = 2$ the mass of the pseudo scalar has not been calculated in [21]. The reason is that it is not clear how to construct a charge conjugation odd operator in a pure gauge $SU(2)$ lattice theory. So it is possible that the situation at $N = 2$ is non generic in this respect. In this case our strong coupling picture should apply at $N > 2$.
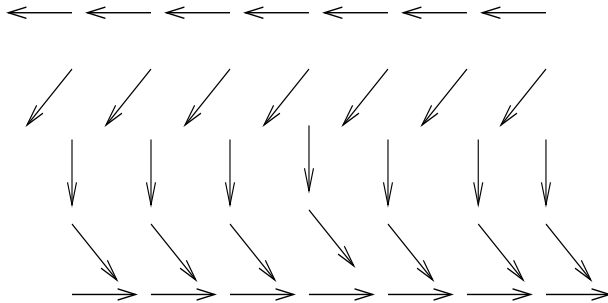
Figure 5: The structure of the string (domain wall) in the regime when the pseudo scalar is lighter than the scalar, $m < M$.

The fact that masses of the scalar and the pseudo scalar particles are interchanged in gluodynamics relative to the weakly coupled regime leads to some interesting qualitative differences [20]. In particular the structure of the confining string and the interaction between the strings differ in some important ways. Let me briefly discuss those.

In the weakly coupled regime the phase of V is much lighter than the radial part. A cartoon of the fundamental string in this situation is depicted in Fig. 5. The radial part ρ being very heavy practically does not change inside the string. The value of ρ in the middle of the string can be estimated from the following simple argument. The width of region where ρ varies from its vacuum value μ to the value ρ_0 in the middle is of the order of the inverse mass of ρ. The energy per unit length that this variation costs is

$$\sigma_\rho \sim M(\mu - \rho_0)^2 + x\frac{m^2}{M}\rho_0^2 \tag{63}$$

where x is a dimensionless number of order unity. The first term is the contribution of the kinetic term of ρ and the second contribution comes from the interaction term between ρ and χ due to the fact that the value of χ in the middle of the string differs from its vacuum value. Our notations are such that m is the mass of the pseudo scalar particle and M is the mass of the scalar. Minimizing this with respect to ρ_0 we find

$$\rho_0 = \mu(1 - x\frac{m^2}{M^2}) \tag{64}$$

Thus even in the middle of the string the difference in the value of ρ and its VEV is second order in the small ratio m/M. Correspondingly the contribution

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of the energy density of ρ to the total energy density is also very small.

$$\sigma_\rho \sim \frac{m}{M} m\mu^2 \tag{65}$$

This is to be compared with the total tension of the string which is contributed mainly by the pseudo scalar phase χ

$$\sigma_\chi \sim m\mu^2 \tag{66}$$

This again we obtain by estimating the kinetic energy of χ on a configuration of width $1/m$ where χ changes by an amount of order 1^i.

Let us now consider the domain wall (or fundamental string) in the opposite regime, that is when the mass of the scalar is much smaller than the mass of the pseudo scalar. The profile of the fields in the wall now is very different. The cartoon of this situation is given on Fig. 6 We will use the same notations, denoting the mass of the pseudo scalar by m and the mass of the scalar by M, but now $m \gg M$. Let us again estimate the string tension and the contributions of the scalar and a pseudo scalar to it. The width of the region in which the variation of ρ takes place is of the order of its inverse mass. The estimate of the energy density of the ρ field is given by the contribution of the kinetic term

$$\sigma_\rho \sim M(\mu - \rho_0)^2 \tag{68}$$

The width of the region in which the phase χ varies is $\sim 1/m$. In this narrow strip the radial field ρ is practically constant and is equal to ρ_0. The kinetic energy of χ therefore contributes

$$\sigma_\chi \sim m\rho_0^2 \tag{69}$$

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[i] The fact that the heavy radial field ρ practically does not contribute to the string tension is natural from the point of view of decoupling. In the limit of infinite mass ρ should decouple from the theory without changing its physical properties. It is however very different from the situation in superconductors. In a superconductor of the second kind, where the order parameter field is much heavier than the photon ($\kappa > \frac{1}{\sqrt{2}}$) the magnetic field and the order parameter give contributions of the same order (up to logarithmic corrections $O(\log \kappa)$) to the energy of the Abrikosov vortex. This is the consequence of the fact that the order parameter itself is forced to vanish in the core of the vortex, and therefore even though it is heavy, its variation inside the vortex is large. An even more spectacular situation arises if we consider a domain wall between two vacuum states in which the heavy field has different values [23]. In this situation the contribution of the heavy field ϕ to the tension would be

