Field Strength Formulation of $SU(2)$ Yang-Mills Theory in the Maximal Abelian Gauge: Perturbation Theory[1]

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Abstract

We present a reformulation of $SU(2)$ Yang-Mills theory in the maximal Abelian gauge, where the non-Abelian gauge field components are exactly integrated out at the expense of a new Abelian tensor field. The latter can be treated in a semiclassical approximation and the corresponding saddle point equation is derived. Besides the non-trivial solutions, which are presumably related to non-perturbative interactions for the Abelian gauge field, the equation of motion for the tensor fields allows for a trivial solution as well. We show that the semiclassical expansion around this trivial solution is equivalent to the standard perturbation theory. In particular, we calculate the one-loop β-function for the running coupling constant in this approach and reproduce the standard result.

1   Introduction

QCD is generally accepted as the theory of strong interactions. It has been successfully tested in the high energy regime, where perturbation theory is applicable. On the other hand the low energy sector of the theory, and in particular the confinement mechanism is not well understood. A theoretically appealing mechanism for confinement has been proposed some time ago by 't Hooft and Mandelstam, who conjectured that confinement is realised by the dual Meissner effect [1]. This confinement scenario assumes that the QCD ground state consist of a condensate of magnetic monopoles (dual superconductor), which squeezes the colour electric field of colour charges into flux tubes.

Magnetic monopoles arise in Yang-Mills theory in the so called Abelian gauges, which were proposed by 't Hooft [2]. Although in these gauges the magnetic monopoles arise as gauge artifacts, they may nevertheless represent the dominant infrared degrees of freedom in these gauges. The realisation of the dual Meissner effect in the Abelian gauges has recently received strong support from lattice calculations [3]. These lattice calculations also show that there is a preferred Abelian gauge, the so-called maximal Abelian gauge. In this gauge one, in fact, observes Abelian dominance and furthermore dominance of magnetic monopoles, i. e. about 92 % of the full string tension comes from the Abelian field configurations and furthermore 95 % of the Abelian string tension is reproduced by the magnetic monopoles alone [4]. The Abelian dominance indicates that the low energy sector of QCD can probably most efficiently be described in terms of an effective Abelian theory[1]. Unfortunately the underlying effective Abelian theory of QCD, which gives rise to the dual Meissner effect, is not known. The dual Meissner effect also seems to be the confinement mechanism in supersymmetric gauge theories [5].

Apparently the dual Meissner effect can be most efficiently described in a dual formulation, which is known to exist for quantum electrodynamics. The transition to the dual theory basically amounts to an interchange of electric and magnetic fields, and moreover to an inversion of the coupling constant.

Unfortunately in the strict sense the dual theory of non-supersymmetric Yang-Mills theory is not known and does perhaps not exist. However, there exists a non-local formulation in terms of "dual potentials" which are introduced as functional Fourier conjugates to the Bianchi forms [6]. Furthermore, within loop space formulation of gauge theory an extended duality transformation for non-Abelian theories has been introduced, which for electrodynamics reduces to the standard duality transformation [7]. Although this extended dual formulation has some attractive features as, for example, it allows to construct explicitly 't Hooft's magnetic disorder operator

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[1]This Abelian theory should account for the non-Abelian sampling of field configurations. Let us emphasise that the Abelian dominance is observed with a non-Abelian sampling of field configurations, i. e. the field configurations are weighted by the full non-Abelian Yang-Mills action.

[8], it seems to be rather inconvenient for calculational purposes.

There have been also attempts to construct the dual theory of QCD phenomenologically [9]. Furthermore, there exist a microscopic approach to the dual description of QCD, which has not been fully appreciated in the past. This is the so-called field strength approach [10], which reformulates the Yang-Mills theory in terms of the field strength tensor and comes very close to a dual description as has been recently discussed [11].

In the present paper by reformulating the field strength approach in the maximal Abelian gauge, we derive an effective Abelian theory.[2] For simplicity we will confine ourself to the gauge group $SU(2)$. In this case we succeed to exactly integrate out the non-Abelian part of the gauge field at the expense of the introduction of an Abelian tensor field. As a first illustration of the emerging effective Abelian theory we calculate the one-loop β-function and reproduce the standard Yang-Mills result.

Let us also mention that recently the field strength approach has been reconsidered from the point of view of the so-called BF-theory [13]. In this context Yang-Mills theory can be regarded as a deformation of the BF-theory. The corresponding Feynman rules and one loop β-function were considered in ref. [14] in the Lorentz gauge, making use of special properties of this gauge, like the finiteness of the gluon-ghost vertex, which is absent in the nonlinear maximal Abelian gauge. In the present paper we will perform a one-loop calculation in the maximal Abelian gauge in a modified field strength approach, thereby showing explicitly that vertex corrections and contributions from the charged gluon propagator cancel on one-loop level. This cancellation is similar to the background field calculation, where only the propagator of the background field contributes to the β-function. In fact, in our case the Abelian gluon field figures in this context as a background field.

The organisation of the paper is as follows: After fixing our conventions, we discuss the Faddeev-Popov gauge fixing for the Maximal Abelian gauge. By introducing an Abelian tensor field $\chi_{\mu\nu}$ similar to the field strength approach [10], we can exactly integrate out the charged gauge field components, resulting in an effective Abelian theory given implicitly as an integral over the tensor field χ and the FP ghosts. We also investigate the general form of the resulting charged gluon propagator. By integrating out the charged gluon fields a good deal of quantum fluctuations has been included and the remaining integral over the tensor field χ can presumably be treated in saddle point approximation. In section 2.3, we derive the corresponding equation of motion for χ and show that, unlike the standard field strength approach, a trivial solution $\chi = 0$ also exists.

In section 3 we examine the nature of the semiclassical expansion around this trivial saddle point. As expected, it corresponds to the standard perturbative expansion. After deriving an explicit form of the effective action to one-loop order, we show in

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[2]Some preliminary results of this approach have been reported in ref. [12].

section 3.2 that there is a cancellation between the charged field propagators and the vertex correction, so that the β-function receives contributions only from the neutral gluon vacuum polarisation, a state of affairs familiar from the background gauge formalism. Finally, we complete the calculation of the one-loop β-function by evaluating the corrections to the neutral gluon propagator. The result coincides with the known expression.

We conclude in section 4 with an outlook on possible non-perturbative applications of our approach.

2   The Effective Abelian Theory

We consider pure $SU(N)$ Yang-Mills theory on an Euclidean four-manifold $\mathcal{M}$ with the action given by

$$S_{\text{YM}}[A] = \frac{1}{4g^2} \int_{\mathcal{M}} d^4x \, F_{\mu\nu}^a(x) F_a^{\mu\nu}(x). \tag{1}$$

Here, $F_{\mu\nu} = F_{\mu\nu}^a T^a = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ denotes the field strength tensor of the gauge field $A_\mu \equiv A_\mu^a T^a$, and the generators T^a $(a = 1, \ldots, N^2 - 1)$ of the Lie-algebra $[T^a, T^b] = f^{abc} T^c$ are taken to be anti-hermitian and normalised according to

$$\text{tr}\left(T^a T^b\right) = -\frac{1}{2}\delta^{ab}. \tag{2}$$

For the case of $SU(2)$, $T^a = \sigma^a/2i$ where σ^a are the Pauli matrices. Under a gauge transformation $U = \exp(-\omega^a T^a)$,

$$A_\mu \xrightarrow{U} A_\mu^U := U A_\mu U^\dagger + U \partial_\mu U^\dagger. \tag{3}$$

For infinitesimal $U \approx 1 - \omega^a T^a$, this means

$$\delta A_\mu^a = f^{abc} A_\mu^b \omega^c + \partial_\mu \omega^a \equiv \hat{D}_\mu^{ab} \omega^b,$$

where we also introduced the covariant derivative

$$D_\mu = \partial_\mu + A_\mu \quad ; \quad \hat{D}_\mu^{ab} = \partial_\mu \delta^{ab} - f^{abc} A_\mu^c \equiv \partial_\mu \delta^{ab} + \hat{A}_\mu^{ab}. \tag{4}$$

Finally, quantities with a hat (ˆ) denote Lie-algebra elements in the adjoint representation, with generators given by $\left[\hat{T}^a\right]_{bc} = -f_{abc}$.

2.1   Maximal Abelian Gauge

The basic idea of Abelian gauges [2] is to remove as many non-Abelian degrees of freedom as possible by partially fixing the gauge, leaving a theory with a residual

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Abelian gauge symmetry. Technically, this is accomplished by the so-called Cartan decomposition.

Every Lie algebra has a largest Abelian subalgebra spanned by a maximal set of commuting generators. Throughout this paper, colour indices a_0, b_0 etc. denote the generators of this Cartan subalgebra, while letters with a bar, $\bar{a}$, $\bar{b}$ etc. are reserved for the remaining generators. For the simple Lie algebras $su(N)$ we can always adjust the Cartan decomposition in such a way that the Abelian generators T^{a_0} are diagonal, and the remaining $T^{\bar{a}}$ have vanishing diagonal elements.

Since the generators of the Cartan algebra commute, their picture under the exponential is an Abelian subgroup H of the corresponding Lie group G, and one finds an analogous decomposition for the group

$$G = H \otimes G/H \quad ; \quad H \equiv \{\exp(-\omega_{a_0} T^{a_0})\}, \tag{5}$$

where the quotient G/H is called coset. For $SU(N)$, there are $N-1$ commuting generators (the rank of the group), and the Cartan subgroup H is a (reducible) representation of $U(1)^{N-1}$. Our goal is to reduce the gauge symmetry to this $U(1)^{N-1}$ by fixing the coset.

In the following, we will mainly consider the case $SU(N=2)$, for which the Cartan decomposition of the gauge field is

$$A_\mu(x) = a_\mu(x) T^3 + \sum_{\bar{a}=1}^{2} A_\mu^{\bar{a}}(x) T^{\bar{a}} \equiv A_\mu^{(n)}(x) + A_\mu^{(ch)}(x). \tag{6}$$

The superscripts $^{(ch)}$ and $^{(n)}$ for "charged" and "neutral" refer to the transformation properties under the residual Abelian gauge group $\Omega = e^{-\omega T^3} \in H$:

$$A_\mu^{(ch)} \longrightarrow \Omega \cdot A_\mu^{(ch)} \cdot \Omega^\dagger \quad ; \quad a_\mu \longrightarrow a_\mu + \partial_\mu \omega. \tag{7}$$

Under this residual $U(1)$, $A_\mu^{(ch)}$ transforms as a charged matter field in the adjoint representation, while the diagonal part $A_\mu^{(n)} = a_\mu T^3$ acts like a photon. Note that the symmetry (7) involves the diagonal part of the *original* gauge group and gluon field. Therefore, it constitutes an *electric* $U(1)$.

