Preprint typeset in JHEP style - HYPER VERSION

LBNL-49311
UFIFT-HEP–01-26

arXiv:hep-ph/0112326 v1 25 Dec 2001

Gluon Chain Model of the Confining Force

Jeff Greensite

Physics and Astronomy Department, San Francisco State University
San Francisco, CA 94117. E-mail: greensit@quark.sfsu.edu
and
Theory Group, Lawrence Berkeley National Laboratory
Berkeley, CA 94720

Charles B. Thorn*

Department of Physics, University of California
Berkeley, CA 94720. Email: thorn@phys.ufl.edu
and
Theory Group, Lawrence Berkeley National Laboratory
Berkeley, CA 94720

Abstract: We develop a picture of the QCD string as a chain of constituent gluons, bound by attractive nearest-neighbor forces which may be treated perturbatively. This picture accounts for both Casimir scaling at large N_c, and the asymptotic center dependence of the static quark potential. We discuss the relevance, to the gluon-chain picture, of recent three-loop results for the static quark potential. A variational framework is presented for computing the minimal energy and wavefunction of a long gluon chain, which enables us to derive both the logarithmic broadening of the QCD flux tube ("roughening"), and the existence of a Lüscher $-c/R$ term in the potential.

Keywords: QCD, Confinement, 1/N Expansion, Nonperturbative Effects.

*Visiting Miller Research Professor, on sabbatical leave from Department of Physics, University of Florida, Gainesville FL 32611.

Contents

1. Introduction

Many years ago [1–4] we suggested a picture of the formation and composition of the QCD string as a linear chain of gluons, which are the perturbative excitations the theory. This picture is motivated by 't Hooft's large-N_c expansion. In particular, a time-slice of a high-order planar diagram for a Wilson loop (Fig. 1) reveals a sequence of gluons, each of which interacts only with its nearest neighbors in the diagram by mainly attractive forces. This immediately leads to the idea that the QCD string is composed of a "chain" of constituent gluons, each held in place by its attraction to its two nearest neighbors in the chain. The challenge in such a model is to understand how the attractive force between gluons, which is essentially due to one-gluon exchange, can manage to hold these massless constituent particles in a bound state. In this article we address this problem, and in particular we show how recent perturbative results for the static quark potential, and the related force renormalization scheme [5], bear on this issue. We propose an appealing way to interpret perturbative QCD that leads to a self-consistent extrapolation of the perturbative force of a quark on an anti-quark to a linear confining force. We do not

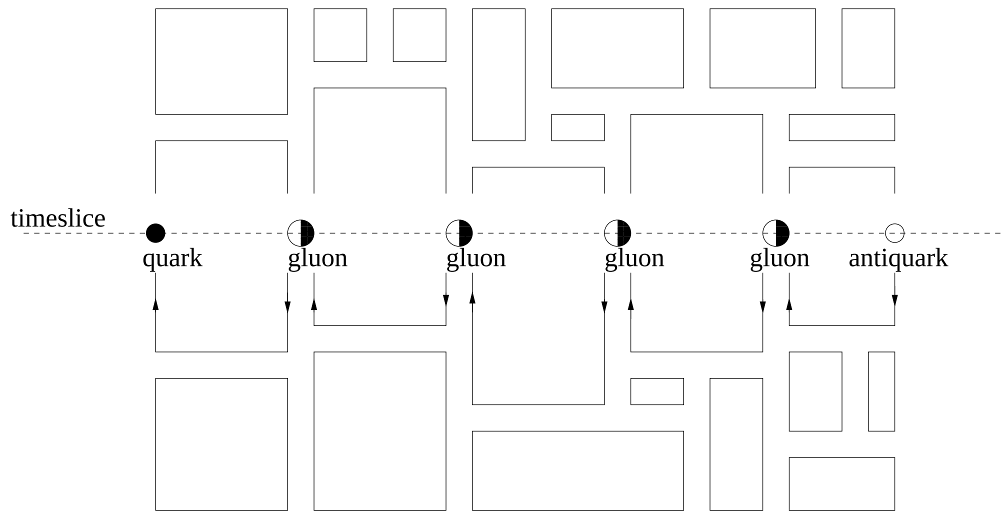

Figure 1: The gluon chain as a time slice of a planar diagram (shown here in double-line notation). A solid hemisphere indicates a quark color index, open hemisphere an antiquark color index.

claim that this in any way *proves* confinement, but rather that it provides a model framework for thinking about the physics of confinement which stays conceptually close to perturbation theory. A computational scheme is presented, involving variational and perturbative elements, which we hope to eventually apply to calculate the ground state (and the corresponding static quark potential) of the gluon chain. Even without a full calculation of this ground state, we are able to use our variational framework to demonstrate both the logarithmic broadening of the QCD flux tube, and the existence of a Lüscher $-c/R$ term at long distances.

Section 2, below, reviews the motivation of the gluon-chain model, with emphasis on how this model accounts for features of the confining force which are problematic for other theories of quark confinement. In section 3 we discuss in more detail the physics of gluon chain formation, and the binding of gluons in the chain. The running coupling is, of course, crucial to this dynamics, and at the coupling strengths relevant to chain formation there are important issues of renormalization scheme dependence that must be confronted. In section 4 we present a variational framework for semi-perturbative calculation, and show how it applies to a string tension calculation. In this section we demonstrate how the gluon-chain model accounts for roughening, and the Lüscher $-c/R$ term in the static quark potential. The last section contains some concluding remarks.

2. The Gluon-Chain Model

Like any theory of quark confinement, the gluon-chain model aims at explaining the linearity of the static quark potential. However, it is now widely recognized that in addition to the linearity feature, there are at least three other properties of the confining potential which a satisfactory theory of confinement is obligated to explain:

- **Casimir Scaling:** Consider the potential between static quarks in a representation r of the gauge group. From the onset of linearity in the potential, to a finite (adjoint string-breaking) scale, the string tension σ_r is proportional to the quadratic Casimir C_r of the representation, i.e.

$$\sigma_r = \frac{C_r}{C_F}\sigma_F \tag{2.1}$$

  where the subscript F denotes the fundamental representation. For rectangular $L \times T$ Wilson loops with L/T fixed, the range of L for which the Casimir scaling law is valid increases logarithmically with N_c.

- **Center Dependence:** Asymptotically, the string tension can depend only on the N-ality of the group representation r, i.e. on its transformation properties under the center subgroup of the gauge group.

- **String Behavior:** The diameter of the color-electric flux tube between static sources is believed to grow logarithmically with the separation L of the sources (roughening), and there is a $-c/L$ contribution to the asymptotic potential (the Lüscher term) which is due to quantum fluctuations of the QCD string, rather than Coulomb attraction of the quarks.

Taken together, this is a challenging set of conditions. The abelian monopole theory [6], for example, has a very hard time accounting for Casimir scaling [7], as well as for the center dependence of certain operators [8]. Instanton [9] and meron [10] mechanisms are consistent with Casimir scaling, but have difficulties with center dependence. The center vortex theory [11] is in perfect accord with center dependence, and it is at least roughly compatible with Casimir scaling, as shown in ref. [12]. On the other hand, the vortex theory does not really explain the high degree of accuracy of the Casimir scaling rule, which has been revealed in numerical simulations [13]. Finally, the string-like behavior of the QCD flux tube seems to pose problems for any theory of confinement based essentially on one-gluon exchange (e.g. the proposal of ref. [14]), as well as for the proposal of stochastic confinement [15].

The gluon-chain model of QCD string formation, on the other hand, meets the conditions listed above in a rather simple and appealing way, as we will discuss below.

2.1 Linear Potential

As a heavy quark-antiquark pair move apart, and their color charge separation increases, we expect that at some point the interaction energy increases rapidly due to

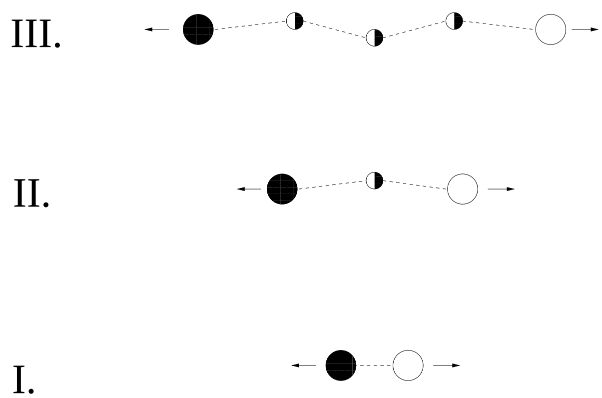

Figure 2: As heavy quarks (large circles) separate, energy is minimized by keeping the average color charge separation below a certain limit. This is achieved by pulling out a sequence of gluons (small circles) between the heavy quarks. Again, solid and open shadings denote quark and antiquark color indices, while dotted lines indicate contracted color indices between a quark and a gluon or between neighboring gluons.

the running coupling (cf. section 3). Eventually it becomes energetically favorable to reduce the effective charge separation by inserting a gluon between the quarks. In the $N_c \to \infty$ limit, the quark and antiquark can only interact with the intermediate gluon, but not directly with each other. As the heavy quarks continue to move apart, the process repeats, and we end up with a chain of gluons, as shown in Fig. 2, in which the average distance between color charges remains fixed, irrespective of the separation of the heavy sources. The energy of the system is approximately NE_{gluon}, where N is the number of gluons in the chain, and E_{gluon} is the kinetic and nearest-neighbor interaction energy per gluon. If the quarks are separated by a distance L, and the number of gluons per unit quark separation ($N/L = 1/R$) is fixed, then

$$E_{chain} \approx NE_{gluon} = \frac{E_{gluon}}{R} L = \sigma L \tag{2.2}$$

where $\sigma = E_{gluon}/R$ is the string tension. The linear growth in the number of constituent gluons is the origin of the linear potential in the gluon-chain model.