$$\sigma_{heavy} = M(\Delta\phi)^2 \tag{67}$$

where $\Delta\phi$ is the difference in the values of ϕ on both sides of the wall. For fixed $\Delta\phi$ the energy density diverges when ϕ becomes heavy. In the present case this does not happen since the two vacua which are separated by the domain wall differ only in VEV of the light field χ and not the heavy field ρ.
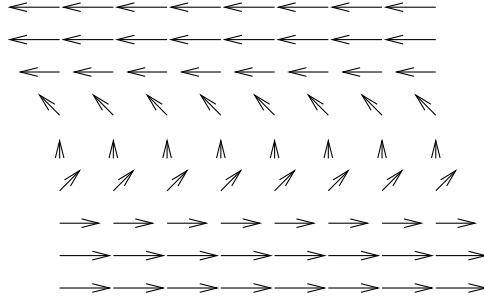
Figure 6: The structure of the string (domain wall) in the regime when the pseudo scalar is heavier than the scalar, $m > M$.

Minimizing the sum of the two contributions with respect to ρ_0 we find

$$\rho_0 \sim \frac{M}{m}\mu \ll \mu \tag{70}$$

And also

$$\begin{aligned} \sigma_\chi &\sim \frac{M}{m}M\mu^2 \\ \sigma_F &= \sigma_\rho \sim M\mu^2 \end{aligned} \tag{71}$$

Now the radial field is very small in the core of the string. The energy density is contributed almost entirely by the scalar rather than by the pseudo scalar field[j].

The extreme situation $m \gg M$ is not realized in the non Abelian gauge theory. From the lattice simulations we know that in reality even in the pure Yang Mills case the ratio between the pseudo scalar and scalar masses is about 1.5 - not a very large number. The analysis of the previous paragraph therefore does not reflect the situation in the strongly coupled regime of the theory. Rather we expect that the actual profile of the string is somewhere in between Fig. 5 and Fig. 6 although somewhat closer to Fig. 6. The widths of the string in terms of the scalar and pseudo scalar fields are of the same order, although the scalar component is somewhat wider. The same goes for the contribution to the string tension. Both glueballs contribute, with the scalar contribution being somewhat larger[k].

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[j]Again this is in agreement with decoupling. The heavier field does not contribute to the energy, even though its values on the opposite sides of the wall differ by $O(1)$. Its contribution to the energy is suppressed by the factor ρ_0^2 which is very small inside the wall.

[k]One expects similar non negligible contributions also from higher mass glueballs which are not taken into account in our effective Lagrangian framework.

The interaction between two domain walls (confining fundamental strings) in the two extreme regimes is also quite different. In the weakly coupled region we can disregard the variation of ρ. For two widely separated parallel strings the interaction energy comes from the kinetic term of χ. This obviously leads to repulsion, since for both strings in the interaction region the derivative of the phase is positive. On the other hand if the pseudo scalar is very heavy the main interaction at large separation is through the "exchange" of the scalar. This interaction is clearly attractive, since if the strings overlap, the region of space where ρ is different from its value in the vacuum is reduced relative to the situation when the strings are far apart.

The situation is therefore very similar to that in superconductivity. The confining strings in the weakly coupled and strongly coupled regimes behave like Abrikosov vortices in the superconductor of the second and first kind respectively. This observation has an immediate implication for the string tension of the adjoint string. As we have discussed in the previous section the phase χ changes from 0 to 2π inside the adjoint string. The adjoint string therefore can be pictured as two fundamental strings running parallel to each other. In the weak coupling regime the two fundamental strings repel each other. The two fundamental strings within the adjoint string therefore will not overlap, and the energy of the adjoint string is twice the energy of the fundamental one.

$$\sigma_{Adj} = 2\sigma_F \qquad (72)$$

In the strongly coupled region the situation is quite different. The strings attract. It is clear that the contribution of ρ to the energy will be minimized if the strings overlap completely. In that case the contribution of ρ in the fundamental and adjoint strings will be roughly the same. There will still be repulsion between the pseudo scalar cores of the two fundamental strings, so presumably the core energy will be doubled. We therefore have an estimate:

$$\sigma_{Adj} = \sigma_F + O(\frac{M}{m}\sigma_F). \qquad (73)$$

Again in the pure Yang - Mills limit the situation is more complicated. The scalar is lighter, therefore the interaction at large distances is attractive. However the pseudo scalar core size and its contribution to the tension is not small. In other words M/m in Eq. 73 is a number of order one.