To fix the coset we need $N(N-1) = 2$ conditions, which are invariant under the Cartan group $U(1)$. Several propositions have been made since 't Hooft's original work [2], but in this paper, for the reasons mentioned in the introduction, we will use the so-called *maximal Abelian gauge* (MAG):

$$\left[D_\mu^{(n)}, A_\mu^{(ch)}\right] = \partial_\mu A_\mu^{(ch)} + \left[A_\mu^{(n)}, A_\mu^{(ch)}\right] \equiv 0. \tag{8}$$

The main motivation for the use of Abelian gauges is, besides the appearance of magnetic monopoles, that it should facilitate integrating out the charged gauge field

components $A_\mu^{(\text{ch})}$ leaving an effective Abelian theory. The quantisation of the latter still requires a gauge fixing for the neutral photon, which in the present paper will be done by the usual Lorentz condition $\partial_\mu A_\mu^{(\text{n})} \equiv 0$. The complete gauge fixing constraints thus read

$$\left[D_\mu^{(\text{n})}, A_\mu\right] = \partial_\mu A_\mu + \left[A_\mu^{(\text{n})}, A_\mu\right] \equiv 0. \tag{9}$$

The standard Faddeev-Popov (or BRST) quantisation of the gauge (9) leads to the following gauge-fixing insertion in the path integral

$$\delta_{\text{gf}} = \delta[\partial_\mu a_\mu] \det (-\Box) \cdot \delta \left[D_\mu^{(\text{n})} A_\mu^{(\text{ch})}\right] \det \mathsf{M}[a, A]. \tag{10}$$

The splitting of the FP determinant in an Abelian and charged part only holds upon using the MAG-condition (8), i.e. the second delta-function in δ_{gf} has to be implemented *exactly*. We may, however, relax the Abelian Lorentz-condition in the usual way by introducing a gauge-fixing term in the action:

$$\delta_{\text{gf}} = \exp\left(-\frac{1}{2\hbar g^2\alpha} \int (\partial \cdot a)^2\right) \det (-\Box) \cdot \delta \left[D_\mu^{(\text{n})} A_\mu^{(\text{ch})}\right] \det \mathsf{M}[a, A] \tag{11}$$

where α is a gauge fixing parameter. Note that the first two factors in (11) represent the standard FP insertion for the Abelian Lorentz gauge, so that the last two terms may be considered as implementation of the MAG (8) alone.

In the $SU(2)$ case, we have

$$\mathsf{M}^{\bar{a}\bar{b}} := -\left[D_\mu^{(\text{n})} D_\mu^{(\text{n})}\right]^{\bar{a}\bar{b}} + \left(A_\mu^{\bar{a}} A_\mu^{\bar{b}} - A_\mu^{\bar{c}} A_\mu^{\bar{c}} \delta^{\bar{a}\bar{b}}\right) \tag{12}$$

Introducing charged ghosts and a multiplier field $\phi^{(\text{ch})}$ to express the MAG delta-function in (11) by its Fourier representation, we can rewrite the generating functional for $SU(2)$ YM-theory in the following form:

$$Z[j] = \int \mathcal{D}(A^{(\text{ch})}, a, \eta, \bar{\eta}, \phi) \exp\left\{-\frac{1}{\hbar}S_{\text{eff}} + \int \left(j_\mu^{\bar{a}} A_\mu^{\bar{a}} + j_\mu^3 a_\mu\right) d^4x\right\} \tag{13}$$

$$S_{\text{eff}} = S_{\text{YM}}[a, A^{(\text{ch})}] + \frac{1}{2\alpha g^2} \int (\partial \cdot a)^2 + \frac{1}{g^2} \int \bar{\eta}^{\bar{a}} \mathsf{M}^{\bar{a}\bar{b}} \eta^{\bar{b}} + \frac{i}{g^2} \int \phi^{\bar{a}} D_{\mu,\bar{a}b}^{(\text{n})} A_\mu^{\bar{b}}.$$

For simplicity, we will from now on use the notation $a_\mu \equiv A_\mu^3$ for the neutral gluon (photon) in $SU(2)$.

2.2 Derivation of the Effective Abelian Theory

In principle, we can map $SU(2)$ YM into an effective Abelian theory by integrating out the charged field components $A^{(\text{ch})}$ from the generating functional (13). Unfortunately, the YM action is quartic in the charged gauge field components, so that

the integration cannot be performed exactly. Furthermore, the singularities of the gauge fixing procedure will not only lead to monopoles in the neutral photon, but will also induce non-trivial topology in $A^{(\text{ch})}$ [15].

To make the dependence on $A^{(\text{ch})}$ more explicit, we decompose the YM field strength in charged and neutral parts:

$$
\begin{aligned}
F_{\mu\nu} &= \left(\partial_\mu A_\nu^{(\text{n})} - \partial_\nu A_\mu^{(\text{n})}\right) + \left(\left[D_\mu^{(\text{n})}, A_\nu^{(\text{ch})}\right] - \left[D_\nu^{(\text{n})}, A_\mu^{(\text{ch})}\right]\right) + \left[A_\mu^{(\text{ch})}, A_\nu^{(\text{ch})}\right] \\
&\equiv f_{\mu\nu} + V_{\mu\nu} + T_{\mu\nu}.
\end{aligned}
\tag{14}
$$

For an arbitrary gauge group $G = SU(N)$, the Abelian field strength $f_{\mu\nu}$ is always neutral, i.e. it lives in the Cartan subalgebra, while $V_{\mu\nu}$ is always charged. The problematic term is the commutator $T_{\mu\nu} = \left[A_\mu^{(\text{ch})}, A_\nu^{(\text{ch})}\right]$, which in general contains both types of generators. However, in the special case $G = SU(2)$ $T_{\mu\nu}$ only lives in the Cartan subalgebra. From the orthogonal normalisation (2) of the generators, this entails

$$
\text{tr}\left(f_{\mu\nu} \cdot V_{\mu\nu}\right) = \text{tr}\left(T_{\mu\nu} \cdot V_{\mu\nu}\right) \equiv 0.
\tag{15}
$$

In other words, this expresses the absence of a charged three gluon vertex in the YM-action for the case of only two colours. From now on, we will therefore focus on the gauge group $SU(2)$. Using the decomposition (14) in the YM action, we find

$$
S_{\text{YM}} = -\frac{1}{2g^2} \int \text{tr}\left\{f_{\mu\nu}f^{\mu\nu} + V_{\mu\nu}V^{\mu\nu} + T_{\mu\nu}T^{\mu\nu} + 2f_{\mu\nu}T^{\mu\nu}\right\} d^4x.
\tag{16}
$$

The V^2-term can be simplified by a partial integration, using the the MAG-condition:

$$
\begin{aligned}
\int \text{tr}\left(V_{\mu\nu}V^{\mu\nu}\right) d^4x &= \int A_\mu^{\bar{a}} \left[\hat{D}_\sigma^{(\text{n})} \hat{D}_\sigma^{(\text{n})}\right]_{\bar{a}\bar{b}} A_\mu^{\bar{b}} d^4x + 2\int \text{tr}\left(T_{\mu\nu}f^{\mu\nu}\right) d^4x + \text{surface} \\
\text{surface} &= 2 \oint \text{tr}\left(V_{\mu\nu} \cdot A_\nu^{(\text{ch})}\right) d\sigma^\mu.
\end{aligned}
\tag{17}
$$

This allows to reformulate the YM action as

$$
\begin{aligned}
S_{\text{YM}} &= \frac{1}{4g^2} \int \left(f_{\mu\nu}^3\right)^2 d^4x + \frac{1}{2g^2} \int A_\mu^{\bar{a}} \left[-\hat{D}_\sigma^{(\text{n})} \hat{D}_\sigma^{(\text{n})}\delta_{\mu\nu} - 2\hat{f}_{\mu\nu}\right]_{\bar{a}\bar{b}} A_\nu^{\bar{b}} + \\
&\quad + \frac{1}{4g^2} \int \left(T_{\mu\nu}^3\right)^2 d^4x + \text{surface}.
\end{aligned}
\tag{18}
$$

The usual approach of treating the $(T_{\mu\nu})^2$ term (which is quartic in the charged gauge field) perturbativelyΦ is probably not appropriate in the low energy regime, where the induced self-interaction of the Abelian field a_μ is expected to be strong in order to trigger monopole condensation. Therefore, some non-perturbative treatment must be adopted, and the structure of the action suggests a path integral linearisation.

For this purpose, let us introduce a neutral, antisymmetric tensor field $\chi_{\mu\nu}^{(n)} = \chi_{\mu\nu} T^3$ by the identity[3]

$$\exp\left\{ -\frac{1}{4\hbar g^2} \int \left(T_{\mu\nu}^3\right)^2 d^4x \right\} = \int \mathcal{D}\chi_{\mu\nu} \exp\left\{ -\frac{1}{4\hbar g^2} \int \left(\chi_{\mu\nu}\right)^2 d^4x + \right.$$
$$\left. + \frac{i}{2\hbar g^2} \int T_{\mu\nu}^3 \chi_{\mu\nu} d^4x \right\}. \qquad (19)$$

Upon inserting (19) and (18) in the generating functional (13), we can formally express the effective Abelian theory as

$$Z[j] = \int \mathcal{D}a_\mu \exp\left\{ -\frac{1}{\hbar}\left(S_0[a] + S_{\mathrm{int}}[a, j^{(\mathrm{ch})}]\right) + \int j_\mu^{(n)} a_\mu\, d^4x \right\}. \qquad (20)$$

Here S_0 is the standard free Maxwell action

$$S_0 = \frac{1}{4g^2} \int f_{\mu\nu} f_{\mu\nu}\, d^4x + \frac{1}{2g^2\alpha} \int \left(\partial_\mu a_\mu\right)^2 d^4x$$
$$= \frac{1}{2g^2} \int a_\mu \left[D_0^{-1}\right]_{\mu\nu} a_\nu\, d^4x + \text{surface} \qquad (21)$$

which defines the tree-level photon propagator

$$\left[D_0\right]_{\mu\nu}(x, y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)} \left[\delta_{\mu\nu} + (\alpha - 1)\frac{p_\mu p_\nu}{p^2}\right] \frac{1}{p^2}. \qquad (22)$$

The remaining term S_{int} describes the self-coupling of the neutral gluons (photons) as well as their coupling to the external charged currents. Both are non-Abelian effects mediated by the exchange of charged gluons, ghosts and multipliers enforcing the gauge constraints. In order to make the following formulae more transparent we will mostly suppress colour, Lorentz and spacetime indices; a bracket $\langle\cdots\rangle$ symbolises summation or integration over all relevant indices. Neglecting surface terms, S_{int} is then formally given by

$$e^{-\frac{1}{\hbar}S_{\mathrm{int}}} = \int \mathcal{D}(A^{(\mathrm{ch})}, \chi, \eta, \bar{\eta}, \phi) \exp\left\{ -\frac{1}{2\hbar g^2}\left\langle A^{(\mathrm{ch})} \cdot \mathsf{H} \cdot A^{(\mathrm{ch})} \right\rangle - \frac{1}{4\hbar g^2}\left\langle \chi \cdot \chi \right\rangle - \right.$$
$$- \frac{1}{\hbar g^2}\left\langle \bar{\eta} \cdot \left(-\hat{D}^{(n)}\hat{D}^{(n)}\right) \cdot \eta \right\rangle + \left\langle A^{(\mathrm{ch})} \cdot \left(j^{(\mathrm{ch})} - \frac{i}{\hbar g^2}\hat{D}^{(n)}\phi\right)\right\rangle +$$
$$\left. + \left\langle \bar{\xi} \cdot \eta + \bar{\eta} \cdot \xi \right\rangle + \left\langle \phi \cdot \varphi \right\rangle \right\}. \qquad (23)$$

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[3]Sums over repeated Lorentz indices run from $1\ldots 4$, but the path integration is only performed over the *independent* components of χ, i.e. $\mathcal{D}\chi = \prod_{\mu<\nu} \mathcal{D}\chi_{\mu\nu}(x)$.