Alternatively, we may understand the linear potential in terms of a constant force. The force between neighboring constituent gluons is dependent on the average separation $R = L/N$ of gluons along the quark-antiquark axis. If this separation remains fixed as L increases, then the average intergluon force remains fixed. The intergluon force, which is the same everywhere in the chain, can be interpreted as a string tension, which is constant irrespective of the quark separation.

2.2 Casimir Scaling

To leading order in N_c, a group character in representation r is given by a product

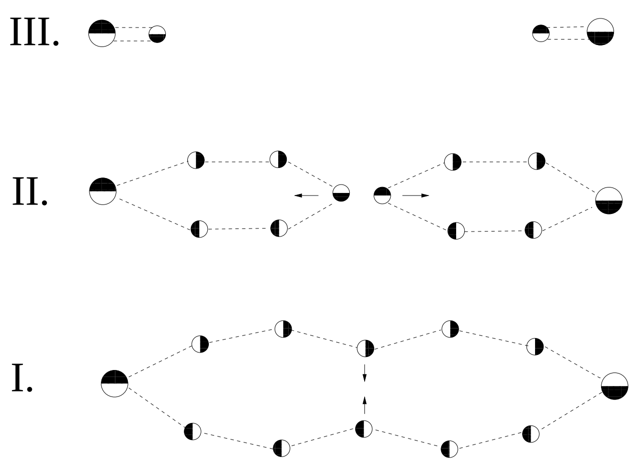

Figure 3: Adjoint string-breaking in the gluon chain model. Two gluons in separate chains (I) scatter by a contact interaction, resulting in the re-arrangement of color indices indicated in II. This corresponds to chains starting and ending on the same heavy source. The chains then contract down to smaller "gluelumps" (III).

of group characters in the fundamental representation

$$\chi_r[g] \propto \Big(\chi_F[g]\Big)^n \Big(\chi_F^*[g]\Big)^{\overline{n}} + \text{ sub-leading terms.} \tag{2.3}$$

By factorization at large-N_c, a Wilson loop in representation r has a string tension

$$\sigma_r = M_r \sigma_F \tag{2.4}$$

at $N_c \to \infty$, where $M_r = n + \overline{n}$. In this limit, the quadratic Casimir is $C_r = M_r N_c/2$. Exact Casimir scaling is therefore a property of the planar limit.

The gluon-chain model, which is motivated by large-N_c considerations, inherits this property. A heavy source in representation r is the terminus of M_r separate gluon chains, one for each of the n quark and $\overline{n}$ antiquark charges in the direct product forming the representation r. Since the chains do not interact in the $N_c = \infty$ limit (the interaction is a non-planar process, as can easily be verified by considering the relevant Feynman diagrams), the total energy is simply the sum of the energies of each of the chains. In this way, Casimir scaling is obtained, at least at large N_c.[1]

2.3 Center Dependence

Asymptotically, the string tension of static sources in a higher color group representation r depends only on the N-ality, rather than the quadratic Casimir, of the

[1]At small N_c, the Casimir is not simply proportional to M_r. On the other hand, at small N_c, interactions between the chains cannot be neglected. We cannot say, at present, whether this model predicts any substantial deviation from Casimir scaling at small N_c.

group representation. Let us consider how this comes about in, e.g., the adjoint representation (the analysis is easily generalized to other representations). Beginning with heavy quark sources in the adjoint representation, there are two gluon chains, as shown in Fig. 3. Nearby gluons in each chain can scatter, e.g. by a contact interaction, and rearrange the sequence of colors as shown in the figure. The result is that two gluon chains transform to two "gluelumps"; the adjoint string is broken (providing the sum of gluelump masses is less than that of the double chain), and the resulting string tension is zero. This is the correct prediction, since the N-ality of the adjoint representation is also zero. However, the scattering process indicated is non-planar, and the transition rate from the two-chain structure to the gluelump structure is $1/N_c^2$ suppressed. In the large-N_c limit, we therefore recover exact factorization and Casimir scaling.

2.4 String Behavior

It should be obvious, just from the figures, that a gluon-chain is a discretized string [2] of some kind, with the constituent gluons playing the role of "string-bits." Therefore it is reasonable to expect that, due to quantum fluctuations of the chain configuration, we should find the logarithmic broadening of the color-electric flux tube with quark separation (roughening), as well as the Lüscher $-c/R$ term term in the static quark potential. These effects are very non-trivial, however, and need to be demonstrated in the context of the gluon-chain model. We will postpone the analysis to section 4.

3. The Force Renormalization Scheme and Gluon Chain Formation

In the remarks above, we have passed lightly over a fundamental issue. Gluons are massless particles. The question is how nearest neighbor forces between constituent gluons actually manage to bind such particles together in a chain. Clearly it is hopeless to find a binding mechanism at very weak coupling $\alpha_s = g^2/4\pi$, since the kinetic energy of a gluon confined to a region of size R is of order $1/R$, whereas the interaction energy (at tree level) is only of order $-\alpha_s/R$. The fact that the effective α_s is not really constant, but grows with distance scale R, is obviously of crucial importance.

Superficially, a coupling which grows monotonically with distance seems exactly what is needed for quark confinement. However, different choices for what the coupling measures lead to drastically different estimates of the quark-antiquark static potential. For example, if we say that $\alpha_s(R)$ is a measure of the potential energy

[2] For a treatment of discretized bosonic strings in light-cone gauge, see ref. [16].

between quark and anti-quark (the “V-scheme”), we would then write the familiar expression for the potential energy of a static quark and antiquark separated by distance R in an SU(N_c) gauge theory

$$V_{q\bar{q}}(R) = -\left(1 - \frac{1}{N_c^2}\right) \frac{N_c \alpha_s^V(R)}{2R}. \tag{3.1}$$

It should be understood that this formula is a *definition* of the running coupling, and there is a complicated relation (beyond two loops) of $\alpha_s^V(R)$ in the V-scheme to the running coupling in, e.g., the $\overline{\text{MS}}$ scheme [5, 17].

If $\alpha_s^V(R)$ grows monotonically with R, as suggested by the first few terms in the renormalization group Gell-Mann-Low function, the potential in eq. (3.1) is actually the *opposite* of what is required for confinement. Assuming that $\alpha_s^V(R)$ grows with R, perhaps even blowing up at some finite R_∞ (the Landau singularity), the potential in eq. (3.1) leads to a force which first weakens, then vanishes at some point, and finally becomes ever more strongly repulsive. The problem is that a monotonically increasing $\alpha_s^V(R)$ tends to drive the potential away from the $V = 0$ axis in the direction of negative V, whereas in fact the static potential crosses the axis at some point and becomes positive. Therefore, as defined in the V-scheme, $\alpha_s^V(R)$ derived from the true static potential cannot possibly grow monotonically. Instead, at some scale, α_s^V must start to become smaller and eventually change sign. There seems to be little hope of relating such behavior to the running coupling, at least up to three loops in perturbation theory.

As long advocated by one of us (C.B.T.), it is less problematic to define the running coupling by the *force* (the derivative of the potential) that the static quark exerts on the static anti-quark [18]

$$|F(R)| = \left(1 - \frac{1}{N_c^2}\right) \frac{N_c \alpha_s(R)}{2R^2} \equiv \left(1 - \frac{1}{N_c^2}\right) \frac{\pi \lambda(R)}{2R^2}, \tag{3.2}$$

where we have defined the ’t Hooft coupling held fixed in the $N_c \to \infty$ limit by $\lambda = N_c \alpha_s / \pi$. In the rest of this paper α_s and λ will always refer to the running coupling defined through the static force. This scheme, also advocated in recent years by Sommer [19], has the obvious advantage that the running coupling so defined doesn’t have to change sign. The corresponding Gell-Mann-Low function, defined as

$$\psi(\lambda(R)) \equiv -R \frac{d\lambda}{dR}, \tag{3.3}$$

is given to three loops for $N_f = 0$ but for any N_c by [5]

$$\psi(\lambda) \equiv -\frac{11}{6}\lambda^2 - \frac{17}{12}\lambda^3 - (3.795...)\lambda^4 - \cdots \tag{3.4}$$

The exact coefficient in the last term is rather cumbersome, so we have merely quoted its numerical value to 3 decimal places.