Thus broadly the relation between the weakly and strongly coupled confining theories is similar to the relation of the superconductors of the second and first kind.

5 Magnetic Z_N in 3+1 dimensions.

The discussion in the previous three sections was in the framework of 2+1 dimensional theories. The reason for this is that magnetic symmetry in 2+1 dimensions is simple. The objects that carry the magnetic charge are particles which are interpolated by local scalar fields, and they are natural building blocks for the effective theory.

In 3+1 dimensions the situation is a little more complicated. The objects that carry magnetic symmetry are strings rather than particles. Still many elements of the 2+1 dimensional discussion can be generalized.

5.1 Magnetic symmetry generator and the vortex operator in 3+1 dimensions.

First of all, 't Hooft's topological argument carries over to 3+1 dimensions immediately. However here it establishes the existence of topologically stable string like objects rather than particles. The gauge transformation $U(x)$ winds once in the group space when we go around any closed curve which encloses the core of such a fluxon.

The symmetry that guarantees stability of these objects is again associated with the magnetic field. In Abelian theories the symmetry is continuous and the relevant conserved current this time is the dual field strength tensor

$$\partial_\mu \tilde{F}_{\mu\nu} = 0 \tag{74}$$

The somewhat unusual property of this current is that it is a tensor and not a vector. Therefore if we want to define a conserved Lorentz scalar charge associated with it, we should integrate the "charge density" not over all the three dimensional space, but rather over a two dimensional space like surface. The conserved charge then is the magnetic flux through such an infinite surface, whose boundary is at infinity

$$\Phi = \int d^2 S_i B_i \tag{75}$$

In an Abelian theory without monopoles this charge does not depend on the integration surface as long as the boundary of the surface remains fixed. The symmetry group element as in 2+1 dimensions is large Wilson loop

$$W_\alpha = \exp\{i\alpha g \oint_{C' \to \infty} dl_i A_i\} \tag{76}$$

The objects that carry this charge are obviously ANO vortices in the Higgs (superconducting) phase.

The operator that creates a closed magnetic vortex along a curve C is constructed as

$$V(C) = \exp\{\frac{2\pi i}{g} \int_S d^2 S^i E^i\} \tag{77}$$

where the integration goes over the area S bounded by the contour C. In this definition g is the smallest electric charge present in the theory. This ensures independence of V on the choice of S, as long as its boundary is C. The argument here is exactly the same as the one that established independence of V in 2+1 dimensions of the choice of the cut in the angle function. With this definition of the 't Hooft loop operator $V(C)$ we can easily calculate its commutator with the spatial Wilson loop, the so-called 't Hooft algebra [12]

$$V^\dagger(C)W_\alpha(C')V(C) = e^{2\pi\alpha in(C,C')}W_\alpha(C')$$
$$W_\alpha(C')V(C)W_\alpha^\dagger(C') = e^{2\pi\alpha in(C,C')}V(C) \tag{78}$$

where $n(C,C')$ is the linking number of the curves C and C'. For an infinite contour C and the Wilson loop along the spatial boundary of the system the linking number is always unity. The $V(C)$ for an infinite loop is therefore an eigenoperator of the magnetic symmetry and is the analog of the vortex operator $V(x)$ in 2+1 dimensions. Any closed vortex loop of fixed size commutes with the Wilson loop if the contour C' is very large. Such a closed loop is thus an analog of the 2+1 dimensional vortex-anti vortex correlator $V(x)V^\dagger(y)$, which also commutes with the global symmetry generator, but has a nontrivial commutator with W_C if C encloses only one of the points x or y.