For later convenience, we have introduced sources $\eta^{\bar{a}}$ and $\varphi^{\bar{a}}$ for the charged ghost $\eta, \bar{\eta}$ and multiplier fields ϕ, respectively. Furthermore, the symmetric operator

$$\mathsf{H}_{\mu\nu}^{\bar{a}\bar{b}} := -\delta_{\mu\nu}\left(\left[D_{\sigma}^{(\mathrm{n})}D_{\sigma}^{(\mathrm{n})}\right]^{\bar{a}\bar{b}} + 2\delta^{\bar{a}\bar{b}}\bar{\eta}^{\bar{c}}\eta^{\bar{c}} - 2\bar{\eta}^{\bar{a}}\eta^{\bar{b}}\right) - 2\hat{f}_{\mu\nu}^{\bar{a}\bar{b}} + i\hat{\chi}_{\mu\nu}^{\bar{a}\bar{b}}. \qquad (24)$$

is the inverse of the charged gluon propagator *before* gauge fixing to the MAG. Note that $\mathsf{H} = \mathsf{H}_0 + \delta\mathsf{H}$, where the zero field expression $\mathsf{H}_0 = -\Box$ and its inverse are well defined. In the present paper, we will use H and its inverse only in the perturbative sense, i.e. $\mathsf{H}^{-1} = \mathsf{H}_0^{-1} - \mathsf{H}_0^{-1} \cdot \delta\mathsf{H} \cdot \mathsf{H}_0^{-1} + \cdots$ which is well defined to any finite order.

At this point, all of the integrations in (23) should be performed in order to obtain the effective photon action. Of course, this cannot be done exactly for *all* fields in (23), but at least for ϕ and $A^{(\mathrm{ch})}$, the integration can be straightforwardly performed. While this is not surprising[4] for $A^{(\mathrm{ch})}$, the exact integral over ϕ is a very pleasant feature. After all, the factorisation (10) of the FP determinant requires an exact implementation of the MAG condition, i.e. any approximation in ϕ would couple charged and neutral entries of the FP determinant.[5] In view of the dual superconductor picture, however, the factorisation (10) is crucial as it makes the residual Abelian $U(1)$ symmetry and its (Lorentz) gauge fixing explicit.

Let us have a closer look at the residual *electric* $U(1)$ symmetry. Under such a diagonal gauge transformation $\omega = e^{-\theta T^3}$ we have

$$\begin{aligned}
a_\mu &\xrightarrow{\omega} a_\mu + \omega \cdot \partial_\mu \omega^\dagger = a_\mu + \partial_\mu \theta \ , \\
j_\mu^{(\mathrm{ch})} &\xrightarrow{\omega} \omega \cdot j_\mu^{(\mathrm{ch})} \cdot \omega^\dagger \ , \\
A_\mu^{(\mathrm{ch})} &\xrightarrow{\omega} \omega \cdot A_\mu^{(\mathrm{ch})} \cdot \omega^\dagger \ , \\
\chi_{\mu\nu} &\xrightarrow{\omega} \chi_{\mu\nu}
\end{aligned} \qquad (25)$$

and all the charged fields $(\eta, \bar{\eta}, \phi)$ and their currents transform similar to $A^{(\mathrm{ch})}$ and $j^{(\mathrm{ch})}$, respectively. Note that the new Abelian tensor field $\chi_{\mu\nu}$ has to be regarded as an *invariant* with respect to the residual $U(1)$. This is clear from the introduction (19) of $\chi_{\mu\nu}$, which shows that it only couples to the commutator $[A_\mu^{(\mathrm{ch})}, A_\nu^{(\mathrm{ch})}]$, which in turn is invariant under Abelian gauge transformations.

The $U(1)$ symmetry (25) is only broken by the Lorentz gauge fixing in (11), i.e. by the gauge fixing term in the Maxwell action S_0 (21). Furthermore, the integration measure in (23) is clearly invariant under the simple gauge rotation (25) of the charged fields. Some of the immediate consequences of this observation are:

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[4]The aim of the path integral linearisation is exactly to make this integration feasible at the expense of a new auxiliary field χ.

[5]A gauge fixing term for the MAG would have the same effect. This is the main reason for our use of the multiplier field ϕ.

- The complicated interaction term S_int from (23) is invariant under the residual Abelian symmetry (25) for the photon and the charged currents.

- The same is true for the exponent in (23) (as can be easily checked by inspection).

- Green's functions from the effective theory (20) obey (Abelian) Ward identities. This has important consequences for the renormalisation in the next chapter.

Returning to the formula (23), we can now perform the integration over $A^{(\mathrm{ch})}$. The resulting exponent is at most quadratic in the multiplier field ϕ, so that an *exact* implementation of the MAG is possible by performing the Gaussian integration over ϕ. We end up with

$$
\begin{aligned}
S_\mathrm{int} \;=\; & -\hbar \int \mathcal{D}(\chi,\eta,\bar\eta)\exp\Bigg\{ -\frac{1}{\hbar}\mathsf{S}[a,\chi,\eta,\bar\eta] + \frac{\hbar g^2}{2}\left\langle j^{(\mathrm{ch})}\cdot \mathsf{D}\cdot j^{(\mathrm{ch})}\right\rangle + \\
& + \frac{\hbar g^2}{2}\left\langle \varphi\cdot \mathsf{L}^{-1}\cdot\varphi\right\rangle + i\hbar g^2\left\langle \varphi\cdot \mathsf{L}^{-1}\hat{D}^{(\mathrm{n})}_\mu \mathsf{H}^{-1}\cdot j^{(\mathrm{ch})}\right\rangle + \left\langle\bar\xi\eta + \bar\eta\xi\right\rangle\Bigg\} \quad (26)
\end{aligned}
$$

where

$$
\mathsf{S}[a,\chi,\eta,\bar\eta] = \frac{1}{4g^2}\int\chi^2 + \frac{1}{g^2}\int\bar\eta\left(-\hat{D}^{(\mathrm{n})}_\sigma\hat{D}^{(\mathrm{n})}_\sigma\right)\eta + \frac{\hbar}{2}\mathrm{Tr}\ln\mathsf{H} + \frac{\hbar}{2}\mathrm{Tr}\ln\mathsf{L} \qquad (27)
$$

is an effective action. Here we have introduced the definitions

$$
\begin{aligned}
\mathsf{L}_{\bar a\bar b} \;&:=\; -\left[\hat{D}^{(\mathrm{n})}\mathsf{H}^{-1}\hat{D}^{(\mathrm{n})}\right]_{\bar a\bar b} \\
\mathsf{D}^{\mu\nu}_{\bar a\bar b} \;&:=\; \left[\mathsf{H}^{-1} + \left(\mathsf{H}^{-1}\hat{D}^{(\mathrm{n})}\right)\cdot \mathsf{L}^{-1}\cdot\left(\hat{D}^{(\mathrm{n})}\mathsf{H}^{-1}\right)\right]^{\mu\nu}_{\bar a\bar b}.
\end{aligned} \qquad (28)
$$

To prevent the equations from becoming cluttered we have used here a shorthand notation suppressing the intermediate indices which are summed over. The detailed definition of these quantities (with the summation indices restored) can be found in appendix A. Let us stress that S_int (26) and in particular the effective action $\mathsf{S}[a,\chi,\eta,\bar\eta]$ from (27) is *invariant* under the Abelian gauge transformation (25) provided the employed regularisation prescription does not spoil this symmetry. From eq. (26) we read off that D and L^{-1} are the propagators of the charged gluons $A^{(\mathrm{ch})}$ and the multiplier field ϕ, respectively.[6]

Some comments on these propagators are in order. At zero fields, we find $\mathsf{L}_0^{-1} = 1$ and $\mathsf{D}_0 = -\Box^{-1}\mathsf{P}_T$, where P_T denotes the usual transversal projector. Both results

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[6]The remarks given above on the perturbative existence of H and its inverse apply to the propagators D and L^{-1} as well.

are, of course, expected. The local form of L_0^{-1} simply reflects the fact that the field ϕ really is a multiplier, not a propagating quantum field. At zero Abelian field, on the other hand, the MAG coincides with the Lorentz gauge for $A^{(\text{ch})}$ which yields the transversal propagator D_0 when implemented exactly.

Prior to integrating out ϕ in (23), the charged gluon propagator defined by the term quadratic in $A^{(\text{ch})}$ would be simply H^{-1}. The subsequent integration over ϕ implements the MAG exactly and converts the charged gluon propagator from H^{-1} to D. Therefore, we expect the complicated form (28) for D (at non-zero fields) to be some kind of *projection* of H^{-1} to the MAG subspace. Indeed, we can rewrite D, using the above introduced compact notation, as

$$D = T^\dagger H^{-1} = H^{-1} T = T^\dagger H^{-1} T \qquad \text{with} \qquad \begin{aligned} T &\equiv 1 + D^{(\text{n})} L^{-1} D^{(\text{n})} H^{-1} \\ T^\dagger &\equiv 1 + H^{-1} D^{(\text{n})} L^{-1} D^{(\text{n})} \end{aligned} \quad (29)$$

As expected, the operator T is a projector, i.e. $T^2 = T$, and represents the identity in the space of charged gluon field configurations satisfying the MAG.

The above discussion can be made more transparent by noting that the MAG can be considered as a generalised Lorentz gauge replacing the ordinary derivative ∂_μ by the covariant one $\hat{D}_\mu^{(\text{n})}$. In the same way as the Lorentz gauge eliminates the longitudinal component of the gauge field, the MAG eliminates the generalised longitudinal charged components $\hat{D}^{(\text{n})} A_L^{(\text{ch})} = \ell A^{(\text{ch})} = 0$, where the covariant longitudinal and transversal projectors are defined by

$$\ell_{\mu\nu}^{ab} = \left(\hat{D}_\mu^{(\text{n})} \frac{1}{\hat{D}^{(\text{n})} \hat{D}^{(\text{n})}} \hat{D}_\nu^{(\text{n})} \right)^{ab} \quad ; \quad t_{\mu\nu}^{ab} = \delta^{ab} \delta_{\mu\nu} - \ell_{\mu\nu}^{ab}. \tag{30}$$

Note also that in the colour subspace of the Cartan algebra these operators reduce to the familiar longitudinal and transversal projectors. It is now easy to see that the propagator (28) for the charged gluon is in fact purely *transversal* in the generalised sense

$$\ell \cdot D = 0 \quad , \quad D \cdot \ell = 0. \tag{31}$$

One could have certainly expected this result, but it is gratifying that it comes out in our approach without being explicitly implemented.