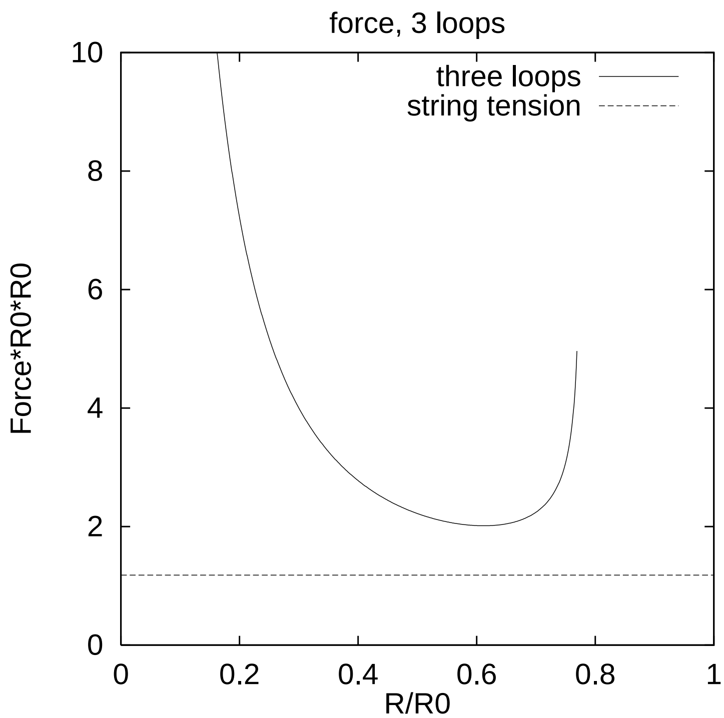

Figure 4: The force between static quarks in units of the Sommer scale $R_0 \approx 0.5$ fm, computed perturbatively to three loops at $N_c = 3$ in the force renormalization scheme [5]. Also shown is the asymptotic string tension of $(430\text{MeV})^2 \times R_0^2$.

Calculating $\lambda(R)$ by truncating ψ at three loops leads to a force that, as R increases, drops to a minimum at R_m given by

$$2\lambda(R_m) + \psi(\lambda(R_m)) = 0, \tag{3.5}$$

after which the force *increases*, blowing up as $(R_\infty - R)^{-1/3}$ at the Landau singularity $R = R_\infty$, as shown (for $N_c = 3$) in Fig. 4. In this figure, R_0 is the Sommer scale $R_0 \approx 0.5$ fm. The corresponding static quark potential, in the force renormalization scheme, is obtained by integrating the force,

$$V_{q\bar{q}}(R) = V_{q\bar{q}}(R_A) + \int_{R_A}^{R} dR \, \Big|F(R)\Big| \tag{3.6}$$

where $R_A \ll R_0$ and $V_{q\bar{q}}(R_A)$ may be estimated at sufficiently small $\alpha_s(R_A)$ by using eq. (3.1), with $\alpha_s^V \approx \alpha_s$. Note that the Landau singularity in the force is integrable as $R \to R_\infty$ from below, and the integrand becomes complex for $R > R_\infty$. Thus the potential curve stops at R_∞ at a finite value. It was found by Necco and Sommer in ref. [5] that the resulting three-loop potential is a surprisingly accurate match to the lattice Monte Carlo result, almost up to the Landau point at $R_\infty/R_0 \approx 0.78$. This three-loop potential (with $R_A = 0.15R_0$) is displayed in Fig. 5.

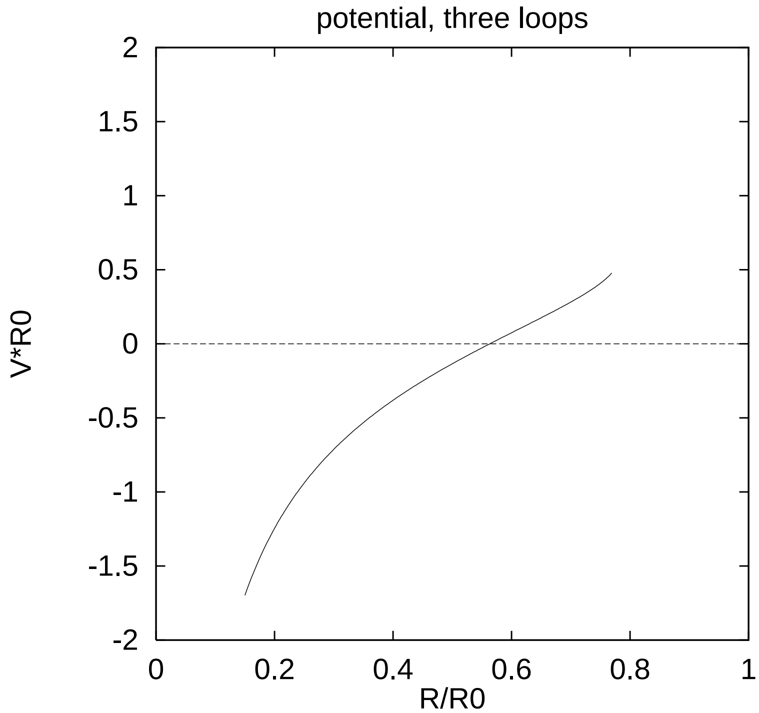

Figure 5: The static quark potential at $N_c = 3$, computed perturbatively to three loops in the force renormalization scheme.

It is clear that a potential of this sort can trap a massless particle, whose kinetic energy $|\mathbf{p}|$ decreases as $1/R$ by the uncertainty principle: Minimizing $1/R + V(R)$ gives

$$\frac{1}{R^2} = V'(R) = \left(1 - \frac{1}{N_c^2}\right) \frac{\pi\lambda(R)}{2R^2}, \tag{3.7}$$

in other words $\lambda(R) = 9/4\pi \approx 0.71$ for $N_c = 3$. The effective coupling required for binding a massless particle is of order $O(1)$, but not huge. There is thus hope that, after taking binding into account with such an extrapolation of the RG improved perturbative force, subsequent corrections may be under control. It follows that, according to the force renormalization scheme, it is possible to have bound states containing massless constituent gluons, such as glueballs and gluon chains. Moreover, there is no indication of any gross failure of perturbation theory up to $R \approx R_m$. On the contrary, as just mentioned, it seems that perturbation theory fits the numerical lattice data quite well.

3.1 Saturation Mechanism

Beyond $R = R_m$ the three loop force increases, and the corresponding potential is concave upwards. In contrast, a confining force is expected to be asymptotically constant: One can say that the perturbative force extrapolated to $R > R_m$ *over-confines*. In fact, there is a rigorous theorem, analogous to the well known convexity of the thermal free energy, which states that the static force never increases with quark separation, or, equivalently, that the static potential is concave downwards [20]. It appears that physics beyond three loops, at $R > R_m$, must supply a mechanism to *weaken* the extrapolated perturbative force law, not strengthen it. In our opinion, this mechanism is supplied by the formation of a gluon chain, as described in section 2. As the quark-antiquark separation increases, it can become energetically favorable for a gluon to materialize between the quark and anti-quark. Because the gluon is in the adjoint representation, its color can be arranged so that it is simultaneously a sink of the color flux from the quark and from the anti-quark, and indeed this is precisely the arrangement of color dictated by the $N_c \to \infty$ limit. The effect of a gluon with this color orientation is to shield the direct force of the quark on the antiquark, and thereby reduce the effective separation of color charge. Further increase of R leads to the successive appearance of more constituent gluons, creating a gluon chain between the quark and anti-quark. Thus the separation of directly interacting color charges never exceeds some maximum value. It is presumably this saturation mechanism which prevents the static force from eventually increasing with quark-antiquark separation, in accordance with the concavity theorem. The same saturation mechanism which bounds the average color charge separation also bounds the corresponding effective coupling. If this maximal effective coupling is not too large, it should be possible to treat the interaction between neighboring gluons by perturbative methods.

In the next section we develop a variational approach to the calculation of the properties of the gluon chain responsible for quenching the increasing force beyond R_m. But it is worth noting here that the form of the perturbative force law shown in Fig. 4, plus the hypothesis of a saturation mechanism, already provides the basis for a zero parameter estimate of the string tension, namely $\sigma \approx F(R_m)$. In other words, we accept the extrapolated perturbative force up to the point where it violates the concavity theorem, after which we replace it with the simplest behavior consistent with the theorem, namely a constant. This reasoning is reminiscent of the Maxwell constuction in statistical physics, which restores convexity to an approximate calculation of the free energy of a Van der Waals gas.

As already noted, there is a theorem which tells us that the increasing part of the force curve in Fig. 4 (at $R > R_m$) cannot apply to the actual force between static quarks. However, if that curve is taken as a paradigm of the nearest-neighbor force binding gluons in the chain (neglecting change in gluon number), then its increasing part may play a role in providing a restoring force towards an equilibrium gluon separation, particularly if the mean intergluon separation in the chain exceeds R_m. In this case, there is a net restoring force on any gluon fluctuating away from the mean separation. Again, the saturation mechanism will prevent any overall increase of the static force with quark separation.