Moving on to non Abelian theories it is *deja vu* all over again. The magnetic symmetry is now discreet. A simple way to understand it is again through monopoles. In an $SU(N)$ theory one finds 't Hooft-Polyakov monopoles of magnetic charge $\frac{2\pi N}{g}$. The elementary magnetic fluxon can therefore disappear in N-plets leaving a monopole-antimonopole pair behind. This means that the magnetic flux is only preserved modulo N, and thus the symmetry is reduced to Z_N. The generator of this symmetry is still the large fundamental Wilson loop. The 't Hooft loop - magnetic vortex creation operator is similar to QED and can be thought just like in 2+1 dimensions as an operator of a singular gauge transformation

$$V(C) = \exp\{\frac{i}{gN} \int d^3x \mathrm{Tr}(D^i\omega_C Y)E^i\} = \exp\{\frac{4\pi i}{gN} \int_S d^2 S^i \mathrm{Tr}(Y E^i)\} \tag{79}$$

with $\omega_C(x)$, the singular gauge function which is equal to the solid angle subtended by C as seen from the point x^l. The function ω is continuous everywhere, except on a surface S bounded by C, where it jumps by 4π. Other than

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^{l}The derivative term $\partial^i\omega$ in this expression should be understood to contain only the smooth

the fact that S is bounded by C, its location is arbitrary. The vortex loop and the spatial Wilson loop satisfy the 't Hooft algebra

$$V^\dagger(C)W(C')V(C) = e^{\frac{2\pi i}{N}n(C,C')}W(C') \qquad (80)$$

Again, for an infinite curve C which goes through the whole system the vortex operator is an eigenoperator of the large Wilson loop and thus the order parameter of the magnetic Z_N symmetry.

So there is a symmetry and there is an order parameter. Except that the symmetry charge is not a volume integral but a surface integral, and the order parameter is not a local field, but a field that creates a string like object. This situation is a little unusual, since strictly speaking there are no phases in such a theory for which the order parameter has a nonvanishing expectation value in the thermodynamic limit. This is due to the fact that in the system infinite in all directions C is necessarily an infinite line. The expectation value $< V(C) >$ clearly vanishes irrespective of whether Z_N is broken or not. The 't Hooft loop along a closed contour on the other hand is never zero, since it is globally invariant under the Z_N transformation.

Nevertheless the behavior of the closed loops does reflect the realization mode of the symmetry, since it is qualitatively different in the two possible phases. Namely vacuum expectation value of a large closed 't Hooft loop (by large, as usual we mean that the linear dimensions of the loop are much larger than the correlation length in the theory) has an area law decay if the magnetic symmetry is spontaneously broken, and perimeter law decay if the vacuum state is invariant under the magnetic Z_N.

To understand the physics of this behavior it is useful to think of the 't Hooft line as built of "local" operators - little "magnetic dipoles". Consider Eq. 79 with the contour C running along the x- axis. Let us mentally divide the line into (short) segments of length 2Δ centered at x_i. Each one of these segments is a little magnetic dipole and the 't Hooft loop is a product of the operators that create these dipoles. The definition of these little dipole operators is somewhat ambiguous but since we only intend to use them here for the purpose of an intuitive argument any reasonable definition will do. It is convenient to define a single dipole operator in the following way

$$D_\Delta(x) = \exp\{i \int d^3y[a_i^+(x + \Delta - y) + a_i^-(x - \Delta - y)]\mathrm{Tr}(YE^i(y))\} \quad (81)$$

where $a_i^\pm(x - y)$ is the c-number vector potential of the Abelian magnetic monopole (antimonopole) of strength $4\pi/gN$. The monopole field correspond-

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part of the derivative and to exclude the contribution due to the discontinuity of ω on the surface S.

ing to a_i contains both, the smooth x_i/x^3 part as well as the Dirac string contribution. The Dirac string of the monopole - antimonopole pair in Eq. 81 is chosen so that is connects the points $x - \Delta$ and $x + \Delta$ along the straight line. The dipole operators obviously have the property

$$D_\Delta(x)D_\Delta(x + 2\Delta) = D_{2\Delta}(x + \Delta) \qquad (82)$$

This is because in the product the smooth field contribution of the monopole in $D_\Delta(x)$ cancels the antimonopole contribution in $D_\Delta(x + 2\Delta)$, while the Dirac string now stretches between the points $(x - \Delta)$ and $(x + 3\Delta)$. When multiplied over the closed contour, the smooth fields cancel out completely, while the surviving Dirac string is precisely the magnetic vortex created by a closed 't Hooft loop operator. The 't Hooft loop can therefore be written as