Finally we can use the generalised projectors (30) to simplify somewhat the above obtained effective action (27). From the definition (28) one immediately derives the relation

$$\text{Tr}_{(c)} \ln L = \text{Tr}_{(c)} \ln \left(-\hat{D}_\mu^{(\text{n})} \hat{D}_\mu^{(\text{n})} \right) + \text{Tr}_{(c,L)} \left(\ell \ln H^{-1} \right) + \text{Tr}_{(c,L)} \left(\ell \ln \ell \right). \tag{32}$$

Here we have explicitly indicated over which discrete indices the trace has to be taken (where "c" and "L" stand for the colour and Lorentz indices, respectively). The last two terms in the effective action (27) can then be combined to

$$\text{Tr}_{(c)} \ln L + \text{Tr}_{(c,L)} \ln H = \text{Tr}_{(c)} \ln \left(-\hat{D}^{(\text{n})} \hat{D}^{(\text{n})} \right) + \text{Tr}_{(c,L)} \left(t \ln H \right) + \text{Tr}_{(c,L)} \left(\ell \ln \ell \right). \tag{33}$$

11

Note that only the generalised transversal part of $\ln \mathsf{H}$ contributes to the effective action. Furthermore the last term in (33) can be interpreted as the entropy of the generalised longitudinal degrees of freedom. In the perturbative calculation performed in section 3 we will also see that the first term in (33) cancels against the FP determinant.

Eq. (20) represents the effective Abelian theory with the the non-Abelian effects contained in the action S_{int} (26). This is as far as one can get exactly. For the remaining integrations in (26) over the auxiliary field χ (and the FP ghosts $\eta, \bar{\eta}$) we have to resort to some approximation. Previous applications of the path integral linearisation method suggest that a semiclassical type of treatment in the auxiliary field χ is appropriate, since a good deal of the non-perturbative quantum effects have already been included by partly integrating out the original gauge fields. We discuss the saddle point equation for χ in the next section.

2.3   The Saddle-Point Equation

In order to perform the remaining integration in (26) we are looking for the stationary points of the action S (27) with respect to the auxiliary fields χ, η and $\bar{\eta}$. The calculation of the corresponding functional derivatives is straightforward (though lengthy) and we find the following equations of motion:

$$
\begin{aligned}
\chi_{\mu\nu}(x) - i\hbar g^2 \epsilon^{\bar{a}\bar{b}3} \, \mathsf{D}^{\bar{a}\bar{b}}_{\mu\nu}(x, x) &= 0 \\
\int \bar{\eta}^{\bar{a}} \cdot \Omega^{\bar{a}\bar{b}}[a, \chi, \eta, \bar{\eta}] \cdot \eta^{\bar{b}} &= 0
\end{aligned}
\tag{34}
$$

Here, the kernel in the ghost eom. is explicitly given by

$$
\Omega^{\bar{a}\bar{b}}(x, y) := \left[\left[-D^{(\mathrm{n})}_{\sigma} D^{(\mathrm{n})}_{\sigma} \right]^{\bar{a}\bar{b}}(x) + \hbar g^2 \mathsf{D}^{\bar{a}\bar{b}}_{\mu\mu}(x, x) - \hbar g^2 \delta^{\bar{a}\bar{b}} \mathsf{D}^{\bar{c}\bar{c}}_{\mu\mu}(x, x) \right] \delta(x, y).
\tag{35}
$$

The gap equations (34) as well as their solutions depend of course on the Abelian gauge field a_μ and in lowest order saddle point approximation the photon interaction S_{int} from (26) is given by the effective action (27) taken at the solution of (34). Thus, the direct application of the saddle point method to the functional integral (26) requires the solution of eq. (34) for *arbitrary* photon fields a_μ.

To obtain the full YM partition function the remaining integral over the Abelian field a_μ has to be carried out, too. In order to treat all fields on equal footing, one may wish to perform the a_μ-integral in saddle point approximation as well. In this case we would have to solve (34) together with the a_μ-equation (obtained by varying $S_0 + S_{\text{int}}$)

$$
a_\mu + g^2 \left[D_0 \right]_{\mu\nu} \frac{\delta S_{\text{int}}}{\delta a_\nu} = 0.
\tag{36}
$$

12

This system of equations is quite complicated to solve, in general, but one easily checks that it has the particular solution $a = \chi = \eta = \bar{\eta} = 0$. The semiclassical expansion around this *trivial* saddle point corresponds to the standard perturbation theory as shown in the next chapter.

Some remarks on the trivial solution are in order. First of all, the ghost eom. *always* (i.e. independently of the value of a and χ) allows for the solution $\eta = \bar{\eta} = 0$ and this is in fact the *only* one if the kernel Ω is invertible. The existence of a non-vanishing ghost condensate $\langle \bar{\eta}\eta \rangle \neq 0$ is not generally ruled out, but in the present paper we are mainly interested in the perturbative aspects of our formulation, and thus we confine ourselves to $\eta = \bar{\eta} = 0$.

With the ghost fields set to zero, the remaining gap equation for χ always allows for the trivial solution $\chi = 0$. To see this, set $\chi = 0$ in the a_μ-equation (36) and find $a_\mu = 0$. This in turn implies that the charged gluon propagator becomes diagonal in colour space $\mathsf{D}^{\bar{a}\bar{b}} \sim \delta^{\bar{a}\bar{b}}$, which also persists at any finite order of a semiclassical expansion for the a_μ-integral. Physically, it means that with this trivial saddle point, the charged gluon never changes its colour while propagating.

Due to its inhomogeneity we expect the χ-equation (34) to allow for non-trivial solutions as well, describing non-perturbative effects in the photon interaction. This is what happens in the so-called field strength approach [10]. However in the present work, we are mainly interested in perturbation theory, and therefore we concentrate on the trivial solution $\chi \equiv 0$. Note that the standard field strength approach leads to a strong coupling expansion which does not allow for an immediate contact to perturbation theory.

In the rest of this paper, we will focus on the detailed description of the semiclassical type of expansion around the trivial saddle point. In particular, we will show how to recover the usual loop expansion in our formulation and we will calculate the β-function for the running coupling constant to one-loop order.

3 The Loop Expansion

In this section we study the generating functional (20,26) in a semiclassical expansion around the trivial saddle point discussed in the last section.

3.1 The Effective Action in a Semiclassical Expansion

Our starting point is the generating functional for the full YM theory, as obtained by inserting the effective photon action (26) in eq. (20):

$$Z[j, \ldots] = \int \mathcal{D}(a, \chi, \eta, \bar{\eta}) \exp\left\{ -\frac{1}{\hbar}(S_0 + \mathsf{S}) + \text{cur.} \right\}.$$

Here, "cur." stands for the current terms in the exponent of (26) plus the $j_\mu^{(\mathrm{n})} \cdot a_\mu$ coupling from (20).

The following calculation is somewhat cumbersome due to the number of integration variables in Z. We will use the superfield notation [16] and collect the integration variables in the symbol $\Theta = \{a, \eta, \bar\eta, \chi\}$ with the corresponding currents denoted by $\theta = \{j^{(\mathrm{n})}, \bar\xi, \xi, \vartheta\}^7$. The generating functional depends on two additional sources $\lambda = \{j_\mu^{(\mathrm{ch})}, \varphi\}$ referring to the fields $\Lambda = \{A_\mu^{(\mathrm{ch})}, \phi\}$ that have been integrated out in our approach. With this compact superfield notation, we can re-express the generating functional as

$$Z[\theta, \lambda] = \int \mathcal{D}\Theta \exp\left\{ -\frac{1}{\hbar g^2} \left(\mathcal{F} + \mathcal{F}_{\mathrm{cur}} \right) \right\}, \tag{37}$$

where $\mathcal{F}[\Theta] = g^2(S_0 + \mathsf{S})$ is the part of the action independent of the currents. The explicit forms for $\mathcal{F}$ and $\mathcal{F}_{\mathrm{cur}}$ can be found in appendix B.

Our goal is to perform the superfield integration in eq. (37) in a semiclassical expansion around the *trivial* saddle point $\bar\Theta$ of the exponent. Since we include the current terms in the action the saddle point itself will be a functional of the currents, $\bar\Theta = \bar\Theta[\theta, \lambda]$. The explicit form of this dependence is not needed, however, for the calculation below. All we require is that at zero currents, the saddle point is also zero: $\bar\Theta[\theta = \lambda = 0] = 0$. The existence of this *trivial* saddle point at zero currents was shown in the last section.

Let us now expand the exponent $\mathcal{F} + \mathcal{F}_{\mathrm{cur}}$ of (37) to second order around the saddle point $\bar\Theta$ and perform the Gaussian superfield integration. Taking the logarithm, $W = -\hbar g^2 \ln Z$, we obtain the generating functional for connected Green's functions:

$$W[\theta, \lambda] = \mathcal{F}[\bar\Theta] + \mathcal{F}_{\mathrm{cur}}[\bar\Theta, \theta, \lambda] + \frac{\hbar g^2}{2} \mathrm{Tr} \ln \left. \frac{\delta^2(\mathcal{F} + \mathcal{F}_{\mathrm{cur}})}{\delta\Theta^2} \right|_{\Theta = \bar\Theta} + \mathcal{O}(\hbar^2). \tag{38}$$

The supertrace term on the rhs. is the next to leading correction in our semiclassical expansion. It contains terms of order $\mathcal{O}(\hbar)$ and $\mathcal{O}(\hbar^2)$, i.e. the expansion around the trivial saddle point is *not* directly organised in powers of $\hbar$. What we claim, however, is that to a given order in the saddle point approximation (SPA), we find *all* the standard loop corrections to this order, plus some higher contributions in $\hbar$. Thus, given the SPA to a fixed order, we can rearrange it in powers of $\hbar$ such that we recover the loop series.

In the formula (38), the explicit dependence of the saddle point $\bar\Theta$ on the currents is needed. This can be avoided by performing a Legendre transformation to obtain the effective action Γ as generating functional for 1PI Green's functions. In principle, this transformation is straightforward: First we define classical superfields Θ, Λ as

---

^{7}A source $\vartheta_{\mu\nu}$ for the neutral field $\chi_{\mu\nu}$ has been introduced for notational reasons.

functionals of the currents:

$$\Theta[\theta, \lambda] = -\frac{\delta W[\theta, \lambda]}{\delta \theta} \quad , \quad \Lambda[\theta, \lambda] = -\frac{\delta W[\theta, \lambda]}{\delta \lambda}. \tag{39}$$

Remember that these superfields stand for the component classical fields $\Theta = \{a_\mu, \eta, \bar{\eta}, \chi_{\mu\nu}\}$ and $\Lambda = \{A_\mu, \phi\}$. With this definition, the effective action can be written as

$$g^2 \Gamma[\Theta, \Lambda] = W[\theta[\Theta, \Lambda], \lambda[\Theta, \Lambda]] + \int \Theta \theta[\Theta, \Lambda] + \int \Lambda \lambda[\Theta, \Lambda]. \tag{40}$$

Following the remarks above, we will rearrange the expression (38) for W in powers of $\hbar$ and calculate the effective action in a $\hbar$-expansion, $\Gamma = \Gamma_0 + \hbar \Gamma_1 + \cdots$. We start by inserting (38) in the definition of the classical superfields (39) and expand in $\hbar$:

$$\Theta[\theta, \lambda] = \bar{\Theta}[\theta, \lambda] + \mathcal{O}(\hbar) \quad , \quad \Lambda[\theta, \lambda] = -\left.\frac{\delta(\mathcal{F} + \mathcal{F}_{\text{cur}})}{\delta \lambda}\right|_{\Theta = \bar{\Theta}} + \mathcal{O}(\hbar). \tag{41}$$

The first equation is what one usually expects: The classical field coincides with the saddle point value to lowest order. This fact can be used to simplify the calculation of the effective action. When inserting our result (38) for W in the definition (40) of Γ and expanding in powers of $\hbar$, we can simply replace $\bar{\Theta} \to \Theta$ to one-loop order and the saddle point $\bar{\Theta}$ drops out of the calculation.