To arrive at an estimate of the string tension, it remains to find $\lambda(R_m)$ and R_m from Eq. 3.5. Truncating ψ at three loops leads to a cubic equation for $\lambda(R_m)$, with the numerical solution $\lambda(R_m) \approx 0.540$, so $(1-1/N_c^2)\pi\lambda(R_m)/2 \approx 0.754$ for $N_c = 3$. This value of λ gives the relative size of the terms entering into Eq. 3.5 as

$$2\lambda - \psi_0\lambda^2 - \psi_1\lambda^3 - \psi_2\lambda^4 = .540(2 - 0.990 - 0.413 - 0.598), \tag{3.8}$$

so it is clear that one is extrapolating beyond the strict validity of perturbation theory. However, if the three loop term turns out to be anomalously large, the prospects for our perturbative approach would be brighter. From the slope of the meson Regge trajectories $\alpha' \approx 0.86\text{GeV}^{-2}$ we have the estimate from data of $\sigma = 1/2\pi\alpha' \approx (430\text{MeV})^2$. This leads to $R_m \approx (495\text{MeV})^{-1} \approx 0.40\text{fm}$. According to Figures 2 and 4 in ref. [5], we can infer that with $\Lambda_{\overline{\text{MS}}} = 238\text{MeV}$, the phenomenologically preferred value, $R_m \approx 0.3\text{fm}$. The string tension for this value would be higher by a factor of $(1.33)^2 \approx 1.76$, *i.e.* $\sigma \approx (570\text{MeV})^2$. Quark production, vetoed in our $N_c \to \infty$ model, is expected to reduce the predicted tension somewhat, but probably not this much. Conversely, the string tension inferred from the meson trajectories would require $\Lambda_{\overline{\text{MS}}} = 177$ MeV.

Since the saturation mechanism must come into play at $R \approx R_m$, where perturbation theory begins to fail, we would conclude that the QCD ground state $\Psi_{q\overline{q}}[A]$ in the presence of static $q\overline{q}$ sources is dominated, at $R \approx R_m$, by a one-constituent gluon component (perturbation theory expands around a state with zero constituent gluons).[3] If that is so, then the average separation of color charge along the axis joining the static sources will be $R_m/2$. Assuming this "axial separation" of color charge remains fixed as quark separation L increases, as it should according to the gluon chain model, then for large separations, at $N_c = 3$, we may estimate $L/N \approx R_m/2 \approx 0.15$ fm, where N is the number of gluons in the chain. This does not mean that the gluon is sharply localized on the axis, however. Indeed, in order to be bound it must be spread to a size r at which the running coupling is of $O(1)$, which means $r > R_m$.

4. A Variational Framework, and String-like Behavior

In order to make the notion of "constituent gluon" precise, it is necessary to work in a Hamiltonian framework, which also means picking a physical gauge, e.g. Coulomb or light-cone gauge. The ground-state wavefunctional in any physical gauge can be expressed in the path-integral form

$$\Psi_0[A(\mathbf{x})] = \int DA_\mu(\mathbf{x}, t<0)\; \delta\Big[F(A)\Big]\Delta_{FP}[A]\; \exp\Big[-\int_{-\infty}^{0} dt L_A\Big] \tag{4.1}$$

[3] An old lattice Monte Carlo investigation of the gluon content of the QCD string (second article of ref. [2]) suggests that the one-constituent gluon component dominates beyond 3 lattice spacings at $\beta = 2.4$, corresponding to $R = 0.73R_0$.

where L_A is the gauge-field Lagrangian, $F(A) = 0$ the gauge condition, and Δ_{FP} the Faddeev-Popov determinant.[4] An excited state can be constructed by multiplying the ground state with some polynomial in the fields, i.e.

$$\Psi_{ex}[A] = Q[A]\Psi_0[A]. \tag{4.2}$$

For example, the wavefunctional appropriate for a glueball state would be

$$\Psi_G[A] = \sum_{N=1}^{N_{max}} \Psi_G^{(N)}[A] \tag{4.3}$$

where $\Psi_G^{(N)}[A]$ is the component of the glueball state with N constituent gluons, which has the general form

$$\begin{aligned}\Psi_G^{(N)}[A] = \Big\{ \int & d\mathbf{x}_1 d\mathbf{x}_2 ... d\mathbf{x}_N \;\, f_{\mu_1\mu_2...\mu_N}(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N) \\ & \text{Tr} A_{\mu_1}(\mathbf{x}_1) A_{\mu_2}(\mathbf{x}_2) ... A_{\mu_N}(\mathbf{x}_N) \Big\} \Psi_0[A]. \end{aligned} \tag{4.4}$$

For our purposes, a "constituent gluon" simply refers to a gluon field operator multiplying the exact ground state. The energy of any excited state (4.2) above the ground-state energy, is given by

$$E = \frac{\langle \Psi_{ex} | H | \Psi_{ex} \rangle}{\langle \Psi_{ex} | \Psi_{ex} \rangle} - \langle H \rangle. \tag{4.5}$$

Defining

$$Q_t \equiv Q[A(\mathbf{x}, t)] \tag{4.6}$$

the energy of the excited state can be computed from a correlation function in the Euclidean-time version of the theory

$$E = -\frac{1}{2} \lim_{T \to 0} \frac{d}{dT} \log \langle Q_T^\dagger Q_{-T} \rangle \tag{4.7}$$

where $\langle Q_T^\dagger Q_{-T} \rangle$ is the Euclidean vacuum expectation value of these operators. In the perturbative calculation of E one need only include the connected diagrams.

Equation (4.7) is the starting point for a field-theoretic variational approach to bound states of constituent quanta [3,21]. The idea is to first make some trial ansatz for $Q[A]$ involving a few parameters, calculate the energy E_{trial} of the trial state perturbatively, via eq. (4.7), and then find the parameter values which minimize this trial energy. The result is a variational estimate for bound-state energy, and an approximation to the corresponding wavefunctional.

[4] We note that in a physical gauge either Δ_{FP} is trivial (as in light-cone gauge), so that there is no need to introduce ghost fields, or else the ghost fields do not propagate in time, as in Coulomb gauge.

In the case of the gluon chain, we take the static sources to be at points $\mathbf{x} = \mathbf{0}$ and $\mathbf{x} = \mathbf{L}$, and consider N-constituent gluon states generated from trial operators of the form

$$Q_{chain}[A] = q^{a_1}(\mathbf{0})\Big\{\int d\mathbf{x}_1 d\mathbf{x}_2 ... d\mathbf{x}_N \ \psi_{\mu_1\mu_2...\mu_N}(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N)$$
$$A^{a_1 a_2}_{\mu_1}(\mathbf{x}_1) A^{a_2 a_3}_{\mu_2}(\mathbf{x}_2) ... A^{a_N a_{N+1}}_{\mu_N}(\mathbf{x}_N)\Big\}\overline{q}^{a_{N+1}}(\mathbf{L}) \tag{4.8}$$

where ψ is a trial N-gluon wavefunction. In the $N_c \rightarrow \infty$ limit, only interactions between nearest-neighbor gluons (in the diagrams) need be taken into account. A consistent chain picture will require that the RG-improved intergluon interaction energy as a function of gluon separation is qualitatively similar to the static quark potential shown in Fig. 5. Whether this happens remains to be seen.

4.1 A simplified model

We will not attempt, in this article, a variational computation of the gluon-chain energy $\mathcal{E}$ in the full field theory. The perturbative evaluation of $\langle Q^\dagger_T Q_{-T}\rangle$ for the gluon chain state is very tedious, especially beyond tree level, and again involves delicate issues of renormalization scheme dependence. However, we can capture most of the important qualitative features of the field theory calculation with a simple quantum-mechanical model, in which the (N-1)-gluon Hamiltonian is taken to be

$$H = \sum_{n=1}^{N-1} \Big|\mathbf{p}_n\Big| + \sum_{n=2}^{N-1} V(\mathbf{x}_n - \mathbf{x}_{n-1}) + V_{qg}(\mathbf{x}_1) + V_{qg}(\mathbf{L} - \mathbf{x}_{N-1}) \tag{4.9}$$

where $V(x)$ is the gluon-gluon interaction energy, while $V_{qg}(x)$ is the interaction energy between a static quark and its neighboring gluon. The restriction to nearest-neighbor interactions is, of course, justified in the large N_c limit. In field theory the interaction between each pair of neighbors is described by renormalization group improved Bethe-Salpeter style ladder exchanges. In addition to the RG improved Coulomb exchanges that build up the static force displayed in Fig. 4, exchange of transverse gluons and contact interactions will add spin and momentum dependence to the interaction. We can anticipate that the most attractive spin channels are those in which neighbor gluons are in a spin 0 state, so that for a chain we should have anti-ferromagnetic order [22]. As we would do in the field theory calculation, we use the variational method in this simplified model to obtain an approximation to the N-gluon ground state energy $\mathcal{E}(L)$. As already mentioned, if the potentials V and V_{qg} have the qualitative behavior shown in Fig. 5, a self-consistent gluon chain will arise.