$$V(C) = \Pi_{x_i} D_\Delta(x_i) \qquad (83)$$

The dipole operator $D(x_i)$ is an eigenoperator of the magnetic flux defined on a surface that crosses the segment $[x_i - \Delta, x_i + \Delta]$. Suppose the magnetic symmetry is broken. Then we expect the dipole operator to have a nonvanishing expectation value[m] $< D >= d(\Delta)$. If there are no massless excitations in the theory, the operators $D(x_i)$ and $D(x_j)$ should be decorrelated if the distance $x_i - x_j$ is greater than the correlation length l. Due to Eq. 83, the expectation value of the 't Hooft loop should therefore roughly behave as

$$< V(C) >= d(l)^{L/l} = \exp\{-\ln\left(\frac{1}{d(l)}\right)\frac{L}{l}\} \qquad (84)$$

where L is the perimeter of the loop. In the system of finite length L_x, the vacuum expectation value of the vortex line which winds around the system in x-direction is therefore finite as in Eq. 84 with $L \to L_x$.

On the other hand in the unbroken phase the VEV of the dipole operator depends on the size of the system in the perpendicular plane L_y. For large L_y it must vanish exponentially as $d = \exp\{-aL_y\}$. So the expectation value of V behaves at finite L_y in the unbroken phase as:

$$< V(C) >= \exp\{-aL_yL_x\} \qquad (85)$$

and vanishes as $L_y \to \infty$. Thus in a system which is finite in x direction, but infinite in y direction, the 't Hooft line in the x direction has a finite VEV in the broken phase and vanishing VEV in the unbroken phase.

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[m]The magnetic dipole operators defined above are strictly speaking not local, since they carry the long range magnetic field of a dipole. However, the dipole field falls off with distance very fast. Therefore even though this fall off is not exponential the slight non locality of D should not affect the following qualitative discussion.

In the limit of the infinite system size $L_x \to \infty$ the VEV obviously vanishes in both phases. This is of course due to the fact that V is a product of infinite number of dipole operators, and this product vanishes even if individual dipole operators have finite VEV[n]. However one can avoid any reference to finite size system and infinite vortex lines by considering closed 't Hooft loops.

For a closed loop with long sides along x axis at $y = 0$ and $y = R$ the above argument leads to the conclusion that in the broken phase V must have a perimeter law, Eq. 84. In the unbroken phase the correlation between the dipoles at $y = 0$ and dipoles at $y = R$ should decay exponentially $< D(0)D(R) > \propto \exp\{-\alpha \frac{R}{l}\}$ and thus

$$< V(C) > = \exp\{-\alpha \frac{LR}{l^2}\} = \exp\{-\alpha \frac{S}{l^2}\} \qquad (86)$$

Thus the perimeter behavior of $< V(C) >$ indicates a vacuum state which breaks spontaneously the magnetic Z_N while the area behavior means that the magnetic Z_N is unbroken.

The magnetic symmetry structure thus generalizes to 3+1 dimensions. Moreover 't Hooft [12] gave an argument according to which Wilson loop and 't Hooft loop can not simultaneously have an area law behavior, at least for $N = 2, 3$. It follows then that in the confining phase the 't Hooft loop must have a perimeter law behavior, which signals spontaneous breaking of the magnetic Z_N. It is reasonable to expect therefore that if we learned how to implement spontaneous breaking of a symmetry of this type in the effective Lagrangian framework, we would have a similar universal and simple picture of confinement as in 2+1 dimensions. Unfortunately we don't know how to do it at this point. One way of going about it would be to go in the direction of a string theory, since the basic operator V creates a string like object. This however seems unnecessarily complicated for two reasons. First because even if such a theory can be constructed it would be difficult to get information out of it. Solving a string theory is not an easy task, especially since the vacuum of such a theory must be nontrivial and contain some sort of string condensate $< V >$. And the second reason is that even in the string theory the low energy dynamics is always described by some effective field theory. So we may just as well go directly for an effective field theory description.