The final step is now to eliminate in Γ the currents θ and λ in favour of the classical fields by inverting (39). Recall that the fields $\Lambda = \{A_\mu^{(\text{ch})}, \phi\}$ have been integrated out, but the generating functional still contains the corresponding currents $\lambda = \{j_\mu^{(\text{ch})}, \varphi\}$. Therefore, the second eq. (39) defines classical fields $\Lambda = \{A_\mu, \phi\}$ which should not be confused with the original integration variables. The current for the auxiliary field χ has been introduced for notational reasons only and the corresponding classical field χ is therefore not an independent quantity. By the path integral linearisation (19), χ is the field conjugate to the commutator term $[A_\mu^{(\text{ch})}, A_\nu^{(\text{ch})}]$. In terms of the classical field A_μ, we therefore expect to lowest order:

$$\chi = i\epsilon^{\bar{a}\bar{b}3} A_\mu^{\bar{a}} A_\nu^{\bar{b}} + \mathcal{O}(\hbar). \tag{42}$$

The explicit derivation of (42) can be found in appendix B, together with the elimination of the currents in Γ. As our final result we find the following structure:

The lowest oder part of the effective action coincides precisely with the initial BRST-action (13), i.e.

$$\Gamma_0[a_\mu, A_\mu, \bar{\eta}, \eta, \phi] = S_{\text{eff}}[a_\mu, A_\mu, \bar{\eta}, \eta, \phi] \qquad \text{see (13) .} \tag{43}$$

15

Since we have completely reformulated the non-Abelian part of the theory, this is a non-trivial test for the calculation. Our main interest, however, lies in the one-loop corrections, which are given by

$$\Gamma_1 = \frac{1}{2}\mathrm{Tr}\ln\mathsf{H} + \frac{1}{2}\mathrm{Tr}\ln\mathsf{L} + \frac{1}{2}\mathrm{Tr}\ln\frac{\delta^2\mathsf{F}}{\delta\Theta^2}\Big|_\Theta . \tag{44}$$

The arguments of the operators in this formula are the classical fields (cf. (42)).
The operator F in the supertrace on the rhs. of (44) is quite complicated, see appendix B. This is mainly due to the fact that it explicitly contains the currents $\lambda = \{j_\mu^{(\mathrm{ch})}, \varphi\}$ as functional of the classical fields, leading to rather messy expressions. We refrain from presenting them in any detail here.
At this point, we can explore the fact that we are expanding around the *trivial* saddle point: At zero currents, the classical fields are zero and vice versa. This means that in the effective action we can simply set the complicated currents $\lambda = \{j_\mu^{(\mathrm{ch})}, \varphi\} = 0$ if the corresponding classical fields A_μ, ϕ vanish. Thus, we restrict ourselves to 1PI diagrams with no external charged gluons and multiplier lines. For the purpose of calculating the β-function, this is sufficient if we choose to define the coupling constant through the ghost-photon-vertex. By BRST-invariance, i.e. the Slavnov-Taylor identities, any other definition must give the same answer. In this special case, we can actually expand the supertrace in (44) in its component blocks with the result[8]

$$\Gamma_1[\mathsf{a}_\mu, \bar{\boldsymbol{\eta}}, \boldsymbol{\eta}, \mathsf{A}_\mu = 0, \boldsymbol{\phi} = 0] = \frac{1}{2}\mathrm{Tr}\ln\mathsf{H} + \frac{1}{2}\mathrm{Tr}\ln\mathsf{L} - \mathrm{Tr}\ln\left[-D_\sigma^{(\mathrm{n})}D_\sigma^{(\mathrm{n})}\right] +$$
$$+ \frac{1}{2}\mathrm{Tr}\ln\left[\left(D_0^{-1}\right)_{\mu\nu} + 2\,\delta_{\mu\nu}\bar{\boldsymbol{\eta}}^{\bar{a}}\boldsymbol{\eta}^{\bar{a}} - 2\,\Delta_{\mu\nu}[\bar{\boldsymbol{\eta}}, \boldsymbol{\eta}, \mathsf{a}]\right] \tag{45}$$

where $\Delta_{\mu\nu}$ is a quadratic form in the ghost fields, which is given in appendix B.
Let us finally consider the case that we only have external photon lines, i.e. examine the effective photon theory. In this case, we only retain the photon field in the effective action and obtain the simple one-loop result

$$\Gamma_0[\mathsf{a}_\mu] = \frac{1}{2g^2}\int \mathsf{a}_\mu\left[D_0^{-1}\right]_{\mu\nu}\mathsf{a}_\nu\, d^4x$$
$$\Gamma_1[\mathsf{a}_\mu] = \frac{1}{2}\mathrm{Tr}\ln\mathsf{H} + \frac{1}{2}\mathrm{Tr}\ln\mathsf{L} - \mathrm{Tr}\ln\left[-D_\sigma^{(\mathrm{n})}D_\sigma^{(\mathrm{n})}\right] \tag{46}$$

with all operators taken at $\chi = \eta = \bar{\eta} = 0$.

---

[8]From the $(\bar{\eta}, \eta)$ block, we obtain the covariant Laplacian, $-\mathrm{Tr}\ln\left(-D^{(\mathrm{n})}D^{(\mathrm{n})}\right)$, where the minus sign originates from the fermion nature of the ghost loop. As mentioned at the end of section 3.2, this "perturbative" FP determinant cancels against the corresponding term in (33).

3.2 The MAG symmetry

The MAG looks very similar to a background gauge with the neutral photon playing the role of the background field. This suggests that the β-function for the running coupling constant only depends on the photon propagator, if one chooses to define the coupling constant through a vertex involving a_μ. In this section, we will prove this statement explicitly for the $\bar{\eta}\eta a_\mu$-vertex,[9] i.e. we show that the one-loop corrections to this vertex are exactly cancelled by the corrections to the ghost propagator, so that the coupling g defined by this vertex is exclusively renormalised by the neutral vacuum polarisation:[10]
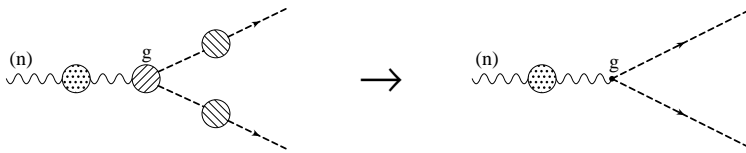
To this end, we keep only the ghost and photon fields in the effective action, i.e. we start from (45).

For the computation, we proceed as follows: The n-point functions are obtained as functional derivatives of the effective action calculated in the last section. For a typical term $\Gamma_1 \simeq \mathrm{Tr}\ln\Omega$ (Ω some operator) this gives the trace of a sequence of *propagators* Ω^{-1} and *vertices* (derivatives of Ω), both taken at zero fields. Introducing momenta by a Fourier representation, all the spacetime integrations in the trace can be performed. In this way we recover the usual Feynman diagrams in momentum space. Some cautionary remarks on the remaining loop integration over the internal momentum are in order:

In the calculation below, we will frequently encounter loop integrations which are not only logarithmically, but also linearly and quadratically divergent (by power counting). The counterterms for this latter divergences are ambiguous, i.e. they depend on the regularisation method, and, even worse, they correspond to terms (e.g. photon mass terms) that are not present in the initial Lagrangian. The reason for this problem is clear: The residual $U(1)$ gauge invariance (25) is "artificially" broken by the regularisation scheme. As a consequence, the Green's functions are not of the form required by the Ward identities.

There are two possibilities to cure this situation: In a regularisation independent scheme like BPHZ, one subtracts off a sufficient number of leading terms in the Taylor expansion around zero momentum. This removes the dangerous ambiguities.

---

[9]We have checked this statement for the $A^{(\mathrm{ch})}A^{(\mathrm{ch})}a$ triple gluon vertex as well. As explained above the explicit calculations are rather cumbersome in this case and were in fact performed with the help of a symbolic algebra program. In order to present the formulae as simple as possible, we take the ghost-photon vertex to define the coupling constant.

[10]Here and in the following the wavy and the dashed lines refer to gluons (neutral or charged as indicated) and ghosts, respectively.

A more obvious method is to use a gauge invariant regularisation scheme from the very beginning. At least within the class of such regularisations, the results should be unambiguous and the dangerous divergences should simply drop out.

We have performed our calculations in two independent, gauge invariant regularisation schemes: In the main text, we will use dimensional regularisation to $d = 4 - \epsilon$ spacetime dimensions. There is one problem with this scheme, namely the fact that quadratic divergences are automatically set to zero. Thus, this scheme does not allow for a check whether the dangerous terms really *cancel*. In order to parametrise the ambiguities (rather then setting them to zero from the outset), we have redone all the calculations using *proper time regularisation*, which preserves gauge invariance. This also gives a check on the scheme independence mentioned above.

Finally we should also remark that formally correct operations with divergent integrals can lead to ambiguous results. While we can always come to finite expressions by a suitable cutoff procedure, some arithmetic operations may be in conflict with the regularisation. In all of our calculations, we have only performed such manipulations that are allowed by the respective regularisation scheme. In practice, the main feature of a gauge invariant regularisation is the possibility to shift the loop momentum.

Some more details on the regularisation procedure as well as some results of the proper time calculation can be found in appendix C.

Let us now present the results of the explicit calculation:

1. Ghost vacuum polarisation: The loop corrections to the ghost propagator are defined by[11]

$$\left\langle \bar{\eta}^{\bar{a}}(x)\eta^{\bar{b}}(y) \right\rangle = \hbar \left[\frac{\delta^2 \Gamma}{\delta_r \eta^{\bar{b}}(y)\delta_l \bar{\eta}^{\bar{a}}(x)} \right]^{-1} = \hbar g^2 \left[\mathsf{M}_0^{-1} - \mathsf{M}_0^{-1} \cdot \Pi^{\text{(gh)}} \cdot \mathsf{M}_0^{-1} + \cdots \right], \quad (47)$$

where the free propagator is $\mathsf{M}_0^{\bar{a}\bar{b}}(p) = \delta^{\bar{a}\bar{b}}/p^2$ (see eq. (12)) and

$$\Pi_{\bar{a}\bar{b}}^{\text{(gh)}}(x,y) = \hbar g^2 \frac{\delta \Gamma_1}{\delta_r \eta^{\bar{b}}(y)\delta_l \bar{\eta}^{\bar{a}}(x)} \bigg|_0 \quad (48)$$

is the one-loop correction.

There are three contributions from (45) that are proportional to $\int \frac{d^d k}{k^2}$ and vanish in dimensional regularisation, namely the H and L contribution and the piece coming from the second term in the last trace (45). In addition there is a contribution from the complicated Δ-term in the last trace (45). In momentum space, it reads

$$\Pi_{\bar{a}\bar{b}}^{\text{(gh)}}(p) = -\hbar g^2 \mu^\epsilon \delta^{\bar{a}\bar{b}} \int \frac{d^d k}{(2\pi)^d} \left[D_0 \right]_{\nu\mu}(k) \frac{(2p+k)_\mu (2p+k)_\nu}{(k+p)^2}$$

---

[11]Here and in the following, all derivatives with respect to Grassmann fields $\bar{\eta}$ or η are taken to be *left* or *right* derivatives, as indicated.