- **Product Ansatz – Relative Coordinates**

We begin with a simple product ansatz for the gluon-chain wavefunctional

$$\Psi(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_{N-1}) = A \prod_{i=1}^{N} \psi(\mathbf{u}_i) \tag{4.10}$$

where the $\{\mathbf{u}_i\}$ are relative coordinates

$$\mathbf{u}_i = \mathbf{x}_i - \mathbf{x}_{i-1} \tag{4.11}$$

and

$$\mathbf{x}_0 \equiv \mathbf{0} \quad , \quad \mathbf{x}_N = \mathbf{L} \tag{4.12}$$

It is convenient to also change integration variables to relative coordinates, and the integration measure becomes

$$dV = \frac{d^3q}{(2\pi)^3} \prod_{i=1}^{N} d^3u_i \exp\left\{ i\mathbf{q} \cdot \left(\sum_{i=1}^{N} \mathbf{u}_i - \mathbf{L} \right) \right\}, \qquad \int dV |\Psi|^2 = 1 \tag{4.13}$$

where integration over q gives a delta function enforcing the constraint

$$\sum_{i=1}^{N} \mathbf{u}_i = \mathbf{L} \equiv N\mathbf{R} \tag{4.14}$$

Defining

$$F(\mathbf{q}) \equiv \int d^3u |\psi(\mathbf{u})|^2 e^{i\mathbf{q}\cdot\mathbf{u}} \tag{4.15}$$

and the Fourier transform of the relative-coordinate wavefunction

$$\phi(\mathbf{k}) = \int \frac{d^3u}{(2\pi)^{3/2}} e^{-i\mathbf{k}\cdot\mathbf{u}} \psi(\mathbf{u}) \tag{4.16}$$

we find for the normalization constant

$$A^{-2} = \int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{L}} F(\mathbf{q})^N \tag{4.17}$$

and the expectation value of the gluon nearest-neighbor potential

$$\langle V(\mathbf{u}_k) \rangle = A^2 \int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{L}} F(\mathbf{q})^{N-1} \int d^3u |\psi(\mathbf{u})|^2 V(\mathbf{u}) e^{i\mathbf{q}\cdot\mathbf{u}} \tag{4.18}$$

For the purpose of computing the expectation value of kinetic energy, it is useful to introduce

$$\begin{aligned}
\langle e^{i\mathbf{y}\cdot\mathbf{p}_k}\rangle &= A^2 \int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{L}} F(\mathbf{q})^{N-2} \\
&\quad \int d^3u d^3u' e^{i\mathbf{q}\cdot(\mathbf{u}+\mathbf{u}')} \psi^*(\mathbf{u})\psi^*(\mathbf{u}')\psi(\mathbf{u}+\mathbf{y})\psi(\mathbf{u}'-\mathbf{y}) \\
&= A^2 \int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{L}} F(\mathbf{q})^{N-2} \\
&\quad \int d^3k d^3k' e^{i\mathbf{y}\cdot(\mathbf{k}-\mathbf{k}')} \phi^*(\mathbf{k}+\mathbf{q})\phi^*(\mathbf{k}'+\mathbf{q})\phi(\mathbf{k})\phi(\mathbf{k}') \qquad (4.19) \\
\langle \delta(\mathbf{p}_k - \mathbf{p})\rangle &= A^2 \int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{L}} F(\mathbf{q})^{N-2} \\
&\quad \int d^3k \phi^*(\mathbf{k}+\mathbf{q}-\mathbf{p})\phi^*(\mathbf{k}+\mathbf{q})\phi(\mathbf{k}-\mathbf{p})\phi(\mathbf{k}). \qquad (4.20)
\end{aligned}$$

In the limit $N \to \infty$ we can evaluate the q integral by a saddle-point technique. Call $\mathbf{q}_0$ the saddle point determined by

$$\frac{\nabla F}{F}(\mathbf{q}_0) = i\frac{\mathbf{L}}{N} = i\mathbf{R}. \qquad (4.21)$$

Then we have in this limit

$$\begin{aligned}
\langle V(\mathbf{u}_k)\rangle &= F(\mathbf{q}_0)^{-1} \int d^3u |\psi(\mathbf{u})|^2 V(\mathbf{u}) e^{i\mathbf{q}_0\cdot\mathbf{u}} \qquad (4.22) \\
\langle e^{i\mathbf{Y}\cdot\mathbf{p}_k}\rangle &= F(\mathbf{q}_0)^{-2} \int d^3u \int d^3u' e^{i\mathbf{q}_0\cdot(\mathbf{u}+\mathbf{u}')} \\
&\quad \psi^*(\mathbf{u})\psi^*(\mathbf{u}')\psi(\mathbf{u}+\mathbf{Y})\psi(\mathbf{u}'-\mathbf{Y}) \qquad (4.23) \\
\langle |\mathbf{p}_k|\rangle &= F(\mathbf{q}_0)^{-2} \int d^3p |\mathbf{p}| \int \frac{d^3Y}{(2\pi)^3} e^{-i\mathbf{Y}\cdot\mathbf{p}} \int d^3u \int d^3u' e^{i\mathbf{q}_0\cdot(\mathbf{u}+\mathbf{u}')} \\
&\quad \psi^*(\mathbf{u})\psi^*(\mathbf{u}')\psi(\mathbf{u}+\mathbf{Y})\psi(\mathbf{u}'-\mathbf{Y}) \\
&= F(\mathbf{q}_0)^{-2} \int d^3p |\mathbf{p}| \int d^3k \\
&\quad \phi^*(\mathbf{k}+\mathbf{q}_0-\mathbf{p})\phi^*(\mathbf{k}+\mathbf{q}_0)\phi(\mathbf{k}-\mathbf{p})\phi(\mathbf{k}). \qquad (4.24)
\end{aligned}$$

Note that in our product ansatz for the trial wavefunction, all dependence on the gluon number in the chain, indicated by the subscript k, has disappeared in $V(\mathbf{u}_k)$ and $|\mathbf{p}_k|$. The total energy of the trial state is then simply [5]

$$\mathcal{E} = N(\langle|\mathbf{p}_k|\rangle + \langle V(\mathbf{u}_k)\rangle). \qquad (4.25)$$

[5] We disregard the distinction between V and V_{qg} at the ends of the chain, on the grounds that this is unimportant for $\mathcal{E}/L$ at large L.

Once this quantity is minimized as a function of the variational parameters in the wavefunction, our estimate for the QCD string tension is the energy per unit length

$$\sigma = \frac{\mathcal{E}}{L} = \frac{\langle|\mathbf{p}_k|\rangle + \langle V(\mathbf{u}_k)\rangle}{R}. \tag{4.26}$$

To go further, we need to choose a definite $\psi(\mathbf{u})$, for example the gaussian

$$\psi(\mathbf{u}) = e^{-u^2/2r^2}. \tag{4.27}$$

In this case there are two variational parameters. One of them is the parameter r in the above gaussian. The other is the number N of gluons in a chain between heavy sources separated by distance L or, equivalently, the distance $R = L/N$. For the gaussian wavepacket we find

$$\begin{aligned} \phi(\mathbf{k}) &= r^3 e^{-k^2r^2/2} \\ F(\mathbf{q}) &= (\pi r^2)^{3/2} e^{-q^2r^2/4}, \qquad \mathbf{q}_0 = -\frac{2i\mathbf{R}}{r^2} \\ \langle V\rangle &= \frac{1}{Rr\sqrt{\pi}}\int_0^\infty u du V(u)\left(e^{-(u-R)^2/r^2} - e^{-(u+R)^2/r^2}\right) \end{aligned} \tag{4.28}$$

for central $V(\mathbf{u})$. In particular we find

$$\left\langle \frac{1}{|\mathbf{u}|}\right\rangle = \frac{1}{Rr\sqrt{\pi}}\int_0^\infty du\left(e^{-(u-R)^2/r^2} - e^{-(u+R)^2/r^2}\right) = \frac{\mathrm{erf}(R/r)}{R}. \tag{4.29}$$

To evaluate the kinetic energy, note that

$$\begin{aligned} &(\mathbf{k}+\mathbf{q}_0-\mathbf{p})^2 + (\mathbf{k}+\mathbf{q}_0)^2 + (\mathbf{k}-\mathbf{p})^2 + (\mathbf{k})^2 \\ &\quad = 4\mathbf{k}^2 + 4\mathbf{k}\cdot(\mathbf{q}_0-\mathbf{p}) + (\mathbf{q}_0-\mathbf{p})^2 + \mathbf{q}_0^2 + \mathbf{p}^2 \\ &\quad = (2\mathbf{k}+\mathbf{q}_0-\mathbf{p})^2 + \mathbf{q}_0^2 + \mathbf{p}^2 \end{aligned} \tag{4.30}$$

so the $\mathbf{k}$ integral is a simple Gaussian leading to

$$\langle|\mathbf{p}_k|\rangle = \left(\frac{r^2}{2\pi}\right)^{3/2}\int d^3p|\mathbf{p}|e^{-r^2p^2/2} = \frac{1}{r}\sqrt{\frac{8}{\pi}}. \tag{4.31}$$

Thus the product ansatz with gaussian wavepackets leads to

$$\frac{\mathcal{E}}{L} = \frac{1}{rR}\sqrt{\frac{8}{\pi}} + \frac{1}{R}\langle V\rangle. \tag{4.32}$$

In particular, taking for V the instantaneous Coulomb potential

$$V(u) = -C_F\frac{\alpha_s}{|u|} \tag{4.33}$$

we have

$$\frac{\mathcal{E}}{L} = \frac{1}{rR}\sqrt{\frac{8}{\pi}} - \frac{C_F\alpha_s}{R^2}\mathrm{erf}\left(\frac{R}{r}\right). \tag{4.34}$$

- **String Wavefunction Ansatz**

The second type of trial state we consider here is the ground state wavefunction of a discretized string. Strings with discrete degrees of freedom were studied some time ago in ref. [16], and we will borrow some results directly from that reference.