Construction of such an effective theory is an interesting open problem. But even though we can not really do it, we can try to learn something by taking hints from the lattice QCD. The spectrum of pure gluodynamics has been quite extensively studied on the lattice. It is known that the two lightest glueballs are the scalar and the tensor ones[9]. We can use this information and

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[n]The VEV of the dipole D must be smaller than one since D is defined as a unitary operator.

try to write down a simple theory that describes these excitations[24]. The object of this exercise is to see whether in the framework of such an effective theory we can identify naturally appearing classical objects that look like confining strings. Let me briefly describe what happens if we do that.

5.2   Effective Lagrangian.

We start therefore by writing down a theory which contains a scalar field σ and a massive symmetric tensor field $G_{\mu\nu} = G_{\nu\mu}$ with a simple interaction.

$$
\begin{aligned}
L & = \frac{1}{4} G_{\lambda\sigma} D^{\lambda\sigma\rho\omega} G_{\rho\omega} + \partial_\mu \sigma \partial^\mu \sigma - 2g^2 v\sigma G^{\mu\nu} G_{\mu\nu} \\
& - g^2 \sigma^2 G^{\mu\nu} G_{\mu\nu} - V(\sigma)
\end{aligned}
\tag{87}
$$

The operator D which appears in the kinetic term of the tensor field is

$$
\begin{aligned}
D^{\lambda\sigma\rho\omega} & = (g^{\lambda\rho} g^{\sigma\omega} + g^{\sigma\rho} g^{\lambda\omega})(\partial^2 + m^2) - 2g^{\lambda\sigma} g^{\rho\omega}(\partial^2 + M^2) \\
& - (\partial^\lambda \partial^\rho g^{\sigma\omega} + \partial^\sigma \partial^\rho g^{\lambda\omega} + \partial^\lambda \partial^\omega g^{\sigma\rho} + \partial^\sigma \partial^\omega g^{\lambda\rho}) \\
& + 2(\partial^\lambda \partial^\sigma g^{\rho\omega} + \partial^\omega \partial^\rho g^{\sigma\lambda})
\end{aligned}
\tag{88}
$$

Several comments are in order here. A general symmetric tensor field has ten components. A massless spin two particle has only two degrees of freedom. The tensor structure of the kinetic term for the massless tensor particles is therefore determined so that it should project out

two components out of $G^{\mu\nu}$. In fact it is easy to check that the kinetic term in Eq. 88 in the massless case ($m^2 = M^2 = 0$) is invariant under the four parameter local gauge transformation

$$
\delta G^{\mu\nu} = \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu
\tag{89}
$$

These four gauge invariances together with four corresponding gauge fixing conditions indeed eliminate eight components of $G^{\mu\nu}$. This gauge invariance is broken by the mass. However the equations of motion even in the massive case lead to four constraints

$$
\partial_\mu G^{\mu\nu} = 0
\tag{90}
$$

This eliminates four degrees of freedom out of ten[o]. In addition the scalar field G^μ_μ decouples from the rest of the dynamics and can be neglected. In fact we have added a parameter M whose only purpose is to make G^μ_μ arbitrarily

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[o]In the interacting theory, Eq. 87, the constraint equations are slightly modified, but they still provide four conditions on the fields.

heavy. All in all therefore we are left with five independent propagating degrees of freedom in $G^{\mu\nu}$, which is the correct number to describe a massive spin two particle.

The scalar glueball self interaction potential $V(\sigma)$ is fairly general and the following discussion will not depend on it. The natural choice to keep in mind is a mass term augmented with the standard triple and quartic self interaction, although one could also consider more complicated logarithmic potential as is becoming a dilaton field.

Our first observation is that the Lagrangian Eq. 87 allows for static classical solutions which are very reminiscent of the Abrikosov - Nielsen - Olesen vortices. Consider a static field configuration of the form

$$G^{ij} = G^{00} = 0$$
$$G^{0i} = G^{i0} \equiv a^i(\vec{x})$$
$$\sigma \equiv \rho(\vec{x}) - v \tag{91}$$