18

$$= \hbar g^2 \, \delta^{\bar{a}\bar{b}} \, \frac{\alpha - 3}{8\pi^2 \epsilon} \, p^2 + \text{finite} \ . \tag{49}$$

where the arbitrary scale μ must be introduced to keep the renormalised coupling constant dimensionless in $d \neq 4$. Note that this non-vanishing contribution can be identified with the diagram
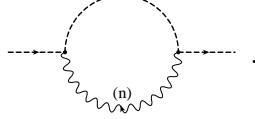
There is a quadratic divergence in (49) which cancels exactly against the three contributions $\sim \int \frac{d^d k}{k^2}$ discussed above, if we parametrise it in a gauge invariant way. See the proper time calculation given in appendix C for an example of such a cancellation.

The finite part in (49) depends on the arbitrary scale μ, whereas the divergent part does not. As a consequence, the corresponding counterterm in the MS-scheme,

$$\delta \Gamma = \frac{1}{g^2 \mu^\epsilon} \int \bar{\eta}^{\bar{a}} \cdot \left[-\Pi^{\bar{a}\bar{b}}_{(\mathrm{gh})} \Big|_{\mathrm{div}} \right] \cdot \eta^{\bar{b}} = \frac{1}{g^2 \mu^\epsilon} \int \bar{\eta}^{\bar{a}} \left(-\Box \right) \eta^{\bar{a}} \cdot \left(\hbar g^2 \frac{3 - \alpha}{8\pi^2 \epsilon} \right) \tag{50}$$

depends on the scale μ only by the overall prefactor. Since (50) is proportional to the free ghost term in the initial BRST action (in d dimensions), we can renormalise *multiplicatively*. Our definitions of renormalisation constants are

$$
\begin{aligned}
g_0 \, \eta_0 &= Z^{1/2}_{\mathrm{gh}} \cdot g \mu^{\epsilon/2} \, \eta \qquad ; \qquad g_0 \, \bar{\eta}_0 = Z^{1/2}_{\mathrm{gh}} \cdot g \mu^{\epsilon/2} \, \bar{\eta} \\
g_0 \, a^{(0)}_\mu &= Z^{1/2}_n \cdot g \mu^{\epsilon/2} \, a_\mu \\
g_0 &= g \mu^{\epsilon/2} \cdot Z_g Z^{-1/2}_n Z^{-1}_{\mathrm{gh}}
\end{aligned}
\tag{51}
$$

where the bare quantities are denoted by an index '0'. In perturbation theory one usually rescales the fields by a factor of g (as compared to our conventions) in order to get rid of the prefactor $1/(g^2 \mu^\epsilon)$ in the action. We have inserted extra factors of g and g_0 in the definition of the wave-function renormalisation such that the choice (51) gives the same Z's as with the perturbative convention. From (50) and (51), we can read off Z_{gh}:

$$Z_{\mathrm{gh}} = 1 - \hbar g^2 \frac{\alpha - 3}{8\pi^2 \epsilon} + \cdots \tag{52}$$

2. Ghost-photon vertex: As usual, we define the vertex through the amputated three-point function
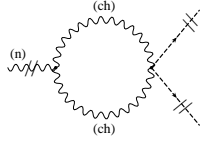
$$- g^2 \left. \frac{\delta^3 \Gamma}{\delta_r \eta^{\bar{b}}(z) \delta_l \bar{\eta}^{\bar{a}}(y) \delta a_\mu(x)} \right|_0 \equiv i \epsilon^{\bar{a}\bar{b}3} \int \frac{d^4(p,q)}{(2\pi)^8} e^{-ip(x-y) - iq(x-z)} \, G_\mu(p,q). \tag{53}$$

19

The bare vertex from the initial BRST-action is

$$G_\mu^{(0)}(p,q) = q_\mu - p_\mu. \tag{54}$$

By power counting, the one-loop corrections to this vertex are linearly divergent. Note however, that the results are unambiguous in any gauge invariant regularisation scheme, i.e. if we can shift the loop momentum in the regularised integral.

As an example, consider the contribution from the first trace in (45). This correction to $G_\mu(p,q)$ can be identified with the following diagram:
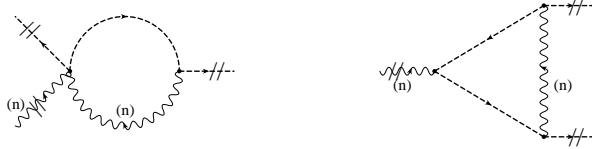

$$= -\hbar g^2 d \int \frac{d^d k}{(2\pi)^d} \frac{(2k+p+q)_\mu}{k^2(k+p+q)^2} = 0. \tag{55}$$

Clearly, this will vanish in *any* regularisation scheme that allows for the shift $k \mapsto p+q-k$ in the loop momentum. Note that the diagram (55) manifestly is a function of $p+q$ and thus *not* proportional to the bare vertex (54) as requested by the Ward identities. It is therefore clear that this diagram has to vanish in any gauge invariant regularisation scheme.

The same analysis also applies to the contribution from the second trace in (45). The calculations are a bit more involved, but the resulting loop integral vanishes identically in any gauge invariant scheme, just as in (55).

Let us now look at the non-vanishing loop corrections to the vertex. They come from the last trace in the effective action (45) and split into two parts which can be identified with the following diagrams:

$$\tag{56}$$

Each of these diagrams is proportional to the bare vertex (54). We find

$$2\hbar g^2 \mu^\epsilon \int \frac{d^d k}{(2\pi)^d} [D_0]_{\mu\nu}(k) \left[\frac{(2p-k)_\nu}{(k-p)^2} - \frac{(2q+k)_\nu}{(k+q)^2} \right] = (q-p)_\mu \cdot \hbar g^2 \frac{-3}{8\pi^2 \epsilon} + \text{finite}$$

for the left diagram. The triangle diagram vanishes in Lorentz gauge for the internal photon line:

$$-\hbar g^2 \mu^\epsilon \int \frac{d^d k}{(2\pi)^d} [D_0]_{\sigma\tau}(k) \frac{(2p-k)_\sigma (2q+k)_\tau (2k+q-p)_\mu}{(k-p)^2(k+q)^2} = (q-p)_\mu \cdot \hbar g^2 \frac{\alpha}{8\pi^2 \epsilon} + \cdots.$$

Note that in the proper time formalism we would find the same results with the substitution $1/\epsilon \leftrightarrow 1/2 \ln(\Lambda^2/p^2)$.

To construct a counterterm from these diagrams in the MS scheme, we add the negative divergences to the initial vertex:

$$G_\mu(p, q) \equiv G_\mu^{(0)}(p, q) \cdot Z_g \qquad ; \qquad Z_g = 1 + \hbar g^2 \frac{3 - \alpha}{8\pi^2\epsilon} + \cdots. \tag{57}$$

The counterterm is then given by Fourier transforming and multiplying with external fields to undo the differentiation in (53). This brings in additional Z-factors and yields the relation (51) between the bare and renormalised coupling constants.

The cancellation condition $Z_g = Z_{\text{gh}}$ means that the coupling constant defined through the ghost-photon vertex is only renormalised due to the photon vacuum polarisation. The latter is therefore sufficient to determine the β-function in the MAG.

3.3   The Photon Vacuum Polarisation

We have seen in the last section that the running coupling constant in the MAG is determined by the renormalisation of the Abelian gluon propagator alone. To calculate the correction to the photon propagator, we only retain the photon field in the one-loop effective action, i.e. we start from (46).

The vacuum polarisation is connected with the full two-point function by

$$\langle a_\mu(x)a_\nu(y)\rangle = \hbar \left[\frac{\delta^2\Gamma}{\delta a_\mu(x)\delta a_\nu(y)}\Big|_0 \right]^{-1} = \hbar \left[\frac{1}{g^2}\left(D_0^{-1}\right)_{\mu\nu} + \frac{1}{g^2}\Pi_{\mu\nu}^{(\text{n})} \right]^{-1}$$

$$= \hbar g^2 \left[D_0 - D_0\Pi^{(\text{n})}D_0 + \cdots \right]_{\mu\nu},$$

which in turn leads to

$$\Pi_{\mu\nu}^{(\text{n})}(x, y) \equiv \hbar g^2 \frac{\delta^2\Gamma_1}{\delta a_\mu(x)a_\nu(y)}\Big|_0. \tag{58}$$

Note that $\Pi^{(\text{n})}{}_{\mu\nu}$ is transversal in any gauge invariant regularisation scheme, as a general consequence of the Ward identities from the residual $U(1)$ symmetry (25). Let us now consider the contribution from the three determinants in the effective photon action (46).

The determinant of the covariant Laplacian is well known from e.g. heat kernel expansions. In momentum space, we find

$$\Pi_{\mu\nu}^{(\text{n})}(p) \stackrel{\text{lapl.}}{=} -\hbar g^2 \mu^\epsilon \kappa \left[2\delta_{\mu\nu} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2} - \int \frac{d^dk}{(2\pi)^d} \frac{(2k + p)_\mu(2k + p)_\nu}{k^2(k + p)^2} \right]$$

$$= -\hbar g^2 \frac{\kappa}{24\pi^2\epsilon} \left[\delta_{\mu\nu}p^2 - p_\mu p_\nu \right] + \text{finite.} \tag{59}$$

21

The colour factor $\kappa = \delta^{\bar{c}\bar{c}} = \epsilon^{\bar{a}\bar{b}3}\epsilon^{\bar{a}\bar{b}3} = 2$ has been introduced for later comparison with the $SU(N)$ result.[12] The formula (59) corresponds to the two diagrams
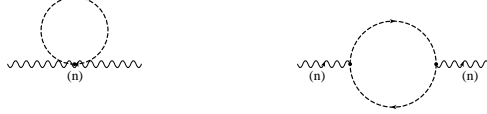
where the first one originates from the non-linearity of the MAG. If we regularise these diagrams using the proper time method (cf. appendix C), we find an exact cancellation of the inherent quadratic divergences between the two diagrams. The remaining logarithmic divergence leads to a transversal result, as expected.

The H-determinant in (46) describes the (unprojected) propagation of charged gluons, as expressed in the diagrams
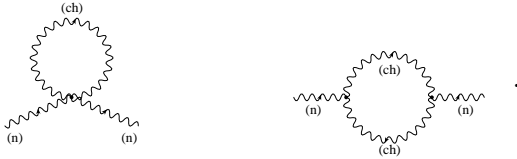
The analytical expression is

$$
\begin{aligned}
\Pi^{(n)}_{\mu\nu}(p) \;\overset{\mathsf{H}}{=}\;& \hbar g^2\mu^\epsilon\kappa\,\delta_{\mu\nu}\int\frac{d^dk}{(2\pi)^d}\frac{d}{k^2} - \\
& -\hbar g^2\mu^\epsilon\frac{\kappa}{2}\int\frac{d^dk}{(2\pi)^d}\frac{8\,\delta_{\mu\nu}p^2 + 4dk_\mu k_\nu + 2d(k_\mu p_\nu + k_\nu p_\mu) + (d-8)p_\mu p_\nu}{k^2(k+p)^2} \\
=\;& -\hbar g^2\kappa\left[\frac{5}{12\pi^2\epsilon}\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right) + \cdots\right].
\end{aligned}
\tag{60}
$$

As before, the proper time calculation of appendix C confirms that the quadratic divergences cancel between the two diagrams, leaving us with the unambiguous (and transversal) result (60).