The ground state of the discrete string is the state annihilated by all the lowering operators for string modes

$$a^i_m \Psi(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_{N-1}) = 0 \qquad \begin{cases} m = 1, 2, .., N-1 \\ i = 1, 2, 3 \end{cases} \tag{4.35}$$

where the position and momentum of the k-th gluon are related to the string modes via

$$\mathbf{x}_k = \frac{\mathbf{L}}{N}k + \sqrt{\frac{2}{NT_0}} \sum_{m=1}^{N-1} \frac{1}{\sqrt{2\omega_m}} \left(\mathbf{a}_m + \mathbf{a}^\dagger_m\right) \sin\left(\frac{m\pi}{N}k\right) \tag{4.36}$$

$$\mathbf{p}_k = -i\sqrt{\frac{2T_0}{N}} \sum_{m=1}^{N-1} \sqrt{\frac{\omega_m}{2}} \left(\mathbf{a}_m - \mathbf{a}^\dagger_m\right) \sin\left(\frac{m\pi}{N}k\right) \tag{4.37}$$

where

$$\omega_m = 2\sin\frac{m\pi}{2N}$$

is the frequency of the mth mode. Then we have

$$\begin{aligned} \langle 0|e^{i\mathbf{x}\cdot\mathbf{p}_k}|0\rangle &= \exp\left(-\mathbf{x}^2\Delta^k_2\right) \\ \langle 0|\delta(\mathbf{p}-\mathbf{p}_k)|0\rangle &= \left(\frac{1}{4\pi\Delta^k_2}\right)^{3/2} \exp\left(-\frac{\mathbf{p}^2}{4\Delta^k_2}\right) \end{aligned} \tag{4.38}$$

$$\begin{aligned} \langle 0|e^{i\mathbf{p}\cdot\mathbf{u}_k}|0\rangle &= \exp\left(i\mathbf{R}\cdot\mathbf{p} - \mathbf{p}^2\Delta^k_1\right) \\ \langle 0|\delta(\mathbf{u}-\mathbf{u}_k)|0\rangle &= \left(\frac{1}{4\pi\Delta^k_1}\right)^{3/2} \exp\left(-\frac{(\mathbf{u}-\mathbf{R})^2}{4\Delta^k_1}\right) \end{aligned} \tag{4.39}$$

where

$$\begin{aligned} \Delta^k_1 &= \frac{1}{NT_0}\sum_{m=1}^{N-1} \sin\frac{m\pi}{2N}\cos^2\left(\frac{m\pi}{N}\left(k+\frac{1}{2}\right)\right) \\ &= \frac{1}{4NT_0}\left\{\cot\frac{\pi}{4N} + \frac{1}{2}\cot\frac{\pi(4k+3)}{4N} - \frac{1}{2}\cot\frac{\pi(4k+1)}{4N}\right\} \qquad (4.40) \\ &\approx \frac{1}{\pi T_0}\left[1 - \frac{1}{(4k+3)(4k+1)}\right] \to \frac{1}{\pi T_0} \qquad (4.41) \\ \Delta^k_2 &= \frac{T_0}{N}\sum_{m=1}^{N-1}\sin\frac{m\pi}{2N}\sin^2\left(\frac{m\pi}{N}k\right) \to \frac{T_0}{\pi} \\ &= \frac{T_0}{4N}\left\{\cot\frac{\pi}{4N} - \frac{1}{2}\cot\frac{\pi(4k+1)}{4N} + \frac{1}{2}\cot\frac{\pi(4k-1)}{4N}\right\} \qquad (4.42) \\ &\approx \frac{T_0}{\pi}\left[1 + \frac{1}{(4k+1)(4k-1)}\right] \to \frac{T_0}{\pi}. \qquad (4.43) \end{aligned}$$

From these expressions we find, for $V(u)$ proportional to $1/|u|$ as in eq. (4.33),

$$\frac{\mathcal{E}}{L} = \frac{1}{R}\frac{4}{\sqrt{\pi}}\sqrt{\Delta_2} - \frac{C_F\alpha_s}{R^2}\text{erf}\left(\frac{R}{2\sqrt{\Delta_1}}\right). \tag{4.44}$$

Having tried two different types of trial wavefunctions, the question is which leads to a lower ground state energy. The answer is that the string wavefunction is the better of the two, at least for inter-gluon potentials proportional to $1/|\mathbf{u}|$. To see this, simply define $r \equiv 2\sqrt{\Delta_1}$, so that

$$\frac{\mathcal{E}}{L} = \frac{1}{rR}\frac{8}{\sqrt{\pi}}\sqrt{\Delta_1\Delta_2} - \frac{C_F\alpha_s}{R^2}\text{erf}\left(\frac{R}{r}\right) \tag{4.45}$$

and, for the string wavefunction, $\Delta_1\Delta_2 = 1/\pi^2$. Comparison with eq. (4.34) shows that for any values of r and R, the string wavefunction has a slightly lower energy than the relative-coordinate product wavefunction.

4.2 String Tension

Without making any assumptions about the inter-gluon potential, the energy per unit length of the gluon-chain in the string-wavefunction ansatz is

$$\frac{\mathcal{E}}{L} = \frac{1}{R}\left[\frac{8}{\pi^{3/2}}\frac{1}{r} + \langle V(\mathbf{u})\rangle\right]. \tag{4.46}$$

As explained in section 2, the minimal value of this energy per unit quark separation, as a function of the parameters r, R, is the variational estimate of the QCD string tension σ in our quantum-mechanical model. Of course, even the simplified model requires as input the intergluon potential $V(\mathbf{u})$, and to get this interaction energy right one should probably use the variational approach in the full field theory. However, even without knowing the intergluon potential precisely, we can make some rough estimates using knowledge of the three-loop static Coulomb potential as input.

First of all, if $V(u)$ were simply due to tree-level effects, then we would have

$$\langle V(\mathbf{u})\rangle \propto \left\langle \frac{1}{|\mathbf{u}|}\right\rangle. \tag{4.47}$$

Likewise, the running coupling is generally regarded as a function of the *inverse* color charge separation [5,17]. Let us therefore assume that $\langle V\rangle$ depends on the variational parameters r, R only through the VEV

$$\frac{1}{s} \equiv \left\langle \frac{1}{|\mathbf{u}|}\right\rangle = \frac{1}{R}\text{erf}\left(\frac{R}{r}\right) \tag{4.48}$$

so that we may write

$$\begin{aligned} V(s) &= \langle V(\mathbf{u})\rangle \\ F(s) &\equiv \frac{dV}{ds} \end{aligned} \tag{4.49}$$

where $F(s)$ can be regarded as the magnitude of a semi-classical "force" between gluons.

Minimizing $\mathcal{E}/L$ results in two conditions. The first, obtained from minimizing wrt R, is

$$\frac{1}{R}\left(\frac{8}{\pi^{3/2}}\frac{1}{r} + \langle V(\mathbf{u})\rangle\right) = \frac{\partial}{\partial R}V(s) \tag{4.50}$$

or, in view of (4.46),

$$\frac{\mathcal{E}}{L} = \frac{\partial}{\partial R}V(s) = \frac{\partial s}{\partial R}F(s). \tag{4.51}$$

This equation has a simple physical interpretation. On the one hand, the QCD string tension in the gluon chain model is simply the energy of the chain per unit quark-antiquark separation, i.e. $\mathcal{E}/L$ on the lhs of (4.51). On the other hand, the "tension" in this system should be related to the change in potential energy of the system with respect to small deformations δR in inter-gluon separation; this is the "restoring force" in the system along the quark-antiquark axis. The minimum condition (4.51) equates these two types of expression for the string tension. The second condition for the minimum is obtained from minimizing $\mathcal{E}/L$ wrt r, which gives

$$\frac{8}{\pi^{3/2}}\frac{1}{r^2} = \frac{\partial s}{\partial r}F(s). \tag{4.52}$$

Solving (4.52) for $F(s)$ and inserting into (4.51) gives an expression for the string tension ($\sigma = \mathcal{E}/L$)

$$\sigma = \frac{\partial s}{\partial R}\left(\frac{\partial s}{\partial r}\right)^{-1}\frac{8}{\pi^{3/2}}\frac{1}{r^2} \tag{4.53}$$

where

$$\begin{aligned}
\frac{\partial s}{\partial R} &= \mathrm{erf}^{-1}\left(\frac{R}{r}\right) - \frac{2}{\sqrt{\pi}}\frac{R}{r}\mathrm{erf}^{-2}(R/r)\exp(-R^2/r^2) \\
\frac{\partial s}{\partial r} &= \frac{2}{\sqrt{\pi}}\frac{R^2}{r^2}\mathrm{erf}^{-2}(R/r)\exp(-R^2/r^2).
\end{aligned} \tag{4.54}$$

Given $F(s)$, we could use (4.51) and (4.52) to determine R and r, and then (4.53) would give the string tension. Of course, this procedure assumes that eqs. (4.51) and (4.52) have a solution at finite r, R. For $F(s)$ determined at tree level, without taking account of loop corrections or the running coupling constant, eq. (4.52) at fixed R will typically have a solution at $r = \infty$ (small α_s), or $r \propto R$ (large α_s). But then (4.51) will only be solved at $R = \infty$ or $R = 0$ respectively. Clearly, incorporating the running coupling is a crucial ingredient to a self-consistent chain solution. Even though the detailed strengths and shapes will be different, we can expect the RG improved forces (with ψ truncated at one, two or three loops), corresponding to V and V_{qg} in attractive channels, to have the same qualitative behavior as Fig. 4: an initial decrease to a nonzero minimum, and a blowup at finite R due to the

Landau singularity. This qualitative behavior is sufficient to guarantee a consistent chain solution. Quantitative accuracy relies on the chance that the effective coupling required for the solution is not too large.