For such a configuration the Lagrangian Eq. 87 reduces to the Lagrangian of the Abelian Higgs model (AHM) in the unitary gauge. This configuration is therefore a solution of equations of motion, provided a^i and ρ solve precisely the same equations as in the AHM with the only modification that the scalar potential is given by $V(\sigma)$. We will call this configuration the tensor flux tube (TFT). In fact it does carry a conserved tensor flux. To see this let us consider the following operator

$$\tilde{F}^{\mu\nu\lambda} = \epsilon^{\mu\nu\rho\sigma}\partial_\rho G_{\sigma\lambda} \tag{92}$$

For single valued field $G_{\mu\nu}$ it satisfies the conservation equation

$$\partial_\mu \tilde{F}^{\mu\nu\lambda} = 0 \tag{93}$$

This is the analog of the homogeneous Maxwell equation of the AHM without monopoles

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{94}$$

For our static TFT solution we have

$$\tilde{F}_{i00} \equiv b_i = \epsilon_{ijk}\partial_j a_k \tag{95}$$

and the flux through the plane perpendicular to the symmetry axis of TFT is

$$\Phi \equiv \int dS_i b_i = 2\pi/g \tag{96}$$

The energy per unit length of the TFT is directly related to the masses of the two glueballs and to the inter glueball coupling constants g^2 and v. This

quantity is a natural candidate for the string tension of the pure Yang - Mills theory.

There is one important proviso to this discussion. As we noted the Lagrangian Eq. 87 reduces for TFT configurations to the AHM Lagrangian in the unitary gauge. The identification of the magnetic flux solutions in the unitary gauge is a little tricky. Naively one would say that they can not exist, since the vector potential is massive but for the magnetic fluxon it should vanish at infinity as a power and not exponentially, since its contour integral does not vanish even for contours taken infinitely far from the location of the fluxon. However one should remember that the mass term for the vector field in unitary gauge allows certain singular configurations. Consider a vector potential configurations which far enough from the origin ($r_{perp} = 0$) vanishes everywhere except on a half plain (let's say $x = 0, y > 0$):

$$a_i = \Phi_z \delta_{1i} \delta(x) \theta(y) \tag{97}$$

It certainly describes a fluxon in the z direction with the flux Φ_z. The singularity in a_i does not contribute to the energy, if the singularity is quantized. The reason is that the vector field mass term is strictly speaking not a simple $m^2 a_i^2$. It really is the remnant of the covariant derivative term of the Higgs field $|D_i H|^2$. This expression when written in terms of the phase ϕ of the Higgs is $g^2 \rho^2 (A_i - \frac{1}{g} \partial_i \phi_{mod 2\pi})^2$. It does not feel jumps in the phase ϕ which are integer multiples of 2π. Since the unitary gauge field a_i is defined shifting A_i by $\frac{1}{g} \partial_i \phi$ the mass term in the AHM Lagrangian is actually

$$m^2 (a^i_{mod \frac{2\pi}{g} \Delta})^2 \tag{98}$$

where Δ is the ultraviolet regulator - lattice spacing. For smooth fields a_i there is no difference between the two mass terms. However Eq. 98 admits singularities of the type of Eq. 97. provided

$$\Phi_z = \frac{2\pi n}{g} \tag{99}$$

with integer n. For these values of the magnetic flux the lowest energy configuration has the form $a_i = \Phi_z \delta_{1i} \delta(x) \theta(y) + b_i(x)$ with b_i a smooth, exponentially decreasing function. The energy per unit length of these configurations is perfectly finite.

In our discussion of the TFT solutions in the effective glueball theory we have assumed that just like in the AHM the mass term (and the interaction terms) of the tensor field allows quantized discontinuities of the form $2\pi\Delta$. This is not all that unnatural. Our effective theory can be thought of as a

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gauge theory with the "spontaneously broken" gauge group of Eq. 89. The analog of the phase field ϕ in our model is played by the gauge parameter Λ_μ of Eq. 89. All that is needed for the discontinuities to be allowed is that Λ_μ (or at least Λ_0) be a phase, or in other words for the gauge group to have a $U(1)$ subgroup. This does not seem to be an extremely unnatural requirement, although to determine whether this is indeed true one would need to have some information about dynamics on distance scales shorter than the inverse glueball mass.