Finally, let us consider the L-determinant in the effective action (46), which describes the effect of the multiplier field, i.e. the projection onto MAG. The calculation turns out to be very tedious, but for completeness, let us quote the final result:

$$
\begin{aligned}
\Pi^{(n)}_{\mu\nu}(p) \;\overset{\mathsf{L}}{=}\;& \frac{\kappa}{2}\hbar g^2\mu^\epsilon\int\frac{d^dk}{(2\pi)^d}\left[\frac{\mathcal{R}_{\mu\nu}(p,k)}{k^2(k+p)^2} + \frac{\mathcal{S}_{\mu\nu}(p,k)}{k^4(k+p)^2} + \frac{\mathcal{T}_{\mu\nu}(p,k)}{k^4(k+p)^4}\right] \\
\mathcal{R}_{\mu\nu}(p,k) \;=\;& -2(k_\mu p_\nu + k_\nu p_\mu) + 2p_\mu p_\nu + 8\delta_{\mu\nu}(k\cdot p) \\
\mathcal{S}_{\mu\nu}(p,k) \;=\;& 8\left[p^2 k_\mu k_\nu - (k\cdot p)(k_\mu p_\nu + k_\nu p_\mu) + \delta_{\mu\nu}(k\cdot p)^2\right] \\
\mathcal{T}_{\mu\nu}(p,k) \;=\;& -p^4 k_\mu k_\nu + p^2(k\cdot p)(k_\mu p_\nu + k_\nu p_\mu) - p_\mu p_\nu(k\cdot p)^2.
\end{aligned}
\tag{61}
$$

---

[12]From $\epsilon^{abc}\epsilon^{abd} = \delta^{cd}\kappa$, the factor κ is simply the quadratic Casimir of the adjoint rep. of $SU(2)$.

All the divergences in this expression cancel, giving a finite result.[13] In particular, the quadratic divergences in this expression,

$$\frac{\kappa}{2}\hbar g^2 \mu^\epsilon \delta_{\mu\nu} \int \frac{d^d k}{(2\pi)^d} \left[\frac{1}{(k+p)^2} + \frac{1}{(k-p)^2} - \frac{2}{k^2} \right]$$

cancel in *any* regularisation scheme that allows to shift the loop momentum. Putting all results together, we obtain for the divergent part of the one-loop photon vacuum polarisation

$$\Pi_{\mu\nu}^{(n)}(p) = -\hbar g^2 \frac{11\kappa}{24\pi^2 \epsilon} \left[\delta_{\mu\nu} p^2 - p_\mu p_\nu \right] + \mathcal{O}(\hbar^2). \tag{62}$$

3.4 The One-Loop Beta-Function

The counterterm for (62) in the MS-scheme is fixed by the requirement that it cancels the divergence in $\Pi^{(n)}$ without any finite part. In general, we will express the action in terms of the renormalised quantities and add the counterterm contribution to obtain finite Green's functions. Alternatively, the Lagrangian can be written in terms of the bare fields $\left\{ a_\mu^{(0)}, \bar\eta_0, \eta_0 \right\}$ and couplings g_0, where g_0 is not running and the residue of the propagator for the bare fields is fixed to 1 by the canonical commutation relations. For the Abelian vacuum polarisation, this results in[14]

$$
\begin{aligned}
\Gamma + \delta\Gamma &= \frac{1}{2g^2\mu^\epsilon} \int \frac{d^4 p}{(2\pi)^4} \, a_\mu(-p) \left[\delta_{\mu\nu} p^2 - p_\mu p_\nu \right] a_\nu(p) \cdot \left(1 + \hbar g^2 \frac{11\kappa}{24\pi^2\epsilon} \right) + \cdots \\
&\stackrel{!}{=} \frac{1}{2g_0^2} \int \frac{d^4 p}{(2\pi)^4} \, a_\mu^{(0)}(-p) \left[D_0^{-1} \right]_{\mu\nu}(p) \, a_\nu^{(0)}(p) + \cdots
\end{aligned}
\tag{63}
$$

and our definitions (51) lead to

$$Z_n = 1 + \hbar g^2 \frac{11\kappa}{24\pi^2 \epsilon} + \cdots. \tag{64}$$

The coupling constant renormalisation is given by the loop corrections to any of the vertices of the theory. For the $\bar\eta\eta a_\mu$-vertex, this results in the relation (51) between the bare and renormalised couplings

$$g_0 = g\mu^{\epsilon/2} \cdot Z_g \, Z_n^{-1/2} \, Z_{\text{gh}}^{-1}. \tag{65}$$

As explained in section 3.2, the cancellation $Z_g = Z_{\text{gh}}$ holds, so that the coupling constant from the $\eta\bar\eta a_\mu$-vertex is determined by the Abelian vacuum polarisation (62) alone:

$$g_0 = g\mu^{\epsilon/2} Z_n^{-1/2} = g\mu^{\epsilon/2} \left[1 - \hbar g^2 \frac{11\kappa}{48\pi^2 \epsilon} + \mathcal{O}(\hbar^2) \right]. \tag{66}$$

---

[13]It might very well be zero in any gauge invariant regularisation, but we have not checked this.

[14]Since the vacuum polarisation is transversal, there is no correction to the longitudinal part of D_0^{-1}. Thus, the gauge fixing parameter α is not renormalised.

By gauge invariance, a similar cancellation must be present in any other definition of the coupling constant (involving a_μ), e.g. the triple gluon vertex. Note that in this respect, the MAG closely resembles the background gauge formalism, with the neutral photon figuring as background field.

From (66) the β-function is easily found as $\epsilon \to 0$:

$$\beta(g) \equiv \mu \frac{\partial g}{\partial \mu} = -\hbar \cdot \beta_0 g^3 + \mathcal{O}(\hbar^2) \qquad ; \qquad \beta_0 = \frac{11\kappa}{48\pi^2} > 0. \qquad (67)$$

This is the known one-loop result for a gauge group $G = SU(2)$.

In summary, we have shown that the semiclassical expansion around the trivial saddle point of our reformulated theory coincides with the standard perturbation theory. In particular, we find that the vacuum polarisation of the Abelian gauge field alone accounts for the asymptotic freedom of $SU(2)$ YM theory in the MAG. The charged field contributions cancel exactly.

4   Conclusions

In this paper we have presented a reformulation of $SU(2)$ Yang-Mills theory in the maximal Abelian gauge where the charged gauge fields $A_\mu^{(\mathrm{ch})}$ are exactly integrated out at the expense of an Abelian tensor field $\chi_{\mu\nu}$. From a physical standpoint, this field (or at least its saddle point value) describes the commutator term of the YM field strength in a non-perturbative manner.

The upshot of this formulation is an effective Abelian theory equivalent to $SU(2)$ YM with monopole-like gauge fixing singularities. The action contains a free Maxwell piece and a manifest $U(1)$ invariant self-interaction S_{int} described in terms of the new field χ (and the FP ghosts for the MAG). We derived the equation of motion for these fields and showed that, unlike the so-called field strength approach [10], there exists a trivial solution with a definite physical meaning: It gives access to the standard loop expansion, i.e. the short distance physics.

The detailed analysis of the semiclassical expansion around this trivial saddle point shows that the Abelian self-interaction alone is responsible for the asymptotic freedom of the full YM theory. This is due to a complete cancellation of all diagrams with external charged legs in the divergent part of the one-loop vertex correction. The coupling constant is only renormalised through the Abelian vacuum polarisation, which leads to the correct value for the one-loop β-function. In this sense, asymptotic freedom in the maximal Abelian gauge can be understood in terms of the Abelian gauge field (photon) alone.

This important result is another attractive feature of MAG. It shows that in this gauge, not only the low-energy confinement properties (string tension) is dominated by the Abelian field configurations (as seen on the lattice [4]), but also the asymptotic freedom. This fact make this gauge indeed very attractive.

24

In the non-perturbative regime, recent lattice calculations indicate that the long distance behaviour of YM theory in MAG comes almost entirely from the Abelian field configurations, i.e. the effective Abelian theory gives rise to absolute colour confinement by a condensation of the aforementioned magnetic monopoles leading to the dual Meissner effect. As yet, we do not have a satisfying theoretical understanding for the suppression of charged gluon dynamics in low energy observables like the Wilson loop (Abelian dominance). The modification of the charged gluon propagator caused by non-trivial solutions of the χ-eom could provide further insight into this question, as will be shown in a forthcoming publication.

A The Propagators D and L

In this appendix we give the full form of the multiplier and charged gluon propagator. This illustrates the shorthand definition given in the main text. Firstly, L^{-1} can be interpreted as propagator of the multiplier field ϕ. Its inverse is given explicitly by

$$\mathsf{L}_{\bar{a}\bar{b}}(x,y) := -\hat{D}^{(\mathrm{n})}_{\mu,\bar{a}\bar{c}}(x)\left[\mathsf{H}^{-1}\right]^{\mu\nu}_{\bar{c}\bar{d}}(x,y)\hat{D}^{(\mathrm{n})}_{\nu,\bar{d}\bar{b}}(y) \equiv -\left[\hat{D}^{(\mathrm{n})}\mathsf{H}^{-1}\hat{D}^{(\mathrm{n})}\right]_{\bar{a}\bar{b}}. \tag{68}$$

The operator D, on the other hand, has been identified in the main text as the charged gluon propagator in MAG. The precise meaning of formula (28) and the subsequent definition of the projectors T, $\mathsf{T}^\dagger$ can be taken from the detailed form

$$\mathsf{D}^{\mu\nu}_{\bar{a}\bar{b}}(x,y) := \left[\mathsf{H}^{-1}\right]^{\mu\nu}_{\bar{a}\bar{b}}(x,y) \;\; + \;\; \int\left[\mathsf{H}^{-1}\right]^{\mu\alpha}_{\bar{a}\bar{c}}(x,x_1)\,D^{(\mathrm{n})}_{\alpha,\bar{c}\bar{d}}(x_1)\,\mathsf{L}^{-1}_{\bar{d}\bar{e}}(x_1,x_2)\cdot$$

$$\cdot \;\; D^{(\mathrm{n})}_{\beta,\bar{e}\bar{f}}(x_2)\left[\mathsf{H}^{-1}\right]^{\beta\nu}_{\bar{f}b}(x_2,y)\,d^4x_1\,d^4x_2. \tag{69}$$

B The Legendre Transformation

Using the superfield formalism of section 3.1, we start with the YM generating functional in the form (37), i.e.

$$Z[\theta,\lambda] = \int \mathcal{D}\Theta\exp\left\{-\frac{1}{\hbar g^2}\left(\mathcal{F}+\mathcal{F}_{\mathrm{cur}}\right)\right\}. \tag{70}$$

The part $\mathcal{F}$ of the exponent independent of the currents has the detailed form

$$\mathcal{F} = \frac{1}{2}\int a_\mu[D_0^{-1}]_{\mu\nu}a_\nu + \frac{1}{4}\int\chi_{\mu\nu}\chi_{\mu\nu} + \int\bar{\eta}\left(-D^{(\mathrm{n})}_\sigma D^{(\mathrm{n})}_\sigma\right)\eta + \frac{1}{2}\mathrm{Tr}\ln\mathsf{H} + \frac{1}{2}\mathrm{Tr}\ln\mathsf{L}. \tag{71}$$