At tree-level, the inter-gluon force $F(s)$ has three components: the Coulomb force, the magnetic exchange force, and the contact interaction. Some previous variational [3] and bag [22] calculations suggest that the magnetic exchange force between neighboring gluons with total spin 0 will be attractive, and roughly equal in magnitude to the Coulomb force, while the contact interaction is repulsive and (for glueball wavefunctions) about 60% as large as the Coulomb force. For an accurate estimate of all contributions, we must actually carry out the calculation for the gluon-chain by the field theory method outlined above. However, based on the work in refs. [3, 22], we expect that the sum of contributions results in a (renormalization-group improved) intergluon force of the form

$$F(s) \approx \kappa C_F \frac{\alpha_s(s)}{s^2} \tag{4.55}$$

where κ is a number which is probably between one and two.[6]

Even without knowing $F(s)$ precisely, we can still use the three-loop perturbative results for the static potential to make some educated guesses about the size of R and r, and arrive at a "ballpark estimate" for σ. At the end of section 3, we argued for $R = R_m/2 \approx 0.3R_0$, on the grounds that a constituent gluon should appear between the quark-antiquark pair when they are $L = R_m$ apart, since this is where the perturbative expansion around a zero constituent gluon state seems to be breaking down. For the r parameter, we just note that it is not reasonable to have s beyond the Landau singularity, because before that distance we would expect more constituent gluons to have appeared and reduced the effective charge separation. On the other hand, the bound state condition will require a running coupling $\alpha_s(s)$ which is $O(1)$, otherwise the gluon kinetic energy dominates, and there is no binding. So s cannot be much less than the Landau singularity. A guess which is perhaps not so far off is $s = R_m = 0.6R_0$ which, when combined with $R = R_m/2 = 0.3R_0$, implies $r = 0.63R_0$. These particular guesses happen to land close to the right answer, i.e.

$$\sigma = \frac{1.26}{R_0^2} \tag{4.56}$$

where the accepted answer for the QCD string tension is $\sigma = (430 \text{ Mev})^2 = 1.18/R_0^2$. This rather close agreement is probably fortuitous, since we are only making guesses for r and R, but it does show that our numbers are in the right ballpark, and there is some hope that the eventual calculation of σ will be in reasonable agreement with the phenomenological value. It is also worth noting that with these parameters, the

[6] At loop level, there will also be corrections to the kinetic energy term, which tend to increase the size of that term [3].

spread in the wavepacket of each constituent gluon (set by s), is roughly twice the size of the average intergluon separation along the quark-antiquark axis which is $R \approx 0.3R_0$. Thus the QCD string is rather thick even before roughening effects come into play.

Continuing a little further, we note that the above choice of parameters R, r implies

$$\begin{aligned} \alpha(s) &= \frac{1}{\kappa C_F}\frac{4}{\pi}e^{R^2/r^2} \\ &= \frac{1.60}{\kappa C_F} \end{aligned} \tag{4.57}$$

If we use the 3-loop running coupling in ref. [5], and $N_c = 3$, then $s \approx 0.6R_0$ would require $\kappa \approx 2$, which is probably a little large. But none of these numbers should be taken very seriously at this stage, and a full field-theoretic calculation using the string wavefunction ansatz is clearly required, before we can draw any quantitative conclusions.

4.3 Roughening and the Lüscher Term

The phenomenon of roughening, *i.e.* the logarithmic growth with L of the transverse size of the gluon chain, requires the long range correlations contained in the string wave function ansatz. For example, the product ansatz would not display this effect. Indeed it is easy to calculate

$$\langle 0|\mathbf{x}_{k\perp}^2|0\rangle = \frac{D-2}{2NT_0}\sum_{m=1}^{N-1}\frac{\sin^2(m\pi k/N)}{\sin(m\pi/2N)} \tag{4.58}$$

$$\sim \frac{D-2}{2\pi T_0}\ln N = \frac{r^2(D-2)}{8}\ln\frac{L}{R}. \tag{4.59}$$

These same correlations also produce certain finite size effects, that are sub-dominant for $L/R \to \infty$, in the trial energy. These come from explicit $1/N$ corrections and from the fact that $\Delta_{1,2}^k$ have mild k dependence.

To estimate these effects we first note some sums:

$$\sum_{k=1}^{N-1}\Delta_2^k = \frac{T_0}{2}\sum_{m=1}^{N-1}\sin\frac{m\pi}{2N} \sim \frac{T_0}{2}\left[\frac{2N}{\pi} - \frac{1}{2} - \frac{\pi}{24N} + \cdots\right] \tag{4.60}$$

$$\sum_{k=1}^{N}\Delta_1^k = \frac{1}{2T_0}\sum_{m=1}^{N-1}\sin\frac{m\pi}{2N} \sim \frac{1}{2T_0}\left[\frac{2N}{\pi} - \frac{1}{2} - \frac{\pi}{24N} + \cdots\right]. \tag{4.61}$$

It is of interest to estimate the finite size contributions to the energy with our string trial wave function. For this purpose the detailed form of the kinetic and potential energy is largely irrelevant and it is more efficient to use an arbitrary kinetic $K(\mathbf{p})$

and potential $V(\mathbf{r})$ energy in our formulas. We then have, after changing integration variables to dimensionless ones:

$$\langle K(\mathbf{p}_k)\rangle = \frac{1}{\pi^{3/2}}\int d^3pe^{-\mathbf{p}^2}K\left(2\mathbf{p}\sqrt{\Delta_2^k}\right) \tag{4.62}$$

$$\langle V(\mathbf{u}_k)\rangle = \frac{1}{\pi^{3/2}}\int d^3ue^{-\mathbf{u}^2}V\left(\mathbf{R}+2\mathbf{u}\sqrt{\Delta_1^k}\right). \tag{4.63}$$

Then calling the limiting values of $\Delta_{1,2}^k \to \Delta_{1,2}$, we can expand the energy to first order in $\Delta_{1,2}^k - \Delta_{1,2}$, and do the sums over k.

$$\begin{aligned}
\mathcal{E} &= \sum_{k=1}^{N-1}\langle K(\mathbf{p}_k)\rangle + \sum_{k=1}^{N}\langle V(\mathbf{u}_k)\rangle \\
&= (N-1)\frac{1}{\pi^{3/2}}\int d^3p\ e^{-\mathbf{p}^2}K\left(2\mathbf{p}\sqrt{\Delta_2}\right) + N\frac{1}{\pi^{3/2}}\int d^3u\ e^{-\mathbf{u}^2}V\left(\mathbf{R}+2\mathbf{u}\sqrt{\Delta_1}\right) \\
&\quad + \sum_{k=1}^{N-1}\frac{\Delta_2^k-\Delta_2}{\Delta_2}\frac{\sqrt{\Delta_2}}{\pi^{3/2}}\int d^3p\ e^{-\mathbf{p}^2}\mathbf{p}\cdot\nabla K \\
&\quad + \sum_{k=1}^{N}\frac{\Delta_1^k-\Delta_1}{\Delta_1}\frac{\sqrt{\Delta_1}}{\pi^{3/2}}\int d^3u\ e^{-\mathbf{u}^2}\mathbf{u}\cdot\nabla V \qquad (4.64)\\
&\sim (N-1)\frac{1}{\pi^{3/2}}\int d^3p\ e^{-\mathbf{p}^2}K\left(2\mathbf{p}\sqrt{\Delta_2}\right) + N\frac{1}{\pi^{3/2}}\int d^3u\ e^{-\mathbf{u}^2}V\left(\mathbf{R}+2\mathbf{u}\sqrt{\Delta_1}\right) \\
&\quad + \left[1-\frac{\pi}{4}-\frac{\pi^2}{48N}\right]\frac{\sqrt{\Delta_2}}{\pi^{3/2}}\int d^3p\ e^{-\mathbf{p}^2}\mathbf{p}\cdot\nabla K \\
&\quad + \left[-\frac{\pi}{4}-\frac{\pi^2}{48N}\right]\frac{\sqrt{\Delta_1}}{\pi^{3/2}}\int d^3u\ e^{-\mathbf{u}^2}\mathbf{u}\cdot\nabla V.
\end{aligned}$$