Continuing the same line of thought, we can identify the objects which should play in this model the same role as magnetic monopoles in the superconductor, that is the objects that are confined by TFT. The fields G and σ can couple to much heavier objects, which from the low energy point of view are basically point like. At these short distances Eq. 93 should be modified, and we can consider the current

$$J^{\nu\lambda} = \partial_\mu \tilde{F}^{\mu\nu\lambda} \tag{100}$$

Again I stress that the current $J^{\nu\lambda}$ must have only very high momentum components $k >> m$, otherwise Eq. 87 will not describe faithfully the low energy sector of the theory. Of course, this situation is very similar to AHM with heavy magnetic monopoles where the homogeneous Maxwell equation is also violated on the distance scales of the order of the monopole size. Now, since $\tilde{F}^{\mu\nu\lambda}$ is antisymmetric under the interchange of μ and ν, our newly born current is conserved

$$\partial_\nu J^{\nu\lambda} = 0 \tag{101}$$

The components of the current which are relevant to our TFT configuration are $J^{\mu 0}$. Imagine that the underlying theory does indeed contain objects that carry the charge $Q = \int d^3x J^{00}$. Then the argument for confinement of these objects is identical to the argument for confinement of magnetic monopoles in AHM. If these objects are identified with heavy quarks this becomes the picture of the QCD confinement from the low energy point of view. Of course this last point is purely speculative and whether these objects exist and have anything to do with quarks remains to be seen.

However what we learn from this discussion is that some kind of confining objects appear under natural assumptions in the effective theory. This is indeed reminiscent of the appearance of confining QCD strings in 2+1 dimensions and gives us hope that the whole approach makes sense. What would be really interesting to see is how the spontaneously broken Z_N structure ties into this effective Lagrangian pattern. One would hope that it should dictate the form of this Lagrangian as well as the angular nature of the "gauge" parameter Λ_μ on a similar level of rigor as in 2+1 dimensions.

6   Discussion.

Th main thesis of these notes is this. Confinement in gluodynamics can be unambiguously characterized as spontaneous breaking of the magnetic Z_N symmetry. This symmetry breaking pattern under some mild assumptions also determines the structure of the low energy effective theory. The basic low energy degrees of freedom are magnetic vortices - the objects that carry the magnetic Z_N charge. Spontaneous breaking of magnetic Z_N is synonymous with condensation of magnetic vortices. In this effective theory confinement is manifested on the classical level in a very simple and straightforward manner. In 2+1 dimensions this picture is well substantiated. In 3+1 dimensions although some elements are in place, there is still work to be done.

We note that a lot of numerical work has been done in the past several years to relate the properties of magnetic vortices with confinement [25],[26]. The properties of vortex distributions seem to correlate very well with the confinement properties of the theory. This should be viewed as confirmation of the relation of the vortex condensation to confinement.

There are many questions one can ask at this junction. Let me mention her just two obvious ones. First, if at zero temperature some global symmetry is spontaneously broken, one expects it to be restored at some finite temperature T_c. Does this Z_N symmetry restoring phase transition exist in gluodynamics? The answer to this question is an emphatic yes. It has been shown in [27] that the restoration of the magnetic symmetry is indeed the deconfining phase transition of gluodynamics. The canonical order parameter for the deconfinement phase transition therefore is the expectation value of the vortex operator. In the high temperature deconfined phase in 2+1 dimensions $< V(x) >= 0$, while in 3+1 dimensions $< V(C) > \propto \exp\{-\alpha S\}$. This has been confirmed numerically[28].

Another interesting question is what happens to the magnetic symmetry when the theory contains dynamical fundamental fields. The answer to this is that the symmetry stays there, but now it does not have a local order parameter. Physically this is because fundamentally charged particles have nontrivial Aharonov-Bohm phase while scattering off a pointlike magnetic vortex created by V. So the properties of the Z_N charge here are somewhat similar to the properties of the electric charge in QED - both are global symmetries and both do not have a local order parameter[p]. Indeed it can be shown[29] that the Z_N symmetry in low energy effective theory has to be gauged. This leads to some interesting dynamics and allows one to construct a bag like picture of baryons[29].

―――――――――――――

[p]Recall that any physical gauge invariant operator that carries electric charge is nonlocal, since it creates a long range electric field.

Still in my mind the most interesting question is how to extend intelligently this picture to 3+1 dimensions and how to relate the magnetic symmetry structure with the expected form of the effective low energy Lagrangian. This ought to be the direction of the main effort in this area.

Acknowledgments

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