Furthermore, using the bracket notation to indicate summation or integration over relevant indices, the current part of the action can be rewritten in components as

$$-\mathcal{F}_{\mathrm{cur}}[\Theta,\theta,\lambda] \;\; = \;\; \left\langle j^{(\mathrm{n})}\cdot a^{(\mathrm{n})}\right\rangle + \left\langle\bar{\xi}\cdot\eta+\bar{\eta}\cdot\xi\right\rangle + \left\langle\vartheta\cdot\chi\right\rangle + \frac{1}{2}\left\langle j^{(\mathrm{ch})}\cdot\mathsf{D}\cdot j^{(\mathrm{ch})}\right\rangle +$$

$$+ \frac{1}{2}\left\langle\varphi\cdot\mathsf{L}^{-1}\cdot\varphi\right\rangle + i\left\langle\varphi\cdot\mathsf{L}^{-1}D^{(\mathrm{n})}\mathsf{H}^{-1}\cdot j^{(\mathrm{ch})}\right\rangle. \tag{72}$$

The calculation of (70) in a semiclassical expansion and the definition of the Legendre transformation to the effective action is given in the main text, see section 3.1. As indicated in formula (40), we should eliminate the currents in Γ in favour of the classical fields. Usually, however, one does not need to do this explicitly to one loop order in Γ:

- The implicit dependence on the currents through the saddle point $\bar{\Theta}$ drops out, since $\bar{\Theta}$ and the classical field Θ differ by a term of order $\mathcal{O}(\hbar)$ which leads to a $\mathcal{O}(\hbar^2)$ effect in the effective action Γ. We can simply replace $\bar{\Theta} \to \Theta$.

- The explicit dependence of Γ on the currents cancels between the term $\mathcal{F}_{\mathrm{cur}}$ in the tree level action and the linear current term in the definition (40) of the effective action. This is only true for the currents θ corresponding to fields that have not yet been integrated out. Clearly, the exact integration over $\Lambda = \{A_\mu^{(\mathrm{ch})}, \phi\}$ in our approach leads to *quadratic* current terms in $\mathcal{F}_{\mathrm{cur}}$ and thus, the currents $\lambda = \{j_\mu^{(\mathrm{ch})}, \varphi\}$ do *not* drop out of the effective action.

In superfield notation, the first two terms in the loop expansion of the effective action are given by

$$g^2\Gamma_0[\Theta, \Lambda] = \mathsf{F}[\Theta, \lambda] + \int \Lambda \lambda \tag{73}$$

$$g^2\Gamma_1[\Theta, \Lambda] = \frac{g^2}{2}\operatorname{Tr}\ln\mathsf{H} + \frac{g^2}{2}\operatorname{Tr}\ln\mathsf{L} + \frac{g^2}{2}\operatorname{Tr}\ln\left.\frac{\delta^2\mathsf{F}[\Theta, \lambda]}{\delta\Theta^2}\right|_\Theta .$$

As mentioned above, the operator F in this equation still contains the currents $\lambda = \{j_\mu^{(\mathrm{ch})}, \varphi\}$ quadratically:

$$\mathsf{F} = \frac{1}{2}\left\langle a \cdot D_0^{-1} \cdot a \right\rangle + \frac{1}{4}\left\langle \chi \cdot \chi \right\rangle + \left\langle \bar{\eta} \cdot \left(-D^{(\mathrm{n})}D^{(\mathrm{n})}\right) \cdot \eta \right\rangle - \tag{74}$$

$$-\frac{1}{2}\left\langle j^{(\mathrm{ch})} \cdot \mathsf{D} \cdot j^{(\mathrm{ch})} \right\rangle - \frac{1}{2}\left\langle \varphi \cdot \mathsf{L}^{-1} \cdot \varphi \right\rangle - i\left\langle \varphi \cdot \mathsf{L}^{-1}D^{(\mathrm{n})}\mathsf{H}^{-1} \cdot j^{(\mathrm{ch})} \right\rangle .$$

Our final task is now to eliminate $\lambda = \{j_\mu^{(\mathrm{ch})}, \varphi\}$ in this expression in favour of the classical fields $\Lambda = \{A_\mu, \phi\}$ by inverting the definition (39). Once again, this only needs to be done to lowest order. It is useful to work in components rather than superfields, with the result

$$\varphi = -i\, D_\mu^{(\mathrm{n})}A_\mu^{(\mathrm{ch})}$$

$$\mathsf{T} \cdot j^{(\mathrm{ch})} = \mathsf{H} \cdot \mathsf{T}^\dagger \cdot A^{(\mathrm{ch})} = \mathsf{T} \cdot \mathsf{H} \cdot A^{(\mathrm{ch})}$$

$$D_\mu^{(\mathrm{n})}\mathsf{H}_{\mu\nu}^{-1} \cdot j_\nu^{(\mathrm{ch})} = (-i)\,\mathsf{L} \cdot \phi + D_\mu^{(\mathrm{n})}A_\mu^{(\mathrm{ch})}. \tag{75}$$

These equations must be used to eliminate $j_\mu^{(\mathrm{ch})}$ and φ from the operator F (leading to rather messy expressions).

26

Note also that the equations (75) can be used to eliminate the classical field χ from the effective action, as explained in the main text. The calculation is straightforward, though lengthy: We replace χ by the saddle point value $\bar{\chi}$, which is allowed at one-loop order. Since $\bar{\chi}$ depends on the currents λ, we can directly use equations (75) to obtain the result (42) quoted in the main text.

Let us close this section by considering the special case when the classical fields $\{A_\mu, \phi\}$ and thus the currents λ vanish. The operator F simplifies to the first line in (74) and the supertrace in the one-loop correction Γ_1 from (73) can actually be decomposed in its component blocks. One finds the result (45) quoted in the main text, where the explicit form of the abbreviation $\Delta_{\mu\nu}$ reads

$$
\begin{aligned}
\Delta_{\mu\nu}(x,y) &= \left\{ 2\bar{\boldsymbol{\eta}}^{\bar{a}}\mathsf{a}_\mu + \epsilon^{\bar{a}\bar{c}3}\left[\left(\partial^x_\mu \bar{\boldsymbol{\eta}}^{\bar{c}}(x)\right) - \bar{\boldsymbol{\eta}}^{\bar{c}}(x)\partial^x_\mu\right]\right\}\left[-D^{(\mathrm{n})}_\sigma D^{(\mathrm{n})}_\sigma\right]^{-1}_{\bar{a}\bar{b}}(x,y) \cdot \\
&\quad \cdot \left\{ 2\boldsymbol{\eta}^{\bar{b}}\mathsf{a}_\nu + \epsilon^{\bar{b}\bar{d}3}\left[\left(\partial^y_\nu \boldsymbol{\eta}^{\bar{d}}(y)\right) - \overleftarrow{\partial^y_\nu}\,\boldsymbol{\eta}^{\bar{d}}(y)\right]\right\}.
\end{aligned}
\tag{76}
$$

C Proper Time Regularisation of Loop Integrals

The basic idea of the proper time method is to represent the typical propagators by a parameter integral and then to cut off the parameter (the *proper time*) instead of the loop integration itself:

$$
\begin{aligned}
\frac{1}{k^2+m^2} &= \int_0^\infty ds\, e^{-s(k^2+m^2)} \longrightarrow \int_{1/\Lambda^2}^\infty ds\, e^{-s(k^2+m^2)} \\
\frac{1}{(k^2+m^2)^2} &= \int_0^\infty ds\, s\, e^{-s(k^2+m^2)} \longrightarrow \int_{1/\Lambda^2}^\infty ds\, s\, e^{-s(k^2+m^2)}.
\end{aligned}
$$

Here, Λ corresponds to a UV momentum cutoff. The obvious advantage of this method is that the loop integration itself is left unchanged and we can perform all the standard manipulations on it. In particular we are allowed to *shift* the loop momentum. Examples are

$$
\begin{aligned}
\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2} &\longrightarrow \frac{\Lambda^2}{16\pi^2} + \cdots \\
\int \frac{d^4k}{(2\pi)^4}\frac{k_\mu k_\nu}{k^2(k+p)^2} &\longrightarrow \frac{\delta_{\mu\nu}p^2 - 4p_\mu p_\nu}{192\pi^2}\ln\frac{p^2}{\Lambda^2} + \frac{\delta_{\mu\nu}\Lambda^2}{32\pi^2} + \cdots \\
\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k+p)^2} &\longrightarrow -\frac{1}{16\pi^2}\ln\frac{p^2}{\Lambda^2} + \cdots.
\end{aligned}
\tag{77}
$$

The omitted terms are finite as $\Lambda \to \infty$. The first two examples clearly show that the proper time method is able to parametrise quadratic divergences in a gauge invariant way.

As an example of the cancellation of such divergences consider the H contribution to the Abelian vacuum polarisation. The relevant term $\Gamma_1 \sim \frac{1}{2}\mathrm{Tr}\ln\mathsf{H}$ of the effective action is separately invariant under the residual $U(1)$ symmetry (25), provided the regularisation of the trace respects this. As a consequence, we expect the contribution to the Abelian vacuum polarisation to be transversal and the dangerous quadratic divergences should cancel exactly.

Let us now check this expectation using the proper time method. The contributing diagrams are (see (60) in the main text)
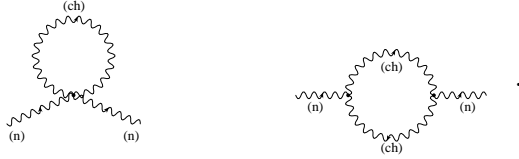
In momentum space, the first diagram reads

$$\hbar g^2 \kappa\, \delta_{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{4}{k^2} = \hbar g^2 \kappa\, \delta_{\mu\nu} \frac{\Lambda^2}{4\pi^2} + \cdots. \tag{78}$$

We observe that the proper time method really *parametrises* the quadratic divergence, whereas this integral would be simply zero in dimensional regularisation. The second diagram above is given by

$$- \hbar g^2 \frac{\kappa}{2} \int \frac{d^4k}{(2\pi)^4} \frac{8\,\delta_{\mu\nu}p^2 + 16 k_\mu k_\nu + 8(k_\mu p_\nu + k_\nu p_\mu) - 4 p_\mu p_\nu}{k^2(k+p)^2} =$$
$$= \hbar g^2 \kappa \cdot 5\, \frac{\delta_{\mu\nu}p^2 - p_\mu p_\nu}{24\pi^2} \ln\frac{p^2}{\Lambda^2} - \hbar g^2 \kappa \cdot \frac{\delta_{\mu\nu}\Lambda^2}{4\pi^2} + \cdots. \tag{79}$$

Clearly the quadratic divergences cancel *exactly* between the two diagrams and the remaining logarithmic divergence has the required transversal form. On comparing with the result of the main text, where we used dimensional regularisation, we observe the correspondence $1/\epsilon \leftrightarrow 1/2\ln(\Lambda^2/p^2)$ between the two procedures.

In all the other diagrams containing quadratic divergences, the calculation (and the correct cancellation) is completely equivalent to the above example.

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