In these expressions ∇ is always the gradient with respect to the *argument* of the function that follows it. Next we minimize $\mathcal{E}/L$ at fixed L with respect to the parameters T_0 and $R = L/N$, and obtain

$$0 = \frac{\sqrt{\Delta_2}}{\pi^{3/2}}\int d^3p\ e^{-\mathbf{p}^2}\mathbf{p}\cdot\nabla K - \frac{\sqrt{\Delta_1}}{\pi^{3/2}}\int d^3u\ e^{-\mathbf{u}^2}\mathbf{u}\cdot\nabla V \tag{4.65}$$

$$\begin{aligned}
0 &= -\frac{1}{R^2}\left[\int d^3p\ e^{-\mathbf{p}^2}K\left(2\mathbf{p}\sqrt{\Delta_2}\right) + \int d^3u\ e^{-\mathbf{u}^2}V\left(\mathbf{R}+2\mathbf{u}\sqrt{\Delta_1}\right)\right] \\
&\qquad + \frac{1}{R}\int d^3u\ e^{-\mathbf{u}^2}\frac{\mathbf{R}}{R}\cdot\nabla V\left(\mathbf{R}+2\mathbf{u}\sqrt{\Delta_1}\right) \qquad (4.66)
\end{aligned}$$

These two equations determine T_0 and R. When both are satisfied, we can simplify our expression for the variational energy:

$$\mathcal{E} = NR\frac{1}{\pi^{3/2}}\int d^3u\ e^{-\mathbf{u}^2}\frac{\mathbf{R}}{R}\cdot\nabla V\left(\mathbf{R}+2\mathbf{u}\sqrt{\Delta_1}\right) - \frac{1}{\pi^{3/2}}\int d^3p\ e^{-\mathbf{p}^2}K\left(2\mathbf{p}\sqrt{\Delta_2}\right)$$

$$+\left[1-\frac{\pi}{2}-\frac{\pi^2}{24N}\right]\frac{\sqrt{\Delta_2}}{\pi^{3/2}}\int d^3p\ e^{-\mathbf{p}^2}\mathbf{p}\cdot\nabla K(2\mathbf{p}\sqrt{\Delta_2}) \tag{4.67}$$

$$=\frac{L}{\pi^{3/2}}\int d^3u\ e^{-\mathbf{u}^2}\frac{\mathbf{R}}{R}\cdot\nabla V\left(\mathbf{R}+2\mathbf{u}\sqrt{\Delta_1}\right)-\frac{\pi R}{24L}\frac{\sqrt{\Delta_2}}{\sqrt{\pi}}\int d^3p\ e^{-\mathbf{p}^2}\mathbf{p}\cdot\nabla K(2\mathbf{p}\sqrt{\Delta_2})$$
$$+\left[1-\frac{\pi}{2}\right]\frac{\sqrt{\Delta_2}}{\pi^{3/2}}\int d^3p\ e^{-\mathbf{p}^2}\mathbf{p}\cdot\nabla K-\int\frac{d^3p}{\pi^{3/2}}\ e^{-\mathbf{p}^2}K\left(2\mathbf{p}\sqrt{\Delta_2}\right). \tag{4.68}$$

The first two terms are the linear potential and Lüscher terms respectively. The last two terms are independent of L and can be absorbed in the source energy. For $K(\mathbf{p})=|\mathbf{p}|$, as appropriate for massless gluons, the integrals in the last three terms can be immediately carried out, yielding

$$\mathcal{E}=\frac{L}{\pi^{3/2}}\int d^3u\ e^{-\mathbf{u}^2}\frac{\mathbf{R}}{R}\cdot\nabla V\left(\mathbf{R}+2\mathbf{u}\sqrt{\Delta_1}\right)$$
$$-\frac{\pi}{24L}\frac{4R}{r\sqrt{\pi}}-\left[1+\frac{\pi}{2}\right]\frac{4}{r\pi^{3/2}}. \tag{4.69}$$

For the bosonic string the factor multiplying $\pi/24L$ in the Luscher term would be $D-2=2$ in 4 dimensional space time. With our estimate above of $R\approx r/2$ this coefficient is estimated to be $2/\sqrt{\pi}\approx 1.13$. But these numbers are far too preliminary to arouse disappointment in such a discrepancy, especially since it is not even settled that the bosonic string result is correct for QCD.

5. Conclusions

We have developed our picture of a gluon chain into a viable calculational framework for the physics of quark confinement which stays close to perturbative ideas. The crucial observation is that the RG improved force law shows behavior which can be interpreted as over-confinement. This shows that perturbative physics might contain the germ of quark confinement which, in combination with the gluon chain saturation mechanism, leads to a concrete proposal for detailed calculations.

We sketched a field theoretic variational approach to the gluon chain wave function. For a preliminary estimate of the the string tension and other features of the QCD string, we studied a simplified model, which replaced field theoretic exchange interactions with a quantum mechanical potential. This exercise shows how the chain model incorporates the important effects of roughening and the Luscher term in the quark potential. In addition, the numerical estimate for the string tension was certainly in the right ballpark.

The next stage of this project is to redo the variational treatment completely in the context of field theory. One issue left unresolved is whether the RG improvement of the interaction between neighbor gluons on the chain shows the same qualitative behavior as that of the RG improved static force. If it does, the consistency of our physical picture will be confirmed. Whether the detailed quantitative predictions of our model will then turn out as accurate as we hope − say to 15 or 20 per cent − is not yet clear.

We should finally note that the gluon chain model, which is formulated in terms of particle-like excitations, is not opposed to a description of quark confinement in terms of some special class of field configurations, e.g. center vortices or monopoles, which dominate the QCD vacuum. Instead, these approaches should be seen as complementing one another. The center dependence of the asymptotic string tension provides a good example. In terms of particle (gluon) excitations, we can easily understand that this dependence is due to string-breaking by particle production, which results in color screening of the higher-representation heavy sources by constituent gluons. In the gluon chain model in particular, the process is illustrated in Fig. 3. On the other hand, if one tries to explain the center dependence of large Wilson loops in terms of field fluctuations affecting the Wilson loop holonomy, then it is clear that the area law must be due exclusively to fluctuations in the loop holonomy among the center elements of the gauge group. This leads (perhaps inevitably) to a picture of the QCD vacuum as being dominated at large scales by center vortex configurations. Thus we have both a particle (chain breaking) and field (center vortex) explanation for the same phemonemon, namely, the N-ality dependence of the asymptotic string tension. These particle/field descriptions need not contradict one another; they are more likely to be dual descriptions of the same underlying physics.

The attractive feature of the gluon chain model is that it offers a simple and concise account of so many features of the QCD confining potential: linearity, Casimir scaling (at large N_c), center dependence, roughening, and the Lüscher term. Beyond that, the model provides a promising framework for quantitative calculation of the string tension and, perhaps, low-lying masses. Whether these calculations are practical, and if so how the results compare with phenomenology, remains to be seen.

Acknowledgments

Our research is supported in part by the U.S. Department of Energy under grant DE-AC03-76SF00098, and in addition by grants DE-FG03-92ER40711 (J.G.), and DE-FG02-97ER41029 (C.B.T). C.B.T. also acknowledges the support of the Miller Institute for Research in Basic Science.

References

[1] C. B. Thorn, Phys. Rev. D19 (1979) 639; D20 (1979) 1435; D20 (1979) 1934.

[2] J. Greensite, Nucl. Phys. B249 (1985) 263; B315 (1989) 663.

[3] J. Greensite and M. Halpern, Nucl. Phys. B271 (1986) 379.

[4] C. B. Thorn, Phys. Rev. D51 (1995) 647.

[5] S. Necco and R. Sommer, hep-ph/0109093.

[6] G. 't Hooft, Nucl. Phys. B190 [FS3] (1981) 455.

[7] L. Del Debbio, M. Faber, J. Greensite, and S. Olejnik, Nucl. Phys. Proc. Suppl. 53 (1997) 141, hep-lat/9607053.

[8] J. Ambjørn, J. Giedt, and J. Greensite, J. High Energy Phys. 02 (2000) 033, hep-lat/9907021.

[9] M. Fukushima, H. Suganuma, H. Toki, Phys. Rev. D60 (1999) 094504, hep-lat/9902005.

[10] J. Steele and J. Negele, Phys. Rev. Lett. 85 (2000) 4207, hep-lat/0007006.

[11] G. 't Hooft, Nucl. Phys. B138 (1978) 1;
G. Mack, in *Recent Developments in Gauge Theories*, edited by G. 't Hooft et al. (Plenum, New York, 1980)

[12] M. Faber, J. Greensite, and S. Olejnik, Phys. Rev. D57 (1998) 2603, hep-lat/9710039.

[13] S. Deldar, Phys. Rev. D62 (2000) 034509;
G. Bali, Phys. Rev. D62 (2000) 114503

[14] D. Zwanziger, Prog. Theor. Phys. Suppl. 131 (1998) 233, hep-th/9802180.

[15] V. Shevchenko and Yu. Simonov, Phys. Rev. Lett. 85 (2000) 1811, hep-ph/0001299.

[16] C. B. Thorn, Nucl. Phys. B263 (1986) 493.

[17] G. Bali, Phys. Lett. B460 (1999) 170, hep-ph/9905387.

[18] G. Grunberg, Phys. Rev. D40 (1989) 680.

[19] R. Sommer, Nucl. Phys. B411 (1994) 839.

[20] C. Bachas, Phys. Rev. D33 (1986) 2723.

[21] J. Greensite and M. Halpern, Nucl. Phys. B259 (1985) 90.

[22] C. B. Thorn, Phys. Rev. D59 (1999) 116011, hep-th/9810066.